On Bases and the Rank-Nullity Theorem

Math 240 — Calculus III

Summer 2015, Session II

Tuesday, July 14, 2015

Agenda

What are bases good for?

Let V be a vector space with basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

- Since the basis is a spanning set, every $\mathbf{v} \in V$ can be written as
$$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n,$$
- Since the basis is independent, if
$$\mathbf{v} = d_1\mathbf{v}_1 + \cdots + d_n\mathbf{v}_n$$
is another way of writing $\mathbf{v}$, then $c_i = d_i$ for each i.

Theorem

Every vector $\mathbf{v} \in V$ can be written **in a unique way** *as a linear combination of basis elements.*

The theorem allows us to identify $\mathbf{v}$ with the vector $(c_1, \ldots, c_n)$.

Examples

Examples

1. What is a basis for the vector space whose vectors are complex numbers and whose scalars are real numbers? Is there more than one "natural" one?
2. In the subspace of $C^0(\mathbb{R})$ spanned by $f_1(x) = 2\sin^2 x$ and $f_2(x) = -5\cos^2 x$, give coordinates for the vector $f(x) = 1$.
3. In the vector space P_2, show that

   $$p_1(x) = 1-x, \quad p_2(x) = 5+3x-2x^2, \quad p_3(x) = 1+2x-x^2$$

   is a basis. Then write the standard basis vectors $e_1(x) = 1$, $e_2(x) = x$, $e_3(x) = x^2$ in terms of $p_1(x)$, $p_2(x)$, and $p_3(x)$.
4. Use a basis to identify the span of $\mathbf{v}_1 = (1, -1, 4)$, $\mathbf{v}_2 = (5, 3, -2)$, $\mathbf{v}_3 = (1, 2, -1)$ with $\mathbb{R}^2$.

Reinterpreting rank

Suppose that we have vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$. Form the matrix $A = [\mathbf{v}_1 \cdots \mathbf{v}_k]$. We saw yesterday that

- if $\operatorname{rank} A < k$ then our vectors are linearly dependent,
- if $\operatorname{rank} A = k$ then the vectors are linearly independent.

Specifically, we get a linear dependency for each independent vector in $\operatorname{nullspace}(A)$. The remaining vectors will be a basis for their span.

Proposition

The rank of a matrix is equal to the dimension of the span of its columns.

Definition

The span of the columns of a matrix A is called its **column space**. It is denoted by $\operatorname{colspace}(A)$.

The Rank-Nullity Theorem

Definition

When A is an $m \times n$ matrix, recall that the null space of A is

$$\operatorname{nullspace}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\} .$$

Its dimension is referred to as the **nullity** of A.

Theorem (Rank-Nullity Theorem)

For any $m \times n$ matrix A,

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n.$$

Homogeneous linear systems

We're now going to examine the geometry of the solution set of a linear system. Consider the linear system

$$A\mathbf{x} = \mathbf{b},$$

where A is $m \times n$.

If $\mathbf{b} = \mathbf{0}$, the system is called **homogeneous**. In this case, the solution set is simply the null space of A.

Any homogeneous system has the solution $\mathbf{x} = \mathbf{0}$, which is called the **trivial solution**. Geometrically, this means that the solution set passes through the origin. Furthermore, we have shown that the solution set of a homogeneous system is in fact a subspace of $\mathbb{R}^n$.

Structure of a homogeneous solution set

Theorem

- *If* $\text{rank}(A) = n$, *then* $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution* $\mathbf{x} = \mathbf{0}$, *so* $\text{nullspace}(A) = \{\mathbf{0}\}$.
- *If* $\text{rank}(A) = r < n$, *then* $A\mathbf{x} = \mathbf{0}$ *has an infinite number of solutions, all of which are of the form*

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_{n-r}\mathbf{x}_{n-r},$$

where $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-r}\}$ *is a basis for* $\text{nullspace}(A)$.

Remark

Such an expression is called the **general solution** to the homogeneous linear system.

Nonhomogeneous linear systems

Now consider a nonhomogeneous linear system

$$A\mathbf{x} = \mathbf{b}$$

where A be an $m \times n$ matrix and $\mathbf{b}$ is not necessarily $\mathbf{0}$.

Theorem

- *If* $\mathbf{b}$ *is not in* $\operatorname{colspace}(A)$*, then the system is inconsistent.*
- *If* $\mathbf{b} \in \operatorname{colspace}(A)$*, then the system is consistent and has*
  - *a unique solution if and only if* $\operatorname{rank}(A) = n$*.*
  - *an infinite number of solutions if and only if* $\operatorname{rank}(A) < n$*.*

Geometrically, a nonhomogeneous solution set is just the corresponding homogeneous solution set that has been shifted away from the origin.

Structure of a nonhomogeneous solution set

Theorem

In the case where $\operatorname{rank}(A) = r < n$ *and* $\mathbf{b} \in \operatorname{colspace}(A)$*, then all solutions are of the form*

$$\begin{aligned} \mathbf{x} &= \underbrace{c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_{n-r}\mathbf{x}_{n-r}}_{\mathbf{x}_c} + \mathbf{x}_p, \\ &= \mathbf{x}_c + \mathbf{x}_p \end{aligned}$$

where $\mathbf{x}_p$ *is any particular solution to* $A\mathbf{x} = \mathbf{b}$ *and* $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-r}\}$ *is a basis for* $\operatorname{nullspace}(A)$*.*

Remark

The above expression is the **general solution** to a nonhomogeneous linear system. It has two components:

- the **complementary solution**, $\mathbf{x}_c$, and
- the **particular solution**, $\mathbf{x}_p$.