7.1

T/F

4. F

7.2

T/F

2. F. $W[\mathbf{x}_2, \mathbf{x}_1](t) = -W[\mathbf{x}_1, \mathbf{x}_2](t)$.

4. F. The Wronskian only needs to be nonzero at a single point in I.

Problems

4. $$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} t & |t| \\ t & t \end{vmatrix} = t\,(t - |t|)$$

   The Wronskian is nonzero at e.g. $t = -1$ since $(-1)\,(-1 - |-1|) = 2$. This tells us that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent on $(-\infty, \infty)$. However, when $t \geq 0$ we get $|t| = t$ so $W[\mathbf{x}_1, \mathbf{x}_2](t) = t(t - t) = 0$. Indeed, when $|t| = t$,

   $$\mathbf{x}_2(t) = \begin{bmatrix} |t| \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = \mathbf{x}_1(t)$$

   so $\mathbf{x}_1$ and $\mathbf{x}_2$ are manifestly dependent on any interval contained in $[0, \infty)$.

7.4

T/F

2. T

6. F. $\lim_{t\to\infty} \mathbf{x}(t) = 0$ if and only if $\mathbf{x}(0)$ is parallel to the eigenvector $\mathbf{v}_1$ that corresponds to λ_1.

Problems

2. Find the eigenvalue/eigenvector pairs of

   $$A = \begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix}.$$

If

$$0 = \begin{vmatrix} -\lambda & -4 \\ 4 & -\lambda \end{vmatrix} = \lambda^2 + 16$$

then $\lambda = \pm 4i$. The eigenvectors corresponding to $\lambda = 4i$ are the null space of $A - 4iI_2$. The row-echelon form of this matrix is

$$\begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

so the eigenvector is $(i, 1)$.

This eigenvalue/eigenvector pair gives the complex solution

$$\mathbf{x}(t) = \begin{bmatrix} ie^{4it} \\ e^{4it} \end{bmatrix} = \begin{bmatrix} i\cos 4t - \sin 4t \\ \cos 4t + i\sin 4t \end{bmatrix}.$$

Two linearly independent real solutions are given by the real and imaginary parts of this complex solution:

$$\mathbf{x}_1(t) = \begin{bmatrix} -\sin 4t \\ \cos 4t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} \cos 4t \\ \sin 4t \end{bmatrix}.$$

Thus, the general solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} -\sin 4t \\ \cos 4t \end{bmatrix} + c_2 \begin{bmatrix} \cos 4t \\ \sin 4t \end{bmatrix} = \begin{bmatrix} -c_1 \sin 4t + c_2 \cos 4t \\ c_1 \cos 4t + c_2 \sin 4t \end{bmatrix}$$

for real numbers c_1 and c_2.

10.

$$A = \begin{bmatrix} 0 & -3 & 1 \\ -2 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

The eigenvalues of A are 2 and -3. Two linearly independent eigenvectors corresponding to 2 are $(1, 0, 2)$ and $(0, 1, 3)$ and an eigenvector for -3 is $(1, 1, 0)$. Each eigenvector turns into a solution to $\mathbf{x}' = A\mathbf{x}$:

$$\mathbf{x}_1(t) = e^{2t} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \mathbf{x}_2(t) = e^{2t} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \ \text{and } \mathbf{x}_3(t) = e^{-3t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Because the eigenvectors are linearly independent, these three solutions are also linearly independent. So the general solution is

$$\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} + c_3 e^{-3t} \\ c_2 e^{2t} + c_3 e^{-3t} \\ (2c_1 + 3c_2) e^{2t} \end{bmatrix}.$$

22. Since A is 2×2 and nondefective, it has two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ corresponding to complex eigenvalues $\lambda_1 = u_1 + iv_1$ and $\lambda_2 = u_2 + iv_2$. (The two eigenvalues may be equal.) We assume that u_1 and u_2 are negative. Since we know the eigenvalues and eigenvectors of A, we can form the general solution to $\mathbf{x}' = A\mathbf{x}$ which is

$$\begin{aligned}\mathbf{x}(t) &= c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 \\ &= c_1 e^{u_1 t}\left(\cos v_1 t + i \sin v_1 t\right)\mathbf{v}_1 + c_2 e^{u_2 t}\left(\cos v_2 t + i \sin v_2 t\right)\mathbf{v}_2.\end{aligned}$$

Now sin and cos take values between -1 and 1, so

$$|\cos v_1 t + i \sin v_1 t| \leq 2 \text{ and } |\cos v_2 t + i \sin v_2 t| \leq 2.$$

Since u_1 and u_2 are negative, $e^{u_1 t}$ and $e^{u_2 t}$ go to 0 as t goes to infinity, so we have

$$\lim_{t\to\infty} c_1 e^{u_1 t}\left(\cos v_1 t + i \sin v_1 t\right) = 0 \text{ and } \lim_{t\to\infty} c_2 e^{u_2 t}\left(\cos v_2 t + i \sin v_2 t\right) = 0$$

and hence

$$\lim_{t\to\infty} \mathbf{x}(t) = 0.$$

7.5

T/F

4. F. The generalized eigenvector $\mathbf{v}_1$ gives rise to the solution $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}_1 + te^{\lambda t}\mathbf{v}_0$.

Problems

2. The matrix

$$A = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}$$

has a single eigenvalue $\lambda = -1$. This eigenvalue has a single eigenvector $\mathbf{v}_0 = (1, -1)$. This eigenvector gives us the solution

$$\mathbf{x}_0(t) = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}.$$

In order to find another linearly independent solution we need to find a generalized eigenvector for the eigenvalue -1. Do this by solving the linear system $(A - \lambda I_2)\mathbf{v}_1 = \mathbf{v}_0$ for $\mathbf{v}_1$ using the value we found for $\mathbf{v}_0$ and its corresponding eigenvalue $\lambda = -1$. Explicitly, the linear system is

$$\begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and one solution is $\mathbf{v}_1 = (-\frac{3}{2}, 1)$. Now we get another solution

$$\mathbf{x}_1(t) = e^{-t}\mathbf{v}_1 + te^{-t}\mathbf{v}_0 = e^{-t}\begin{bmatrix} -\frac{3}{2} + t \\ 1 - t \end{bmatrix}.$$

The general solution we end up with is

$$\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -\frac{3}{2} + t \\ 1 - t \end{bmatrix} = \begin{bmatrix} (c_1 - \frac{3}{2}c_2)e^{-t} + c_2 t e^{-t} \\ (-c_1 + c_2)e^{-t} - c_2 t e^{-t} \end{bmatrix}.$$

4.

$$\det(A - \lambda I_3) = \lambda^2(2 - \lambda)$$

so the eigenvalues of A are 0 and 2. For $\lambda = 2$ we get the eigenvalue $(3, 2, 4)$ and, since the multiplicity is 1, there are no generalized eigenvalues. Looking at $\lambda = 0$, we get a single eigenvector $\mathbf{v}_0 = (1, 0, 2)$. Since 0 has multiplicity 2, we know we can find a generalized eigenvector. Solving $A\mathbf{v}_1 = \mathbf{v}_0$ gives us $\mathbf{v}_1 = (-\frac{1}{2}, 1, 0)$. Now we have three linearly independent solutions

$$\mathbf{x}_0(t) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \mathbf{x}_1(t) = \begin{bmatrix} -\frac{1}{2} + t \\ 1 \\ 2t \end{bmatrix}, \text{ and } \mathbf{x}_2(t) = e^{2t} \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}.$$

Thus, the general solution is

$$\mathbf{x}(t) = \begin{bmatrix} c_1 + (t - \frac{1}{2})c_2 + 3c_3 e^{2t} \\ c_2 + 2c_3 e^{2t} \\ 2c_1 + 2c_2 t + 4c_3 e^{2t} \end{bmatrix}.$$

14. First, find the general solution to $\mathbf{x}' = A\mathbf{x}$. This matrix has a single eigenvalue which is $\lambda = -3$ corresponding to the eigenvector $\mathbf{v}_0 = (1, 1)$. Then solve $(A + 3I_2)\,\mathbf{v}_1 = \mathbf{v}_0$ to get the generalized eigenvector $\mathbf{v}_1 = (2, 1)$. The general solution is

$$\mathbf{x}(t) = e^{-3t} \begin{bmatrix} c_1 + c_2(2 + t) \\ c_1 + c_2(1 + 2) \end{bmatrix}.$$

To finish solving the initial value problem, set $\mathbf{x}(0)$ equal to the given vector $\mathbf{x}_0$. Set $t = 0$ to get

$$\mathbf{x}(0) = \begin{bmatrix} c_1 + 2c_2 \\ c_1 + c_2 \end{bmatrix}$$

and thus we want to solve the linear system

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

The solution to the system is $c_1 = -2$ and $c_2 = 1$, so the solution to the initial value problem is

$$\mathbf{x}(t) = e^{-3t} \begin{bmatrix} t \\ t - 1 \end{bmatrix}.$$