Vector Differential Equations: Defective Coefficient Matrix and Matrix Exponential Solutions

Math 240 — Calculus III

Summer 2015, Session II

Monday, July 27, 2015

Agenda

1. Vector differential equations: defective coefficient matrix

2. Matrix exponential solutions

Introduction

We've learned how to find a matrix S so that $S^{-1}AS$ is almost a diagonal matrix. Recall that diagonalization allows us to solve linear systems of diff. eqs. because we can solve the equation

$$y' = ay.$$

Jordan form will instead give us small systems that look like

$$\begin{aligned} y_1' &= ay_1 + y_2, \\ y_2' &= ay_2. \end{aligned}$$

Is there an obvious solution?

$$y_1(t) = e^{at} \text{ and } y_2(t) = 0.$$

One we didn't already know? Yes!

$$y_1(t) = te^{at} \text{ and } y_2(t) = e^{at}.$$

Write this in the vector form

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = e^{at}\left(t\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).$$

2×2 defective systems

Switching back to the standard basis, these are the solutions

$$\mathbf{x}_1(t) = e^{at}\mathbf{v}_1 \text{ and } \mathbf{x}_2(t) = e^{at}(t\mathbf{v}_1 + \mathbf{v}_2)$$

where $\mathbf{v}_1, \mathbf{v}_2$ is a chain of generalized eigenvectors.

Example

Find the general solution to

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}.$$

1. The single eigenvalue is $\lambda = 3$.
2. Chain of generalized e-vectors is $\mathbf{v}_1 = (1, 3)$, $\mathbf{v}_2 = (0, 1)$.
$$(A - 3I)\mathbf{v}_1 = \mathbf{0} \text{ and } (A - 3I)\mathbf{v}_2 = \mathbf{v}_1.$$
3. Fundamental set of solutions is therefore
$$\mathbf{x}_1(t) = e^{3t}\mathbf{v}_1 \text{ and } \mathbf{x}_2(t) = e^{3t}\left(t\mathbf{v}_1 + \mathbf{v}_2\right).$$

Longer chains

What about chains of generalized eigenvectors longer than 2?

If A is an $n \times n$ matrix with eigenvalue λ and chain of generalized eigenvectors

$$\mathbf{v}_1 = (A - \lambda I)^{p-1}\mathbf{v}, \qquad \mathbf{v}_2 = (A - \lambda I)^{p-2}\mathbf{v}, \quad \ldots$$
$$\mathbf{v}_{p-1} = (A - \lambda I)\mathbf{v}, \qquad \mathbf{v}_p = \mathbf{v},$$

check that the following are solutions to $\mathbf{x}' = A\mathbf{x}$:

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1$$
$$\mathbf{x}_2(t) = e^{\lambda t}\left(\mathbf{v}_2 + t\mathbf{v}_1\right)$$
$$\vdots$$
$$\mathbf{x}_p(t) = e^{\lambda t}\left(\mathbf{v}_p + t\mathbf{v}_{p-1} + \cdots + \frac{1}{(p-1)!}t^{p-1}\mathbf{v}_1\right)$$

Longer chains

We should also check that $\{\mathbf{x}_1(t), \ldots, \mathbf{x}_p(t)\}$ is independent. We know that $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is independent, that is,

$$\det\left(\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}\right) \neq 0.$$

Theorem

The set $\{\mathbf{x}_1(t), \ldots, \mathbf{x}_p(t)\}$ is a linearly independent subset of $V_n(I)$.

Thus, we can construct a fundamental set of solutions by applying the foregoing construction to each chain of generalized eigenvectors.

Example

Find the general solution to $\mathbf{x}' = A\mathbf{x}$ if

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}.$$

1. Only eigenvalue is $\lambda = 1$.
2. On Thursday we found the chain
$$\mathbf{v}_1 = (-2, 0, 1),\ \mathbf{v}_2 = (0, -1, 0),\ \mathbf{v}_3 = (-1, 0, 0).$$
3. Thus, solutions are
$$\begin{aligned} \mathbf{x}_1(t) &= e^t \mathbf{v}_1, \\ \mathbf{x}_2(t) &= e^t \left(\mathbf{v}_2 + t\mathbf{v}_1 \right), \\ \mathbf{x}_3(t) &= e^t \left(\mathbf{v}_3 + t\mathbf{v}_2 + \tfrac{1}{2}t^2 \mathbf{v}_1 \right). \end{aligned}$$

Example

Find the general solution to $\mathbf{x}' = A\mathbf{x}$ if

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

1. Eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 5$.
2. Eigenvectors and generalized eigenvectors are

$$A\mathbf{e}_1 = 2\mathbf{e}_1, \quad A\mathbf{e}_2 = 2\mathbf{e}_2 + \mathbf{e}_1, \quad A\mathbf{e}_3 = 5\mathbf{e}_3,$$
$$A\mathbf{e}_4 = 5\mathbf{e}_4, \quad A\mathbf{e}_5 = 5\mathbf{e}_5 + \mathbf{e}_4, \quad A\mathbf{e}_6 = 5\mathbf{e}_6 + \mathbf{e}_5.$$

3. Our fundamental set of solutions is

$$\mathbf{x}_1(t) = e^{2t}\mathbf{e}_1, \quad \mathbf{x}_2(t) = e^{2t}\left(\mathbf{e}_2 + t\mathbf{e}_1\right), \quad \mathbf{x}_3(t) = e^{5t}\mathbf{e}_3,$$
$$\mathbf{x}_4(t) = e^{5t}\mathbf{e}_4, \quad \mathbf{x}_5(t) = e^{5t}\left(\mathbf{e}_5 + t\mathbf{e}_4\right),$$
$$\mathbf{x}_6(t) = e^{5t}\left(\mathbf{e}_6 + t\mathbf{e}_5 + \tfrac{1}{2}t^2\mathbf{e}_4\right).$$

What is the matrix exponential, again?

Recall that, if A is an $n \times n$ matrix of constants, then

$$e^{At} = I_n + At + \frac{1}{2}(At)^2 + \frac{1}{2 \cdot 3}(At)^3 + \cdots + \frac{1}{k!}(At)^k + \cdots$$

is a matrix function called the **matrix exponential function**.

Theorem

If A is diagonalizable, with $S^{-1}AS = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, then

$$e^{At} = S \operatorname{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t}) S^{-1}.$$

How is this relevant to differential equations? Differentiating term by term, we find that

$$\frac{d}{dt} e^{At} = A e^{At}.$$

Matrix exponential solutions

Theorem

If $\mathbf{b}$ *is any constant vector, the initial value problem* $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{b}$ *is solved uniquely by* $\mathbf{x}(t) = e^{At}\mathbf{b}$.

Example

Solve the above initial value problem with

$$A = \begin{bmatrix} -2 & -7 \\ -1 & 4 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} -10 \\ 2 \end{bmatrix}.$$

You determined for homework that $S^{-1}AS = \operatorname{diag}(5, -3)$, with $S = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$, $\mathbf{v}_1 = (-1, 1)$, $\mathbf{v}_2 = (7, 1)$. Thus,

$$e^{At} = S \begin{bmatrix} e^{5t} & 0 \\ 0 & e^{-3t} \end{bmatrix} S^{-1} = \begin{bmatrix} \frac{1}{8}e^{5t} + \frac{7}{8}e^{-3t} & -\frac{7}{8}e^{5t} + \frac{7}{8}e^{-3t} \\ -\frac{1}{8}e^{-5t} + \frac{1}{8}e^{-3t} & \frac{7}{8}e^{5t} + \frac{1}{8}e^{-3t} \end{bmatrix}$$

and

$$\mathbf{x} = e^{At} \begin{bmatrix} -10 \\ 2 \end{bmatrix} = \begin{bmatrix} -3e^{5t} - 7e^{-3t} \\ 3e^{5t} - e^{-3t} \end{bmatrix}.$$

Now do it backwards

This theorem can be used "backwards" to determine the matrix exponential function by solving a vector differential equation.

Example

Determine e^{At} if $A = \begin{bmatrix} 6 & -8 \\ 2 & -2 \end{bmatrix}$.

Find the JCF of A: $J = S^{-1}AS$ where

$$S = \begin{bmatrix} 4 & 1 \\ 2 & 0 \end{bmatrix} \text{ and } J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

This leads to the fundamental set of solutions

$$\mathbf{x}_1(t) = e^{2t} \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \ \mathbf{x}_2(t) = e^{2t} \begin{bmatrix} 1 + 4t \\ 2t \end{bmatrix}.$$

Then, if $X(t) = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}$, we have $X' = AX$ and $X(0) = S$. So $e^{At}X(0) = X(t)$, and thus

$$e^{At} = X(t)X(0)^{-1} = \begin{bmatrix} (1+4t)e^{2t} & -8te^{2t} \\ 2te^{2t} & (1-4t)e^{2t} \end{bmatrix}.$$

Fundamental matrix

Definition

If $\mathbf{x}' = A\mathbf{x}$ is a vector differential equation and $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a fundamental set of solutions then the corresponding **fundamental matrix** is

$$X(t) = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix}.$$

Theorem

If A is an $n \times n$ matrix and $X(t)$ is any fundamental matrix for the equation $\mathbf{x}' = A\mathbf{x}$ then the matrix exponential function can be calculated by

$$e^{At} = X(t)\left(X(0)\right)^{-1}.$$