Vector Spaces

Math 240 — Calculus III

Summer 2013, Session II

Wednesday, July 17, 2013

Agenda

1. Definition
2. Properties of vector spaces
3. Set notation
4. Subspaces

Motivation

We know a lot about Euclidean space. There is a larger class of objects that behave like vectors in $\mathbb{R}^n$. What do all of these objects have in common?

Vector addition: a way of combining two vectors, $\mathbf{u}$ and $\mathbf{v}$, into the single vector $\mathbf{u} + \mathbf{v}$

Scalar multiplication: a way of combining a scalar, k, with a vector, $\mathbf{v}$, to end up with the vector $k\mathbf{v}$

A **vector space** is any set of objects with a notion of addition and scalar multiplication that behave like vectors in $\mathbb{R}^n$.

Examples of vector spaces

Real vector spaces

- $\mathbb{R}^n$ (the archetype of a vector space)
- $\mathbb{R}$ — the set of real numbers
- $M_{m\times n}(\mathbb{R})$ — the set of all $m \times n$ matrices with real entries for fixed m and n. If $m = n$, just write $M_n(\mathbb{R})$.
- P_n — the set of polynomials with real coefficients of degree at most n
- P — the set of all polynomials with real coefficients
- $C^k(I)$ — the set of all real-valued functions on the interval I having k continuous derivatives

Complex vector spaces

- $\mathbb{C}$, $\mathbb{C}^n$
- $M_{m\times n}(\mathbb{C})$

Definition

Definition

A **vector space** consists of a set of scalars, a nonempty set, V, whose elements are called **vectors**, and the operations of vector addition and scalar multiplication satisfying

1. *Closure under addition:* For each pair of vectors $\mathbf{u}$ and $\mathbf{v}$, the sum $\mathbf{u}+\mathbf{v}$ is an element of V.
2. *Closure under scalar multiplication:* For each vector $\mathbf{v}$ and scalar k, the scalar multiple $k\mathbf{v}$ is an element of V.
3. *Commutativity of addition:* For all $\mathbf{u},\mathbf{v}\in V$, we have $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
4. *Associativity of addition:* For all $\mathbf{u},\mathbf{v},\mathbf{w}\in V$, we have $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
5. *Existence of a zero vector:* There is a vector $\mathbf{0}\in V$ satisfying $\mathbf{v}+\mathbf{0}=\mathbf{v}$ for all $\mathbf{v}\in V$.

Definition

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A **vector space** consists of a set of scalars, a nonempty set, V, whose elements are called **vectors**, and the operations of vector addition and scalar multiplication satisfying

6. *Existence of additive inverses:* For each $\mathbf{v} \in V$, there is a vector $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
7. *Unit property:* For all vectors $\mathbf{v}$, we have $1\mathbf{v} = \mathbf{v}$.
8. *Associativity of scalar multiplication:* For all vectors $\mathbf{v}$ and scalars r, s, we have $(rs)\mathbf{v} = r(s\mathbf{v})$.
9. *Distributive property of scalar multiplication over vector addition:* For all vectors $\mathbf{u}$ and $\mathbf{v}$ and scalars r, we have $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$.
10. *Disributive property of scalar multiplication over scalar addition*: For all vectors $\mathbf{v}$ and scalars r and s, we have $(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$.

Example

Let's verify that $M_2(\mathbb{R})$ is a vector space.

1. From the definition of matrix addition, we know that the sum of two 2×2 matrices is also a 2×2 matrix.
2. From the definition of scalar-matrix multiplication, we know that multiplying a 2×2 matrix by a scalar results in a 2×2 matrix.
3. Given two 2×2 matrices

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix},$$

their sum is

$$\begin{aligned} A + B &= \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{bmatrix} \\ &= \begin{bmatrix} b_1 + a_1 & b_2 + a_2 \\ b_3 + a_3 & b_4 + a_4 \end{bmatrix} = B + A. \end{aligned}$$

Example

Let's verify that $M_2(\mathbb{R})$ is a vector space.

4. Given three 2×2 matrices

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix},$$

we have

$$\begin{aligned}(A+B)+C &= \begin{bmatrix} (a_1+b_1)+c_1 & (a_2+b_2)+c_2 \\ (a_3+b_3)+c_3 & (a_4+b_4)+c_4 \end{bmatrix} \\ &= \begin{bmatrix} a_1+(b_1+c_1) & a_2+(b_2+c_2) \\ a_3+(b_3+c_3) & a_4+(b_4+c_4) \end{bmatrix} \\ &= A+(B+C).\end{aligned}$$

5. If $A \in M_2(\mathbb{R})$ then $A + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = A$, so the zero vector in $M_2(\mathbb{R})$ is $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Example

Let's verify that $M_2(\mathbb{R})$ is a vector space.

6. The additive inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $-A = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$ because

$$A + (-A) = \begin{bmatrix} a + (-a) & b + (-b) \\ c + (-c) & d + (-d) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}.$$

7. If A is any matrix, then obviously $1A = A$.
8. Given a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and scalars r and s, we have

$$(rs)A = \begin{bmatrix} (rs)a & (rs)b \\ (rs)c & (rs)d \end{bmatrix} = \begin{bmatrix} r(sa) & r(sb) \\ r(sc) & r(sd) \end{bmatrix}$$
$$= r \begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix} = r(sA).$$

Example

Let's verify that $M_2(\mathbb{R})$ is a vector space.

9. Given matrices $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ and a scalar r, we have

$$r(A+B) = \begin{bmatrix} r(a_1+b_1) & r(a_2+b_2) \\ r(a_3+b_3) & r(a_4+b_4) \end{bmatrix}$$
$$= \begin{bmatrix} ra_1+rb_1 & ra_2+rb_2 \\ ra_3+rb_3 & ra_4+rb_4 \end{bmatrix} = rA+rB.$$

10. Given a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and scalars r and s, we have

$$(r+s)A = \begin{bmatrix} (r+s)a & (r+s)b \\ (r+s)c & (r+s)d \end{bmatrix}$$
$$= \begin{bmatrix} ra+sa & rb+sb \\ rc+sc & rd+sd \end{bmatrix} = rA+sA.$$

Additional properties of vector spaces

The following properties are consequences of the vector space axioms.

- The zero vector is unique.
- $0\mathbf{u} = \mathbf{0}$ for all $\mathbf{u} \in V$.
- $k\mathbf{0} = \mathbf{0}$ for all scalar k.
- The additive inverse of a vector is unique.
- For all $\mathbf{u} \in V$, its additive inverse is given by $-\mathbf{u} = (-1)\mathbf{u}$.
- If k is a scalar and $\mathbf{u} \in V$ such that $k\mathbf{u} = \mathbf{0}$ then either $k = 0$ or $\mathbf{u} = \mathbf{0}$.

Aside: set notation

Definition

Let V be a set. We write the subset of V satisfying some conditions as

$$S = \{\mathbf{v} \in V : \text{conditions on } \mathbf{v}\}.$$

Examples

1. The plane $-3x + 2y + z = 4$ can be written
$$\left\{(x, y, z) \in \mathbb{R}^3 : -3x + 2y + z = 4\right\}.$$
2. The line perpendicular to this plane passing through the point $(1, 0, 0)$ can be written
$$\left\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = (1 - 3r, 2r, r), r \in \mathbb{R}\right\}$$
or
$$\left\{(1 - 3r, 2r, r) \in \mathbb{R}^3 : r \in \mathbb{R}\right\}.$$

Practice problem

If A is an $m \times n$ matrix, verify that

$$V = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$$

is a vector space.

$\mathbb{R}^n$ is a vector space. V is a subset of $\mathbb{R}^n$ and also a vector space. One vector space inside another?!?

What about

$$W = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\}$$

where $\mathbf{b} \neq \mathbf{0}$?

Definition

Definition

Suppose V is a vector space and S is a nonempty subset of V. We say that S is a **subspace** of V if S is a vector space under the same addition and scalar multiplication as V.

Examples

1. Any vector space has two **improper** subspaces: $\{\mathbf{0}\}$ and the vector space itself. Other subspaces are called **proper**.
2. The solution set of a homogeneous linear system is a subspace of $\mathbb{R}^n$. This includes all lines, planes, and hyperplanes through the origin.
3. The set of polynomials in P_2 with no linear term forms a subspace of P_2. In turn, P_2 is a subspace of P.
4. $C^k(I)$ is a subspace of $C^\ell(I)$ for all intervals I and all $k \geq \ell$.

Criteria for subspaces

Checking all 10 axioms for a subspace is a lot of work. Fortunately, it's not necessary.

Theorem

If V is a vector space and S is a nonempty subset of V then S is a subspace of V if and only if S is closed under the addition and scalar multiplication in V.

Remark

Don't forget the "nonempty." It's often quicker and easier to just check that $\mathbf{0} \in S$.

Example

Let S denote the set of real symmetric $n \times n$ matrices. Let's check that S is a subspace of $M_n(\mathbb{R})$.

First, write S as

$$S = \left\{ A \in M_n(\mathbb{R}) : A^T = A \right\}.$$

Now, check three things:

1. $\mathbf{0} \in S$: Obvious.
2. If $A, B \in S$ then $A + B \in S$:

$$(A+B)^T = A^T + B^T = A + B$$

3. If $A \in S$ and k is a scalar then $kA \in S$:

$$(kA)^T = kA^T = kA$$

It's a subspace!

The null space of a matrix

Definition

If A is an $m \times n$ matrix, the solution space of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ is called the **null space** of A.

$$\text{nullspace}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$$

Remarks

- The null space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$.
- The null space of a matrix with complex entries is defined analogously, replacing $\mathbb{R}$ with $\mathbb{C}$.
- As noted before, the solution set of a nonhomogeneous equation ($A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} \neq \mathbf{0}$) is not a subspace since it does not contain $\mathbf{0}$.

Differential equation example

Show that the set of all solutions to the differential equation

$$y'' + a_1(x)y' + a_2(x)y = 0$$

on an interval I is a subspace of $C^2(I)$.

The set of solutions to a homogeneous linear differential equation is called the **solution space**.

Span

Here's another way to construct subspaces:

Definition

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ a set of vectors in a vector space V. A **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is an expression of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n,$$

where $c_1, \ldots, c_n$ are scalars. The **span** of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is the set of all linear combinations of them.

$$\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = \{c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n \in V : c_1, \ldots, c_n \in \mathbb{R}\}$$

Example

The span of a single, nonzero vector is a line through the origin.

$$\operatorname{span}\{\mathbf{v}\} = \{t\mathbf{v} \in V : t \in \mathbb{R}\}$$

Span

Theorem

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors in a vector space V. The span of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a subspace of V.

Question

What's the span of $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (2, -1)$ in $\mathbb{R}^2$?