Green's Theorem and Parameterized Surfaces

Math 240 — Calculus III

Summer 2013, Session II

Tuesday, July 2, 2013

Agenda

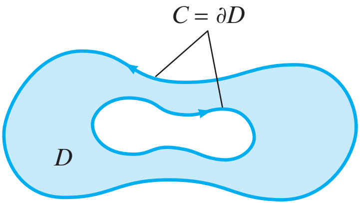
1. Green's Theorem
   - Calculating area
2. Parameterized Surfaces
   - Tangent and normal vectors
   - Tangent planes

Green's theorem

Theorem

Let D be a closed, bounded region in $\mathbb{R}^2$ whose boundary $C = \partial D$ consists of finitely many simple, closed C^1 curves. Orient C so that D is on the left as you traverse C. If $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ is a C^1 vector field on D then

$$\oint_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$



Example

Let $\mathbf{F} = xy\,\mathbf{i} + y^2\mathbf{j}$ and let D be the first quadrant region bounded by the line $y = x$ and the parabola $y = x^2$. Let's calculate $\oint_{\partial D} \mathbf{F} \cdot d\mathbf{s}$ in two ways.

First, we can calculate it directly.
Parameterize ∂D using two pieces:

$$C_1 : \begin{cases} x = t \\ y = t^2 \end{cases} \quad \text{and} \quad C_2 : \begin{cases} x = 1 - t \\ y = 1 - t \end{cases}$$

with t varying from 0 to 1 for each.
The integral is

$$\begin{aligned} \oint_{\partial D} \mathbf{F} \cdot d\mathbf{s} &= \int_{C_1} xy\,dx + y^2 dy + \int_{C_2} xy\,dx + y^2 dy \\ &= \int_0^1 \left(t^3 + 2t^5\right) dt + \int_0^1 2(1-t)^2(-dt) = -\tfrac{1}{12}. \end{aligned}$$

Example

Let $\mathbf{F} = xy\,\mathbf{i} + y^2\mathbf{j}$ and let D be the first quadrant region bounded by the line $y = x$ and the parabola $y = x^2$. Let's calculate $\oint_{\partial D} \mathbf{F} \cdot d\mathbf{s}$ in two ways.

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \int_{C_1} xy\,dx + y^2 dy + \int_{C_2} xy\,dx + y^2 dy$$
$$= \int_0^1 \left(t^3 + 2t^5\right) dt + \int_0^1 2(1-t)^2(-dt) = -\tfrac{1}{12}.$$

Now, let's do the calculation using Green's theorem.

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D \left[\frac{\partial}{\partial x} y^2 - \frac{\partial}{\partial y}(xy)\right] dx\,dy$$
$$= \int_0^1 \int_{x^2}^{x} -x\,dy\,dx = \int_0^1 x^3 - x^2\,dx = -\tfrac{1}{12}.$$

Using Green's theorem to calculate area

Recall that, if D is any plane region, then

$$\text{Area of } D = \int_D 1\, dx\, dy.$$

Thus, if we can find a vector field, $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$, such that $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$, then we can use

$$\oint_{\partial D} M\, dx + N\, dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$
$$= \iint_D 1\, dx\, dy = \text{area of } D$$

to calculate the area of D via a *line integral*!

Here are three such (of many):

$$\mathbf{F} = x\,\mathbf{j}, \quad \mathbf{F} = -y\,\mathbf{i}, \quad \text{or } \mathbf{F} = \tfrac{1}{2}\left(-y\,\mathbf{i} + x\,\mathbf{j}\right).$$

Using Green's theorem to calculate area

Theorem

Suppose D is a plane region to which Green's theorem applies and $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j}$ is a C^1 vector field such that $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ is identically 1 on D. Then the area of D is given by

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{s}$$

where ∂D is oriented as in Green's theorem.

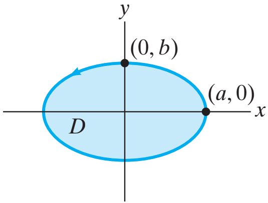
Our three examples from the previous slide yield

$$\text{Area of } D = \begin{cases} \oint_{\partial D} x\,dy \\ \oint_{\partial D} -y\,dx \\ \oint_{\partial D} \frac{1}{2}\,(-y\,dx + x\,dy) \end{cases} .$$

Using Green's theorem to calculate area

Example

We can calculate the area of an ellipse using this method.



The ellipse can be parameterize by

$$\mathbf{x}(t) = (a\cos t, b\sin t), \text{ with } 0 \le t \le 2\pi.$$

Now our theorem tells us that the area of the ellipse is

$$\begin{aligned}\frac{1}{2}\int_{\mathbf{x}} -y\,dx + x\,dy &= \frac{1}{2}\int_0^{2\pi} \left(ab\sin^2 t + ab\cos^2 t\right) dt \\ &= \frac{1}{2}\int_0^{2\pi} ab\,dt = \pi ab.\end{aligned}$$

Parameterized surfaces

Definition

Let D be a plane region that consists of an open set together with some or all of its boundary. A **parameterized surface** in $\mathbb{R}^3$ is a continuous map $\mathbf{X} : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ that is one-to-one on D, except possible along ∂D.

There is a subtle difference between the mapping, $\mathbf{X}$, and its image $\mathbf{X}(D)$, which is just a set of points.

Definition

We refer to $\mathbf{X}(D)$ as the **underlying surface** of $\mathbf{X}$, or the surface **parameterized** by $\mathbf{X}$.

We use bold letters (e.g. $\mathbf{X}$, $\mathbf{Y}$) to represent *parameterized surfaces* and unbold, upper-case letters (e.g. S, T) to represent the *underlying surfaces*.

Parameterized surfaces

Examples

1. The parameterization $\mathbf{X}:\mathbb{R}^2\to\mathbb{R}^3$ defined by
$$\mathbf{X}(s,t)=s(\mathbf{i}-\mathbf{j})+t(\mathbf{i}+2\mathbf{k})+3\mathbf{j}$$
determines a plane.
2. Let $D=[0,2\pi)\times[0,\pi]$ and consider $\mathbf{X}:D\to\mathbb{R}^3$ given by
$$\mathbf{X}(s,t)=(\cos s)(\sin t)\,\mathbf{i}+(\sin s)(\sin t)\,\mathbf{j}+(\cos t)\,\mathbf{k}.$$
3. The equations
$$\begin{cases} x=\cos s \\ y=\sin s \qquad 0\le s\le 2\pi \\ z=t \end{cases}$$
satisfy $x^2+y^2=1$, so they parameterize a cylinder.

Tangent and normal vectors

Definition

Given a parameterization $\mathbf{X}(s,t) = (x(s,t), y(s,t), z(s,t))$, the **tangent vector with respect to** s is

$$\mathbf{T}_s = \frac{\partial \mathbf{X}}{\partial s} = \frac{\partial x}{\partial s}\mathbf{i} + \frac{\partial y}{\partial s}\mathbf{j} + \frac{\partial z}{\partial s}\mathbf{k}.$$

Similarly, the **tangent vector with respect to** t is

$$\mathbf{T}_t = \frac{\partial \mathbf{X}}{\partial t} = \frac{\partial x}{\partial t}\mathbf{i} + \frac{\partial y}{\partial t}\mathbf{j} + \frac{\partial z}{\partial t}\mathbf{k}.$$

The **standard normal vector** is

$$\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t.$$

Tangent and normal vectors

Example

The equation $z^2 = x^2 + y^2$ defines a cone in $\mathbb{R}^3$.
It can be parameterized by

$$\mathbf{X} = s(\cos t)\,\mathbf{i} + s(\sin t)\,\mathbf{j} + s\,\mathbf{k},$$

with t varying from 0 to 2π. We have

$$\mathbf{T}_s = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + \mathbf{k} \text{ and } \mathbf{T}_t = -s(\sin t)\,\mathbf{i} + s(\cos t)\,\mathbf{j}.$$

Therefore,

$$\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 1 \\ -s\sin t & s\cos t & 0 \end{vmatrix}$$
$$= -s(\cos t)\,\mathbf{i} - s(\sin t)\,\mathbf{j} + s\,\mathbf{k}.$$

Tangent planes

Definition

We say that a parameterized surface is **smooth** if the parameterization is C^1 and if it has a nonzero normal vector at every point.

Definition

Let $\mathbf{X}$ be a parameterized surface smooth at the point $\mathbf{X}(s_0, t_0)$. The **tangent plane** to the surface parameterized by $\mathbf{X}$ is the plane that passes through $\mathbf{X}(s_0, t_0)$ and has normal vector $\mathbf{N}(s_0, t_0)$. It is given by the equation

$$\mathbf{N}(s_0, t_0) \cdot (\mathbf{x} - \mathbf{X}(s_0, t_0)) = 0.$$

If $\mathbf{X}(s_0, t_0) = (x_0, y_0, z_0)$ and $\mathbf{N}(s_0, t_0) = A\,\mathbf{i} + B\,\mathbf{j} + C\,\mathbf{k}$ then the equation can also be written

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Tangent planes

Example

Recall the parameterized cone

$$\mathbf{X}(s,t) = s(\cos t)\,\mathbf{i} + s(\sin t)\,\mathbf{j} + s\,\mathbf{k}$$

from the previous example. At the point $(0,1,1) = \mathbf{X}\left(1, \frac{\pi}{2}\right)$, our previous calculation gives us

$$\mathbf{T}_s = (0,1,1),\ \mathbf{T}_t = (-1,0,0),\ \text{and } \mathbf{N} = (0,-1,1).$$

Hence, the equation for the tangent plane is

$$0(x-0) - 1(y-1) + 1(z-1) = 0,$$

which simplifies to

$$z = y.$$