# Nonhomogeneous Linear Differential Equations

Math 240 — Calculus III

Summer 2015, Session II

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# Introduction

We have now learned how to solve homogeneous linear differential equations

$$P(D)y = 0$$

when $P(D)$ is a polynomial differential operator. Now we will try to solve nonhomogeneous equations

$$P(D)y = F(x).$$

Recall that the solutions to a nonhomogeneous equation are of the form

$$y(x) = y_c(x) + y_p(x),$$

where $y_c$ is the general solution to the associated homogeneous equation and $y_p$ is a particular solution.

# Overview

The technique proceeds from the observation that, if we know a polynomial differential operator $A(D)$ so that

$$A(D)F = 0,$$

then applying $A(D)$ to the nonhomogeneous equation

$$P(D)y = F \tag{1}$$

yields the homogeneous equation

$$A(D)P(D)y = 0. \tag{2}$$

A particular solution to (1) will be a solution to (2) that is not a solution to the associated homogeneous equation $P(D)y = 0$.

## Example

Determine the general solution to

$$(D+1)(D-1)y = 16e^{3x}.$$

1. The associated homogeneous equation is $(D+1)(D-1)y = 0$. It has the general solution $y_c(x) = c_1e^x + c_2e^{-x}$.
2. Recognize the nonhomogeneous term $F(x) = 16e^{3x}$ as a solution to the equation $(D-3)y = 0$.
3. The differential equation

$$(D-3)(D+1)(D-1)y = 0$$

   has the general solution $y(x) = c_1e^x + c_2e^{-x} + c_3e^{3x}$.
4. Pick the **trial solution** $y_p(x) = c_3e^{3x}$. Substituting it into the original equation forces us to choose $c_3 = 2$.
5. Thus, the general solution is

$$y(x) = y_c(x) + y_p(x) = c_1e^x + c_2e^{-x} + 2e^{3x}.$$

# Annihilators and the method of undetermined coefficients

This method for obtaining a particular solution to a nonhomogeneous equation is called the **method of undetermined coefficients** because we pick a trial solution with an unknown coefficient. It can be applied when

1. the differential equation is of the form
$$P(D)y = F(x),$$
where $P(D)$ is a polynomial differential operator,
2. there is another polynomial differential operator $A(D)$ such that
$$A(D)F = 0.$$

A polynomial differential operator $A(D)$ that satisfies $A(D)F = 0$ is called an **annihilator** of $F$.

# Finding annihilators

Functions that can be annihilated by polynomial differential operators are exactly those that can arise as solutions to constant-coefficient homogeneous linear differential equations. We have seen that these functions are

1. $F(x) = cx^k e^{ax}$,
2. $F(x) = cx^k e^{ax} \sin bx$,
3. $F(x) = cx^k e^{ax} \cos bx$,
4. linear combinations of 1–3.

If the nonhomogeneous term is one of 1–3, then it can be annihilated by something of the form $A(D) = (D - r)^{k+1}$, with $r = a$ in 1 and $r = a + bi$ in 2 and 3. Otherwise, annihilators can be found by taking successive derivatives of $F$ and looking for linear dependencies.

## Example

Determine the general solution to

$$(D-4)(D+1)y = 16xe^{3x}.$$

1. The general solution to the associated homogeneous equation $(D-4)(D+1)y = 0$ is $y_c(x) = c_1e^{4x} + c_2e^{-x}$.
2. An annihilator for $16xe^{3x}$ is $A(D) = (D-3)^2$.
3. The general solution to $(D-3)^2(D-4)(D+1)y = 0$ includes $y_c$ and the terms $c_3e^{3x}$ and $c_4xe^{3x}$.
4. Using the trial solution $y_p(x) = c_3e^{3x} + c_4xe^{3x}$, we find the values $c_3 = -3$ and $c_4 = -4$.
5. The general solution is

$$y(x) = y_c(x) + y_p(x) = c_1e^{4x} + c_2e^{-x} - 3e^{3x} - 4xe^{3x}.$$

## Example

Determine the general solution to

$$(D-2)y = 3\cos x + 4\sin x.$$

1. The associated homogeneous equation, $(D-2)y=0$, has the general solution $y_c(x) = c_1 e^{2x}$.
2. Look for linear dependencies among derivatives of $F(x) = 3\cos x + 4\sin x$. Discover the annihilator $A(D) = D^2 + 1$.
3. The general solution to $(D^2+1)(D-2)y = 0$ includes $y_c$ and the additional terms $c_2 \cos x + c_3 \sin x$.
4. Using the trial solution $y_p(x) = c_2 \cos x + c_3 \sin x$, we obtain values $c_2 = -2$ and $c_3 = -1$.
5. The general solution is

$$y(x) = c_1 e^{2x} - 2\cos x - \sin x.$$

# Motivating example

## Example

Find the general solution to

$$y'' + y' - 6y = 4\cos 2x.$$

1. Recall from yesterday that the complementary function is $y_c(x) = c_1 e^{-3x} + c_2 e^{2x}$.
2. The right-hand side would be annihilated by $D^2 + 4$.
3. Since $\pm 2i$ is not already a root of the auxiliary polynomial, use the trial solution $y_p(x) = c_3 \cos 2x + c_4 \sin 2x$.
4. Plugging $y_p$ into the original equation yields $c_3 = -\frac{5}{13}$ and $c_4 = \frac{1}{13}$.
5. The general solution is

$$y(x) = c_1 e^{-3x} + c_2 e^{2x} - \frac{5}{13}\cos 2x + \frac{1}{13}\sin 2x.$$

# Motivating example

## Example

Find the general solution to

$$y'' + y' - 6y = 4e^{2ix}.$$

1. The complementary function is $y_c(x) = c_1e^{-3x} + c_2e^{2x}$.
2. If we're using complex numbers, use the trial solution $y_p(x) = c_3e^{2ix}$.
3. Plugging $y_p$ into the original equation yields $c_3 = -\frac{1}{13}(5 + i)$.
4. Thus, the general solution is

$$y(x) = c_1e^{-3x} + c_2e^{2x} - \frac{1}{13}(5 + i)e^{2ix}.$$

# Complex-valued trial solutions

Which problem was easier? Depends on your opinion of complex numbers, but the second only involved one unknown coefficient while the first had two. So it may be advantageous, when the nonhomogeneous term is $cx^k e^{ax}\cos bx$ or $cx^k e^{ax}\sin bx$, to change it to $cx^k e^{(a+bi)x}$, solve, and take the real or imaginary part.

# Rigor

## Theorem

*If* $y(x) = u(x) + iv(x)$ *is a complex-valued solution to*

$$P(D)y = F(x) + iG(x),$$

*then*

$$P(D)u = F(x) \quad \textit{and} \quad P(D)v = G(x).$$

## Proof.

If $y(x) = u(x) + iv(x)$, then

$$P(D)y = P(D)(u + iv) = P(D)u + iP(D)v.$$

Equating real and imaginary parts gives

$$P(D)u = F(x) \quad \text{and} \quad P(D)v = G(x).$$

$\mathcal{Q.E.D.}$

# Method

Solutions to the nonhomogeneous polynomial differential equations

$$P(D)y = cx^k e^{ax}\cos bx \quad \text{and} \quad P(D)y = cx^k e^{ax}\sin bx,$$

may be found by solving the complex equation

$$P(D)z = cx^k e^{(a+bi)x}$$

and then taking the real and imaginary parts, respectively, of the solution $z(x)$.

**Bonus**

Solve two equations at once!

## Example

Solve $y'' - 2y' + 5y = 8e^x \sin 2x$.

1. The complementary function is $y_c(x) = e^x(c_1 \cos 2x + c_2 \sin 2x)$.
2. Instead, solve $z'' - 2z' + 5z = 8e^{(1+2i)x}$.
3. Since $1 + 2i$ is a root of the auxiliary polynomial, use the trial solution $z_p(x) = c_3 x e^{(1+2i)x}$.
4. Plugging $z_p$ into $z'' - 2z' + 5z = 8e^{(1+2i)x}$ yields $c_3 = -2i$.
5. Thus, the particular solution is
$$z_p(x) = -2ixe^{(1+2i)x} = -2xe^x(-\sin 2x + i \cos 2x).$$
6. To get $8e^x \sin 2x$ on the right-hand side, take the imaginary part
$$y_p(x) = \mathrm{Im}(z_p) = -2xe^x \cos 2x.$$
7. The general solution is
$$y(x) = e^x(c_1 \cos 2x + c_2 \sin 2x) - 2xe^x \cos 2x.$$