Matrices

Math 240 — Calculus III

Summer 2015, Session II

Thursday, July 2, 2015

Agenda

1. Things you should already know
2. Matrices
   - Matrix functions
3. Matrix algebra
   - Matrix function algebra
4. Some definitions

Vectors in $\mathbb{R}^n$

Definition

A **vector** in $\mathbb{R}^n$ consists of a list of n real numbers.

Example

The list $(5, \frac{2}{3}, e, -3)$ is a vector in $\mathbb{R}^4$.

We can add vectors and multiply them by scalars.

Examples

$$(4, -7, 2) + (-1, 7, 3) = (3, 0, 5)$$

$$-3(4, -7, 2) = (-12, 21, -6)$$

Vector functions

Definition

A **path** in $\mathbb{R}^n$ is a vector whose entries are functions of a single variable. It can also be seen as a function of a single variable whose output is a vector in $\mathbb{R}^n$. For this reason, it can also be called a **vector-valued function** (or simply a **vector function**).

Example

$\mathbf{x}(t) = \left(\sin 4t, \frac{e^t}{t}\right)$ is a vector function, $\mathbf{x} : \mathbb{R} \to \mathbb{R}^2$. And we can take its derivative:

$$\frac{d\mathbf{x}}{dt} = \left(4\cos 4t, \frac{t-1}{t^2}e^t\right).$$

Matrices

Definition

An $m \times n$ **matrix** is a rectangular array of numbers arranged in m horizontal rows and n vertical columns. These numbers are called the **entries** or **elements** of the matrix.

Example

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

is an $m \times n$ matrix. It can be written more succinctly as $A = [a_{ij}]$.

Two matrices are equal if they have the same size (identical numbers of rows and columns) and the same entries.

Row and column vectors

Definition

A $1 \times n$ matrix is called a **row** n**-vector**, or simply a **row vector**. An $n \times 1$ matrix is called a **column** n**-vector**, or a **column vector**. The elements of a such a vector are its **components**.

Examples

1. The matrix $\mathbf{a} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{5} & \frac{4}{7} \end{bmatrix}$ is a row 3-vector.
2. $\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 4 \end{bmatrix}$ is a column 4-vector.

Row and column vectors

Any matrix can be written as a list of row or column vectors.

Example

The matrix

$$A = \begin{bmatrix} -2 & 1 & 3 & 4 \\ 1 & 2 & 1 & 1 \\ 3 & -1 & 2 & 5 \end{bmatrix}$$

has three row 4-vectors:

$$\mathbf{a}_1 = \begin{bmatrix} -2 & 1 & 3 & 4 \end{bmatrix},$$
$$\mathbf{a}_2 = \begin{bmatrix} 1 & 2 & 1 & 1 \end{bmatrix}, \text{ and}$$
$$\mathbf{a}_3 = \begin{bmatrix} 3 & -1 & 2 & 5 \end{bmatrix}$$

and we can write

$$A = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix}.$$

Row and column vectors

Any matrix can be written as a list of row or column vectors.

Example

The matrix

$$A = \begin{bmatrix} -2 & 1 & 3 & 4 \\ 1 & 2 & 1 & 1 \\ 3 & -1 & 2 & 5 \end{bmatrix}$$

has four column 3-vectors:

$$\mathbf{b}_1 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \ \mathbf{b}_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \text{ and } \mathbf{b}_4 = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}$$

and we can write

$$A = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{bmatrix}.$$

Matrix functions

Definition

A **matrix function** is like a matrix, but replaces numbers with functions of a single real variable. **Column vector functions** and **row vector functions** are analogously defined.

Example

$A(t)$ is a 2×3 matrix function:

$$A(t) = \begin{bmatrix} t^3 & t - \cos t & \frac{5}{t} \\ e^{t^2} & \ln(t+1) & te^t \end{bmatrix}.$$

The matrix function is only defined for values of t such that *all* elements are defined. In this example, $A(t)$ is defined for values of t such that $t \neq 0$ *and* $t + 1 > 0$.

Matrix addition

Definition

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices with the *same dimensions*, their sum is

$$A + B = [a_{ij} + b_{ij}].$$

Similarly, their difference is

$$A - B = [a_{ij} - b_{ij}].$$

Example

We have

$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & -5 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 5 \\ -5 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 8 \\ -1 & -3 & 7 \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & -5 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 5 \\ -5 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -2 \\ 9 & -7 & -7 \end{bmatrix}.$$

Scalar multiplication

A **scalar** is a real or complex number, as opposed to a vector or matrix.

Definition

If A is a matrix and s a scalar, then the product of s with A is the matrix obtained by multiplying every element of A by s. Symbolically, if $A = [a_{ij}]$ then $sA = [sa_{ij}]$.

Examples

If $A = \begin{bmatrix} 2 & -1 \\ 4 & 6 \end{bmatrix}$ then $5A = \begin{bmatrix} 10 & -5 \\ 20 & 30 \end{bmatrix}$.

If A and B are matrices with the *same dimensions* then

$$A - B = A + (-1)B.$$

Matrices behave like you expect

Matrix addition, subtraction, and scalar multiplication have familiar properties:

- $A + B = B + A$
- $A + (B + C) = (A + B) + C$
- $1A = A$
- $s(A + B) = sA + sB$
- $(s + t)A = sA + tA$
- $s(tA) = (st)A = (ts)A = t(sA)$

$$\mathbf{0} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}$$

- $A + \mathbf{0} = A$
- $A - A = \mathbf{0}$
- $0A = \mathbf{0}$

. . . but matrix multiplication does not!

Matrix multiplication

Example

$P =$

	widget A	widget B
sprockets	5	6
belts	2	5
propellers	3	0

$O =$

	customer 1	customer 2
widget A	2	0
widget B	1	3

$PO =$

	customer 1	customer 2
sprockets	16	18
belts	9	15
propellers	6	0

Matrix multiplication

Definition

Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{jk}]$ be an $n \times p$ matrix. Their product is the $m \times p$ matrix

$$AB = [c_{ik}] \text{ where } c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}.$$

If write write A as a matrix of rows and B as a matrix of columns,

$$A = \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{bmatrix} \text{ and } B = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_p \end{bmatrix},$$

then we can express their product using the vector dot product

$$AB = [\mathbf{a}_i \cdot \mathbf{b}_k].$$

Other perspectives

Earlier, we saw the matrix-column vector product

$$\begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_m \\ | & | & & | \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = b_1\mathbf{a}_1 + b_2\mathbf{a}_2 + \cdots + b_m\mathbf{a}_m = A\mathbf{b}.$$

Matrix multiplication involves multiple columns on the right.

$$A \begin{bmatrix} | & | & & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \\ | & | & & | \end{bmatrix} = AB$$

Other perspectives

Alternatively, we can work with rows. The matrix-row vector product is

$$\begin{bmatrix} b_1 & b_2 & \cdots & b_m \end{bmatrix} \begin{bmatrix} \text{——} & \mathbf{a}_1 & \text{——} \\ \text{——} & \mathbf{a}_2 & \text{——} \\ & \vdots & \\ \text{——} & \mathbf{a}_m & \text{——} \end{bmatrix} = \sum b_i \mathbf{a}_i = \mathbf{b}A.$$

Then we can mutiply matrices by having multiple rows on the left.

$$\begin{bmatrix} \text{——} & \mathbf{b}_1 & \text{——} \\ \text{——} & \mathbf{b}_2 & \text{——} \\ & \vdots & \\ \text{——} & \mathbf{b}_m & \text{——} \end{bmatrix} A = \begin{bmatrix} \text{——} & \mathbf{b}_1 A & \text{——} \\ \text{——} & \mathbf{b}_2 A & \text{——} \\ & \vdots & \\ \text{——} & \mathbf{b}_m A & \text{——} \end{bmatrix} = AB$$

Familiar properties of matrix multiplication

In most ways matrix multiplication behaves like multiplication of scalars:

- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $(A + B)C = AC + BC$
- $(sA)B = s(AB) = A(sB)$

Definition

The **identity matrix**, I_n (or just I), is the $n \times n$ diagonal matrix with ones on the main diagonal.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{etc.}$$

If A is an $m \times n$ matrix then

$$AI_n = A \quad \text{and} \quad I_m A = A.$$

Matrix multiplication is not commutative

If A and B are $n \times n$ matrices, it is not always true that $AB = BA$.

Example

If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$ then

$$AB = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 3 & -4 \end{bmatrix}$$

but

$$BA = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 9 \\ 3 & 1 \end{bmatrix}.$$

Matrix function algebra

All of the operations we discussed can be applied to matrix functions.

In the case of scalar multiplication, a matrix function can be multiplied by any *scalar function*.

Example

If $s(t) = e^t$ and $A(t) = \begin{bmatrix} -2+t & e^{2t} \\ 4 & \cos t \end{bmatrix}$, their product is

$$s(t)A(t) = \begin{bmatrix} e^t(-2+t) & e^{3t} \\ 4e^t & e^t \cos t \end{bmatrix}.$$

Matrix function algebra

Additionally, we can do calculus with matrix functions!

Definition

Suppose $A(t) = [a_{ij}(t)]$ is a matrix function. Its **derivative** is

$$\frac{dA}{dt} = \left[\frac{da_{ij}(t)}{dt}\right]$$

and its **integral** over the interval $[a, b]$ is

$$\int_a^b A(t)\,dt = \left[\int_a^b a_{ij}(t)\,dt\right].$$

Theorem (Matrix product rule)

If A and B are differentiable matrix functions and the product AB is defined then

$$\frac{d}{dt}(AB) = A\frac{dB}{dt} + \frac{dA}{dt}B.$$

Matrix function algebra

Example

Let $A(t) = \begin{bmatrix} 2t & 1 \\ 6t^2 & 4e^{2t} \end{bmatrix}$. We have

$$\frac{dA}{dt} = \begin{bmatrix} 2 & 0 \\ 12t & 8e^{2t} \end{bmatrix}$$

and

$$\int_0^1 A(t)\,dt = \begin{bmatrix} 1 & 1 \\ 2 & 2e^2 - 2 \end{bmatrix}.$$

Transpose

Definition

If A is the matrix $A = [a_{ij}]$, the **transpose** of A is the matrix $A^T = [a_{ji}]$.

If A is an $m \times n$ matrix then A^T is an $n \times m$ matrix.

Example

Suppose A is the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Then

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Types of matrices

Square An $n \times n$ matrix is called a **square matrix** since it has the same number of rows and columns. The elements a_{ii} make up the **main diagonal**.

Triangular A square matrix is called **upper triangular** if

$$a_{ij} = 0 \text{ whenever } i > j,$$

that is, it has only zeros below the main diagonal. A **lower triangular** matrix is a square matrix with only zeros *above* the main diagonal, that is,

$$a_{ij} = 0 \text{ whenever } i < j.$$

Diagonal A **diagonal matrix** is a square matrix whose only nonzero entries lie along the main diagonal, that is,

$$a_{ij} = 0 \text{ whenever } i \neq j.$$

Types of matrices

Symmetric A matrix satisfying $A^T = A$ is called a **symmetric matrix**.

Skew-symmetric A matrix that satisfies $A^T = -A$ is called **skew-symmetric**.

Notice that

- both symmetric and skew-symmetric matrices must be square (because if A is $m \times n$ then A^T is $n \times m$),
- a skew-symmetric matrix must have zeros along its main diagonal (because $a_{ii} = -a_{ii}$).