Constant-Coefficient Linear Differential Equations

Math 240 — Calculus III

Summer 2013, Session II

Monday, August 5, 2013

Agenda

1. Homogeneous constant-coefficient linear differential equations

2. Nonhomogeneous constant-coefficient linear differential equations

Introduction

Last week we found solutions to the linear differential equation

$$y'' + y' - 6y = 0$$

of the form $y(x) = e^{rx}$. In fact, we found all solutions.
This technique will often work. If $y(x) = e^{rx}$ then

$$y'(x) = re^{rx}, \quad y''(x) = r^2e^{rx}, \quad \ldots, \quad y^{(n)}(x) = r^ne^{rx}.$$

So if $r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n = 0$ then $y(x) = e^{rx}$ is a solution to the linear differential equation

$$y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = 0.$$

Today we'll develop this approach more rigorously.

The auxiliary polynomial

Consider the homogeneous linear differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

with *constant coefficients* a_i. Expressed as a linear differential operator, the equation is $P(D)y = 0$, where

$$P(D) = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n.$$

Definition

A linear differential operator with constant coefficients, such as $P(D)$, is called a **polynomial differential operator**. The polynomial

$$P(r) = r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n$$

is called the **auxiliary polynomial**, and the equation $P(r) = 0$ the **auxiliary equation**.

The auxiliary polynomial

Example

The equation $y'' + y' - 6y = 0$ has auxiliary polynomial

$$P(r) = r^2 + r - 6.$$

Examples

Give the auxiliary polynomials for the following equations.

1. $y'' + 2y' - 3y = 0$ $\qquad r^2 + 2r - 3$
2. $(D^2 - 7D + 24)y = 0$ $\qquad r^2 - 7r + 24$
3. $y''' - 2y'' - 4y' + 8y = 0$ $\qquad r^3 - 2r^2 - 4r + 8$

The roots of the auxiliary polynomial will determine the solutions to the differential equation.

Polynomial differential operators commute

The key fact that will allow us to solve constant-coefficient linear differential equations is that polynomial differential operators commute.

Theorem

If $P(D)$ and $Q(D)$ are polynomial differential operators, then

$$P(D)Q(D) = Q(D)P(D).$$

Proof.

For our purposes, it will suffice to consider the case where P and Q are linear. $\mathcal{Q.E.D.}$

Commuting polynomial differential operators will allow us to turn a root of the auxiliary polynomial into a solution to the corresponding differential equation.

Linear polynomial differential operators

In our example,

$$y'' + y' - 6y = 0,$$

with auxiliary polynomial

$$P(r) = r^2 + r - 6,$$

the roots of $P(r)$ are $r = 2$ and $r = -3$. An equivalent statement is that $r - 2$ and $r + 3$ are linear factors of $P(r)$.

The functions $y_1(x) = e^{2x}$ and $y_2(x) = e^{-3x}$ are solutions to

$$y_1' - 2y_1 = 0 \quad \text{and} \quad y_2' + 3y_2 = 0,$$

respectively.

Theorem

The general solution to the linear differential equation

$$y' - ay = 0$$

is $y(x) = ce^{ax}$.

Theorem

Suppose $P(D)$ and $Q(D)$ are polynomial differential operators

$$P(D)y_1 = 0 = Q(D)y_2.$$

If $L = P(D)Q(D)$, then

$$Ly_1 = 0 = Ly_2.$$

Proof.

$$\begin{aligned} P(D)Q(D)y_2 &= P(D)\Big(Q(D)y_2\Big) = P(D)0 = 0 \\ P(D)Q(D)y_1 &= Q(D)P(D)y_1 \\ &= Q(D)\Big(P(D)y_1\Big) = Q(D)0 = 0 \end{aligned}$$

$\mathcal{Q.E.D.}$

Example

The theorem implies that, since

$$(D-2)y_1 = 0 \quad \text{and} \quad (D+3)y_2 = 0,$$

the functions $y_1(x) = e^{2x}$ and $y_2(x) = e^{-3x}$ are solutions to

$$y'' + y' - 6y = (D^2 + D - 6)y = (D-2)(D+3)y = 0.$$

Linear polynomial differential operators

Furthermore, solutions produced from different roots of the auxiliary polynomial are independent.

Example

If $y_1(x) = e^{2x}$ and $y_2(x) = e^{-3x}$, then

$$\begin{aligned} W[y_1, y_2](x) &= \begin{vmatrix} e^{2x} & e^{-3x} \\ 2e^{2x} & -3e^{-3x} \end{vmatrix} \\ &= e^{-x} \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5e^{-x} \neq 0. \end{aligned}$$

Distinct linear factors

If we can factor the auxiliary polynomial into distinct linear factors, then the solutions from each linear factor will combine to form a fundamental set of solutions.

Example

Determine the general solution to $y'' - y' - 2y = 0$.

The auxiliary polynomial is

$$P(r) = r^2 - r - 2 = (r-2)(r+1).$$

Its roots are $r_1 = 2$ and $r_2 = -1$. The functions $y_1(x) = e^{2x}$ and $y_2(x) = e^{-x}$ satisfy

$$(D-2)y_1 = 0 = (D+1)y_2.$$

Therefore, y_1 and y_2 are solutions to the original equation. Since we have 2 solutions to a 2^{nd} degree equation, they constitute a fundamental set of solutions; the general solution is

$$y(x) = c_1 e^{2x} + c_2 e^{-x}.$$

Multiple roots

What can go wrong with this process? The auxiliary polynomial could have a multiple root. In this case, we would get one solution from that root, but not enough to form the general solution. Fortunately, there are more.

Theorem

The differential equation $(D-r)^m y = 0$ has the following m linearly independent solutions:

$$e^{rx},\ xe^{rx},\ x^2e^{rx},\ \ldots,\ x^{m-1}e^{rx}.$$

Proof.

Check it. $\mathcal{Q.E.D.}$

Multiple roots

Example

Determine the general solution to $y'' + 4y' + 4y = 0$.

1. The auxiliary polynomial is $r^2 + 4r + 4$.
2. It has the multiple root $r = -2$.
3. Therefore, two linearly independent solutions are
$$y_1(x) = e^{-2x} \quad \text{and} \quad y_2(x) = xe^{-2x}.$$
4. The general solution is
$$y(x) = e^{-2x}(c_1 + c_2 x).$$

Complex roots

What happens if the auxiliary polynomial has complex roots?
Can we recover real solutions? Yes!

Theorem

If $P(D)y = 0$ is a linear differential equation with real constant coefficients and $(D - r)^m$ is a factor of $P(D)$ with $r = a + bi$ and $b \neq 0$, then

1. *$P(D)$ must also have the factor $(D - \overline{r})^m$,*
2. *this factor contributes the complex solutions*
$$e^{(a \pm bi)x}, \ xe^{(a \pm bi)x}, \ \ldots, \ x^{m-1}e^{(a \pm bi)x},$$
3. *the real and imaginary parts of the complex solutions are linearly independent real solutions*
$$x^k e^{ax} \cos bx \quad \text{and} \quad x^k e^{ax} \sin bx$$
for $k = 0, 1, \ldots, m - 1$.

Complex roots

Example

Determine the general solution to $y'' + 6y' + 25y = 0$.

1. The auxiliary polynomial is $r^2 + 6r + 25$.
2. Its has roots $r = -3 \pm 4i$.
3. Two independent real-valued solutions are
$$y_1(x) = e^{-3x} \cos 4x \quad \text{and} \quad y_2(x) = e^{-3x} \sin 4x.$$
4. The general solution is
$$y(x) = e^{-3x}(c_1 \cos 4x + c_2 \sin 4x).$$

Segue

We have now learned how to solve homogeneous linear differential equations

$$P(D)y = 0$$

when $P(D)$ is a polynomial differential operator. Now we will try to solve nonhomogeneous equations

$$P(D)y = F(x).$$

Recall that the solutions to a nonhomogeneous equation are of the form

$$y(x) = y_c(x) + y_p(x),$$

where y_c is the general solution to the associated homogeneous equation and y_p is a particular solution.

Overview

The technique proceeds from the observation that, if we know a polynomial differential operator $A(D)$ so that

$$A(D)F = 0,$$

then applying $A(D)$ to the nonhomogeneous equation

$$P(D)y = F \tag{1}$$

yields the homogeneous equation

$$A(D)P(D)y = 0. \tag{2}$$

A particular solution to (1) will be a solution to (2) that is not a solution to the associated homogeneous equation $P(D)y = 0$.

Example

Determine the general solution to

$$(D+1)(D-1)y = 16e^{3x}.$$

1. The associated homogeneous equation is $(D+1)(D-1)y = 0$. It has the general solution $y_c(x) = c_1e^x + c_2e^{-x}$.
2. Recognize the nonhomogeneous term $F(x) = 16e^{3x}$ as a solution to the equation $(D-3)y = 0$.
3. The differential equation

   $$(D-3)(D+1)(D-1)y = 0$$

   has the general solution $y(x) = c_1e^x + c_2e^{-x} + c_3e^{3x}$.
4. Pick the **trial solution** $y_p(x) = c_3e^{3x}$. Substituting it into the original equation forces us to choose $c_3 = 2$.
5. Thus, the general solution is

   $$y(x) = y_c(x) + y_p(x) = c_1e^x + c_2e^{-x} + 2e^{3x}.$$

Annihilators and the method of undetermined coefficients

This method for obtaining a particular solution to a nonhomogeneous equation is called the **method of undetermined coefficients** because we pick a trial solution with an unknown coefficient. It can be applied when

1. the differential equation is of the form
$$P(D)y = F(x),$$
where $P(D)$ is a polynomial differential operator,
2. there is another polynomial differential operator $A(D)$ such that
$$A(D)F = 0.$$

A polynomial differential operator $A(D)$ that satisfies $A(D)F = 0$ is called an **annihilator** of F.

Finding annihilators

Functions that can be annihilated by polynomial differential operators are exactly those that can arise as solutions to constant-coefficient homogeneous linear differential equations. We have seen that these functions are

1. $F(x) = cx^k e^{ax}$,
2. $F(x) = cx^k e^{ax} \sin bx$,
3. $F(x) = cx^k e^{ax} \cos bx$,
4. linear combinations of 1–3.

If the nonhomogeneous term is one of 1–3, then it can be annihilated by something of the form $A(D) = (D - r)^{k+1}$, with $r = a$ in 1 and $r = a + bi$ in 2 and 3. Otherwise, annihilators can be found by taking successive derivatives of F and looking for linear dependencies.

Example

Determine the general solution to

$$(D-4)(D+1)y = 16xe^{3x}.$$

1. The general solution to the associated homogeneous equation $(D-4)(D+1)y = 0$ is $y_c(x) = c_1e^{4x} + c_2e^{-x}$.
2. An annihilator for $16xe^{3x}$ is $A(D) = (D-3)^2$.
3. The general solution to $(D-3)^2(D-4)(D+1)y = 0$ includes y_c and the terms c_3e^{3x} and c_4xe^{3x}.
4. Using the trial solution $y_p(x) = c_3e^{3x} + c_4xe^{3x}$, we find the values $c_3 = -3$ and $c_4 = -4$.
5. The general solution is

$$y(x) = y_c(x) + y_p(x) = c_1e^{4x} + c_2e^{-x} - 3e^{3x} - 4xe^{3x}.$$

Example

Determine the general solution to

$$(D-2)y = 3\cos x + 4\sin x.$$

1. The associated homogeneous equation, $(D-2)y = 0$, has the general solution $y_c(x) = c_1 e^{2x}$.
2. Look for linear dependencies among derivatives of $F(x) = 3\cos x + 4\sin x$. Discover the annihilator $A(D) = D^2 + 1$.
3. The general solution to $(D^2+1)(D-2)y = 0$ includes y_c and the additional terms $c_2 \cos x + c_3 \sin x$.
4. Using the trial solution $y_p(x) = c_2 \cos x + c_3 \sin x$, we obtain values $c_2 = -2$ and $c_3 = -1$.
5. The general solution is

$$y(x) = c_1 e^{2x} - 2\cos x - \sin x.$$