Linear Systems

Math 240 — Calculus III

Summer 2013, Session II

Thursday, July 11, 2013

Agenda

1. Linear systems
    - Solutions to linear systems
    - Differential linear systems
2. Solving linear systems

Linear systems

Definition

An $m \times n$ **system of linear equations** is a

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

in **unknowns** $x_1, \ldots, x_n$ with known values for the **coefficients** a_{ij} and the **constants** b_i.

It can be written more concisely as the vector equation

$$A\mathbf{x} = \mathbf{b},$$

where $A = [a_{ij}]$ is the $m \times n$ **coefficient matrix**, and $\mathbf{b} = [b_i]$ and $\mathbf{x} = [x_i]$ are column vectors called the **constant vector** and the **unknown vector**, respectively.

Linear systems

Example

The following linear system

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 3 \\ 3x_1 - 2x_2 + 9x_3 &= -1 \end{aligned}$$

can also be written

$$\begin{bmatrix} 1 & 1 & -2 \\ 3 & -2 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

or

$$A\mathbf{x} = \mathbf{b}.$$

Solutions to linear systems

Definition

A **solution** to a linear system is a choice of scalar values for the unknowns that satisfies every equation. The collection of all solutions of a particular system is its **solution set**.

Example

The system

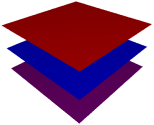
$$\begin{aligned} x_1 + x_2 - 2x_3 &= 3 \\ 3x_1 - 2x_2 + 9x_3 &= -1 \end{aligned}$$

from the previous slide has the solution $x_1 = 1$, $x_2 = 2$, $x_3 = 0$. Its solution set is the line

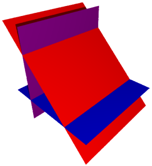
$$\mathbf{x} = (1 - t, 2 + 3t, t) \text{ with } t \in \mathbb{R}.$$

Solutions to linear systems

Geometrically, each equation in an $m \times n$ linear system defines a *hyperplane* in $\mathbb{R}^n$. A solution to the system is a point common to all of the m hyperplanes in the system.

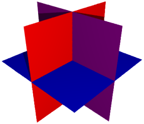


Three parallel planes (no intersection): no solution

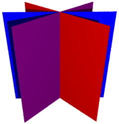


No common intersection: no solution

A system with no solutions is called **inconsistent**.



Planes intersect at a point: a unique solution



Planes intersect in a line: an infinite number of solutions

We say that a system with *at least one* solution is **consistent**.

Linear systems of differential equations

We can write linear differential equations in a similar way. The system

$$\begin{aligned} \frac{dx_1}{dt} &= \quad 3tx_1 + \quad 9x_2 + 6e^t \\ \frac{dx_2}{dt} &= (2+t)x_1 - 7e^{t^2}x_2 + 3e^t \end{aligned}$$

can be written in the matrix form

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x}(t) + \mathbf{b}(t),$$

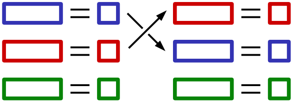
where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ A(t) = \begin{bmatrix} 3t & 9 \\ 2+t & -7e^{t^2} \end{bmatrix}, \text{ and } \mathbf{b}(t) = e^t \begin{bmatrix} 6 \\ 3 \end{bmatrix}.$$

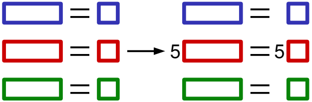
Operations on linear systems

Which operations on a system of linear equations do not change the solution set?

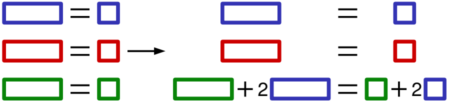
- exchange two equations (or any permutation)

E.g. 

- multiply an equation by a nonzero scalar

E.g. 

- add one equation to another (or add a scalar multiple)

E.g. 

Elementary row operations

When we write a linear system in matrix form, $A\mathbf{x} = \mathbf{b}$, these actions correspond to operations on the rows of the matrix.

Definition

The **augmented matrix** associated to the linear system $A\mathbf{x} = \mathbf{b}$ is the matrix obtained by adding the column vector $\mathbf{b}$ as a new column last column of A. Explicitly, if we write A as a list of columns $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ then the augmented matrix is

$$A^\# = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}.$$

Example

The augmented matrix associated to the linear system

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 2 \\ 2x_1 - 5x_2 + 3x_3 &= 6 \\ 4x_1 + 6x_2 - 7x_3 &= 8 \end{aligned} \quad \text{is} \quad A^\# = \begin{bmatrix} 1 & 2 & 4 & 2 \\ 2 & -5 & 3 & 6 \\ 4 & 6 & -7 & 8 \end{bmatrix}.$$

Elementary row operations

When we write a linear system in matrix form, $A\mathbf{x} = \mathbf{b}$, these actions correspond to operations on the rows of the matrix.

- P_{ij}: **P**ermute the ith and jth rows

$$\begin{bmatrix} 1 & 2 & 4 & 2 \\ 2 & -5 & 3 & 6 \\ 4 & 6 & -7 & 8 \end{bmatrix} \xrightarrow{P_{12}} \begin{bmatrix} 2 & -5 & 3 & 6 \\ 1 & 2 & 4 & 2 \\ 4 & 6 & -7 & 8 \end{bmatrix}$$

- $M_i(k)$: **M**ultiply the ith row by the nonzero scalar k

$$\begin{bmatrix} 1 & 2 & 4 & 2 \\ 2 & -5 & 3 & 6 \\ 4 & 6 & -7 & 8 \end{bmatrix} \xrightarrow{M_2(5)} \begin{bmatrix} 1 & 2 & 4 & 2 \\ 10 & -25 & 15 & 30 \\ 4 & 6 & -7 & 8 \end{bmatrix}$$

- $A_{ij}(k)$: **A**dd k times the ith row to the jth row

$$\begin{bmatrix} 1 & 2 & 4 & 2 \\ 2 & -5 & 3 & 6 \\ 4 & 6 & -7 & 8 \end{bmatrix} \xrightarrow{A_{13}(2)} \begin{bmatrix} 1 & 2 & 4 & 2 \\ 2 & -5 & 3 & 6 \\ 6 & 10 & 1 & 12 \end{bmatrix}$$

Back substitution

What are the “simple” linear systems we want to reduce to?

Example

Consider the following linear system.

$$\begin{aligned} x_1 + x_2 - x_3 &= 4 && (1) \\ x_2 - 3x_3 &= 5 && (2) \\ x_3 &= 2 && (3) \end{aligned}$$

Equation 3 says that $x_3 = 2$. Plugging this into equation 2 yields $x_2 = 5 + 3x_3 = 11$, and then use equation 1 to find $x_1 = 4 - x_2 + x_3 = -5$. Thus, the solution to the linear system is $(-5, 11, 2)$.

This technique in which equations are solved from last to first by substituting in the values known so far is called **back substitution**.

Row-echelon form

What characteristics must the augmented matrix of a linear system have in order to do back substitution?

Definition

A matrix is in **row-echelon form** (REF) if

- any row consisting of all zeros is at the bottom,
- the leftmost non-zero entry in any row is 1 (called a **leading 1**),
- in two consecutive rows, the leading 1 in the lower row appears to the right of the leading 1 in the upper row.

It is in **reduced row-echelon form** (RREF) if in addition

- every leading 1 is the only non-zero entry in its column.

Row-echelon form

Examples

The following matrices are in row-echelon form:

$$\begin{bmatrix}1 & -2 & 3 & 7\\ 0 & 1 & 5 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \quad \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \text{ and } \begin{bmatrix}1 & -7 & 6 & 5 & 9\\ 0 & 0 & 1 & 2 & 5\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

whereas these are not:

$$\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 2\\ 0 & 1 & -1\end{bmatrix} \text{ and } \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}.$$

These matrices are in reduced row-echelon form:

$$\begin{bmatrix}1 & 3 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \quad \begin{bmatrix}1 & -1 & 7 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \quad \begin{bmatrix}1 & 0 & 5 & 3\\ 0 & 1 & 2 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}, \text{ and } I_n.$$

Row-echelon form

Theorem

Any matrix can be reduced to row-echelon form using elementary row operations (P_{ij}, $M_i(k)$, and $A_{ij}(k)$).

Example

Reduce the following matrix to row-echelon form.

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & -1 & 2 & 1 \\ -4 & 6 & -7 & 1 \\ 2 & 0 & 1 & 3 \end{bmatrix} \xrightarrow[\substack{A_{13}(4) \\ A_{14}(-2)}]{\substack{P_{12} \\ A_{12}(-2)}} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & 1 \\ 0 & 2 & 1 & 5 \\ 0 & 2 & -3 & 1 \end{bmatrix}$$

$$\xrightarrow[\substack{A_{23}(-2) \\ A_{24}(-2)}]{A_{32}(-1)} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -6 & -4 \\ 0 & 0 & 13 & 13 \\ 0 & 0 & 9 & 9 \end{bmatrix} \xrightarrow[A_{34}(-9)]{M_3\left(\frac{1}{13}\right)} \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -6 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row-echelon form

Algorithm for reducing a matrix to REF

1. Start with an $m \times n$ matrix, A. If $A = 0$, go to step 7.
2. Determine the leftmost nonzero column (this is called a **pivot column**, and the topmost position in this column is called a **pivot position**, or simply a **pivot**).
3. Use elementary row ops to put a 1 in the pivot position.
4. Use elementary row ops to put 0s below the pivot position.
5. If there are no more nonzero rows below the pivot position go to step 7, otherwise go to step 6.
6. Apply steps 2–5 to the submatrix consisting of the rows that lie below the pivot position.
7. The matrix is in row-echelon form.

Gaussian elimination

Example

In the previous example we obtained the REF augmented matrix

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -6 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{aligned} x_1 - x_2 + 2x_3 &= 1, \\ x_2 - 6x_3 &= -4, \\ x_3 &= 1. \end{aligned}$$

We can do back substitution to find

$$\begin{aligned} x_3 &= 1, \\ x_2 &= -4 + 6x_3 = 2, \\ \text{and } x_1 &= 1 + x_2 - 2x_3 = 1. \end{aligned}$$

So the solution is $(1, 2, 1)$.

The method of solving a linear system by reducing the augmented matrix to REF and then using back substitution is called **Gaussian elimination**.