Vector Differential Equations: Nondefective Coefficient Matrix

Math 240 — Calculus III

Summer 2015, Session II

Wednesday, July 22, 2015

Agenda

1. Solving linear systems by diagonalization
   Real eigenvalues
   Complex eigenvalues

Introduction

The results discussed yesterday apply to any old vector differential equation

$$\mathbf{x}' = A\mathbf{x}.$$

In order to make some headway in solving them, however, we must make a simplifying assumption:

The coefficient matrix A consists of real *constants*.

Diagonalization

Recall that an $n \times n$ matrix A may be diagonalized if and only if it is nondefective.

When this happens, we can solve the homogeneous vector differential equation

$$\mathbf{x}' = A\mathbf{x}.$$

If $S^{-1}AS = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, then

$$\mathbf{x} = S\mathbf{y}, \text{ where } \mathbf{y} = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix}.$$

Example

Solve the linear system

$$\begin{aligned} x_1' &= 2x_1 + x_2, \\ x_2' &= -3x_1 - 2x_2. \end{aligned}$$

1. Turn it into the vector differential equation
$$\mathbf{x}' = A\mathbf{x}, \quad \text{where } A = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}.$$
2. The characteristic polynomial of A is $\lambda^2 - 1$.
3. Eigenvalues are $\lambda = \pm 1$.
4. Eigenvectors are
$$\begin{aligned} \lambda_1 = 1: &\qquad \mathbf{v}_1 = (-1, 1), \\ \lambda_2 = -1: &\qquad \mathbf{v}_2 = (-1, 3). \end{aligned}$$
5. We have
$$\mathbf{y} = \begin{bmatrix} c_1 e^t \\ c_2 e^{-t} \end{bmatrix}, \text{ so } \mathbf{x} = \begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix} \mathbf{y} = \begin{bmatrix} -c_1 e^t - c_2 e^{-t} \\ c_1 e^t + 3c_2 e^{-t} \end{bmatrix}.$$

Vector formulation

The change of basis matrix S is

$$S = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix},$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are n linearly independent eigenvectors of A. Hence,

$$\begin{aligned} \mathbf{x} = S\mathbf{y} &= c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n \\ &= c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n. \end{aligned}$$

Check if these n solutions are linearly independent:

$$\begin{aligned} W[\mathbf{x}_1, \ldots, \mathbf{x}_n] &= \det\left(\begin{bmatrix} e^{\lambda_1 t}\mathbf{v}_1 & e^{\lambda_2 t}\mathbf{v}_2 & \cdots & e^{\lambda_n t}\mathbf{v}_n \end{bmatrix}\right) \\ &= e^{(\lambda_1 + \cdots + \lambda_n)t} \det\left(\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}\right) \\ &\neq 0. \end{aligned}$$

They are linearly independent, therefore a fundamental set of solutions.

General solution

Theorem

Suppose A is an $n \times n$ matrix of real constants. If A has n real linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not necessarily distinct), then the vector functions $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ defined by

$$\mathbf{x}_k(t) = e^{\lambda_k t}\mathbf{v}_k, \quad \text{for } k = 1, 2, \ldots, n$$

are a fundamental set of solutions to $\mathbf{x}' = A\mathbf{x}$ on any interval. The general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n.$$

Example

Find the general solution to $\mathbf{x}' = A\mathbf{x}$ if

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 4 & -3 \\ -2 & 2 & -1 \end{bmatrix}.$$

1. Characteristic polynomial is $-(\lambda+1)(\lambda-2)^2$.
2. Eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 2$.
3. Eigenvectors are

$$\begin{aligned} \lambda_1 = -1: &\qquad \mathbf{v}_1 = (1,1,1), \\ \lambda_2 = 2: &\qquad \mathbf{v}_2 = (1,1,0), \qquad \mathbf{v}_3 = (-3,0,2). \end{aligned}$$

4. Fundamental set of solution is

$$\mathbf{x}_1(t) = e^{-t}\begin{bmatrix}1\\1\\1\end{bmatrix},\ \mathbf{x}_2(t) = e^{2t}\begin{bmatrix}1\\1\\0\end{bmatrix},\ \mathbf{x}_3(t) = e^{2t}\begin{bmatrix}-3\\0\\2\end{bmatrix}.$$

5. So general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + c_3\mathbf{x}_3(t).$$

Complex eigenvalues

What happens when A has complex eigenvalues?

If $u = a + ib$ and $v = a - ib$ then

$$a = \frac{u+v}{2} \quad \text{and} \quad b = \frac{u-v}{2i}.$$

Theorem

Let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ be real-valued vector functions. If

$$\mathbf{w}_1(t) = \mathbf{u}(t) + i\mathbf{v}(t) \quad \textit{and} \quad \mathbf{w}_2(t) = \mathbf{u}(t) - i\mathbf{v}(t)$$

are complex conjugate solutions to $\mathbf{x}' = A\mathbf{x}$, then

$$\mathbf{x}_1(t) = \mathbf{u}(t) \quad \textit{and} \quad \mathbf{x}_2(t) = \mathbf{v}(t)$$

are themselves real valued solutions of $\mathbf{x}' = A\mathbf{x}$.

Complex eigenvalue example

Example

Find the general solution to $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

1. Characteristic polynomial is $\lambda^2 + 1$.
2. Eigenvalues are $\lambda = \pm i$.
3. Eigenvectors are $\mathbf{v} = (1, \pm i)$.
4. Linearly independent solutions are
$$\mathbf{w}(t) = e^{\pm it} \begin{bmatrix} 1 \\ \pm i \end{bmatrix} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \pm i \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$
5. Yields the two linearly independent real solutions
$$\mathbf{x}_1(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}.$$

The formula

Let's derive the explicit form of the real solutions produced by a pair of complex conjugate eigenvectors.

Suppose $\lambda = a + ib$ is an eigenvalue of A, with $b \neq 0$, corresponding to the eigenvector $\mathbf{r} + i\mathbf{s}$. This produces the complex solution

$$\begin{aligned}\mathbf{w}(t) &= e^{(a+ib)t}(\mathbf{r} + i\mathbf{s}) \\ &= e^{at}(\cos bt + i \sin bt)(\mathbf{r} + i\mathbf{s}) \\ &= e^{at}(\cos bt\, \mathbf{r} - \sin bt\, \mathbf{s}) + ie^{at}(\sin bt\, \mathbf{r} + \cos bt\, \mathbf{s}).\end{aligned}$$

Thus, the two real-valued solutions to $\mathbf{x}' = A\mathbf{x}$ are

$$\begin{aligned}\mathbf{x}_1(t) &= e^{at}(\cos bt\, \mathbf{r} - \sin bt\, \mathbf{s}), \\ \mathbf{x}_2(t) &= e^{at}(\sin bt\, \mathbf{r} + \cos bt\, \mathbf{s}).\end{aligned}$$

Remark

The conjugate eigenvalue $a - ib$ and eigenvector $\mathbf{r} - i\mathbf{s}$ would result in the same pair of real solutions.