CONTINUOUS SPATIAL SEMIGROUPS OF COMPLETELY POSITIVE MAPS OF $\mathfrak{B}(\mathfrak{H})$

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Abstract. This paper concerns the structure of strongly continuous one parameter semigroups of completely positive contractions of $\mathfrak{B}(\mathfrak{H}) = \mathfrak{B}(\mathfrak{K}\otimes L^2(0,\infty))$ which are intertwined by translation. These are called CP-flows over $\mathfrak{K}$. Using Bhat's dilation result each unital CP-flow over $\mathfrak{K}$ dilates to an E_o-semigroup of $\mathfrak{B}(\mathfrak{H}_1)$ where $\mathfrak{H}_1$ can be considered to contain $\mathfrak{B}(\mathfrak{K}\otimes L^2(0,\infty))$. Every spatial E_o-semigroup is cocycle conjugate to one dilated from a CP-flow. Each CP-flow is determined by its associated boundary weight map which determines the generalized boundary representation. The index of the E_o-semigroup dilated from a CP-flow is calculated. Machinery for determining whether two CP-flow dilate to cocycle conjugate E_o-semigroups is developed. This paper is available via http://nyjm.albany.edu:8000/j/2003/9-13.html

I. Introduction.

The goal of this paper is the construction of new spatial E_o-semigroups of $\mathfrak{B}(\mathfrak{H})$. An E_o-semigroup of $\mathfrak{B}(\mathfrak{H})$ is a strongly continuous one parameter semigroup of $*$-endomorphisms of $\mathfrak{B}(\mathfrak{H})$. An E_o-semigroup is spatial if there is a one parameter semigroup of intertwining isometries. If there are enough intertwining semigroups to reconstruct the E_o-semigroup the semigroup is said to be completely spatial. The first examples of spatial E_o-semigroups were given in [P1] and later Arveson [A1] defined and completely classified the completely spatial E_o-semigroups. The index first introduced and the additivity property suggested in [P1] and later correctly defined and proved to be additive under tensoring by Arveson [A2] is a complete cocycle conjugacy invariant for the completely spatial E_o-semigroups. In [P2] an example of a non spatial E_o-semigroups was first constructed and recently Tsirelson [T2] has constructed a one parameter family of non isomorphic product systems of type III in the context of Arveson's theory of continuous tensor products of Hilbert spaces and from Arveson's representation theorem this implies the existence of a one parameter family of non cocycle conjugate non spatial E_o-semigroups.

The first example of a spatial E_o-semigroup which is not completely spatial was constructed in [P4]. Now Tsirelson [T1] has constructed a one parameter family of non isomorphic product systems of type II and by Arveson's theory of product systems this implies the existence of a one parameter family of non cocycle conjugate spatial E_o-semigroups of $\mathfrak{B}(\mathfrak{H})$.

In this paper we develop a way of constructing spatial E_o-semigroups of $\mathfrak{B}(\mathfrak{H})$. This method can in principle construct all spatial E_o-semigroups (see Theorem

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4.0A). We use the new technology developed by Bhat. Bhat showed in [Bh] that every unital CP-semigroup of $\mathfrak{B}(\mathfrak{K})$ can be dilated to an E_o-semigroup of $\mathfrak{B}(\mathfrak{H})$ where $\mathfrak{H}$ can be thought of as a larger Hilbert space containing $\mathfrak{K}$. A CP-semigroup of $\mathfrak{B}(\mathfrak{K})$ is a strongly continuous one parameter semigroup of completely positive contractions of $\mathfrak{B}(\mathfrak{K})$. Since CP-semigroups are much easier to construct than E_o-semigroups Bhat's result is extremely useful in constructing E_o-semigroups. In this paper we study CP-flows over a Hilbert space $\mathfrak{K}$. A CP-flow is CP-semigroup of $\mathfrak{K} \otimes L^2(0,\infty)$ which is intertwined by translation on $L^2(0,\infty)$. We believe this is the simplest object from which one can construct via Bhat's dilation all the spatial E_o-semigroups. We show how each CP-flow over $\mathfrak{K}$ is determined from a boundary weight. We show how to calculate the index of the E_o-semigroup obtained by dilation.

Although we construct no new examples of spatial E_o-semigroups we consider the results of this paper to be a big success. In a subsequent paper we will discuss the classification of E_o-semigroups obtained from CP-flows in the case where $\mathfrak{K}$ is one dimensional. In the case when $\mathfrak{K}$ is two dimensional all sorts of new and interesting problems arise. Since most of the basic problems reduce to questions about completely positive maps of the two by two matrices into themselves we believe these problems are tractable. The reason we have not given applications of CP-flows to constructing E_o-semigroups of $\mathfrak{B}(\mathfrak{H})$ in this paper is that so many different approaches suggest themselves that we are not sure which direction is best. We can assure the reader that CP-flows lead to barrel loads of examples and we believe that these examples will lead the way into developing a classification of spatial E_o-semigroups of $\mathfrak{B}(\mathfrak{H})$.

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II. Background, Definitions and Generators of Semigroups

All Hilbert spaces which will be denoted by the characters such as $\mathfrak{H}, \mathfrak{K}$ and $\mathfrak{M}$ are assumed to be separable unless otherwise stated. On Hilbert spaces we use the physicist's inner product (f,g) which is linear in g and conjugate linear in f. If $\mathfrak{H}$ is a Hilbert space we denote by $\mathfrak{B}(\mathfrak{H})$ the set of all bounded linear operators on $\mathfrak{H}$ and by $\mathfrak{B}(\mathfrak{H})_*$ the pre dual of $\mathfrak{B}(\mathfrak{H})$. Every element $\rho \in \mathfrak{B}(\mathfrak{H})_*$ can be represented in the form $\rho(A) = \Sigma_{i=1}^{\infty}(f_i, Ag_i)$ where $\Sigma_{i=1}^{\infty}\|f_i\|\,\|g_i\| < \infty$. If $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are two Hilbert spaces we denote by $\mathfrak{B}(\mathfrak{H}_1,\mathfrak{H}_2)$ the space of bounded linear operators from $\mathfrak{H}_2$ to $\mathfrak{H}_1$. Note if $A \in \mathfrak{B}(\mathfrak{H}_1,\mathfrak{H}_2)$ then $A^*A \in \mathfrak{B}(\mathfrak{H}_2)$ and $AA^* \in \mathfrak{B}(\mathfrak{H}_1)$.

Definition 2.1. We say α is an E_o-semigroup of a von Neumann algebra M with unit I if the following conditions are satisfied.

(i) α_t is a $*$-endomorphism of M for each $t \geq 0$.
(ii) α_o is the identity endomorphism and $\alpha_t \circ \alpha_s = \alpha_{t+s}$ for all $s,t \geq 0$.
(iii) For each $f \in M_*$ (the pre dual of M) and $A \in M$ the function $f(\alpha_t(A))$ is a continuous function of t.
(iv) $\alpha_t(I) = I$ for each $t \geq 0(\alpha_t$ preserves the unit of $M)$.

Definition 2.2. Suppose α and β are E_o-semigroups $\mathfrak{B}(\mathfrak{H}_1)$ and $\mathfrak{B}(\mathfrak{H}_2)$. We say α and β are conjugate denoted $\alpha \approx \beta$ if there is $*$-isomorphism ϕ of $\mathfrak{B}(\mathfrak{H}_1)$ onto

$\mathfrak{B}(\mathfrak{H}_2)$ so that $\phi \circ \alpha_t = \beta_t \circ \phi$ for all $t \geq 0$. We say α and β are cocycle conjugate denoted $\alpha_t \sim \beta_t$ if α' and β are conjugate where α and α' differ by a unitary cocycle (i.e., there is a strongly continuous one parameter family of unitaries $U(t)$ on $\mathfrak{B}(\mathfrak{H}_1)$ for $t \geq 0$ satisfying the cocycle condition $U(t)\alpha_t(U(s)) = U(t+s)$ for all $t, s \geq 0$ so that $\alpha'_t(A) = U(t)\alpha_t(A)U(t)^{-1}$ for all $A \in \mathfrak{B}(\mathfrak{H}_1)$ and $t \geq 0$).

Definition 2.3. Suppose α is an E_o-semigroup of $\mathfrak{B}(\mathfrak{H})$. We say α is spatial if there exists a strongly continuous one parameter semigroup of isometries $U(t) \in \mathfrak{B}(\mathfrak{H})$ which intertwine α_t, i.e., $U(t)A = \alpha_t(A)U(t)$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$.

The property of being spatial is a cocycle conjugacy invariant. If there are enough intertwining semigroups to reconstruct the E_o-semigroup we say the semigroup is completely spatial. In [A1] Arveson classified the completely spatial E_o-semigroups of $\mathfrak{B}(\mathfrak{H})$. He showed that each completely spatial E_o-semigroup is cocycle conjugate to a CAR flow of rank n for $n = 1, 2, \cdots$ and $n = \infty$. The CAR flows are E_o-semigroups of $\mathfrak{B}(\mathfrak{H})$ constructed using representations of the CAR algebra.

The E_o-semigroups of $\mathfrak{B}(\mathfrak{H})$ themselves form a semigroup and the appropriate group operation is tensoring. If α and β are E_o-semigroups of $\mathfrak{B}(\mathfrak{H})$ and $\mathfrak{B}(\mathfrak{K})$, respectively, then one can form a new semigroup $\gamma = \alpha \otimes \beta$ which acts on the tensor product space $\mathfrak{H} \otimes \mathfrak{K}$. Specifically, we define $\gamma_t(A \otimes B) = \alpha_t(A) \otimes \beta_t(B)$. In [A2] Arveson showed the index is additive (i.e., the index of γ is the sum of the index of α and the index of β). One of the important results of the theory of E_o-semigroups obtained by Arveson is that if σ is a one parameter group of $*$-automorphisms of $\mathfrak{B}(\mathfrak{H})$ (i.e., $\sigma_t(A) = U(t)AU(t)^{-1}$ with $U(t)$ a strongly continuous one parameter unitary group) then σ acts like the unit under tensoring. This means that if α is an E_o-semigroup and σ is one parameter group of $*$-automorphisms then α is cocycle conjugate to $\alpha \otimes \sigma$. Another result we state as a theorem (see Theorem 2.9 of [P4]) so we can refer to it later is that the restriction of an E_o-semigroup to an invariant subspace yields an E_o-semigroup which is cocycle conjugate to the original E_o-semigroup.

Theorem 2.4. *Suppose α is a proper E_o-semigroup of $\mathfrak{B}(\mathfrak{H})$ (so $\alpha_t(\mathfrak{B}(\mathfrak{H})) \neq \mathfrak{B}(\mathfrak{H})$ for $t > 0$) and $E \in \mathfrak{B}(\mathfrak{H})$ is an hermitian projection which is invariant under α_t (i.e., $\alpha_t(E) = E$ for all $t \geq 0$). Let $\mathfrak{M}$ be the range of E and let Q_E be the set of all operators $A \in \mathfrak{B}(\mathfrak{H})$ so that $A = EAE$. Note Q_E is $*$-isomorphic with $\mathfrak{B}(\mathfrak{M})$ the algebra of all bounded operators on $\mathfrak{M}$ and note if $A \in Q_E$ then $\alpha_t(A) \in Q_E$ for all $t \geq 0$. Let β be the restriction of α to Q_E so $\beta_t(A) = \alpha_t(A)$ for all $A \in Q_E$. Then β is an E_o-semigroup of $\mathfrak{B}(\mathfrak{M})$ which is cocycle conjugate to α.*

We assemble some of the standard facts about the semigroups of contractions. Suppose X is a Banach space and $t \to S(t)$ is a strongly continuous one parameter semigroup of contractions of X into itself where by strong continuity we mean $\|S(t)x - x\| \to 0$ as $t \to 0+$ for each $x \in X$. The generator T of S is the linear operator from the domain $\mathfrak{D}(T)$ into X given by

$$Tx = \lim_{x \to 0+} t^{-1}(S(t)x - x)$$

and the domain is the set of $x \in X$ so that the limit exists in the sense of norm convergence. The domain $\mathfrak{D}(T)$ is norm dense in X and the generator T is closed which means that if $x_n \in \mathfrak{D}(T)$ for $n = 1,2,\cdots$, $\|x_n - x\| \to 0$ and $\|Tx_n - y\| \to 0$ as $n \to \infty$ then $x \in \mathfrak{D}(T)$ and $Tx = y$.

Definition 2.5. A densely defined operator T on a Banach space X is said to be dissipative if for each f in the domain of T there is a linear functional F in the unit ball of the dual of X (so that $|F(h)| \leq \|h\|$ for all $h \in X$) so that $F(f) = \|f\|$ and $\mathrm{Re}(F(Tf)) \leq 0$).

Lemma 2.6. *If T is densely defined dissipative operator on a Banach space X then T is closable and its closure is dissipative.*

For the proof see Lemma 3.1.14 of Bratteli and Robinson [BR].

Theorem 2.7. *(Lumer-Phillips) If T is a closed densely defined dissipative operator on a Banach space X and the range of $(\lambda I - T)$ is dense in X for some real $\lambda > 0$ then T is the generator of a strongly continuous one parameter semigroup of contractions. Conversely, if T is the generator of a strongly continuous one parameter semigroup of contractions then T is a closed dissipative operator and the range of $(\lambda I - T)$ is all of X for every real $\lambda > 0$.*

For the proof see Theorem 3.1.16 of [BR].

If T is the generator of a strongly continuous one parameter semigroup of contractions we often refer to $(I - T)^{-1}$ as the resolvent of T. Note the resolvent is a one to one mapping of X onto the domain $\mathfrak{D}(T)$.

Theorem 2.8. *Suppose T is the generator of a strongly continuous one parameter semigroup $S(t)$ of contractions of a Banach space X and $t \to x(t)$ is a differentiable map of $[0, s]$ into the domain $\mathfrak{D}(T)$ and*

$$\frac{d}{dt}x(t) = Tx(t)$$

Then $x(t) = S(t)x(0)$ for $t \in [0, s]$.

Proof. Assume the hypothesis and notation of the theorem. Let $y(t) = S(t)x(0) - x(t)$. Then $y(t)$ is differentiable and

$$\frac{d}{dt}y(t) = Ty(t)$$

for $t \in [0, s]$. We show $\|y(t)\|$ is non increasing. Suppose $t \in [0, s)$ and $h > 0$ and $t + h \leq s$. Since T is dissipative there is an element $F_t \in X^*$ of norm one with $F_t(y(t)) = \|y(t)\|$ and $\mathrm{Re}(\mathrm{F}_t(Ty(t))) \leq 0$ for each $t \in [0, s]$. Then we have

$$\begin{aligned}\|y(t + h)\| - \|y(t)\| =& Re(F_{t+h}(y(t + h)) - F_t(y(t))) \\ =& Re((F_{t+h} - F_t)(y(t)) + hRe(F_{t+h}(Ty(t))) \\ &+ Re(F_{t+h}(y(t + h) - y(t) - hTy(t))).\end{aligned}$$

Now the first of the three terms is non positive and the third is $o(h)$. So all we need to show is that the limit superior of $\mathrm{Re}(\mathrm{F}_{t+h}(Ty(t))$ as $h \to 0+$ is not positive. Suppose the limit superior is a positive number λ. Then there is a decreasing sequence of positive numbers h_n so that $h_n \to 0$ and $Re(F_{t+h_n}(Ty(t))) \to \lambda$ as $n \to \infty$. Let F be a weak limit point of the sequence F_{t+h_n}. Since the $\|F_{t+h_n}\| = 1$ we have $\|F\| \leq 1$. We have

$$\begin{aligned} F(y(t)) =&F(y(t) - y(t + h_n)) + (F - F_{t+h_n})(y(t + h_n) - y(t)) \\ &+ (F - F_{t+h_n})(y(t)) + \|y(t + h_n)\| \end{aligned}$$

As $n \to \infty$ the first two terms tend to zero and the last term tends to $\|y(t)\|$. Since F is a weak limit point there is a subsequence of the sequence h_n converging to zero so that the third term tends to zero for the subsequence. Hence, we have $F(y(t)) = \|y(t)\|$. Also we have $Re(F(Ty(t)) = \lambda$. Let $g(s) = F(S(s)y(t))$ for $s \geq 0$. Since $y(t) \in \mathfrak{D}(T)$ we have g is differentiable and $g'(s) = F(S(s)Ty(t))$. We have $\|S(s)y(t)\| \geq Re(g(s))$ and since $g(0) = \|y(t)\|$ and $Re(g'(0)) = \lambda > 0$ we have $\|S(s)y(t)\| > \|y(t)\|$ for some $s > 0$. But this contradicts the fact that $S(s)$ is a contraction. Hence, we have

$$\limsup_{h \to 0+} (\|y(t + h)\| - \|y(t)\|)/h \leq 0$$

and it follows that $\|y(t)\|$ is a non increasing function of t. Since $\|y(0)\| = 0$ we have $y(t) = 0$ for all $t \in [0, s]$. Hence, $x(t) = S(t)x(0)$ for $t \in [0, s]$. $\square$

Theorem 2.9. *Suppose T is the generator of a strongly continuous one parameter semigroup of contractions Θ_t of $\mathfrak{B}(\mathfrak{H})_*$. Then Θ_t is positivity preserving (i.e., $\Theta_t(\rho) \geq 0$ if $\rho \geq 0$ for all $t \geq 0$ and $\rho \in \mathfrak{B}(\mathfrak{H})_*$) if and only if $\rho - \lambda T\rho \geq 0$ implies $\rho \geq 0$ for all $\lambda \in (0, 1)$ and $\rho \in \mathfrak{D}(T)$.*

Proof. This result can be dug out of Chapter 3 of [BR] (Bratelli and Robinson work with groups but the arguments work for semigroups). A sketch of the proof is as follows. Assume the hypothesis of the theorem and Θ_t is positivity preserving. Then for $\lambda > 0$ we have

$$(I - \lambda T)^{-1} = \frac{1}{\lambda} \int_0^\infty e^{-t/\lambda} \Theta_t \, dt$$

so $(I - \lambda T)^{-1}$ is positivity preserving for $\lambda > 0$. Hence $\rho - \lambda T\rho \geq 0$ implies $\rho \geq 0$ for all $\lambda > 0$ and $\rho \in \mathfrak{D}(T)$.

Conversely, suppose $\rho - \lambda T\rho \geq 0$ implies $\rho \geq 0$ for all $\lambda \in (0, 1)$ and $\rho \in \mathfrak{D}(T)$. Then $(I - \lambda T)^{-1}$ is positivity preserving for all $\lambda \in (0, 1)$. As shown in calculations in Chapter 3 of [BR] we have

$$\Theta_t(\rho) = \exp(tT)(\rho) = \lim_{n \to \infty} (I - (t/n)T)^{-n} \rho$$

for each $\rho \in \mathfrak{B}(\mathfrak{H})_*$ and $t > 0$. Since $(I - (t/n)T)^{-1}$ is positivity preserving for $n > t^{-1}$*we* see Θ_t is the limit of positivity preserving maps and, hence, Θ_t is positivity preserving. $\square$

We will occasionally need the following lemma.

Lemma 2.10. *Suppose $\rho \in \mathfrak{B}(\mathfrak{H})_*$ and $E \in \mathfrak{B}(\mathfrak{H})$ is an orthogonal projection. Let $\rho_1(A) = \rho(EAE)$ for $A \in \mathfrak{B}(\mathfrak{H})$. Then*

$$\|\rho - \rho_1\|^2 \leq 2\|\rho\|^2 - 2\|\rho_1\|^2$$

Proof. We prove the lemma for the case when $\|\rho\| = 1$. The general case then follows by linearity. Suppose $\rho \in \mathfrak{B}(\mathfrak{H})_*$ and $\|\rho\| = 1$. Then ρ can be written in the form

$$\rho(A) = \sum_{i=1} \lambda_i (f_i, Ag_i)$$

where the $f's$ and $g's$ form an orthonormal set of vectors and the λ_i are positive numbers which sum to one. Let π be the countable direct sum of identity representations of $\mathfrak{B}(\mathfrak{H})$ and let

$$F = \sqrt{\lambda_1} f_1 \oplus \sqrt{\lambda_2} f_2 \oplus \cdots \qquad \text{and} \qquad G = \sqrt{\lambda_1} g_1 \oplus \sqrt{\lambda_2} g_2 \oplus \cdots$$

Then we have $\rho(A) = (F, \pi(A)G)$ and $\|F\| = \|G\| = 1$. Let $\rho_1(A) = \rho(EAE)$ for all $A \in \mathfrak{B}(\mathfrak{H})$. We have $\rho_1(A) = (\pi(E)F, \pi(A)\pi(E)G)$ and, hence, $\|\rho_1\| \leq \|\pi(E)F\|\,\|\pi(E)G\|$. Suppose $A \in \mathfrak{B}(\mathfrak{H})$ and $\|A\| \leq 1$. Then

$$\begin{aligned} |\rho(A) - \rho_1(A)| =& |(F, \pi(A)(I - \pi(E))G) + ((I - \pi(E))F, \pi(A)\pi(E)G)| \\ \leq& \|G - \pi(E)G\| + \|F - \pi(E)F\|\,\|\pi(E)G\| \end{aligned}$$

Now $\|F - \pi(E)F\|^2 = (F, (I - \pi(E))F) = 1 - \|\pi(E)F\|^2$ and, similarly we have $\|G - \pi(E)G\|^2 = 1 - \|\pi(E)G\|^2$. Combining these with the above inequality we have

$$|\rho(A) - \rho_1(A)| \leq \sqrt{1 - \|\pi(E)G\|^2} + \sqrt{\|\pi(E)G\|^2 - \|\pi(E)G\|^2\|\pi(E)F\|^2}.$$

Since $\sqrt{x} + \sqrt{y} \leq \sqrt{2x + 2y}$ for all $x, y \in [0, \infty)$ it follows that

$$|\rho(A) - \rho_1(A)| \leq \sqrt{2 - 2\|\pi(E)G\|^2\|\pi(E)F\|^2}.$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ with $\|A\| \leq 1$. Hence, we have

$$\|\rho - \rho_1\|^2 \leq 2 - 2\|\pi(E)G\|^2\|\pi(E)F\|^2 \leq 2 - 2\|\rho_1\|^2$$

Where the second inequality in the line above follows from the inequality $\|\rho_1\| \leq \|\pi(E)F\|\,\|\pi(E)G\|$. Hence, we have proved the lemma for the case $\|\rho\| = 1$. For the general case we first note that if $\|\rho\| = 0$ then $\rho = \rho_1 = 0$ and the conclusion of the lemma follows trivially. Then if $\|\rho\| > 0$ we simply apply the above inequality to the functionals $\|\rho\|^{-1}\rho$ and $\|\rho\|^{-1}\rho_1$ and the inequality of the lemma follows. $\square$

III. Subordinates of CP-Semigroups.

In this section we are interested in the order structure of CP-semigroups and the E_o-semigroups they induce from a result of Bhat [Bh].

Definition 3.1. A CP-semigroup α of $\mathfrak{B}(\mathfrak{H})$ is a one parameter semigroup of completely positive contractions α_t of $\mathfrak{B}(\mathfrak{H})$ into itself which are strongly continuous in the sense that $\|\alpha_t(A)f - Af\| \to 0$ as $t \to 0+$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $f \in \mathfrak{H}$.

We are particularly interested in the order structure for CP-semigroups where the order structure is in the sense of completely positive maps. A mapping ϕ from one operator algebra $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{H})$ completely positive if

$$\sum_{i,j=1}^{n} (f_i, \phi(A_i^* A_j) f_j) \geq 0$$

for $A_i \in \mathfrak{A},\ f_i \in \mathfrak{H}$ for $i = 1, 2, \cdots, n$ and $n = 1, 2, \cdots$. An important result of Stinespring [St] states that if $\mathfrak{A}$ has a unit I and ϕ completely positive from $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{H})$ there is a $*$-representation π of $\mathfrak{A}$ on $\mathfrak{B}(\mathfrak{K})$ and a operator V from $\mathfrak{H}$ to $\mathfrak{K}$ so that

$$\phi(A) = V^* \pi(A) V$$

for all $A \in \mathfrak{A}$ and the vectors $\pi(A)Vf$ for $A \in \mathfrak{A}$ and $f \in \mathfrak{H}$ span $\mathfrak{K}$. Furthermore, this representation π is determined by these requirements up to unitary equivalence. Note that for a completely positive map ϕ we have $\|\phi\| = \|\phi(I)\|$. By requiring $\mathfrak{A}$ to have a unit we insure ϕ is bounded. Also, if γ is a second completely positive map of $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{H})$ and the mapping $A \to \phi(A) - \gamma(A)$ for $A \in \mathfrak{A}$ is completely positive then there is a unique positive operator $C \in \pi(\mathfrak{A})'$(C commutes with $\pi(A)$ for all $A \in \mathfrak{A}$) so that $\gamma(A) = V^* C \pi(A) V$ for all $A \in \mathfrak{A}$.

Another result concerning completely positive maps which we will often use is that if ϕ is a completely positive contraction from a C^*-algebra $\mathfrak{A}$ to $\mathfrak{B}(\mathfrak{H})$ and $S \in \mathfrak{A}$ is a contraction ($\|S\| \leq 1$) then if $\phi(S)$ is an isometry then $\phi(AS) = \phi(A)\phi(S)$ for all $A \in \mathfrak{A}$ and if $\phi(S^*)$ is an isometry then $\phi(SA) = \phi(S)\phi(A)$ for all $A \in \mathfrak{A}$. This result follows easily from the Stinespring construction.

Arveson has described the completely positive maps of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$ which we describe in the following definition. We use Arveson's characterization to defines the rank of such a map. In this section we denote by $\mathfrak{B}(\mathfrak{K}, \mathfrak{H})$ the space of bounded linear operators from the Hilbert space $\mathfrak{H}$ to the Hilbert space $\mathfrak{K}$. Note that if $A \in \mathfrak{B}(\mathfrak{K}, \mathfrak{H})$ then $A^* \in \mathfrak{B}(\mathfrak{H}, \mathfrak{K})$.

Definition 3.2. Suppose ϕ is a completely positive σ-weakly continuous contraction of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$. Arveson has shown that a completely positive σ-weakly continuous contraction ϕ of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$ is of the form

$$\phi(A) = \sum_{i=1}^{r} C_i A C_i^*$$

for $A \in \mathfrak{B}(\mathfrak{H})$ where r is a non-negative integer or a countable infinity and the $C_i \in \mathfrak{B}(\mathfrak{K}, \mathfrak{H})$ are linearly independent over $\ell^2(\mathbb{N})$ which means that for every square summable sequence $z_i \in \mathbb{C}$ for $i \in [1, r+1)$ if C is the operator given by

$$C = \sum_{i=1}^{r} z_i C_i$$

(one can show the sum converges in norm) then $C = 0$ if and only if each $z_i = 0$ for $i \in [1, r+1)$. If ϕ is expressed in terms of a second linearly independent set of operators C_i' the number of terms r' for the second sum is the same. We call r the rank of ϕ.

We have the notion of when map ϕ dominates γ. Sometimes it is useful to have a word for the maps γ which are dominated by ϕ. We call these maps subordinates of ϕ.

Definition 3.3. Suppose ϕ is a σ-weakly continuous completely positive map of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$. Then γ is a subordinate of ϕ if γ is a completely positive map of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$ and the mapping for $A \in \mathfrak{B}(\mathfrak{H})$ given by $A \to \phi(A) - \gamma(A)$ is completely positive. In this situation we say ϕ dominates γ. The fact that ϕ dominates γ or what is the same thing that γ is a subordinate of ϕ is denoted $\phi \geq \gamma$. (Note γ is automatically σ-weakly continuous.) Suppose α is a CP-semigroup of $\mathfrak{B}(\mathfrak{H})$. Then β is a subordinate of α if β is a CP-semigroup and $A \to \alpha_t(A) - \beta_t(A)$ for $A \in \mathfrak{B}(\mathfrak{H})$ is completely positive for all $t \geq 0$ (i.e., $\alpha_t \geq \beta_t$ for each $t \geq 0$). Again we may express this same notion by saying α dominates β and this is denoted by writing $\alpha \geq \beta$.

Suppose ϕ is a σ-weakly continuous complete positive map of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$ and ϕ is given as in Definition 3.2. Then the extremal subordinates of ϕ are of the form $\gamma(A) = CAC^*$ for $A \in \mathfrak{B}(\mathfrak{H})$ with

$$C = \sum_{i=1}^{r} z_i C_i \qquad \text{with} \qquad \sum_{i=1}^{r} |z_i|^2 = 1$$

We see that the extremal subordinates of ϕ are isomorphic to the rank one projections in a r-dimensional Hilbert space.

An important result of Bhat [Bh] is that each unital CP-semigroup α can be dilated to an E_o-semigroup α^d and if the dilation is minimal then α^d is unique up to cocycle conjugacy. The relation between the CP-semigroup α of $\mathfrak{B}(\mathfrak{H})$ and the minimal dilation α^d which is an E_o-semigroup of $\mathfrak{B}(\mathfrak{H}_1)$ is given by

$$\alpha_t(A) = W^* \alpha_t^d(WAW^*)W$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ where W is an isometry of $\mathfrak{H}$ into $\mathfrak{H}_1$ so that WW^* is an increasing projection for α^d (i.e., $\alpha_t^d(WW^*) \geq WW^*$ for all $t \geq 0$) and α^d is minimal over the range of WW^*. We use Arveson's definition of minimal [A6] which is equivalent to Bhat's definition but easier to state which means the linear span of vectors of the form

$$\alpha_{t_1}^d(WA_1W^*)\alpha_{t_2}^d(WA_2W^*) \cdots \alpha_{t_n}^d(WA_nW^*)Wf$$

with $f \in \mathfrak{H}$, $A_i \in \mathfrak{B}(\mathfrak{H})$, $t_i \geq 0$ for $i = 1, \cdots, n$ and $n = 1, 2, \cdots$ is dense in $\mathfrak{H}_1$. Arveson showed that α^d is minimal if and only if the operators $\alpha_t^d(WAW^*)$ for $A \in \mathfrak{B}(\mathfrak{H})$ generate $\mathfrak{B}(\mathfrak{H}_1)$ so every vector is cyclic for the $\alpha_t^d(WAW^*)$.

The minimal dilation α^d is determined by α up to conjugacy. Because of the importance of this construction we briefly describe the situation. Suppose α^d in a minimal dilation of the unital CP-semigroup of $\mathfrak{B}(\mathfrak{H})$ to an E_o-semigroup α^d of $\mathfrak{B}(\mathfrak{H}_1)$ and W is an isometry of $\mathfrak{H}$ into $\mathfrak{H}_1$ so that α^d is minimal over the range of W and

$$\alpha_t(A) = W^*\alpha_t^d(WAW^*)W$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. The key to understanding why α determines α^d is seeing how the expression

$$\begin{aligned}\Xi =&\Xi(A_1, \cdots, A_n, t_1, \cdots, t_n)\\ =&W^*\alpha_{t_1}^d(WA_1W^*)\alpha_{t_2}^d(WA_2W^*)\cdots\alpha_{t_n}^d(WA_nW^*)W\end{aligned}$$

is computable from α. Let us first take the case of two terms with $t_1 \geq t_2$ so we set $s = t_2$ and $t = t_1 - t_2$. Then we have

$$\begin{aligned}W^*\alpha_{t_1}^d(WAW^*)\alpha_{t_2}^d(WBW^*)W =&W^*\alpha_{t+s}^d(WAW^*)\alpha_s^d(WBW^*)W\\ =&W^*\alpha_s^d(\alpha_t^d(WAW^*)WBW^*)W\\ =&W^*\alpha_s^d(WW^*)\alpha_s^d(\alpha_t^d(WAW^*)WBW^*)W\\ =&W^*\alpha_s^d(WW^*\alpha_t^d(WAW^*)WBW^*)W\\ =&W^*\alpha_s^d(W\alpha_t(A)BW^*)W = \alpha_s(\alpha_t(A)B)\end{aligned}$$

where in the third line we used the fact that $\alpha_s^d(WW^*) \geq WW^*$ so we have $W^* = W^*\alpha_s^d(WW^*)$. So for $t_1 \geq t_2$ we have

$$W^*\alpha_{t_1}^d(WA_1W^*)\alpha_{t_2}^d(WA_2W^*)W = \alpha_{t_2}(\alpha_{t_2-t_1}(A_1)A_2)$$

And when $t_1 \leq t_2$ a similar calculation shows

$$W^*\alpha_{t_1}^d(WA_1W^*)\alpha_{t_2}^d(WA_2W^*)W = \alpha_{t_1}(A_1\alpha_{t_1-t_2}(A_2))$$

So in general we have

$$W^*\alpha_{t_1}^d(WA_1W^*)\alpha_{t_2}^d(WA_2W^*)W = \alpha_{t_o}(\alpha_{t_1-t_o}(A_1)\alpha_{t_2-t_o}(A_2))$$

where $t_o = \min(t_1, t_2)$. For the case of n terms we find

$$\begin{aligned}&W^*\alpha_{t_1}^d(WA_1W^*)\alpha_{t_2}^d(WA_2W^*)\cdots\alpha_{t_n}^d(WA_nW^*)W\\ &\qquad=W^*\alpha_s^d(\alpha_{t_1-s}^d(WA_1W^*)\alpha_{t_2-s}^d(WA_2W^*)\cdots\alpha_{t_n-s}^d(WA_nW^*))W\end{aligned}$$

where $s = \min(t_1, t_2, \cdots, t_n)$. So if t_k is the minimum of the $t's$ we have

$$\begin{aligned}&W^*\alpha_{t_1}^d(WA_1W^*)\alpha_{t_2}^d(WA_2W^*)\cdots\alpha_{t_n}^d(WA_nW^*)W =\\ &\quad W^*\alpha_s(\alpha_{t_1-s}^d(WA_1W^*)\cdots\alpha_{t_{k-1}-s}^d(WA_{k-1}W^*)WA_kW^*\alpha_{t_{k+1}}^d(WA_{k+1}W^*)\\ &\qquad\qquad\cdots\alpha_{t_n-s}^d(WA_nW^*))W\end{aligned}$$

In the expression above we can replace W^* by $W^*\alpha_s(WW^*)$ on the left and W by $\alpha_s(WW^*)W$ on the right. Then we have

$$W^*\alpha_{t_1}^d(WA_1W^*)\alpha_{t_2}^d(WA_2W^*)\cdots\alpha_{t_n}^d(WA_nW^*)W = W^*\alpha_s^d(WXA_kYW^*)W$$

where

$$X = W^*\alpha_{t_1-s}^d(WA_1W^*)\cdots\alpha_{t_{k-1}-s}^d(WA_{k-1}W^*)W$$

and

$$Y = W^*\alpha_{t_{k+1}-s}^d(WA_{k+1}W^*)\cdots\alpha_{t_n-s}^d(WA_nW^*)W$$

Note the expressions for X and Y involve an expression Ξ with a smaller number of terms. One sees that by using this procedure repeatedly one can successively reduce the number to terms until the number of terms is two or less. In this way one can evaluate Ξ in solely in terms of α. To give an example, for the product of four terms with $0 \leq t_2 \leq t_1 \leq t_3 \leq t_4$ we find

$$\begin{aligned} W^*\alpha_{t_1}^d(WA_1W^*)&\alpha_{t_2}^d(WA_2W^*)\alpha_{t_3}^d(WA_3W^*)\alpha_{t_4}^d(WA_4W^*)W \\ &= \alpha_{t_2}(\alpha_{t_1-t_2}(A_1)A_2\alpha_{t_3-t_2}(A_3\alpha_{t_4-t_3}(A_4))) \end{aligned}$$

In [P4] we introduced the notion of a local cocycle for E_o-semigroups. If α is an E_o-semigroup of $\mathfrak{B}(\mathfrak{H})$ then $t \to S(t)$ is a cocycle if $S(t)$ is strongly continuous in t, $S(0) = I$ and $S(t)$ satisfies the cocycle identity $S(t)\alpha_t(S(s)) = S(t+s)$ for all $s,t \geq 0$. The family $S(t)$ is a local cocycle if $S(t)$ is a cocycle and $S(t)$ commutes with $\alpha_t(A)$ for all $A \in \mathfrak{B}(\mathfrak{H})$ (i.e., $S(t) \in \alpha_t(\mathfrak{B}(\mathfrak{H}))'$ for all $t \geq 0$). In Theorem 4.9 of [P4] it was shown that the projection valued local cocycles with the order relation $E(t) \geq F(t)$ for all $t \geq 0$ form a complete lattice which is a cocycle conjugacy invariant. The same argument shows that the positive local cocycles with the obvious order relation are a cocycle conjugacy invariant. What is of interest is that the positive contractive local cocycles of an E_o-semigroup α are in one to one correspondence with the subordinates of α.

Theorem 3.4. *Suppose α is an E_o-semigroup of $\mathfrak{B}(\mathfrak{H})$. Suppose β is a subordinate of α. Then there is a local cocycle $t \to C(t)$ with $0 \leq C(t) \leq I$ for $t \geq 0$ and $\beta_t(A) = C(t)\alpha_t(A)$ for all $t \geq 0$ and $A \in \mathfrak{B}(\mathfrak{H})$. Conversely, if $t \to C(t)$ is a local cocycle with $0 \leq C(t) \leq I$ for $t \geq 0$ and if $\beta_t(A) = C(t)\alpha_t(A)$ for all $t \geq 0$ and $A \in \mathfrak{B}(\mathfrak{H})$ then β is a subordinate of α. Furthermore, the local cocycles $t \to C(t)$ with $0 \leq C(t) \leq I$ for all $t \geq 0$ with the obvious order relation are a cocycle conjugacy invariant.*

Proof. Suppose α is an E_o-semigroup of $\mathfrak{B}(\mathfrak{H})$ and β is subordinate of α. Suppose $t > 0$. Since $\alpha_t \geq \beta_t$ and α_t is a $*$-isomorphism we have from the Stinespring results concerning completely positive maps that $\beta_t(A) = \alpha_t(A)C(t)$ with $C(t) \in \alpha_t(\mathfrak{B}(\mathfrak{H}))'$ for each $A \in \mathfrak{B}(\mathfrak{H})$ and $0 \leq C(t) \leq I$. Since β is a semigroup we have for $t,s \geq 0$ that

$$C(t+s) = \beta_{t+s}(I) = \beta_t(\beta_s(I)) = \beta_t(C(s)) = C(t)\alpha_t(C(s)).$$

Conversely suppose $t \to C(t)$ is a local cocycle with $0 \leq C(t) \leq I$ for all $t \geq 0$. Then for each $t > 0$ we have

$$\alpha_t(A) - \beta_t(A) = (I - C(t))\alpha_t(A) = (I - C(t))^{\frac{1}{2}}\alpha_t(A)(I - C(t))^{\frac{1}{2}}$$

Hence, the map $A \to \alpha_t(A) - \beta_t(A)$ is completely positive for $t \geq 0$.

Next suppose α and β are cocycle conjugate E_o-semigroups of $\mathfrak{B}(\mathfrak{H}_1)$ and $\mathfrak{B}(\mathfrak{H}_2)$, respectively. This means there is a unitary operator $W \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ so that

$$\beta_t(A) = W^* S(t)\alpha_t(WAW^*)S(t)^{-1}W$$

for all $A \in \mathfrak{B}(\mathfrak{H}_2)$ and $t \geq 0$ where $t \to S(t)$ is an α unitary cocycle. Suppose $t \to C(t)$ is a local cocycle for α with $0 \leq C(t) \leq I$ for all $t \geq 0$. Let $D(t) = W^* S(t)C(t)S(t)^{-1}W$. We have $0 \leq D(t) \leq I$ for all $t \geq 0$ and

$$\begin{aligned} D(t)\beta_t(D(s)) =& D(t)W^* S(t)\alpha_t(WD(s)W^*)S(t)^{-1}W \\ =& D(t)W^* S(t)\alpha_t(S(s)C(s)S(s)^{-1})S(t)^{-1}W \\ =& W^* S(t)C(t)\alpha_t(S(s))\alpha_t(C(s))S(t+s)^{-1}W \\ =& W^* S(t)\alpha_t(S(s))C(t)\alpha_t(C(s))S(t+s)^{-1}W \\ =& W^* S(t+s)C(t+s)S(t+s)^{-1}W = D(t+s) \end{aligned}$$

for all $t, s \geq 0$. Hence, $t \to D(t)$ is a cocycle for β. Next note that

$$\begin{aligned} D(t)\beta_t(A) =& W^* S(t)C(t)S(t)^{-1}WW^* S(t)\alpha_t(WAW^*)S(t)^{-1}W \\ =& W^* S(t)C(t)\alpha_t(WAW^*)S(t)^{-1}W \\ =& W^* S(t)\alpha_t(WAW^*)C(t)S(t)^{-1}W \\ =& W^* S(t)\alpha_t(WAW^*)S(t)^{-1}WW^* S(t)C(t)S(t)^{-1}W \\ =& \beta_t(A)D(t) \end{aligned}$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. Hence $t \to D(t)$ is a local cocycle for β. Hence, the cocycle conjugacy produces mapping from each positive contractive local cocycle $t \to C(t)$ for α to a positive contractive local cocycle $D(t)$ for β. Since this mapping is invertible and preserves order it follows that we have an order isomorphism from the local cocycles for α onto the local cocycles for β and from what we have shown above this gives and order isomorphism of the subordinates of α with the subordinates of β. $\square$

Theorem 3.5. *Suppose α is a unital CP-semigroup of $\mathfrak{B}(\mathfrak{H})$ and α^d in a minimal dilation of α to an E_o-semigroup of $\mathfrak{B}(\mathfrak{H}_1)$. Then there is an order isomorphism of the subordinates of α with the subordinates of α^d given as follows. Suppose the relation between α_t and α_t^d is given by*

$$\alpha_t(A) = W^* \alpha_t^d(WAW^*)W$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$ where W is an isometry of $\mathfrak{H}$ into $\mathfrak{H}_1$ with WW^* an increasing projection for α^d and α^d is minimal over the range of WW^*. Suppose γ is a subordinate of α^d and $C(t) = \gamma_t(I)$ for $t \geq 0$. Then β the subordinate of α associated with γ under this isomorphism is given by

$$\beta_t(A) = W^* \alpha_t^d(WAW^*)C(t)W$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ *and* $t \geq 0$.

Proof. Assume the hypothesis and notation of the theorem. Suppose γ is a subordinate of α^d and $C(t) = \gamma_t(I)$ for all $t \geq 0$. Let β_t be as given in the statement of the theorem. Note

$$\beta_t(A) = W^*\alpha_t^d(WAW^*)C(t)W = W^*C(t)^{\frac{1}{2}}\alpha_t^d(WAW^*)C(t)^{\frac{1}{2}}W$$

and

$$\begin{aligned}\alpha_t(A) - \beta_t(A) =& W^*\alpha_t^d(WAW^*)(I - C(t))W \\ =& W^*(I - C(t))^{\frac{1}{2}}\alpha_t^d(WAW^*)(I - C(t))^{\frac{1}{2}}W\end{aligned}$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. From this it follows that $A \to \beta_t(A)$ and $A \to \alpha_t(A) - \beta_t(A)$ are completely positive maps for all $t \geq 0$. Note that β_t is a semigroup since

$$\begin{aligned}\beta_t(\beta_s(A)) =& W^*\alpha_t^d(WW^*\alpha_s^d(WAW^*)C(s)WW^*)C(t)W \\ =& W^*\alpha_t^d(WW^*)\alpha_{t+s}^d(WAW^*)\alpha_t^d(C(s))\alpha_t^d(WW^*)C(t)W \\ =& W^*\alpha_t^d(WW^*)\alpha_{t+s}^d(WAW^*)C(t)\alpha_t^d(C(s))\alpha_t^d(WW^*)W \\ =& W^*\alpha_t^d(WW^*)\alpha_{t+s}^d(WAW^*)C(t+s)\alpha_t^d(WW^*)W \\ =& W^*\alpha_{t+s}^d(WAW^*)C(t+s)W = \beta_{t+s}(A)\end{aligned}$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t, s \geq 0$. (Note above we used the fact that $\alpha_t^d(WW^*) \geq WW^*$ which implies $W^*\alpha_t^d(WW^*) = W^*$ and $\alpha_t^d(WW^*)W = W$ for all $t \geq 0$.) Hence, β is a subordinate of α.

Next we show the mapping just described from $C(t)$ to β is one to one. In fact we will show that the local cocycle $C(t)$ can be reconstructed from β. We will show how expressions like

$$\Xi = W^*\alpha_{t_1}^d(WA_1W^*)\cdots\alpha_{t_n}^d(WA_nW^*)C(t_o)\alpha_{s_1}^d(WB_1W^*)\cdots\alpha_{s_m}^d(WB_mW^*)W$$

can be calculated from α and β. Since α^d is minimal over the range of W it follows that these expression determine $C(t)$. We begin with some low order terms. Suppose $0 \leq t \leq s$ and $A \in \mathfrak{B}(\mathfrak{H})$. Then

$$\begin{aligned}W^*C(t)\alpha_s^d(WAW^*)W =& W^*\alpha_s^d(WAW^*)C(t)W \\ =& W^*\alpha_t^d(WW^*)C(t)\alpha_s^d(WAW^*)\alpha_t^d(WW^*)W \\ =& W^*C(t)\alpha_t^d(WW^*)\alpha_s^d(WAW^*)\alpha_t^d(WW^*)W \\ =& W^*C(t)\alpha_t^d(WW^*\alpha_{s-t}^d(WAW^*)WW^*)W \\ =& W^*C(t)\alpha_t^d(W\alpha_{s-t}(A)W^*)W \\ =& \beta_t(\alpha_{s-t}(A))\end{aligned}$$

Suppose $0 \leq s \leq t$ and $A \in \mathfrak{B}(\mathfrak{H})$. Then

$$\begin{aligned} W^*C(t)\alpha_s^d(WAW^*)W =& W^*C(s)\alpha_s^d(C(t-s))\alpha_s^d(WAW^*)W \\ =& W^*\alpha_s^d(WW^*)C(s)\alpha_s^d(C(t-s))\alpha_s^d(WAW^*)W \\ =& W^*C(s)\alpha_s^d(WW^*C(t-s)WAW^*)W \\ =& W^*C(s)\alpha_s^d(W\beta_{t-s}(I)AW^*)W \\ =& \beta_s(\beta_{t-s}(I)A). \end{aligned}$$

Repeating the computation with $C(t)$ on the right gives

$$W^*\alpha_s^d(WAW^*)C(t)W = \beta_s(A\beta_{t-s}(I)).$$

Next we show how an expression Ξ given above can be calculated in terms of a Ξ with fewer terms. Consider Ξ given above. Let $t = \min(t_1, \cdots, t_m, t_o, s_1, \cdots, s_m)$. Suppose $t = t_k$. Note that $C(t_o) = C(t)\alpha_t^d(C(t_o - t))$ and $C(t)$ commutes with $\alpha_s^d(WAW^*)$ for $s \geq t$ and $A \in \mathfrak{B}(\mathfrak{H})$. Using this we find the expression for Ξ above can be reduced as follows.

$$\begin{aligned} \Xi = W^*C(t)\alpha_t^d(\alpha_{t_1'}^d(WA_1W^*)\cdots\alpha_{t_n'}^d(WA_nW^*)C(t_o')\alpha_{s_1'}^d(WB_1W^*)\cdots \\ \cdots\alpha_{s_m'}^d(WB_mW^*))W \end{aligned}$$

for $t_i' = t_i - t$ and $s_j' = s_j - t$ for $i = 0, 1, \cdots, n$ and $j = 1, \cdots, m$. Since $\alpha_t^d(WW^*)W = W$, $W^*\alpha_t^d(WW^*) = W^*$ and $\alpha_t^d(WW^*)$ commutes with $C(t)$ we have

$$\begin{aligned} \Xi = W^*C(t)\alpha_t^d(WW^*\alpha_{t_1'}^d(WA_1W^*)\cdots \\ \cdots\alpha_{t_{k-1}'}^d(WA_{k-1}W^*)WA_kW^*\alpha_{t_{k+1}'}^d(WA_{k+1}W^*)\cdots \\ \cdots\alpha_{t_n'}^d(WA_nW^*)C(t_o')\alpha_{s_1'}^d(WB_1W^*)\cdots \\ \cdots\alpha_{s_m'}^d(WB_mW^*)WW^*)W \end{aligned}$$

Let

$$X = W^*\alpha_{t_1'}^d(WA_1W^*)\cdots\alpha_{t_{k-1}'}^d(WA_{k-1}W^*)W$$

and

$$\begin{aligned} Y = W^*\alpha_{t_{k+1}'}^d(WA_{k+1}W^*)\cdots\alpha_{t_n'}^d(WA_nW^*)C(t_o')\alpha_{s_1'}^d(WB_1W^*)\cdots \\ \cdots\alpha_{s_m'}^d(WB_mW^*)W \end{aligned}$$

Note $X, Y \in \mathfrak{B}(\mathfrak{H})$ and X is computable in terms of α as described at the beginning of this section and Y is of the form Ξ with a smaller number of terms. Then we have

$$\Xi = W^*C(t)\alpha_t^d(WXA_kYW^*)W = \beta_t(XA_kY)$$

Hence, we have shown that in this case Ξ can be computed from a knowledge of β and Ξ with fewer terms.

Next suppose $t = t_o$. Then $C(t_o) = C(t)$ commutes with all the other terms and we have

$$\Xi = W^*C(t)\alpha_t^d(\alpha_{t_1'}^d(WA_1W^*)\cdots\alpha_{t_n'}^d(WA_nW^*)\alpha_{s_1'}^d(WB_1W^*)\cdots \\ \cdots\alpha_{s_m'}^d(WB_mW^*))W$$

for $t_i' = t_i - t$ and $s_j' = s_j - t$ for $i = 1, \cdots, n$ and $j = 1, \cdots, m$. Since $\alpha_t^d(WW^*)W = W$, $W^*\alpha_t^d(WW^*) = W^*$ and $\alpha_t^d(WW^*)$ commutes with $C(t)$ we have

$$\begin{aligned}\Xi =& W^*C(t)\alpha_t^d(WW^*\alpha_{t_1'}^d(WA_1W^*)\cdots \\ &\cdots\alpha_{t_n'}^d(WA_nW^*)\alpha_{s_1'}^d(WB_1W^*)\cdots\alpha_{s_m'}^d(WB_mW^*)WW^*)W\end{aligned}$$

Let

$$Z = W^*\alpha_{t_1'}^d(WA_1W^*)\cdots\alpha_{t_n'}^d(WA_nW^*)\alpha_{s_1'}^d(WB_1W^*)\cdots\alpha_{s_m'}^d(WB_mW^*)W$$

Note $Z \in \mathfrak{B}(\mathfrak{H})$ and Z can be computed from a knowledge of α. Hence, we have

$$\Xi = W^*C(t)\alpha_t^d(WZW^*)W = \beta_t(Z)$$

Hence, we have shown in this case Ξ can be calculated from a knowledge of β and α.

Finally suppose $t = s_k$. Then the same sort of calculation we did for $t = t_k$ shows that Ξ can be computed from a knowledge of α and β and Ξ with fewer terms. Hence, we have shown in all cases Ξ can be computed from a knowledge of α, β and Ξ with fewer terms. Then by iteration we can reduce the number of terms in Ξ until the number of terms is down to two or one where we have shown how to compute these terms from a knowledge of α and β. Hence, we have shown that all the terms Ξ can be computed from a knowledge of α and β. Since α^d is minimal over the range of W it follows that for each $t > 0$ we can compute $(F, C(t)G)$ where F and G are linear combinations of vectors of the form

$$\alpha_{t_1}^d(WA_1W^*)\cdots\alpha_{t_n}^d(WA_nW^*)Wf$$

with $f \in \mathfrak{H}$, $A_i \in \mathfrak{B}(\mathfrak{H})$, $t_i \geq 0$ for $i = 1, \cdots, n$. Since the closed span of such vectors is all of $\mathfrak{H}_1$ it follows that $C(t)$ is determined from a knowledge of α and β. Hence, the mapping $C(t) \to \beta_t$ from positive contractive local cocycles to subordinates of α is one to one.

Next we show the mapping is onto (i.e., it has range all subordinates of α). Suppose then that β is a subordinate of α. Suppose $t > 0$. Since we have $\alpha_t(A) = W^*\alpha_t^d(WAW^*)W$ for $A \in \mathfrak{B}(\mathfrak{H})$ and α_t^d is a $*$-representation of $\mathfrak{B}(\mathfrak{H}_1)$ the mapping $A \to \alpha_t^d(WAW^*)$ is a $*$-representation of $\mathfrak{B}(\mathfrak{H})$ on $\alpha_t^d(WW^*)\mathfrak{M}_t$ where $\mathfrak{M}_t$ is the closed linear span of $\{\alpha_t^d(WAW^*)Wf,\ f \in \mathfrak{H},\ A \in \mathfrak{B}(\mathfrak{H})\}$ and let ϕ_t be the restriction of $A \to \alpha_t^d(WAW^*)$ to $\mathfrak{M}_t$ so $\phi_t(A)f = \alpha_t^d(WAW^*)f$ for all $f \in \mathfrak{M}_t$ and $A \in \mathfrak{B}(\mathfrak{H})$. Note $\alpha_t(A) = W^*\phi_t(A)W$ for $A \in \mathfrak{B}(\mathfrak{H})$ and the span of $\phi_t(A)Wf$ for $A \in \mathfrak{B}(\mathfrak{H})$ and $f \in \mathfrak{H}$ is dense in $\mathfrak{M}_t$. Since $\alpha_t \geq \beta_t$ it follows from the

Stinespring analysis of completely positive maps that there is a unique operator $Z(t) \in \phi_t(\mathfrak{B}(\mathfrak{H}))'$ so that $\beta_t(A) = W^*\phi_t(A)Z(t)W$ for all $A \in \mathfrak{B}(\mathfrak{H})$.

To proceed further we need to know the relation between the unit $I \in \mathfrak{B}(\mathfrak{H}_1)$ and WW^*. Since WW^* is an increasing projection for α^d we have $\alpha_t^d(WW^*) \geq \alpha_s^d(WW^*)$ for $t \geq s$. Note either $WW^* = \alpha_t^d(WW^*)$ for all $t \geq 0$ or $\alpha_t^d(WW^*) \neq \alpha_s^d(WW^*)$ if $t \neq s$. In the first case α_t is already an E_o-semigroup and in this case the theorem is trivial and in the second case $I - WW^*$ is of infinite rank since $\alpha_{n+1}^d(WW^*) - \alpha_n^d(WW^*)$ for $n = 1, 2, \cdots$ is a sequence of non zero orthogonal projections less than $I - WW^*$. Hence, $I - WW^*$ is of rank zero or infinity. In the rank zero case the theorem is trivial so we assume $I - WW^*$ is of infinite rank. Then there exist an infinite sequence of partial isometries $E_{i1} \in \mathfrak{B}(\mathfrak{H}_1)$ so that $E_{i1}^*E_{i1} = E_{11} = WW^*$ and $E_{i1}^*E_{j1} = 0$ for $i \neq j$ for $i, j = 1, 2, \cdots$ and

$$\sum_{i=1}^{\infty} E_{i1}E_{i1}^* = I$$

Let $E_{ij} = E_{i1}E_{j1}^*$ for $i, j = 1, 2, \cdots$. Note the E_{ij} form a set of matrix units in $\mathfrak{B}(\mathfrak{H}_1)$. Since $\mathfrak{M}_t$ is a subspace of $\mathfrak{H}_1$ any operator $A \in \mathfrak{B}(\mathfrak{M}_t)$ can be interpreted as an operator A^1 in $\mathfrak{B}(\mathfrak{H}_1)$ by considering A^1 in $\mathfrak{B}(\mathfrak{H}_1)$ to be given by $(f, A^1g) = (f', Ag')$ where f' and g' are the orthogonal projections of f and g onto $\mathfrak{M}_t$. Then let

$$Y(t) = \sum_{i=1}^{\infty} \alpha_t^d(E_{i1})Z(t)\alpha_t^d(E_{1i})$$

where we interpret $Z(t) \in \mathfrak{B}(\mathfrak{M}_t)$ as an operator in $\mathfrak{B}(\mathfrak{H}_1)$ in the manner just described. We will show $W(t) \in \alpha_t^d(\mathfrak{B}(\mathfrak{H}_1))'$. This is seen as follows. From the formula given above one easily checks that $Y(t)$ commutes with the $\alpha_t^d(E_{ij})$ for $i, j = 1, 2, \cdots$. Let F_{ij} for $i, j = 1, \cdots, r$ be a complete set of matrix units for $\mathfrak{B}(\mathfrak{H})$ (note r is the dimension of $\mathfrak{H}$). Then $G_{(in)(jm)} = E_{i1}WF_{nm}W^*E_{1j}$ for $i, j = 1, 2, \cdots$ and $n, m = 1, \cdots, r$ form a complete set of matrix units for $\mathfrak{B}(\mathfrak{H}_1)$. Then $C \in \alpha_t^d(\mathfrak{B}(\mathfrak{H}_1))'$ if and only if C commutes with $\alpha_t^d(G_{(in)(jm)})$ for all values of the indices. Since $Y(t)$ commutes with $\alpha_t^d(E_{ij})$ for all $i, j = 1, 2, \cdots$ we have $Y(t) \in \alpha_t^d(\mathfrak{B}(\mathfrak{H}_1))'$ if and only if $Y(t)$ commutes with $\alpha_t^d(WF_{nm}W^*)$ for $n, m = 1, \cdots, r$. Suppose $1 \leq n, m < r + 1$. Since $Z(t) \in \phi_t(\mathfrak{B}(\mathfrak{H}))'$ we have for $f \in \mathfrak{M}_t$

$$\begin{aligned}\alpha_t^d(WF_{nm}W^*)Y(t)f =& \alpha_t^d(WF_{nm}W^*)Z(t)f = \phi_t(F_{nm})Z(t)f \\ =& Z(t)\phi_t(F_{nm})f = Z(t)\alpha_t^d(WF_{nm}W^*)f \\ =& Y(t)\alpha_t^d(WF_{nm}W^*)f\end{aligned}$$

and since $\alpha_t^d(WF_{nm}W^*) = \alpha_t^d(WF_{mn}^*W^*)$ maps $\mathfrak{M}_t$ into itself $\alpha_t^d(WF_{mn}W^*)$ maps $\mathfrak{M}_t^\perp$ the orthogonal complement of $\mathfrak{M}_t$ into itself we have for $f \in \mathfrak{M}_t^\perp$

$$\alpha_t^d(WF_{nm}W^*)Y(t)f = \alpha_t^d(WF_{nm}W^*)Z(t)f = 0$$

and

$$Y(t)\alpha_t^d(WF_{nm}W^*)f = Z(t)\alpha_t^d(WF_{nm}W^*)f = 0$$

Since these $Y(t)$ and $\alpha_t^d(WF_{nm}W^*)$ commute when applied to $f \in \mathfrak{M}_t$ and to $f \in \mathfrak{M}_t^\perp$ for all $n, m = 1, \cdots, r$ we have $Y(t) \in \alpha_t^d(\mathfrak{B}(\mathfrak{H}_1))'$. We are now we are prepared to define a local cocycle $C(t)$. We first define $C(t)$ for t a dyadic rational (i.e., $t = m2^{-n}$ with n and m integers). Suppose $t = m2^{-n}$. For n, m and p positive integers consider the operators

$$C(m,n,p) = Y(s)\alpha_s^d(Y(s))\alpha_{2s}^d(Y(s)) \cdots \alpha_{qs}^d(Y(s))$$

where $s = 2^{-p}$ and $q = m2^{\max(p-n,0)} - 1$. Since the unit ball of $\mathfrak{B}(\mathfrak{H}_1)$ is σ-weakly compact the above sequence of operators has a weak limit point as $p \to \infty$. Since $\mathfrak{H}_1$ is separable there is a subsequence of the above operators which converge weakly to a limit. Since the dyadic rationals are countable there is by the diagonal sequence argument a sequence $\{p_k : k = 1, 2, \cdots\}$ tending to infinity so that $C(m, n, p_k)$ converges σ-weakly to a limit as $k \to \infty$ for all positive integers m and n. We define

$$C(m2^{-n}) = \lim_{k \to \infty} C(m, n, p_k).$$

Since $Y(s) \in \alpha_s^d(\mathfrak{B}(\mathfrak{H}_1))'$ for $s \geq 0$ a routine computation shows $C(m, n, p) \in \alpha_{(m2^{-n})}(\mathfrak{B}(\mathfrak{H}_1))'$ for all $p > 0$. Hence, $C(t) \in \alpha_t(\mathfrak{B}(\mathfrak{H}_1))'$ for $t = m2^{-n}$. Next we show $C(t)$ is a cocycle. Before we begin we note that although multiplication is not jointly continuous in the σ-weak topology in our case it is. Note that if M is a type I factor then the mapping $C = AB$ for $A \in M$ and $B \in M'$ is jointly continuous in the σ-weak operator topology. To see this note we can represent our Hilbert space $\mathfrak{K}$ as the tensor product of $\mathfrak{K}_1 \otimes \mathfrak{K}_2$ and represent M as $\mathfrak{B}(\mathfrak{K}_1)$ and M' as $\mathfrak{B}(\mathfrak{K}_2)$ so we can express elements of M in the form A $\otimes I$ and elements of M' as $I \otimes B$. Note that for product vectors $F = f_1 \otimes f_2$ and $G = g_1 \otimes g_2$ we have

$$(F, CG) = (F, (A \otimes B)G) = (f_1, Ag_1)(f_2, Bg_2)$$

and we see the above expression is jointly continuous in the σ-weak operator topology. Since linear combinations of product vectors $f_1 \otimes f_2$ are dense in $\mathfrak{K}$ it follows that multiplication is jointly continuous in A and B for $A \in M$ and $B \in M'$. The same argument shows multiplication is jointly continuous in n variables (i.e., $C = A_1A_2 \cdots A_n$) where $A_i \in M_i$ with the M_i mutually commuting type I factors for $i = 1, \cdots, n$. Since α_t^d is σ-weakly continuous for each $t \geq 0$ we have expression of the form

$$A_1\alpha_{t_1}(A_2)\alpha_{t_1+t_2}(A_3) \cdots \alpha_{t_1+\cdots+t_{n-1}}(A_n)$$

with $A_i \in \alpha_{t_i}(\mathfrak{B}(\mathfrak{H}_1))'$ are jointly continuous in the A_i in the σ-weak topology.

We now show $C(t)$ satisfies the cocycle condition on the dyadic rationals. Suppose $t = m2^{-n}$ and $s = k2^{-j}$. Note that for k sufficiently large we have

$$C(m, n, p_k)\alpha_t^d(C(k, j, p_k)) = C(m2^{q-m} + k2^{q-k}, q, p_k).$$

As $k \to \infty$ the three terms above tend σ-weakly to $C(t)$, $\alpha_t(C(s))$ and $C(t+s)$ and since multiplication is jointly continuous in this situation we have $C(t)\alpha_t^d(C(s)) = C(t + s)$.

Next we show $W^*C(t)\alpha_t^d(WAW^*)W = \beta_t(A)$ for t a dyadic rational and $A \in \mathfrak{B}(\mathfrak{H})$. Suppose $t = m2^{-n}$. Now we have for k sufficiently large (so that $p_k \geq n$) we have

$$C(m,n,p)\alpha_t^d(WAW^*) = Y(s)\alpha_s^d(Y(s))\alpha_{2s}^d(Y(s))\cdots\alpha_{(q-1)s}^d(Y(s))\alpha_{qs}^d(WAW^*)$$

where $s = 2^{-p_k}$ and $q = m2^{p_k-n}$. Since $P = WW^*$ is an increasing projection for α^d and the fact that $Y(s) \in \alpha_s^d(\mathfrak{B}(\mathfrak{H}))'$ it follows that

$$\begin{aligned}
&W^*C(m,n,p_k)\alpha_t^d(WAW^*)W \\
&\quad= W^*Y(s)\alpha_s^d(PY(s))\alpha_{2s}^d(PY(s))\cdots\alpha_{qs-s}^d(PY(s))\alpha_{qs}^d(WAW^*)W \\
&\quad= W^*Y(s)\alpha_s^d(PY(s))\alpha_{2s}^d(PY(s))\cdots\alpha_{qs-s}^d(PY(s)\alpha_s^d(WAW^*)P)W \\
&\quad= W^*Y(s)\alpha_s^d(PY(s))\alpha_{2s}^d(PY(s))\cdots\alpha_{qs-2s}^d(Y(s))\alpha_{qs-s}^d(W\beta_s(A)W^*)W \\
&\quad= W^*Y(s)\alpha_s^d(PY(s))\alpha_{2s}^d(PY(s))\cdots\alpha_{qs-2s}^d(Y(s)\alpha_s^d(W\beta_s(A)W^*)P)W \\
&\quad= W^*Y(s)\alpha_s^d(PY(s))\alpha_{2s}^d(PY(s))\cdots\alpha_{qs-2s}^d(W\beta_{2s}(A)W^*)W \\
&\qquad\qquad\cdots\cdots\cdot \\
&\quad= W^*Y(s)\alpha_s^d(PY(s)\alpha_s^d(W\beta_{qs-2s}(A)W^*)P)W \\
&\quad= W^*Y(s)\alpha_s^d(W\beta_{qs-s}(A)W^*)W = \beta_{qs}(A) = \beta_t(A).
\end{aligned}$$

Since $C(n,m,p_k) \to C(t)$ as $k \to \infty$ we have $WC(t)\alpha_t^d(WAW^*)W = \beta_t(A)$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and t a non negative dyadic rational. We now want to extend these results from dyadic rationals to the real numbers. Since α_t^d is σ-strongly continuous in t and $C(t+s) = C(t)\alpha_t(C(s))$ all for t and s non negative dyadic rationals all we need is to show $C(t) \to I$ σ-weakly as $t \to 0+$ in the dyadic rationals. (Note since $C(t+s) = C(t)\alpha_t(C(s)) \geq C(t)$ for s and t positive dyadic rationals it follows that $C(t)$ is decreasing in t so if $C(t)$ converges weakly to I as $t \to 0+$ it converges strongly.) Now we showed earlier how expressions of the form

$$\Xi = W^*\alpha_{t_1}^d(WA_1W^*)\cdots\alpha_{t_n}^d(WA_nW^*)C(t_o)\alpha_{s_1}^d(WB_1W^*)\cdots\alpha_{s_m}^d(WB_mW^*)W$$

can be computed from a knowledge of α and β and if we restrict the variables t_i and s_j to dyadic rationals the same rules apply and we can compute these terms in terms of α and β. We do not actually have to carry out these computations in detail to see that for an expression Ξ above there will be a finite number of β_{t_j} expressions with $t_j \leq t_o$ and since $\beta_s(A) \to A$ σ-strongly as $s \to 0+$ it follows that as $t_o \to 0+$ these expressions behave so that the limit will be the expression for Ξ with $C(t_o)$ replaced by the unit I. Hence, we have $(F, C(t)G) \to (F, G)$ as $t \to 0+$ (t a dyadic rational) for F and G finite linear combination of vectors of the form

$$\alpha_{t_1}^d(WA_1W^*)\cdots\alpha_{t_n}^d(WA_nW^*)Wf$$

with $f \in \mathfrak{H}$ and $A_i \in \mathfrak{B}(\mathfrak{H})$ for $i = 1, \cdots, n$. Since α^d is minimal over the range of W these vectors are dense in $\mathfrak{H}_1$ and since the $C(t)$ are all of norm less than one we have $C(t) \to I$ weakly as $t \to 0+$ with t a dyadic rational. As we have seen this implies $C(t)$ is continuous in t so we can extend $C(t)$ to all the non negative reals by continuity and the cocycle condition $C(t)\alpha_t^d(C(s)) = C(t+s)$ and the relation $\beta_t(A) = W^*C(t)\alpha_t^d(A)W$ holds for all $A \in \mathfrak{B}(\mathfrak{H})$ and for all $t,s \in [0,\infty)$. Hence, we have shown for a subordinate β of α there is a subordinate γ of α^d (and from what we showed before it follows that γ is unique) so that $\gamma_t(A) = C(t)\alpha_t(A)$ with $C(t)$ a local cocycle and $\beta_t(A) = W^*C(t)\alpha_t^d(A)W$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. It is routine to show the isomorphism $\gamma \leftrightarrow \beta$ is an order isomorphism so the proof of the theorem is complete. □

The next lemma gives a way to determine if one CP-semigroup dominates another.

Lemma 3.6. *Suppose α and β are CP-semigroups of $\mathfrak{B}(\mathfrak{H})$. Let Θ be the semigroup of $\mathfrak{B}(\mathfrak{H} \oplus \mathfrak{H})$ given by*

$$\Theta_t(\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}) = \begin{bmatrix} \alpha_t(X_{11}) & \beta_t(X_{12}) \\ \beta_t(X_{21}) & \beta_t(X_{22}) \end{bmatrix}$$

where $X_{ij} \in \mathfrak{B}(\mathfrak{H})$ for $i,j = 1,2$. Then $\alpha \geq \beta$ if and only if Θ_t is completely positive for each $t \geq 0$.

Proof. Suppose α and β are CP-semigroups of $\mathfrak{B}(\mathfrak{H})$ and Θ is defined as above. Note Θ is a semigroup. Using the notation above we have

$$\Theta_t(\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}) = \begin{bmatrix} \alpha_t(X_{11}) - \beta_t(X_{11}) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \beta_t(X_{11}) & \beta_t(X_{12}) \\ \beta_t(X_{21}) & \beta_t(X_{22}) \end{bmatrix}$$

for $t \geq 0$. Hence, if $\alpha \geq \beta$ the above equations shows that Θ_t is the sum of two completely positive maps and, therefore, is completely positive. Conversely, suppose Θ_t is completely positive for each $t \geq 0$. Suppose $t \geq 0$ and $A_i \in \mathfrak{B}(\mathfrak{H})$ and $f_i \in \mathfrak{H}$ and let

$$B_i = \begin{bmatrix} A_i & A_i \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad F_i = \begin{bmatrix} f_i \\ -f_i \end{bmatrix}$$

for $i = 1, \cdots, n$. Then we have

$$\sum_{i,j=1}^{n} (f_i, (\alpha_t(A_i^*A_j) - \beta_t(A_i^*A_j))f_j) = \sum_{i,j=1}^{n} (F_i, \Theta_t(B_i^*B_j)F_j) \geq 0$$

where the last inequality follows from the fact that Θ_t is completely positive. Hence, $A \to \alpha_t(A) - \beta_t(A)$ is completely positive for each $t \geq 0$. □

When A. Connes introduced the notion of outer conjugacy [Co] which we now call cocycle conjugacy one of the important observations Connes made was that two automorphisms α and β of a factor R are outer conjugate if and only if there is an automorphism Θ of $M_2 \otimes R$ of the form

$$\Theta(\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}) = \begin{bmatrix} \alpha(X_{11}) & \gamma(X_{12}) \\ \gamma^*(X_{21}) & \beta(X_{22}) \end{bmatrix}$$

We will make frequent use of Connes' observation in developing criteria for determining when the minimal dilations of two unital CP-semigroups are cocycle conjugate. We introduce the following notation. If $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are Hilbert spaces then an elements $X \in \mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ (all bounded operators on the direct sum of $\mathfrak{H}_1$ and $\mathfrak{H}_2$) can be represented in matrix form as follows.

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

where $X_{ij} \in \mathfrak{B}(\mathfrak{H}_i, \mathfrak{H}_j)$ for $i, j = 1, 2$.

Definition 3.7. Suppose α and β are unital CP-semigroups of $\mathfrak{B}(\mathfrak{H}_1)$ and $\mathfrak{B}(\mathfrak{H}_2)$, respectively. We say γ is a corner from α to β if γ_t is mapping of $\mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ into itself so that the mapping Θ_t given by

$$\Theta_t(\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}) = \begin{bmatrix} \alpha_t(X_{11}) & \gamma_t(X_{12}) \\ \gamma_t^*(X_{21}) & \beta_t(X_{22}) \end{bmatrix}$$

for $t \geq 0$ where $X_{ij} \in \mathfrak{B}(\mathfrak{H}_i, \mathfrak{H}_j)$ for $i, j = 1, 2$ and $\gamma_t^*(X_{21}) = \gamma_t(X_{21}^*)^*$ is a CP-semigroup of $\mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$. Suppose γ is a corner from α to β and Θ is the CP-semigroup defined above and Θ' is a subordinate of Θ where the mapping Θ_t' is given

$$\Theta_t'(\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}) = \begin{bmatrix} \alpha_t'(X_{11}) & \gamma_t(X_{12}) \\ \gamma_t^*(X_{21}) & \beta_t'(X_{22}) \end{bmatrix}$$

for $t \geq 0$ where $X_{ij} \in \mathfrak{B}(\mathfrak{H}_i, \mathfrak{H}_j)$ for $i, j = 1, 2$ and $\gamma_t^*(X_{21}) = \gamma_t(X_{21}^*)^*$. Then we say γ is a maximal corner from α to β if for every subordinate Θ' we have $\alpha' = \alpha$ and we say γ is a hyper maximal corner from α to β if for every subordinate Θ' we have $\alpha' = \alpha$ and $\beta' = \beta$.

We note that if γ is a corner from α to β then γ^* is a corner from β to α and γ is hyper maximal if and only both γ and γ^* are maximal.

Lemma 3.8. *Suppose α and β are E_o-semigroups of $\mathfrak{B}(\mathfrak{H}_1)$ and $\mathfrak{B}(\mathfrak{H}_2)$, respectively. Then α and β are cocycle conjugate if and only if there is a corner γ from α to β so that Θ_t defined by*

$$\Theta_t(\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}) = \begin{bmatrix} \alpha_t(X_{11}) & \gamma_t(X_{12}) \\ \gamma_t^*(X_{21}) & \beta_t(X_{22}) \end{bmatrix}$$

where $X_{ij} \in \mathfrak{B}(\mathfrak{H}_i, \mathfrak{H}_j)$ for $i, j = 1, 2$ and for $t \geq 0$ is an E_o-semigroup of $\mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$.

Proof. Suppose α_t and β_t are E_o-semigroups of $\mathfrak{B}(\mathfrak{H}_1)$ and $\mathfrak{B}(\mathfrak{H}_2)$ which are cocycle conjugate. Then there is an α_t unitary cocycle $S(t)$ and a unitary operator $W \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ so that $\beta_t(A) = W^*S(t)\alpha_t(WAW^*)S(t)^*W$ for all $A \in \mathfrak{B}(\mathfrak{H}_2)$ and $t \geq 0$. Define Θ_t by

$$\Theta_t(\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}) = \begin{bmatrix} \alpha_t(X_{11}) & \alpha_t(X_{12}W^*)S(t)^*W \\ W^*S(t)\alpha_t(WX_{21}) & W^*S(t)\alpha_t(WX_{22}W^*)S(t)^*W \end{bmatrix}$$

where $X_{ij} \in \mathfrak{B}(\mathfrak{H}_i, \mathfrak{H}_j)$ for $i, j = 1, 2$. A routine computation shows that Θ_t is an E_o-semigroup of $\mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ satisfying the conclusion of the theorem.

Conversely, suppose α_t and β_t are E_o-semigroups of $\mathfrak{B}(\mathfrak{H}_1)$ and $\mathfrak{B}(\mathfrak{H}_2)$ and Θ_t is an E_o-semigroup of $\mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ of the form given in the statement of the theorem. Let E_i be the hermitian projection of $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ onto $\mathfrak{H}_i$. So $E_1 + E_2 = I$ the unit in $\mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ and $\Theta_t(E_i) = E_i$ for $i = 1, 2$ and $t \geq 0$. From Theorem 2.4 we have Θ_t is cocycle conjugate to α_t and β_t since α_t and β_t are obtained from Θ_t by restricting Θ_t to $E_1\mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)E_1 = \mathfrak{B}(\mathfrak{H}_1)$ and $E_2\mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)E_2 = \mathfrak{B}(\mathfrak{H}_2)$, respectively. Since α_t and β_t are both cocycle conjugate with Θ_t they are cocycle conjugate with each other. $\square$

Lemma 3.9. *Suppose α and β are $*$-endomorphisms of $\mathfrak{B}(\mathfrak{H}_1)$ and $\mathfrak{B}(\mathfrak{H}_2)$, respectively. Suppose Θ is a completely positive mapping of $\mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ into itself of the form*

$$\Theta(\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}) = \begin{bmatrix} \alpha(X_{11}) & \gamma(X_{12}) \\ \gamma^*(X_{21}) & \beta(X_{22}) \end{bmatrix}$$

Then

$$\gamma(X_{11}X_{12}X_{22}) = \alpha(X_{11})\gamma(X_{12})\beta(X_{22}) \tag{3.1}$$

for all $X_{ij} \in \mathfrak{B}(\mathfrak{H}_i, \mathfrak{H}_j)$ for $i, j = 1, 2$.

Proof. Suppose α and β are $*$-endomorphisms of $\mathfrak{B}(\mathfrak{H}_1)$ and $\mathfrak{B}(\mathfrak{H}_2)$, respectively, and Θ is a completely positive mapping of $\mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ into itself of the form given above. Then from the Stinespring construction there is a $*$-representation π of $\mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ on a Hilbert space $\mathfrak{H}_3$ and operator $V \in \mathfrak{B}(\mathfrak{H}_3, \mathfrak{H}_1 \oplus \mathfrak{H}_2)$ so that $\Theta(A) = V^*\pi(A)V$ for all $A \in \mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ and $\mathfrak{H}_3$ is the closed span of vectors of the form $\pi(A)Vf$ with $A \in \mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ and $f \in \mathfrak{H}_1 \oplus \mathfrak{H}_2$. Let P_i be the orthogonal projection of $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ onto $\mathfrak{H}_i$. Given the form of Θ given above we see that $\Theta(P_i)P_i = \Theta(P_i)$ for $i = 1, 2$. Suppose $A \in \mathfrak{B}(\mathfrak{H}_1)$ and $B \in \mathfrak{B}(\mathfrak{H}_2)$ and

$$X = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Since $\Theta(X)^*\Theta(X) = \Theta(X^*X)$ we have $V^*\pi(X)^*VV^*\pi(X)V = V^*\pi(X)^*\pi(X)V$. Since V is a contraction we have $VV^*\pi(X)V = \pi(X)V$ and $V^*\pi(X) = V^*\pi(X)VV^*$ for all X of the above form. Suppose $T \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$. Then we have

$$\Theta(XTX) = V^*\pi(XTX)V = V^*\pi(X)VV^*\pi(T)VV^*\pi(X)V = \Theta(X)\Theta(T)\Theta(X).$$

Hence $\gamma(ATB) = \alpha(A)\gamma(T)\beta(B)$ for all $A \in \mathfrak{B}(\mathfrak{H}_1)$ and $B \in \mathfrak{B}(\mathfrak{H}_2)$. $\square$

Next we show that a mapping satisfying (3.1) is automatically σ-strongly continuous.

Lemma 3.10. *Suppose α and β are $*$-endomorphisms of $\mathfrak{B}(\mathfrak{H}_1)$ and $\mathfrak{B}(\mathfrak{H}_2)$, respectively. Suppose γ is a linear mapping of $\mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ into itself satisfying (3.1). Then γ and γ^* are σ-strongly continuous.*

Proof. Suppose the hypothesis of the lemma is satisfied. First let us assume the dimension of $\mathfrak{H}_1$ does not exceed the dimension of $\mathfrak{H}_2$. Then there is isometry W

on $\mathfrak{H}_1$ into $\mathfrak{H}_2$. Suppose ω is a normal state of $\mathfrak{B}(\mathfrak{H}_2)$. Then for $T \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ we have $\gamma(T) = \gamma(W^*WT) = \gamma(W^*)\beta(WT)$ and

$$\begin{aligned}\omega(\gamma(T)^*\gamma(T)) =&\omega(\beta(WT)^*\gamma(W^*)^*\gamma(W^*)\beta(WT)) \leq \|\gamma(W^*)\|^2\omega(\beta(WT)^*\beta(WT))\\ =&\|\gamma(W^*)\|^2\omega(\beta(T^*W^*WT)) = \|\gamma(W^*)\|^2\omega(\beta(T^*T))\\ =&\|\gamma(W^*)\|^2\hat{\beta}(\omega)(T^*T)\end{aligned}$$

Since β is σ-weakly continuous $\hat{\beta}(\omega)$ is normal so we have $\omega(\gamma(T)^*\gamma(T)) < \epsilon$ provided $\|\gamma(W^*)\|^2\hat{\beta}(\omega)(T^*T) < \epsilon$ so γ is σ-strongly continuous.

Next suppose the dimension of $\mathfrak{H}_2$ does not exceed the dimension of $\mathfrak{H}_1$. Then there is isometry W on $\mathfrak{H}_2$ into $\mathfrak{H}_1$. Suppose ω is a normal state of $\mathfrak{B}(\mathfrak{H}_2)$. Then for $T \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ we have $\gamma(T) = \gamma(TW^*W) = \alpha(TW^*)\gamma(W)$ and

$$\omega(\gamma(T)^*\gamma(T)) = \omega(\gamma(W)^*\alpha(TW^*)^*\alpha(TW)\gamma(W)) = \omega(\gamma(W)^*\alpha(W^*T^*TW)\gamma(W))$$

Now ρ defined by $\rho(A) = \omega(\gamma(W)^*A\gamma(W))$ for $A \in \mathfrak{B}(\mathfrak{H}_1)$ is σ-weakly continuous we have

$$\omega(\gamma(T)^*\gamma(T)) = \rho(\alpha(W^*T^*TW)) = \hat{\alpha}(\rho)(W^*T^*TW)$$

Since α is σ-weakly continuous $\hat{\alpha}(\rho)$ is σ -weakly continuous. Since the mappings $A \rightarrow W^*AW$ is σ-weakly continuous we have $A \rightarrow \hat{\alpha}(\rho)(W^*AW)$ is σ-weakly continuous. Then $\omega(\gamma(T)^*\gamma(T)) < \epsilon$ if $\hat{\alpha}(\rho)(W^*T^*TW) < \epsilon$ so γ is σ-strongly continuous.

The proof that γ^* is σ-strongly continuous is the same as the proof for γ except that the roles of α and β are interchanged. $\square$

Lemma 3.11. *Suppose α and β are $*$-endomorphisms of $\mathfrak{B}(\mathfrak{H}_1)$ and $\mathfrak{B}(\mathfrak{H}_2)$, respectively. Suppose γ is a linear mapping of $\mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ into itself satisfying (3.1). Suppose $W \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ is a rank one operator normalized so that $\|W\| = 1$ (so W is a partial isometry). Then $\gamma(W) = \alpha(WW^*)\gamma(W)\beta(W^*W)$. Conversely, suppose $S \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ and $S = \alpha(WW^*)S\beta(W^*W)$. Then there is a unique linear mapping of $\mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ into itself satisfying (3.1) so that $\gamma(W) = S$.*

Proof. Suppose α, β and W satisfying the conditions of the lemma. Suppose γ is a linear mapping of $\mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ into itself satisfying (3.1). Then

$$\gamma(W) = \gamma(WW^*WW^*W) = \alpha(WW^*)\gamma(W)\beta(W^*W).$$

Conversely, suppose is $S \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ and $S = \alpha(WW^*)S\beta(W^*W)$. Let $\{e_i : i = 1, 2, \cdots\}$ and $\{f_j : j = 1, 2, \cdots\}$ be an orthonormal bases for $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively, so that $Wf_1 = e_1$. We define matrix units $T_{ij}f = (f_j, f)e_i$ and $F_{ij}f = (f_j, f)f_i$ for all $f \in \mathfrak{H}_2$ and $E_{ij}f = (e_j, f)e_i$ for all $f \in \mathfrak{H}_1$. We define $\gamma(T_{ij}) = \alpha(E_{i1})S\beta(F_{1j})$ for all i and j in their appropriate range. Now suppose T is a finite linear combination of the T_{ij} and we define $\gamma(T)$ by linearity as

$$T = \sum_{i,j=1} t_{ij}T_{ij} \qquad \text{so} \qquad \gamma(T) = \sum_{i,j=1} t_{ij}\alpha(E_{i1})S\beta(F_{1j})$$

Suppose ω is a normal state of $\mathfrak{B}(\mathfrak{H}_2)$. Then we have

$$\begin{aligned}\omega(\gamma(T)^*\gamma(T)) &= \sum_{i,j,n,m=1} \overline{t_{nm}}t_{ij}\omega(\beta(F_{n1})S^*\alpha(E_{1m})\alpha(E_{i1})S\beta(F_{1j}))\\ &= \sum_{j,n,m=1} \overline{t_{nm}}t_{mj}\omega(\beta(F_{n1})S^*S\beta(F_{1j}))\\ &= \sum_{m=1}\omega(\beta(X_m)^*S^*S\beta(X_m)) \leq \|S^*S\| \sum_{m=1}\omega(\beta(X_m^*X_m))\end{aligned}$$

where

$$X_m = \sum_{j=1} t_{mj}F_{1j}.$$

We have

$$T^*T = \sum_{m=1} X_m^*X_m$$

so we have $\omega(\gamma(T)^*\gamma(T)) \leq \|S\|^2\omega(\beta(T^*T))$. Suppose $T \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$. Let

$$E_n = \sum_{i=1}^n E_{ii} \qquad \text{and} \qquad F_n = \sum_{i=1}^n F_{ii} \qquad \text{and} \qquad T_n = E_nTF_n$$

Then $T_n \to T$ in the σ-strong topology as $n \to \infty$. Since the mapping $C \to \beta(C)$ is continuous in the σ-strong topology and since

$$\omega(\gamma(T_n - T_m)^*\gamma(T_n - T_m)) \leq \|S\|^2\omega(\beta((T_n - T_m)^*(T_n - T_m)))$$

we have $\gamma(T_n)$ converges to a limit which we call $\gamma(T)$ σ-strongly as $n \to \infty$. We have by direct calculation that $\gamma(E_{ij}T_{nm}F_{rs}) = \alpha(E_{ij})\gamma(T_{nm})\beta(F_{rs})$ and by σ-strong continuity and linearity this relation extends to the relation $\gamma(ATB) = \alpha(A)\gamma(T)\beta(B)$ for all $A \in \mathfrak{B}(\mathfrak{H}_1)$, $B \in \mathfrak{B}(\mathfrak{H}_2)$ and $T \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$.

If γ' is a second mapping satisfying (3.1) and such that $\gamma'(W) = S$. Then recalling the construction of γ we see that $\gamma'(T) = \gamma(T)$ for all T which are finite linear combinations of the T_{ij}. From the previous lemma we know that γ' is σ-strongly continuous and so $\gamma' = \gamma$. Hence, the mapping γ satisfying the stated conditions is unique. □

Lemma 3.12. *Suppose α and β are unital $*$-endomorphisms of $\mathfrak{B}(\mathfrak{H}_1)$ and $\mathfrak{B}(\mathfrak{H}_2)$, respectively. Suppose Θ is a completely positive mapping of $\mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ into itself of the form*

$$\Theta(\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22}\end{bmatrix}) = \begin{bmatrix} \alpha(X_{11}) & \gamma(X_{12}) \\ \gamma^*(X_{21}) & \beta(X_{22})\end{bmatrix}$$

Suppose $W \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ is unitary and $\gamma(W) \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ is also unitary. Then Θ is a unital $*$-endomorphism of $\mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ into itself. Conversely, if Θ is a unital $*$-endomorphism of $\mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ into itself then $\gamma(W)$ is unitary for every unitary operator $W \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$.

Proof. Suppose the hypothesis and notation of the lemma is satisfied. Below we define S and compute $\Theta(S)$

$$\Theta(S) = \Theta(\begin{bmatrix} 0 & W \\ W^* & 0\end{bmatrix}) = \begin{bmatrix} 0 & \gamma(W) \\ \gamma(W)^* & 0\end{bmatrix}.$$

Since $\gamma(W)$ is unitary $\Theta(S)$ is unitary. As we pointed out at the beginning of this section it then follows that $\Theta(XS) = \Theta(X)\Theta(S)$ and $\Theta(SX) = \Theta(S)\Theta(X)$ for all $X \in \mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$. Applying this to the case where X has entries $A \in \mathfrak{B}(\mathfrak{H}_1)$ and $B \in \mathfrak{B}(\mathfrak{H}_2)$ in the upper left hand corner and lower right hand corner, respectively, and the zero operator in the off diagonal entries we find $\gamma(AW) = \alpha(A)\gamma(W)$, $\gamma(WB) = \gamma(W)\beta(B)$, $\gamma^*(BW) = \beta(B)\gamma(W)^*$ and $\gamma^*(WA) = \gamma(W)^*\alpha(A)$ for all $A \in \mathfrak{B}(\mathfrak{H}_1)$ and $B \in \mathfrak{B}(\mathfrak{H}_2)$. Now suppose $A \in \mathfrak{B}(\mathfrak{H}_1)$, $B \in \mathfrak{B}(\mathfrak{H}_2)$ and $T \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$. Then we have

$$\begin{aligned}\gamma(ATB) =&\gamma(ATBW^*W) = \alpha(ATBW^*)\gamma(W) = \alpha(A)\alpha(TBW^*)\gamma(W)\\ =&\alpha(A)\gamma(TBW^*W) = \alpha(A)\gamma(TB) = \alpha(A)\gamma(WW^*TB)\\ =&\alpha(A)\gamma(W)\beta(W^*TB) = \alpha(A)\gamma(W)\beta(W^*T)\beta(B)\\ =&\alpha(A)\gamma(WW^*T)\beta(B) = \alpha(A)\gamma(T)\beta(B)\end{aligned}$$

A similar calculation shows that if $A \in \mathfrak{B}(\mathfrak{H}_1)$, $B \in \mathfrak{B}(\mathfrak{H}_2)$ and $T \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ then $\gamma^*(BT^*A) = \beta(B)\gamma^*(T^*)\alpha(A) = \beta(B)\gamma(T)^*\alpha(A)$. Now suppose $X_{ij} \in \mathfrak{B}(\mathfrak{H}_i, \mathfrak{H}_j)$ for $i, j = 1, 2$. Then we have

$$\begin{aligned}\gamma(X_{12})\gamma^*(X_{21}) =&\gamma(X_{12}W^*W)\gamma^*(W^*WX_{21}) = \alpha(X_{12}W^*)\gamma(W)\gamma(W)^*\alpha(WX_{21})\\ =&\alpha(X_{12}W^*)\alpha(WX_{21}) = \alpha(X_{12}W^*WX_{21}) = \alpha(X_{12}X_{21})\end{aligned}$$

and

$$\begin{aligned}\gamma^*(X_{21})\gamma(X_{12}) =&\gamma^*(X_{21}WW^*)\gamma(WW^*X_{12}) = \beta(X_{21}W)\gamma(W)^*\gamma(W)\beta(W^*X_{12})\\ =&\beta(X_{21}W)\beta(W^*X_{12}) = \beta(X_{21}WW^*X_{12}) = \beta(X_{21}X_{12})\end{aligned}$$

Using the facts that α and β are $*$-endomorphisms and the properties of γ and γ^* established above it now just a matrix computation to show that $\Theta(X)\Theta(Y) = \Theta(XY)$ for all $X, Y \in \mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$.

Conversely, suppose Θ is a unital $*$-endomorphism of $\mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ and $W \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$. It is now just a routine computation to show that $\gamma(W)$ is unitary. $\square$

Theorem 3.13. *Suppose α and β are unital CP-semigroups of $\mathfrak{B}(\mathfrak{H}_1)$ and $\mathfrak{B}(\mathfrak{H}_2)$ with minimal dilations α^d and β^d to E_o-semigroups of $\mathfrak{B}(\mathfrak{H}_{11})$ and $\mathfrak{B}(\mathfrak{H}_{21})$, respectively. Then α^d and β^d are cocycle conjugate if and only if there is a hyper maximal corner γ from α to β where hyper maximal corners were defined in Definition 3.7.*

Proof. Assume the notation given in the statement of the theorem and assume α^d and β^d are cocycle conjugate. The relation between the CP-semigroup α of $\mathfrak{B}(\mathfrak{H}_1)$ and the minimal dilation α^d which is an E_o-semigroup of $\mathfrak{B}(\mathfrak{H}_{11})$ is given by

$$\alpha_t(A) = W_1^*\alpha_t^d(W_1AW_1^*)W_1$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ where W_1 is an isometry of $\mathfrak{H}$ into $\mathfrak{H}_1$ so that $W_1W_1^*$ is an increasing projection for α^d (i.e., $\alpha_t^d(W_1W_1^*) \geq W_1W_1^*$ for all $t \geq 0$) and α^d is minimal over the range of $W_1W_1^*$ and the relation between β and β^d is the same with W_1 replaced with W_2. Since α^d and β^d are cocycle conjugate there is by Lemma 3.8 a corner γ^d from α^d to β^d so that the mapping Θ^d given by

$$\Theta_t^d(\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}) = \begin{bmatrix} \alpha_t^d(X_{11}) & \gamma_t^d(X_{12}) \\ \gamma_t^{*d}(X_{21}) & \beta_t^d(X_{22}) \end{bmatrix}$$

where X_{ij} is a bounded operator from $\mathfrak{H}_{j1}$ to $\mathfrak{H}_{i1}$ for $t \geq 0$ is an E_o-semigroup of $\mathfrak{B}(\mathfrak{H}_{11} \oplus \mathfrak{H}_{21})$. Let W be the isometry from $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ to $\mathfrak{H}_{11} \oplus \mathfrak{H}_{21}$ given by $W\{f, g\} = \{W_1 f, W_2 g\}$ for $f \in \mathfrak{H}_1$ and $g \in \mathfrak{H}_2$. Then since $W_1 W_1^*$ is an increasing projection for α^d and $W_2 W_2^*$ is an increasing projection for β^d we have

$$\Theta_t^d(WW^*) = \Theta_t^d(\begin{bmatrix} W_1 W_1^* & 0 \\ 0 & W_2 W_2^* \end{bmatrix}) = \begin{bmatrix} \alpha_t^d(WW^*) & 0 \\ 0 & \beta_t^*(W_2 W_2^*) \end{bmatrix} \geq WW^*$$

for each $t \geq 0$ so WW^* is an increasing projection for Θ^d. Note that since α^d is minimal over the range of W_1 and β^d is minimal over the range of W_2 we see Θ^d is minimal over the range of W. Let Θ be given by $\Theta_t(A) = W^* \Theta_t^d(WAW^*)W$ for $A \in \mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ and $t \geq 0$. We see that Θ is of the form

$$\Theta_t(\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}) = \begin{bmatrix} \alpha_t(X_{11}) & \gamma_t(X_{12}) \\ \gamma_t^*(X_{21}) & \beta_t(X_{22}) \end{bmatrix}$$

where $X_{ij} \in \mathfrak{B}(\mathfrak{H}_i, \mathfrak{H}_j)$ for $i, j = 1, 2$ and $\gamma_t(X_{12}) = W_1^* \gamma_t^d(W_1 X_{12} W_2^*) W_2$ for $X_{12} \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ and $t \geq 0$. Now suppose Θ' is a subordinate of the form given in the statement of the theorem. Then from Theorem 3.5 there is a subordinate Θ'^d of Θ^d so that

$$\Theta'_t(A) = W^* \Theta_t'^d(WAW^*)W$$

for all $A \in \mathfrak{B}(\mathfrak{H}_{11} \oplus \mathfrak{H}_{21})$ and $t \geq 0$ and from Theorem 3.4 there is a local cocycle C so that $\Theta_t'^d(A) = \Theta_t^d(A)C(t)$ for all $A \in \mathfrak{B}(\mathfrak{H}_{11} \oplus \mathfrak{H}_{21})$ and $t \geq 0$. Now $C(t)$ can be written in matrix form so

$$C(t) = \begin{bmatrix} C_{11}(t) & C_{12}(t) \\ C_{21}(t) & C_{22}(t) \end{bmatrix}$$

for $t \geq 0$. Writing out the equation $C(t)\Theta_t^d(X) = \Theta_t^d(X)C(t)$ in matrix form one obtains four equations with four variables X_{ij} for $i, j =$ 1,2. Examination of these equation yields that facts $C_{12}(t) = 0$, $C_{21}(t) = 0$ and $C_{22}(t) = \gamma_t^d(S)^* C_{11}(t) \gamma_t^d(S)$ where S is a unitary form $\mathfrak{H}_{21}$ to $\mathfrak{H}_{11}$. (Note $\gamma_t(S)$ is also a unitary from $\mathfrak{H}_{21}$ to $\mathfrak{H}_{11}$ follows from the fact that Θ^d is a unital E_o-semigroup.) Since $\Theta_t'^d(A) = C(t)\Theta_t^d(A)$ for $A \in \mathfrak{B}(\mathfrak{H}_{11} \oplus \mathfrak{H}_{21})$ and the corner of Θ' is γ by assumption we have $\gamma_t(A) = W_1^* C_{11}(t) \gamma_t^d(W_1 A W_2^*) W_2 = W_1^* \gamma_t^d(W_1 A W_2^*) W_2$ for all $t \geq 0$ and all bounded linear operators $A \in \mathfrak{B}(\mathfrak{H}_1, \mathfrak{H}_2)$. Consider the somewhat complicated expression below.

$$\begin{aligned} \Xi = W_1^* \Theta_{t_1}^d(W_1 A_1 W_1^*) \cdots \Theta_{t_n}^d(W_1 A_n W_1^*) C(t) \Theta_t^d(\Theta_{s_1}^d(W_1 B_1 W_1^*) \cdots \\ \cdots \Theta_{s_m}^d(W_1 B_m W_1^*) W_1 A W_2^* \Theta_{x_1}^d(W_2 R_1 W_2^*) \cdots \\ \cdots \Theta_{x_p}^d(W_2 R_p W_2^*)) \Theta_{y_1}^d(W_2 S_1 W_2^*) \cdots \Theta_{y_q}^d(W_2 S_q W_2^*) W_2 \end{aligned}$$

for A, A_i, B_j, R_k, $S_l \in \mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ and $t > 0$ and $t_i \geq 0$, $s_j \geq 0$, $x_k \geq 0$, $y_l \geq 0$ for i, j, k and l in there respective ranges. First we note that from a knowledge of all such terms we can compute $C_{11}(t)$. This can be seen by noting that in the above expression the linear combinations of the terms in the brackets following $C(t)\Theta_t^d$ are σ-strongly dense in the space of linear operators from $\mathfrak{H}_{21}$ to $\mathfrak{H}_{11}$ so we can compute $C(t)\Theta_t^d(X_{12})$ for X_{12} any operator from $\mathfrak{H}_{21}$ to $\mathfrak{H}_{11}$. Since this operator is determined by $C_{11}(t)$ it follows that we can compute $C_{11}(t)$ from a knowledge of the above terms.

Next note that

$$
\begin{aligned}
&C(t)\Theta_t(\Theta_{s_1}^d(W_1B_1W_1^*)\cdots \\
&\qquad \cdots\Theta_{s_m}^d(W_1B_mW_1^*)W_1AW_2^*\Theta_{x_1}^d(W_2R_1W_2^*)\cdots\Theta_{x_p}^d(W_2R_pW_2^*)) \\
&\qquad = C(t)\Theta_{s_1'}^d(W_1B_1W_1^*)\cdots \\
&\qquad \cdots\Theta_{s_m'}^d(W_1B_mW_1^*)\Theta_t^d(W_1AW_2^*)\Theta_{x_1'}^d(W_2R_1W_2^*)\cdots\Theta_{x_p'}^d(W_2R_pW_2^*) \\
&\qquad = \Theta_{s_1'}^d(W_1B_1W_1^*)\cdots \\
&\qquad \cdots\Theta_{s_m'}^d(W_1B_mW_1^*)C(t)\Theta_t^d(W_1AW_2^*)\Theta_{x_1'}^d(W_2R_1W_2^*)\cdots\Theta_{x_p'}^d(W_2R_pW_2^*)
\end{aligned}
$$

where a prime on a variable means the unprimed variable plus t (e.g., $s_i' = s_i + t$). Then we see that Ξ can be expressed in the simpler form

$$
\begin{aligned}
\Xi = W_1^*\Theta_{t_1}^d(W_1A_1W_1^*)\cdots\Theta_{t_n}^d(W_1A_nW_1^*)C(t)\Theta_t^d(W_1AW_2^*)\Theta_{s_1}^d(W_2B_1W_2^*)\cdots \\
\cdots\Theta_{s_n}^d(W_2B_nW_2^*)W_2
\end{aligned}
$$

where the new $A's$ are made of the original $A's$ and $B's$ and the new $B's$ are made up of the original $R's$ and $S's$ and the new $t's$ are made up of the original t and $s's$ and the new $s's$ are made up of the original t and $x's$. Now in calculating Ξ by the methods described earlier we see that Ξ can be calculated from a knowledge of

$$
W_1^*\Theta_t^d(W_1AW_1^*)W_1,\ W_2^*\Theta_t^d(W_2AW_2^*)W_2
$$

and

$$
W_1^*\Theta_t'^d(W_1^*AW_2)W_2 = W_1^*C(t)\Theta_t^d(W_1^*AW_2)W_2
$$

for $A \in \mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ and $t \geq 0$. Now the first two are given by $\alpha_t(A)$ and $\beta_t(A)$, respectively. And the third one is given by $\gamma_t(A)$ by the assumption of the theorem. Hence, Ξ is computable from α, β and γ. Now if we calculate the expression for Ξ and replace $C(t)$ with the unit we get the same expression. Since in calculating an expression which determines $C_{11}(t)$ we get the same expression if we replace $C_{11}(t)$ with the unit it follows that $C_{11}(t) = I$. At this point we can only conclude $C_{11}(t) = I$ because in these expressions we have restricted our attention to terms where $C(t)$ lies between vectors in $\mathfrak{H}_{11}$. Now we have seen that $C_{12}(t) = 0$ and $C_{21}(t) = 0$ and

$$
C_{22}(t) = \gamma_t^d(S)^*C_{11}(t)\gamma_t^d(S) = \gamma_t^d(S)^*\gamma_t^d(S) = \beta_t^d(S^*S) = \beta_t^d(I) = I.
$$

Hence, $C(t) = I$ so $\Theta' = \Theta$.

Now we prove the reverse implication. Suppose γ is a corner from α to β satisfying the condition of the theorem. Suppose Θ is given in terms of α, β and γ as in Definition 3.7 and Θ^d is the minimal dilation of Θ to an E_o-semigroup of $\mathfrak{H}_3$ so we have a isometry $W \in \mathfrak{B}(\mathfrak{H}_3, \mathfrak{H}_1 \oplus \mathfrak{H}_2)$ so WW^* is an increasing projection for Θ^d and Θ^d is minimal over the range of W and $\Theta_t(A) = W^*\Theta_t^d(WAW^*)W$ for all $A \in \mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ and $t \geq 0$. Let P_1 and P_2 be the projections of $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ onto $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. Let $W_1 = WP_1$ and $W_2 = WP_2$. Since α is unital and α is the top left corner of Θ we have $\Theta_t(P_1) = P_1$ so we have

$$W_1W_1^*\Theta_t^d(W_1W_1^*)W_1W_1^* = W_1W_1^*\Theta_t^d(WP_1W^*)W_1W_1^* = W_1\Theta_t(P_1)W_1^* = W_1W_1^*$$

for $t \geq 0$ so $W_1W_1^*$ is an increasing projection for Θ^d. Since β is unital we have by the same argument that $W_2W_2^*$ is an increasing projection for Θ^d. Next we note that $\Theta_t(W_1W_1^*)\Theta_s(W_2W_2^*) = 0$ for all $s, t \geq 0$. To see this first note that

$$\Theta_t(W_1W_1^*)W_2W_2^*\Theta_t(W_1W_1^*) \leq \Theta_t(W_1W_1^*W_2W_2^*W_1W_1^*) = 0$$

for all $t \geq 0$. So $\Theta_t(W_1W_1^*)W_2W_2 = 0$ for all $t \geq 0$. The same argument shows $\Theta_t(W_2W_2^*)W_1W_1 = 0$ for all $t \geq 0$. Then we have for $0 \leq t \leq s$ that

$$\Theta_t(W_1W_1^*)\Theta_s(W_2W_2^*) = \Theta_t(W_1W_1^*\Theta_{s-t}(W_2W_2^*)) = 0.$$

A similar argument gives the result for $0 \leq s \leq t$. Hence, $\Theta_t(W_1W_1^*)\Theta_s(W_2W_2^*) = 0$ for all $t, s \geq 0$. Let $\mathfrak{N}_i$ be the closed subspace of $\mathfrak{H}_3$ spanned by the vectors

$$Y_i(\Xi) = \Theta_{t_1}^d(W_iA_1W_i^*) \cdots \Theta_{t_n}^d(W_iA_nW_i^*)W_if$$

for $i = 1, 2$, $f \in \mathfrak{H}_1 \oplus \mathfrak{H}_2$, $t_k \geq 0$, $A_k \in \mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ for $k = 1, \cdots, n$ and $n = 1, 2, \cdots$. When we refer to $Y_i(\Xi)$ we mean the vector above. We give this vector a name so we do not have to repeatedly repeat all the quantifiers associated with this vector. Since $\Theta_t^d(W_iW_i^*)Y_i(\Xi) = Y_i(\Xi)$ for $t \geq t_1$ for $i = 1, 2$ and $\Theta_t^d(W_1W_1^*)$ and $\Theta_t^d(W_2W_2^*)$ have orthogonal ranges it follows that $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are orthogonal subspaces. Let $\mathfrak{N}$ be the span of $\mathfrak{N}_1$ and $\mathfrak{N}_2$ and let Q, Q_1 and Q_2 be the orthogonal projections of $\mathfrak{H}_3$ onto $\mathfrak{N}$, $\mathfrak{N}_1$ and $\mathfrak{N}_2$, respectively. We show Q_1 is an increasing projection for Θ^d. Consider the vector $Y_1(\Xi)$ above. Note $Q_1Y_1(\Xi) = Y_1(\Xi)$ for all such vectors $Y_1(\Xi)$. Let $s = \min(t_1, \cdots, t_n)$. Suppose $0 \leq t \leq s$. Then

$$\begin{aligned}\Theta_t^d(Q_1)Y_1(\Xi) =& \Theta_t^d(Q_1\Theta_{t_1'}^d(W_1A_1W_1^*) \cdots \Theta_{t_n'}^d(W_1A_nW_1^*))W_1f \\ =& \Theta_t^d(Q_1\Theta_{t_1'}^d(W_1A_1W_1^*) \cdots \Theta_{t_n'}^d(W_1A_nW_1^*)W_1W_1^*)W_1f\end{aligned}$$

where $t_k' = t_k - t$ for $k = 1, \cdots, n$. Since

$$Q_1\Theta_{t_1'}^d(W_1A_1W_1^*) \cdots \Theta_{t_n'}^d(W_1A_nW_1^*)W_1 = \Theta_{t_1'}^d(W_1A_1W_1^*) \cdots \Theta_{t_n'}^d(W_1A_nW_1^*)W_1$$

we have $\Theta_t^d(Q_1)Y_1(\Xi) = Y_1(\Xi)$ for $0 \leq t \leq s$. Now suppose $t \geq s$ and $t_k = s$. Then we have

$$\Theta_t^d(Q_1)Y_1(\Xi) = \Theta_s^d(\Theta_{t'}^d(Q_1)\Theta_{t_1'}^d(W_1A_1W_1^*) \cdots W_1A_kW_1^* \cdots \Theta_{t_n'}^d(W_1A_nW_1^*))W_1f$$

where $t' = t - s$ and $t'_j = t_j - s$ for $j = 1, \cdots, n$. Hence, $\Theta_t^d(Q_1)Y_1(\Xi) = Y_1(\Xi)$ provided

$$\begin{aligned}\Theta_{t'}(Q_1)\Theta^d_{t'_1}(W_1A_1W_1^*)\cdots\Theta^d_{t'_{k-1}}(W_1A_{k-1}W_1^*)W_1& \\ =\Theta^d_{t'_1}(W_1A_1W_1^*)\cdots\Theta^d_{t'_{k-1}}(W_1A_{k-1}W_1^*)W_1.&\end{aligned}$$

And using this reduction formula repeatedly we can reduce to only one term so we have $\Theta_t^d(Q_1)Y_1(\Xi) = Y_1(\Xi)$ if

$$\Theta_x^d(Q_1)\Theta_y^d(W_1BW_1^*)W_1 = \Theta_y^d(W_1BW_1^*)W_1$$

for all $x, y \geq 0$ and $B \in \mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$. We have already shown that if $x \leq y$ the above equality holds so we consider the case $0 \leq y < x$. But then we have

$$\begin{aligned}\Theta_x^d(Q_1)\Theta_y^d(W_1BW_1^*)W_1 =&\Theta_y^d(\Theta^d_{x-y}(Q_1)W_1BW_1^*))W_1 \\ =&\Theta_y^d(W_1BW_1^*)W_1.\end{aligned}$$

Since $Q_1\Theta^d_{x-y}(W_1BW_1^*)W_1 = \Theta^d_{x-y}(W_1BW_1^*)W_1$ we have proved the above equality for all $x, y \geq 0$ and, hence, $\Theta_t^d(Q_1)Y_1(\Xi) = Y_1(\Xi)$ for all $t \geq 0$ and vectors $Y_1(\Xi)$. Hence, $\Theta_t^d(Q_1) \geq Q_1$ and Q_1 is an increasing projection for Θ^d. The same argument shows Q_2 is an increasing projection for Θ^d. It follows that $Q = Q_1 + Q_2$ is an increasing projection for Θ^d. Now let Θ^b be the CP-semigroup of $\mathfrak{N}$ given by the compression of Θ^d to $\mathfrak{N}$ so $\Theta_t^b(A) = Q\Theta_t^d(A)Q$ for all $A \in \mathfrak{B}(\mathfrak{N})$ where we identify $\mathfrak{B}(\mathfrak{N})$ with the hereditary subalgebra of $\mathfrak{B}(\mathfrak{H}_3)$ of all operators $A \in \mathfrak{B}(\mathfrak{H}_3)$ so that $A = QAQ$.

We see that Θ^b is an intermediate CP-semigroup between Θ and Θ^d. Note that corresponding to the decomposition $\mathfrak{N} = \mathfrak{N}_1 \oplus \mathfrak{N}_2$ we have a matrix decomposition of Θ^b in the form

$$\Theta_t^b(\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22}\end{bmatrix}) = \begin{bmatrix} \alpha_t^d(X_{11}) & \eta_t(X_{12}) \\ \eta_t^*(X_{21}) & \beta_t^d(X_{22})\end{bmatrix}$$

where X_{ij} is a bounded operator from $\mathfrak{N}_j$ to $\mathfrak{N}_i$ for $t \geq 0$. Checking the construction of the minimal dilation we see that the upper left hand corner above is α^d the minimal dilation of α to an E_o-semigroup. Similarly the lower right hand corner is β^d the minimal dilation of β. Also, one checks that the minimal dilation of Θ^b to a E_o-semigroup is Θ^d. From Theorem 3.5 we have there is an order isomorphism from the subordinates of Θ to the subordinates of Θ^d and an order isomorphism from the subordinates of Θ^b to the subordinates of Θ^d and, therefore, there is an order isomorphism from the subordinates of Θ to the subordinates of Θ^b. Suppose S is a unitary operator from $\mathfrak{N}_2$ to $\mathfrak{N}_1$. For each $t \geq 0$ we define

$$\Theta_t^c(\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22}\end{bmatrix}) = \begin{bmatrix} \alpha_t^d(X_{11}) & \eta_t(X_{12}) \\ \eta_t^*(X_{21}) & \eta_t(S)^*\alpha_t^d(SX_{22}S^*)\eta_t(S)\end{bmatrix}$$

where X_{ij} is a bounded operator from $\mathfrak{N}_j$ to $\mathfrak{N}_i$ for $i, j = 1, 2$. In the following argument when we write X_{ij} we mean an arbitrary bounded linear operator from

$\mathfrak{N}_j$ to $\mathfrak{N}_i$ so we will not continually write out the specification for X_{ij}. Similarly when we write t we mean an arbitrary $t \geq 0$. We will show Θ^c is a subordinate of Θ^b. Note that η_t satisfies (3.1). In the calculations below we will use this fact repeatedly. First note the bottom right term above can be rewritten as follows.

$$\eta_t(S)^*\alpha_t^d(SAS^*)\eta_t(S) = \eta_t(S)^*\eta_t(SA) = \eta_t(S)^*\eta_t(S)\beta_t^d(A)$$

for all $A \in \mathfrak{B}(\mathfrak{N}_2)$. Also we have

$$\begin{aligned}\eta_t(S)^*\alpha_t^d(SAS^*)\eta_t(S) =&(\alpha_t^d(SAS^*)^*\eta_t(S))^*\eta_t(S)\\ =&\eta_t(SA^*)^*\eta_t(S) = (\eta_t(S)\beta_t^d(A^*))^*\eta_t(S)\\ =&\beta_t^d(A)\eta_t(S)^*\eta_t(S)\end{aligned}$$

for all $A \in \mathfrak{B}(\mathfrak{N}_2)$. It follows that $\eta_t(S)^*\eta_t(S) \in \beta_t^d(\mathfrak{B}(\mathfrak{N}_2))'$.

Next we show that Θ^c is a semigroup. The top diagonal and the off diagonal terms in Θ^c are the same as Θ^b and since Θ^b is a semigroup these terms satisfy the semigroup property. We only need to check the semigroup property for the bottom right term in Θ^c. Suppose $s, t \geq 0$. Then we have

$$\begin{aligned}&\eta_t(S)^*\eta_t(S)\beta_t^d(\eta_s(S)^*\eta_s(S)\beta_s(A))\\ &\qquad=\eta_t(S)^*\eta_t(S)\beta_t^d(\eta_s(S)^*\eta_s(S))\beta_{t+s}(A)\\ &\qquad=\eta_t(S)^*\eta_t(S\eta_s(S)^*\eta_s(S))\beta_{t+s}(A)\\ &\qquad=\eta_t(S)^*\alpha_t^d(S\eta_s(S)^*)\eta_t(\eta_s(S))\beta_{t+s}(A)\\ &\qquad=\eta_t(S)^*\alpha_t^d(S\eta_s(S)^*)\eta_{t+s}(S)\beta_{t+s}(A)\\ &\qquad=(\alpha_t^d(\eta_s(S)S^*)\eta_t(S))^*\eta_{t+s}(S)\beta_{t+s}(A)\\ &\qquad=\eta_t(\eta_s(S))^*\eta_{t+s}(S)\beta_{t+s}(A)\\ &\qquad=\eta_{t+s}(S)^*\eta_{t+s}(S)\beta_{t+s}(A)\end{aligned}$$

for $A \in \mathfrak{B}(\mathfrak{N}_2)$. Hence, the bottom right term satisfies the semigroup property so Θ^c is a semigroup. Next we show Θ^c is completely positive. We will need an alternate expression for $\eta_t(X_{12})$. Note that

$$\eta_t(X_{12}) = \eta_t(X_{12}S^*S) = \alpha_t^d(X_{12}S^*)\eta_t(S).$$

Also we have

$$\eta_t^*(X_{21}) = \eta_t(X_{21}^*S^*S)^* = (\alpha_t^d(X_{21}^*S^*)\eta_t(S))^* = \eta_t(S)^*\alpha_t^d(SX_{21})$$

Recalling how we defined Θ_t^c we have

$$\Theta_t^c(\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22}\end{bmatrix}) = \begin{bmatrix} \alpha_t^d(X_{11}) & \alpha_t^d(X_{12}S^*)\eta_t(S) \\ \eta_t(S)^*\alpha_t^d(SX_{21}) & \eta_t(S)^*\alpha_t^d(SX_{22}S^*)\eta_t(S)\end{bmatrix}$$

where we have inserted the alternate expressions for $\eta_t(X_{12})$ and its adjoint. We show this map is completely positive by writing it as the product of three completely positive maps. Let

$$R = \begin{bmatrix} I & 0 \\ 0 & S\end{bmatrix} \qquad \text{and} \qquad T = \begin{bmatrix} I & 0 \\ 0 & \eta_t(S)\end{bmatrix}$$

and Δ is the mapping

$$\Delta(\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}) = \begin{bmatrix} \alpha_t^d(X_{11}) & \alpha_t^d(X_{12}) \\ \alpha_t^d(X_{21}) & \alpha_t^d(X_{22}) \end{bmatrix}$$

for $X_{ij} \in \mathfrak{B}(\mathfrak{N}_1)$ for $i,j = 1,2$. Then one calculates that $\Theta_t^c(X) = T^*\Delta(RXR^*)T$ for all $X \in \mathfrak{B}(\mathfrak{N})$ so Θ_t^c is the product of three completely positive maps so Θ_t^c is completely positive and Θ^c is a CP-semigroup. Note that Θ^c is a subordinate of Θ^b since

$$\Theta_t^b(\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}) - \Theta_t^c(\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 0 & (I - \eta_t(S)\eta_t(S))\beta_t(X_{22}) \end{bmatrix}$$

and since $\eta_t(S)^*\eta_t(S) \in \beta_t^d(\mathfrak{B}(\mathfrak{N}_2))'$ we have

$$(I - \eta_t(S)^*\eta_t(S))\beta_t^d(X_{22}) = (I - \eta_t(S)^*\eta_t(S))^{\frac{1}{2}}\beta_t^d(X_{22})(I - \eta_t(S)^*\eta_t(S))^{\frac{1}{2}}$$

which makes it clear that the map $X \to \Theta_t^b - \Theta_t^c$ is completely positive. Hence, Θ^c is a subordinate of Θ^b. Since the subordinates of Θ^b are order isomorphic with the subordinates of Θ, there is a subordinate Θ' of Θ corresponding to Θ^c. Since the off diagonal elements of Θ^c equal the off diagonal elements of Θ^b it follows that the off diagonal elements of Θ' match those of Θ. By the assumption of the theorem we have $\Theta' = \Theta$ and by the order isomorphism we have $\Theta^c = \Theta^b$. Hence, we have $\eta_t(S)^*\eta_t(S) = I$ for all $t \geq 0$.

Now let Θ_t^a be given by

$$\Theta_t^a(\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}) = \begin{bmatrix} \eta_t(S)\beta_t^d(S^*X_{11}S)\eta_t(S)^* & \eta_t(X_{12}) \\ \eta_t^*(X_{21}) & \beta_t^d(X_{22}) \end{bmatrix}.$$

Repeating the argument we made for Θ^c we find Θ^a is a subordinate of Θ^b and this time we find $\eta_t(S)\eta_t(S)^* = I$. Note essentially all we are doing in this new argument is interchanging the roles of α and β. Hence, $\eta_t(S)$ is unitary for all $t \geq 0$ and from Lemma 3.12 we find Θ_t^b is a unital $*$-endomorphism of $\mathfrak{B}(\mathfrak{N})$ and from Lemma 3.8 we have α^d and β^d are cocycle conjugate. $\square$

The previous theorem shows the importance of analyzing corners between CP-semigroup. This brings up the question if α is a unital CP-semigroup what do the corners from α to α correspond to for the dilated E_o-semigroup. As we will see these corners correspond to contractive local cocycles. We will also consider matrices of corners.

Definition 3.14. Suppose α is a CP-semigroup of $\mathfrak{B}(\mathfrak{H})$ and n is a positive integer. We say Θ is a positive $(n \times n)$-matrix of corners from α to α if Θ is a matrix with coefficients $\theta^{(ij)}$ where the $\theta^{(ij)}$ are strongly continuous semigroups of $\mathfrak{B}(\mathfrak{H})$ for $i,j = 1, \cdots, n$ so that Θ is a CP-semigroup of $\mathfrak{B}(\oplus_{i=1}^n \mathfrak{H})$ into itself and the diagonal entries of Θ are subordinates of α.

Definition 3.15. Suppose α^d is a E_o-semigroup of $\mathfrak{B}(\mathfrak{H})$ and n is a positive integer. We say C is a positive $(n \times n)$-matrix of α^d local cocycles if the coefficients C_{ij} of C are contractive local cocycles for α^d for $i, j = 1, \cdots, n$ and the matrix $C(t)$ whose entries are $C_{ij}(t)$ is positive for all $t \geq 0$.

We remark how the cocycle condition fits nicely with the notion of a positive matrix of local cocycles. It is well known that if A and B are positive matrices with coefficients $\{a_{ij}\}$ and $\{b_{ij}\}$ in the complex numbers then the matrix C with coefficients $\{a_{ij}b_{ij}\}$ (C is known as the Schur product of A and B) is positive. The same is true if the coefficients $a_{ij} \in \mathfrak{A}$ where $\mathfrak{A}$ is algebra of operators on a Hilbert space and $b_{ij} \in \mathfrak{A}'$ the commutant of $\mathfrak{A}$. We see then that if $C(t)$ is a positive matrix with coefficients which are operators in $\alpha_t^d(\mathfrak{B}(\mathfrak{H}))'$ and $C(s)$ is a positive matrix with coefficients in $\mathfrak{B}(\mathfrak{H})$ and if $C(t+s)$ is a matrix with coefficients $C_{ij}(t)\alpha_t^d(C_{ij}(s))$ then $C(t+s)$ is a positive matrix. It follows then that in order to check that C is a positive matrix of local cocycles it is only necessary to check the positivity of $C(t)$ for small t.

Theorem 3.16. *Suppose α is a unital CP-semigroup of $\mathfrak{B}(\mathfrak{H})$ and α^d is its Bhat dilation to an E_o-semigroup α^d of $\mathfrak{B}(\mathfrak{H}_1)$. The relation between α and α^d is given by*

$$\alpha_t(A) = W^*\alpha_t^d(WAW^*)W$$

for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$ where W is an isometry from $\mathfrak{H}$ to $\mathfrak{H}_1$ and α^d is minimal over the range of W.

Suppose n is a positive integer and Θ is positive $(n \times n)$-matrix of corners from α to α. Then there is a unique positive $(n \times n)$-matrix C of contractive local cocycles C_{ij} for α^d for $i, j = 1, \cdots, n$ so that

$$\theta_t^{(ij)}(A) = W^*C_{ij}(t)\alpha_t^d(WAW^*)W$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. Conversely, if C is a positive $(n \times n)$-matrix of contractive local cocycles for α^d then the matrix Θ whose coefficients $\theta^{(ij)}$ are given above is a positive $(n \times n)$-matrix of corners from α to α.

Proof. Assume the set up and notation of the theorem. Suppose C is a positive $(n \times n)$-matrix of α^d local cocycles with coefficients C_{ij} for $i, j = 1, \cdots, n$ and $\theta_t^{(ij)}$ are given in terms of the C_{ij} as given in the statement of the theorem. First we check that Θ is a semigroup. To do this we need to show that the coefficients are a semigroup. To save writing subscripts in our calculations suppose i and j are integers in the interval $[1, n]$ and $C(t) = C_{ij}(t)$ and $\gamma_t = \theta_t^{ij}$ for $t \geq 0$. We have

$$\begin{aligned}
\gamma_t(\gamma_s(A)) &= W^*C(t)\alpha_t^d(WW^*C(s)\alpha_s(WAW^*)WW^*)W \\
&= W^*C(t)\alpha_t^d(WW^*)\alpha_t^d(C(s))\alpha_{t+s}^d(WAW^*)\alpha_t^d(WW^*)W \\
&= W^*\alpha_t^d(WW^*)C(t)\alpha_t^d(C(s))\alpha_{t+s}^d(WAW^*)\alpha_t^d(WW^*)W \\
&= W^*C(t+s)\alpha_{t+s}^d(WAW^*)W = \gamma_{t+s}(A)
\end{aligned}$$

for all $t, s \geq 0$ and $A \in \mathfrak{B}(\mathfrak{H})$ where we have used the facts that WW^* is an increasing projection for α^d and $C(t)$ is local. Hence, Θ is a semigroup. Let Θ_t be the family of mappings in the statement of the theorem and W_1 be the mapping of $\oplus_{i=1}^n \mathfrak{H}$ into $\oplus_{i=1}^n \mathfrak{H}_1$ given by $W_1\{f_1, \cdots, f_n\} = \{Wf_1, \cdots, Wf_n\}$. Then we have $\Theta_t = W_1^*\Xi_t W_1$ where Ξ_t are operators on $\oplus_{i=1}^n \mathfrak{H}_1$ with coefficients $\Xi_t^{ij}(A_{ij}) = C_{ij}(t)\alpha_t^d(WA_{ij}W^*)$ for $A_{ij} \in \mathfrak{B}(\mathfrak{H})$ for $i, j = 1, \cdots, n$. We show Ξ_t is completely positive for $t \geq 0$. Suppose $t \geq 0$. The matrix $C(t)$ with coefficients $C_{ij}(t) \in \alpha_t^d(\mathfrak{B}(\mathfrak{H}_1))'$ is positive. For $x \in [0, 1]$ we have

$$(1-x)^{\frac{1}{2}} = 1 - (1/2)x - (1/8)x^2 - (1/16)x^3 - (5/128)x^4 - \cdots$$

where the series converges absolutely in the closed interval. Let $X = I - C(t)$. Then we have

$$C(t)^{\frac{1}{2}} = (I - X)^{\frac{1}{2}} = I - (1/2)X - (1/8)X^2 - \cdots$$

where the series converges in norm. Since $C_{ij}(t) \in \alpha_t^d(\mathfrak{B}(\mathfrak{H}_1))'$ we have $D(t) = C(t)^{\frac{1}{2}}$ has coefficients $D_{ij}(t) \in \alpha_t^d(\mathfrak{B}(\mathfrak{H}_1))'$. Since $C(t) = D(t)^*D(t)$ we have

$$\begin{aligned}\Xi_t^{ij}(A_{ij}) = C_{ij}(t)\alpha_t^d(WA_{ij}W^*) &= \sum_{k=1}^n D_{ki}(t)^*D_{kj}(t)\alpha_t^d(WA_{ij}W^*) \\ &= \sum_{k=1}^n D_{ki}(t)^*\alpha_t^d(WA_{ij}W^*)D_{kj}(t)\end{aligned}$$

for $A_{ij} \in \mathfrak{B}(\mathfrak{H})$ for $i, j = 1, \cdots, n$. Hence, Ξ_t is the sum of n completely positive maps and since $\Theta_t = W_1^*\Xi_t W_1$ is follows that Θ is a CP-semigroup so Θ is a positive $(n \times n)$-matrix of corners from α to α.

Conversely, suppose Θ is a positive $(n \times n)$-matrix of corners from α to α. The proof of the exitance and uniqueness of the positive $(n \times n)$-matrix C of contractive local cocycles for α^d virtually a repetition of the proof in Theorem 3.5. The uniqueness of the matrix coefficients $C_{ij}(t)$ is the same as the proof of the uniqueness of the positive cocycle $C(t)$ in Theorem 3.5. The proof of the existence of the $(n \times n)$-matrix C of contractive local cocycles for α^d is the same as the proof of the existence of the positive contractive cocycle $C(t)$ in Theorem 3.5 with one complication which we explain. Recall in the proof of Theorem 3.5 we found an operator $Z(t) \in \phi_t(\mathfrak{B}(\mathfrak{H}))'$ so that $\beta_t(A) = W^*\phi_t(A)Z(t)W$ for $A \in \mathfrak{B}(\mathfrak{H})$ where ϕ_t was the restriction of $A \to \alpha_t^d(WAW^*)$ to $\mathfrak{M}_t$ which was the closed linear span of $\{\alpha_t^d(WAW^*)Wf : f \in \mathfrak{H},\ A \in \mathfrak{B}(\mathfrak{H})\}$. In our present case we find the same operator $Z(t)$ which is now a positive $(n \times n)$-matrix of elements $\phi_t(\mathfrak{B}(\mathfrak{H}))'$. The existence of $Z(t)$ in the proof of theorem 3.5 was assured by Stinespring analysis of completely positive maps. The existence of the matrix $Z(t)$ in our case follows from the following mild generalization of the Stinespring analysis which is the following.

Suppose η is a completely positive unital map of a C^*-algebra $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{H})$ and π is the Stinespring representation of $\mathfrak{A}$ on $\mathfrak{B}(\mathfrak{H}_1)$ determined by η by the requirement $\eta(A) = V^*\pi(A)V$ for $A \in \mathfrak{A}$ and V is an isometry from $\mathfrak{H}$ to $\mathfrak{H}_1$ and the linear span of the vectors $\pi(A)Vf$ for $A \in \mathfrak{A}$ and $f \in \mathfrak{H}$ is dense in $\mathfrak{H}_1$. Now suppose γ is positive $(n \times n)$-matrix of corners from η to η where we take the notion of positive from Definition 3.14. Then there is a unique positive $(n \times n)$-matrix of contractive operators C_{ij} in $\pi(\mathfrak{A})'$ so that $\gamma_{ij}(A) = V^*C_{ij}\pi(A)V$ for $A \in \mathfrak{A}$ and $i,j = 1,\cdots,n$. Unfortunately, we do not have a reference for this exact result but it a fairly routine argument.

Using this result we construct $Z(t)$ which is now a positive matrix with coefficients in $\phi_t(\mathfrak{B}(\mathfrak{H}))'$. Then following the argument in Theorem 3.5 we construct $Y(t)$ which is now a positive $(n \times n)$-matrix with coefficients in $\alpha_t^d(\mathfrak{B}(\mathfrak{H}))'$. Then following the argument in Theorem 3.5 we construct the positive $(n \times n)$-matrix $C(t)$ for t a dyadic rational and then show $C(t)$ is continuous and can be extended to all real positive t thereby producing the positive $(n\times n)$-matrix of local cocycles. $\square$

Corollary 3.17. *Suppose α is a unital CP-semigroup of $\mathfrak{B}(\mathfrak{H})$ and α^d is its Bhat dilation and the relation between α and α^d is as given in the previous theorem. Suppose θ is a corner from α to α and C is the local contractive cocycle for α^d associated with θ. Then $C(t)$ is an isometry for all $t \geq 0$ if and only if θ is maximal and $C(t)$ is unitary for all $t \geq 0$ if and only if θ is hyper maximal.*

Proof. Assume the set up and notation of the corollary. Let Θ be the (2×2)-matrix of semigroups so that the diagonal semigroups are α and the (12) entry is θ and the (21) entry is θ^* and let C be the positive (2×2)-matrix of local α^d cocycles associated with Θ by the previous theorem. Suppose Θ' a subordinate of Θ whose corner is θ and let C' be the positive (2×2)-matrix associated with Θ'. One checks that $0 \leq C'_{11}(t) \leq I$, $0 \leq C'_{22}(t) \leq I$, $C'_{12}(t) = C_{12}(t)$ and $C'_{21}(t) = C_{21}(t) = C_{12}(t)^*$ for all $t \geq 0$. A matrix computation shows that $C'(t)$ given below satisfies

$$0 \leq C'(t) = \begin{bmatrix} C_{12}(t)^*C_{12}(t) & C_{12}(t) \\ C_{12}(t)^* & I \end{bmatrix} \leq \begin{bmatrix} I & C_{12}(t) \\ C_{12}(t)^* & I \end{bmatrix} = C(t)$$

for $t \geq 0$. Hence, if $C_{12}(t)$ is not an isometry then the top left entry of the above matrix in not the unit so θ is not maximal. Conversely, suppose $C_{12}(t)$ is an isometry for all $t \geq 0$. If $C'(t)$ is positive for all $t \geq 0$ we have

$$0 \leq \begin{bmatrix} C'_{11}(t) & C_{12}(t) \\ C_{12}(t)^* & C'_{22}(t) \end{bmatrix} \leq \begin{bmatrix} C'_{11}(t) & C_{12}(t) \\ C_{12}(t)^* & I \end{bmatrix}$$

for all $t \geq 0$. A straight forward computation shows matrix on the right above is positive if and only if $C'_{11}(t) \geq I$ and since $C'_{11}(t) \leq I$ we have $C'_{11}(t) = I$ for all $t \geq 0$. Hence, θ is maximal. Now θ is hyper maximal if and only if both θ and θ^* are maximal so θ is hyper maximal if and only if $C(t)$ is unitary for all $t \geq 0$. $\square$

IV. CP-Flows.

We consider the problem of finding all strongly continuous semigroups of completely positive contractions of the space of all bounded operators on $\mathfrak{K} \otimes L^2(0,\infty)$ into itself which intertwine with the semigroup of right translation on $\mathfrak{K}\otimes L^2(0,\infty)$. As we will see this is a problem in finding an extension of the differential operator $d = d/dx$. The importance of this problem is that every E_o-semigroup can be induced using the Bhat minimal dilation [Bh] from such a semigroup. We call such semigroups CP-flows over $\mathfrak{K}$ where $\mathfrak{K}$ is a separable Hilbert space.

Definition 4.0. Suppose $\mathfrak{K}$ is a separable Hilbert space and $\mathfrak{H} = \mathfrak{K} \otimes L^2(0,\infty)$ and $U(t)$ is right translations of $\mathfrak{H}$ by $t \geq 0$. Specifically, we may realize $\mathfrak{H}$ as the space of $\mathfrak{K}$-valued Lebesgue measurable functions with inner product

$$(f,g) = \int_0^\infty \overline{f(x)}g(x)\,dx$$

for $f, g \in \mathfrak{H}$. The action of $U(t)$ on an element $f \in \mathfrak{H}$ is given by $(U(t)f)(x) = f(x-t)$ for $x \in [t,\infty)$ and $(U(t)f)(x) = 0$ for $x \in [0,t)$. A semigroup α is a CP-flow over $\mathfrak{K}$ if α is a CP-semigroup of $\mathfrak{B}(\mathfrak{H})$ which is intertwined by the translation semigroup $U(t)$, i.e., $U(t)A = \alpha_t(A)U(t)$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. A semigroup α is a CP_κ-flow over $\mathfrak{K}$ where $\kappa \geq 0$ if α is intertwined by the translation semigroup $U(t)$ and the semigroup $A \to e^{-\kappa t}\alpha_t(A)$ is a CP-semigroup of $\mathfrak{B}(\mathfrak{H})$. The constant κ is called a growth bound for α.

Then next theorem shows that every spacial E_o-semigroup is cocycle conjugate to an E_o-semigroup which is also a CP-flow so and complete classification of CP-flows yields a complete classification of spatial E_o-semigroups.

Theorem 4.0A. *Every spatial E_o-semigroup of $\mathfrak{B}(\mathfrak{H})$ is cocycle conjugate to an E_o-semigroup which is also a CP-flow.*

Proof. Suppose α is a spatial E_o-semigroup of $\mathfrak{B}(\mathfrak{H})$ and V is a one parameter semigroup of isometries that intertwine α. For each one parameter semigroup of isometries V acting on $\mathfrak{H}$ there is the Wold decomposition of $\mathfrak{H} = \mathfrak{H}_o \oplus \mathfrak{K} \otimes L^2(0,\infty)$ so that $V(t)$ is unitary on $\mathfrak{H}_o$ and $V(t)$ is the right shift on is $\mathfrak{K} \otimes L^2(0,\infty)$ for each $t \geq 0$. Note $V(t)V(t)^* \to P$ as $t \to \infty$ where P is the projection onto $\mathfrak{H}_o$ so if $\|V(t)^*f\| \to 0$ as $t \to \infty$ for each $f \in \mathfrak{H}$ then α is a CP-flow since $V(t)$ is the right shift on $\mathfrak{H} = \mathfrak{K} \otimes L^2(0,\infty)$ for each $t \geq 0$. To prove the theorem we need to show that every spatial E_o-semigroup is cocycle conjugate to a E_o-semigroup which is intertwined by a semigroup V with the above property. From Theorem 2.13 of [P4] it follows that every spatial E_o-semigroup is cocycle conjugate to a spatial E_o-semigroup in standard form where an E_o-semigroup α is in standard form if it has a pure absorbing state ω_o which means that if ρ is any normal state of $\mathfrak{B}(\mathfrak{H})$ then $\rho(\alpha_t(A)) \to \omega_o(A)$ as $t \to \infty$ for all $A \in \mathfrak{B}(\mathfrak{H})$. It follows that an absorbing state is invariant (i.e., $\omega_o(\alpha_t(A)) = \omega_o(A)$ for all $t \geq 0$ and $A \in \mathfrak{B}(\mathfrak{H})$). Since ω_o is pure it follows that there is a unit vector $f_o \in \mathfrak{H}$ so that $\omega_o(A) = (f_o, Af_o)$ for all $A \in \mathfrak{B}(\mathfrak{H})$. One defines a strongly continuous one parameter semigroup of isometries $U(t)$ by the relation

$$U(t)Af_o = \alpha_t(A)f_o$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. The semigroup U intertwines α. It follows from proof of Theorem 2.13 in [P4] that $U(t)$ is a pure shift on the orthogonal complement of f_o so the Hilbert space $\mathfrak{H} = \mathfrak{H}_o \oplus \mathfrak{H}_1$ decomposes into a direct sum of the one dimensional subspace $\mathfrak{H}_o$ spanned by f_o and the orthogonal complement $\mathfrak{H}_1$ and the semigroup U decomposes as a pure shift on the orthogonal complement $\mathfrak{H}_1$ and U just the identity on $\mathfrak{H}_o$ (i.e., $V(t)f_o = f_o$ for $t \geq 0$). We will show that we can perturb the E_o-semigroup α and obtain an E_o-semigroup β which is cocycle conjugate with α and β is intertwined by a semigroup of pure shifts so β is a CP-flow.

Now since U is a pure shift on $\mathfrak{H}_1$ we can represent $\mathfrak{H}_1$ as $\mathfrak{K}_1 \otimes L^2(0,\infty)$ and V acts by translation. We can pick a unit vector $h_1 \in \mathfrak{K}_1$ and then let h be the vector in $\mathfrak{H}_1$ defined by the $\mathfrak{K}_1$ valued function $h(x) = \sqrt{2}h_1 e^{-x}$ for $x \geq 0$. To specify the vector h without referring to this representation of vectors as functions we can simply say $h \in \mathfrak{H}$ is a unit vector so that $U(t)^*h = e^{-t}h$ for all $t \geq 0$. Now let H be the skew hermitian operator giving by

$$\text{Hf} \quad = \tfrac{1}{2}(h,f)f_o - \tfrac{1}{2}(f_o,f)h$$

for all $f \in \mathfrak{H}$. Let $-d$ be the generator of $U(t)$ so $U(t) = e^{-td}$ for $t \geq 0$. Let δ be the generator of α and let

$$\delta_1(A) = \delta(A) + HA - AH$$

for all $A \in \mathfrak{D}(\delta)$. Now by Theorem 2.8 of [P3] (restated as Theorem 2.10 in [P4]) δ_1 is the generator on an E_o-semigroup β which is cocycle conjugate to α and β is intertwined by the semigroup of isometries $V(t) = \exp(-td_1) = \exp(-t(d-H))$ for $t \geq 0$ (where $d_1 = d - H$). We show V is a pure shift. Suppose $f \in \mathfrak{H}$. We note that for each $t \geq 0$ we can uniquely decompose $V(t)f$ in the form

$$V(t)f = a(t)f_o + b(t)h + g(t)$$

where $g(t)$ is orthogonal to both f_o and h. Note $a(t) = (f_o, V(t)f)$ and $b(t) = (h, V(t)f)$. Note $f_o, h \in \mathfrak{D}(d_1^*)$ and

$$d_1^* f_o = d^* f_o - H^* f_o = -\tfrac{1}{2}h \qquad \text{and} \qquad d_1^* h = d^* h - H^* h = h + \tfrac{1}{2}f_o.$$

Then we can differentiate $a(t)$ and $b(t)$ and obtain the equations

$$\frac{d}{dt}a(t) = -(d_1^* f_o, V(t)f) = \tfrac{1}{2}(h, V(t)f) = \tfrac{1}{2}b(t)$$

and

$$\frac{d}{dt}b(t) = -(d_1^* h, V(t)f) = ((-h - \tfrac{1}{2}f_o), V(t)f) = -b(t) - \tfrac{1}{2}a(t)$$

for $t \geq 0$. Solving these coupled differential equations one finds that

$$a(t) = (a + \tfrac{1}{2}(a+b)t)e^{-\frac{1}{2}t} \qquad \text{and} \qquad b(t) = (b - \tfrac{1}{2}(a+b)t)e^{-\frac{1}{2}t}$$

for $t \geq 0$ where $a = a(0) = (f_o, f)$ and $b = b(0) = (h, f)$.

Now let $\mathfrak{M}$ be the two dimensional subspace of $\mathfrak{H}$ spanned by f_o and h and let P be the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{M}$. We see from the above equations that if $f \in \mathfrak{M}^\perp$ (the orthogonal complement of $\mathfrak{M}$) then $V(t)f \in \mathfrak{M}^\perp$ for all $t \geq 0$ (since if $a = b = 0$ then $a(t) = b(t) = 0$ for all $t \geq 0$). Next we note that if $f \in \mathfrak{D}(d)$ and $f \in \mathfrak{M}^\perp$ then $d_1 f = df$. Hence, for $f \in \mathfrak{D}(d)$ and $f \in \mathfrak{M}^\perp$ we have $U(t)f = V(t)f$ for all $t \geq 0$. This extends to all $f \in \mathfrak{M}^\perp$ by continuity. Armed with these facts can now prove $V(t)$ is a pure shift for each $t > 0$.

We have shown that

$$V(t) = V(t)P + V(t)(I - P) = V(t)P + U(t)(I - P)$$

for $t \geq 0$. Taking adjoints we have

$$V(t)^* = PV(t)^* + (I - P)U(t)^*$$

for $t \geq 0$. Since U is a pure shift on $f_o^\perp$ and $(I-P)U(t)^* f_o = 0$ for all $t \geq 0$ it follows that $\|(I - P)U(t)^* f\| \to 0$ as $t \to \infty$ for all $f \in \mathfrak{H}$. Then we have $\|V(t)^* f\| \to 0$ as $t \to \infty$ for all $f \in \mathfrak{H}$ if and only if $\|PV(t)^* f\| \to 0$ as $t \to \infty$ for all $f \in \mathfrak{H}$. Now from the equations for $a(t)$ and $b(t)$ we have

$$\begin{aligned}\|PV(t)^* f\|^2 &= |(f_o, V(t)^* f)|^2 + |(h, V(t)^* f)|^2 = |a(t)|^2 + |b(t)|^2 \\ &\leq (1 + t^2)(|a|^2 + |b|^2)e^{-t} = (1 + t^2)\|Pf\|^2 e^{-t}\end{aligned}$$

for $t \geq 0$. Hence, $\|PV(t)^* f\| \to 0$ as $t \to \infty$ for all $f \in \mathfrak{H}$ so $V(t)$ is a pure shift for each $t > 0$. Hence, β is a CP-flow. $\square$

The problem we pose is to describe all CP-flows over $\mathfrak{K}$. In the following when working with a CP-flow over $\mathfrak{K}$ we will assume that $\mathfrak{H} = \mathfrak{K} \otimes L^2(0, \infty)$ and $U(t)$ is the right translation operator described above. Note that a CP_o-flow is a CP-flow. We will prove that every CP_κ-flow is a CP-flow. Also when we write CP_κ-flow in the sequel we assume automatically assume that $\kappa \geq 0$ and $\mathfrak{K}$ is a separable Hilbert space.

Lemma 4.1. *Suppose α is a one parameter semigroup of positive linear mappings of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{H})$ and the semigroup $U(t)$ of isometries intertwine α in that $U(t)A = \alpha_t(A)U(t)$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. Let $E(t) = I - U(t)U(t)^*$. Then for $A \in \mathfrak{B}(\mathfrak{H})$ we have $E(s)\alpha_t(A) = \alpha_t(A)E(s)$ for all s and t with $0 \leq s \leq t < \infty$ and*

$$\alpha_t(A) = U(t)AU(t)^* + E(t)\alpha_t(A)E(t) \tag{4.1}$$

Proof. Suppose α satisfies the hypothesis of the lemma. Since α_t is positive we have $\alpha_t(A^*) = \alpha_t(A)^*$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and all $t \geq 0$. Since $U(t)$ intertwines we have $U(t)A = \alpha_t(A)U(t)$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and taking adjoints and replacing A by A^* we have $AU(t)^* = U(t)^*\alpha_t(A)$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. It follows then that

$$\alpha_s(A)U(s)U(s)^* = U(s)AU(s)^* = U(s)U(s)^*\alpha_s(A)$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ and $s \geq 0$. Since for $s \leq t < \infty$ we have $\alpha_t(A) = \alpha_s(\alpha_{t-s}(A))$ it follows that $E(s)\alpha_t(A) = E(s)\alpha_t(A)$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and s and t satisfying $0 \leq s \leq t < \infty$. The last line of the lemma follows from the computation

$$\alpha_t(A) = \alpha_t(A)U(t)U(t)^* + \alpha_t(A)E(t)^2 = U(t)AU(t)^* + E(t)\alpha_t(A)E(t)$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. $\square$

Lemma 4.2. *Suppose α is a CP_κ-flow over $\mathfrak{K}$ and recall $U(t)$ are the right translations on $\mathfrak{H} = \mathfrak{K} \otimes L^2(0,\infty)$. Let δ be the generator of α and $-d$ be the generator of $U(t)$(d is the differential operator d/dx with the boundary condition that $f(0) = 0$). Then each $A \in \mathfrak{D}(\delta)$ has property that $A\mathfrak{D}(d) \subset \mathfrak{D}(d)$ and $A\mathfrak{D}(d^*) \subset \mathfrak{D}(d^*)$ and for $f \in \mathfrak{D}(d)$ and $g \in \mathfrak{D}(d^*)$ and $A \in \mathfrak{D}(\delta)$ we have*

$$\delta(A)f = Adf - dAf \qquad \text{and} \qquad \delta(A)g = -Ad^*g + d^*Ag \tag{4.2}$$

Proof. Assume the hypothesis and notation of the lemma. Suppose $A \in \mathfrak{D}(\delta)$ and $f \in \mathfrak{D}(d)$. Using the fact that $U(t)A = \alpha_t(A)U(t)$ we have

$$\begin{aligned} t^{-1}(U(t) - I)Af =& t^{-1}(\alpha_t(A) - A)f + t^{-1}A(U(t) - I)f \\ &+ t^{-1}(\alpha_t(A) - A)(U(t) - I)f \end{aligned}$$

Since the first two terms on the right hand side of the above equation converges to $\delta(A)f$ and $-Adf$ respectively and the third term converges to zero it follows that $Af \in \mathfrak{D}(d)$ and $-dAf = \delta(A)f - Adf$. Hence, we have proved the first equation of the conclusion of the lemma. Now continuing to suppose $f \in \mathfrak{D}(d)$ and $A \in \mathfrak{D}(\delta)$ and suppose $g \in \mathfrak{D}(d^*)$. Then we have $-(dAf, g) = (\delta(A)f, g) - (Adf, g)$. Since $A \in \mathfrak{D}(\delta)$ implies $A^* \in \mathfrak{D}(\delta)$ and $\delta(A^*) = \delta(A)^*$ we can replace A by A^* and taking adjoints we find $(df, Ag) = (f, Ad^*g) + (f, \delta(A)g)$. Since this is true for all $f \in \mathfrak{D}(d)$ we have $Ag \in \mathfrak{D}(d^*)$ and $d^*Ag = Ad^*g + \delta(A)g$. $\square$

We introduce a $*$-derivation δ_1 which is an extension of δ.

Definition 4.3. Let δ_1 be the linear mapping of the domain $\mathfrak{D}(\delta_1)$ into $\mathfrak{B}(\mathfrak{H})$ where $\mathfrak{D}(\delta_1)$ consisting of all $A \in \mathfrak{B}(\mathfrak{H})$ so that $A\mathfrak{D}(d) \subset \mathfrak{D}(d)$, $A\mathfrak{D}(d^*) \subset \mathfrak{D}(d^*)$ and there is a $B \in \mathfrak{B}(\mathfrak{H})$ so that $Bf = d^*Af - Ad^*f$ for all $f \in \mathfrak{D}(d^*)$. If $A \in \mathfrak{B}(\mathfrak{H})$ satisfies the above requirements then $\delta_1(A) = B$.

Lemma 4.4. *The domain $\mathfrak{D}(\delta_1)$ is a $*$-algebra which is σ-strongly dense in $\mathfrak{B}(\mathfrak{H})$ and δ_1 is a σ-weakly closed $*$-derivation of $\mathfrak{D}(\delta_1)$ into $\mathfrak{B}(\mathfrak{H})$. If δ is a generator of strongly continuous one parameter semigroup α of completely positive maps of $\mathfrak{B}(\mathfrak{H})$ into itself satisfying the hypothesis of Lemma 4.2 then δ_1 is an extension of δ in that $\mathfrak{D}(\delta_1) \supset \mathfrak{D}(\delta)$ and $\delta_1(A) = \delta(A)$ for all $A \in \mathfrak{D}(\delta)$.*

Proof. Suppose $h, g \in \mathfrak{D}(d)$ and $Xf = (g, f)h$ for all $f \in \mathfrak{H}$. Note $X\mathfrak{D}(d) \subset \mathfrak{D}(d)$ and $X\mathfrak{D}(d^*) \subset \mathfrak{D}(d) \subset \mathfrak{D}(d^*)$. Let $Yf = -(g, f)dh - (dg, f)h$ for all $f \in \mathfrak{H}$. Then one checks that $X \in \mathfrak{D}(\delta_1)$ and $\delta_1(X) = Y$. It follows then that $\mathfrak{D}(\delta_1)$ contains all finite linear combinations of operators of the form X just given. Since $\mathfrak{D}(d)$ is dense in $\mathfrak{H}$ these operators are σ-strongly dense in the finite rank operators. Since the finite rank operators are σ-strongly dense in $\mathfrak{B}(\mathfrak{H})$ we have $\mathfrak{D}(\delta_1)$ is σ-strongly dense in $\mathfrak{B}(\mathfrak{H})$.

Suppose $A \in \mathfrak{D}(\delta_1)$ and $\delta_1(A) = B$. Then $Bf = d^*Af - Ad^*f$ for all $f \in \mathfrak{D}(d^*)$. Suppose $g \in \mathfrak{D}(d)$. Then $(A^*g, d^*f) = (g, (d^*A - B)f) = ((A^*dg - B^*g), f)$ for all $f \in \mathfrak{D}(d^*)$. Hence, $A^*g \in \mathfrak{D}(d^{**}) = \mathfrak{D}(d)$ and we have shown that $A^*\mathfrak{D}(d) \subset \mathfrak{D}(d)$. Since $A\mathfrak{D}(d) \subset \mathfrak{D}(d)$ and $-d^* \supset d$ we have $Bf = Adf - dAf$ for all $f \in \mathfrak{D}(d)$. Suppose $g \in \mathfrak{D}(d^*)$. Then we have $(A^*g, df) = (g, (dA + B)f) = ((A^*d^*g + B^*g), f)$

for all $g \in \mathfrak{D}(d)$. Hence, we have shown that $A^*\mathfrak{D}(d^*) \subset \mathfrak{D}(d^*)$ and $B^*g = d^*A^*g - A^*d^*g$. for $g \in \mathfrak{D}(d^*)$. Hence, $A \in \mathfrak{D}(\delta_1)$ implies $A^* \in \mathfrak{D}(\delta_1)$ and $\delta_1(A^*) = \delta_1(A)^*$.

It is a routine computation to show that if $A, B \in \mathfrak{D}(\delta_1)$ then $AB \in \mathfrak{D}(\delta_1)$ and $\delta_1(AB) = \delta_1(A)B + A\delta_1(B)$ so we have that $\mathfrak{D}(\delta_1)$ is a $*$-algebra and δ_1 is a $*$-derivation of $\mathfrak{D}(\delta_1)$ into $\mathfrak{B}(\mathfrak{H})$. If δ is the generator of Lemma 4.2 it follows that $\mathfrak{D}(\delta_1) \supset \mathfrak{D}(\delta)$ and $\delta_1(A) = \delta(A)$ for all $A \in \mathfrak{D}(\delta)$.

Finally, we show that δ_1 is σ-weakly closed. Suppose then that $A_n \in \mathfrak{D}(\delta_1)$ and $\delta_1(A_n) = B_n$ and $A_n \to A$ and $B_n \to B$ σ-weakly as $n \to \infty$. Then $A_n^* \to A^*$ and $\delta_1(A_n^*) \to B^*$ σ-weakly as $n \to \infty$. Suppose $f \in \mathfrak{D}(d)$ and $g \in \mathfrak{D}(d^*)$. Then $(Af, d^*g) = \lim_{n\to\infty}(A_nf, d^*g) = \lim_{n\to\infty}(A_ndf, g) - (B_nf, g) = ((Adf - Bf), g)$. Hence, $A\mathfrak{D}(d) \subset \mathfrak{D}(d^{**}) = \mathfrak{D}(d)$. Suppose $f \in \mathfrak{D}(d^*)$ and $g \in \mathfrak{D}(d)$. Then $(Af, dg) = \lim_{n\to\infty}(A_nf, dg) = \lim_{n\to\infty}(f, A_n^*dg) = \lim_{n\to\infty}(f, (dA_n^* + B_n^*)g) = \lim_{n\to\infty}((A_nd^* + B_n)f, g) = ((Ad^*f + Bf), g)$. Hence, we have $Af \in \mathfrak{D}(d^*)$ and $Bf = d^*Af - Ad^*f$ for all $f \in \mathfrak{D}(d^*)$. Hence, we have $A \in \mathfrak{D}(\delta_1)$ and $\delta_1(A) = B$. Hence, δ_1 is σ-weakly closed. $\square$

We define the boundary representation π_o of $\mathfrak{D}(\delta_1)$. As is well (see [DS], Lemma 10, p.1227) known each element $f \in \mathfrak{D}(d^*)$ can be uniquely decomposed in the form $f = f_o + f_+$ with $f_o \in \mathfrak{D}(d)$ and $f_+ \in \mathfrak{D}(d^*)$ and $d^*f_+ = f_+$. The vector f_+ is given by $f_+(x) = e^{-x}f(0)$. Note that since f is differentiable f can be represented by a continuous $\mathfrak{K}$-valued function $f(x)$ and when we write $f(0)$ we are of course referring to a representation of f by a continuous function. We introduce the inner product $\langle f, g\rangle$ on $\mathfrak{D}(d^*)$ by the relation

$$\langle f, g\rangle = (d^*f, g) + (f, d^*g)$$

Note that if $f, g \in \mathfrak{D}(d^*)$ then $\langle f, g\rangle = (f(0), g(0))$ so $\langle \cdot, \cdot\rangle$ is an inner product in $\mathfrak{D}(d^*)$ mod $\mathfrak{D}(d)$. Now if $A \in \mathfrak{D}(\delta_1)$ we have $A\mathfrak{D}(d) \subset \mathfrak{D}(d)$ and $A\mathfrak{D}(d^*) \subset \mathfrak{D}(d^*)$. It follows that if $f \in \mathfrak{D}(d^*)$ then $(Af)(0)$ only depends on $f(0)$. The mapping $f(0) \to (Af)(0)$ is called the boundary representation of π_o of $\mathfrak{D}(\delta_1)$. One sees that π_o is a $*$-mapping and π_o is a representation of $\mathfrak{D}(\delta_1)$ since for $A, B \in \mathfrak{D}(\delta_1)$ and $f \in \mathfrak{D}(d^*)$ we have $\pi_o(AB)f(0) = (ABf)(0) = \pi_o(A)(Bf)(0) = \pi_o(A)\pi_o(B)f(0)$. Note π_o is unital in that $\pi_o(I) = I$. We show π_o is a contraction of $\mathfrak{D}(\delta_1)$ into $\mathfrak{B}(\mathfrak{K})$. Since $\mathfrak{D}(\delta_1)$ is not a C^*-algebra this in not an immediate consequence of the fact that π_o is unital. For $\lambda > 0$ let $f_\lambda = e^{-\lambda x}k$ where $k \in \mathfrak{K}$ is a unit vector and suppose $A \in \mathfrak{D}(\delta_1)$. Note $\|f_\lambda\| = 1/\sqrt{2\lambda}$. Then we have

$$\begin{aligned}\|\pi_o(A)k\|^2 =&\langle f_\lambda, A^*Af_\lambda\rangle = (d^*f_\lambda, A^*Af_\lambda) + (f_\lambda, d^*A^*Af_\lambda)\\ =&(d^*f_\lambda, A^*Af_\lambda) + (f_\lambda, A^*Ad^*f_\lambda) + (f_\lambda, \delta(A^*A)f_\lambda)\\ =&2\lambda\|Af_\lambda\|^2 + (f_\lambda, \delta(A^*A)f_\lambda) \le \|A\|^2 + (2\lambda)^{-1}\|\delta(A^*A)\|\end{aligned}$$

Taking the limit as $\lambda \to \infty$ we have $\|\pi_o(A)\| \le \|A\|$ for all $A \in \mathfrak{D}(\delta_1)$.

Definition 4.5. The mapping π_o defined above is called the boundary representation of $\mathfrak{D}(\delta_1)$ on $\mathfrak{K}$.

If one looks for the solutions to the equation $\delta_1(A) = A$ one is lead to the operators $\Lambda(B)$ defined below.

Definition 4.6. For $\lambda \geq 0$ and $A \in \mathfrak{B}(\mathfrak{K})$ we define $\Lambda_\lambda(A)$ on $\mathfrak{H} = \mathfrak{K} \otimes L^2(0,\infty)$ by the relation $(\Lambda_\lambda(A)f)(x) = e^{-\lambda x}Af(x)$ for all $f \in \mathfrak{H}$. If we write $\Lambda(A)$ with no subscript we mean $\Lambda_1(A)$ (i.e., $\lambda = 1$) and we simply write Λ for $\Lambda(I) = \Lambda_1(I)$.

Note that for $\lambda \geq 0$ the mapping $A \to \Lambda_\lambda(A)$ is a contraction of $\mathfrak{B}(\mathfrak{K})$ into $\mathfrak{B}(\mathfrak{H})$. One easily checks that

$$\begin{aligned}\Lambda_\lambda(A)^* =& \Lambda_\lambda(A^*)\\ U(t)\Lambda_\lambda(A) =& e^{\lambda t}\Lambda_\lambda(A)U(t)\\ U(t)^*\Lambda_\lambda(A) =& e^{-\lambda t}\Lambda_\lambda(A)U(t)^*\end{aligned}$$

for $A \in \mathfrak{B}(\mathfrak{K})$ and $t \geq 0$. Note that for $\lambda, \mu \geq 0$ we have

$$\Lambda_\lambda(A)\Lambda_\mu(B) = \Lambda_{\lambda+\mu}(AB)$$

for $A, B \in \mathfrak{B}(\mathfrak{K})$.

Lemma 4.7. *If δ_1 is the $*$-derivation defined in Definition 4.3 and $\lambda \geq 0$ then $\delta_1(A) = \lambda A$ if and only if $A = \Lambda_\lambda(B)$ for some $B \in \mathfrak{B}(\mathfrak{K})$.*

Proof. If $B \in \mathfrak{B}(\mathfrak{K})$ one sees immediately that $\Lambda_\lambda(B) \in \mathfrak{D}(\delta_1)$ and $\delta_1(\Lambda_\lambda(B)) = \lambda\Lambda_\lambda(B)$. Conversely, suppose $A \in \mathfrak{D}(\delta_1)$ and $\delta_1(A) = \lambda A$. For $s > 0$ let $\mathfrak{D}_s$ be the subspace of all $f \in \mathfrak{D}(d^*)$ so that $d^*f = sf$. It is well known that $\mathfrak{D}_s$ consists of all vectors $f \in \mathfrak{H}$ of the form $f(x) = e^{-sx}f_o$ where $f_o \in \mathfrak{K}$. Suppose $f \in \mathfrak{D}_1$. Since $\delta_1(A) = \lambda A$ we have from the definition of δ_1 that $\lambda Af = \delta_1(A)f = -Ad^*f + d^*Af$ and, hence, $d^*Af = (1+\lambda)Af$. Hence, A maps $\mathfrak{D}_1$ into $\mathfrak{D}_{1+\lambda}$. Since the mapping $f \to f(0)$ is continuous and has a continuous inverse both for $f \in \mathfrak{D}_1$ and for $f \in \mathfrak{D}_{1+\lambda}$ it follows that if $f(x) = e^{-x}k$ with $k \in \mathfrak{K}$ then $(Af)(x) = e^{-(1+\lambda)x}Bk$ where B is a bounded linear operator determined by A. Suppose B is this operator determined by A. Let $C = A - \Lambda_\lambda(B)$. We claim $C = 0$.

We have $C \in \mathfrak{D}(\delta_1)$ and $\delta_1(C) = \lambda C$ and $Cf = 0$ for $f \in \mathfrak{D}_1$. From the definition of δ_1 we have $Cf = \delta_1(C)f = -dCf + Cdf$ for all $f \in \mathfrak{D}(d)$. Now suppose $f \in \mathfrak{D}(d)$. Let $g(t) = e^{\lambda t}CU(t)f$. Since $U(t)f \in \mathfrak{D}(d)$ for all $t \geq 0$ and $-d$ is the generator of $U(t)$ we have

$$\begin{aligned}\frac{d}{dt}g(t) =& \lambda g(t) - e^{\lambda t}CdU(t)f\\ =& \lambda g(t) - e^{\lambda t}dCU(t)f - \lambda e^{\lambda t}CU(t)f = -dg(t).\end{aligned}$$

Since $-d$ is the generator of $U(t)$ we have $g(t) = U(t)g(0) = U(t)Cf$ or $U(t)Cf = e^{\lambda t}CU(t)f$ for all $f \in \mathfrak{D}(d)$ and $t \geq 0$. For each fixed t both sides of this equation are norm continuous in f so we can extend this equation to all $f \in \mathfrak{H}$. In particular we can apply this equation to vectors g_1 given by $g(x) = e^{-x}k$ in $\mathfrak{D}_1$ and since $Cf = 0$ for $f \in \mathfrak{D}_1$ we have $CU(t)g_1 = 0$ for all $k \in \mathfrak{K}$ and $t \geq 0$. Note $(g_1 - e^{-t}U(t)g_1)(x) = q_t(x)k$ where $q_t(x) = e^{-x}$ for $x \in [0,t]$ and $q_t(x) = 0$ for $x > t$. The linear span of the functions q_t are dense in $L^2(0,\infty)$ for if $h \in L^2(0,\infty)$ where orthogonal to all the q_t we would have $e^{-x}h(x) = 0$ almost everywhere. Since the linear span of the q_t are dense in $L^2(0,\infty)$ we have the linear span of the vectors $U(t)g_1$ with $g_1(x) = e^{-x}k$ with $k \in \mathfrak{K}$ and $t \geq 0$ are dense in $\mathfrak{H}$. Hence, $Ch = 0$ for a dense set of vectors so $C = 0$. Hence, $A = \Lambda_\lambda(B)$. $\square$

Next we introduce the operator Γ which solves the equation $A - \delta_1(A) = B$.

Definition 4.8. Suppose $\lambda > 0$. For $c > 0$ we define

$$\Gamma_\lambda^c(A) = \int_0^c \lambda e^{-\lambda t} U(t)AU(t)^*\, dt \qquad \text{and} \qquad \Gamma_\lambda(A) = \int_0^\infty \lambda e^{-\lambda t} U(t)AU(t)^*\, dt$$

for all $A \in \mathfrak{B}(\mathfrak{H})$. If we write $\Gamma(A)$ with no subscript we mean $\Gamma_1(A)$ ($\lambda = 1$). When we write Γ_λ we always assume $\lambda > 0$.

We see Γ_λ is a everywhere defined bounded operator and Γ_λ is σ-weakly continuous since Γ_λ^c is σ-weakly continuous and $\Gamma_\lambda^c(A)$ converges in norm to $\Gamma_\lambda(A)$ as $c \to \infty$.

Lemma 4.9. *Suppose $\lambda > 0$. For $A \in \mathfrak{B}(\mathfrak{H})$ we have $\Gamma_\lambda(A) \in \mathfrak{D}(\delta_1)$ and*

$$\Gamma_\lambda(A) - \lambda^{-1}\delta_1(\Gamma_\lambda(A)) = A.$$

For $A \in \mathfrak{D}(\delta_1)$ we have

$$\Gamma_\lambda(A - \lambda^{-1}\delta_1(A)) = A - \Lambda_\lambda(\pi_o(A)).$$

Proof. Suppose Γ_λ is as defined above. Suppose $A \in \mathfrak{B}(\mathfrak{H})$. Then we have

$$\Gamma_\lambda(A) - e^{-\lambda t}U(t)\Gamma_\lambda(A)U(t)^* = \int_0^t \lambda e^{-\lambda s} U(s)AU(s)^*\, ds.$$

Dividing by t and taking the limit as $t \to 0+$ we find.

$$t^{-1}(U(t)\Gamma_\lambda(A)U(t)^* - \Gamma_\lambda(A)) \to \lambda(\Gamma_\lambda(A) - A)$$

in the strong operator topology as $t \to 0+$. Now suppose $f \in \mathfrak{D}(d)$. Then

$$\begin{aligned} t^{-1}(U(t) - I)\Gamma_\lambda(A)f =& t^{-1}U(t)\Gamma_\lambda(A)U(t)^*(U(t) - I)f \\ &+ t^{-1}(U(t)\Gamma_\lambda(A)U(t)^* - \Gamma_\lambda(A))f. \end{aligned}$$

As $t \to 0+$ the first term on the right hand side converges to $-\Gamma_\lambda(A)df$ and the second term converges to $\lambda(\Gamma_\lambda(A)f - Af)$. Hence, $\Gamma_\lambda(A)f \in \mathfrak{D}(d)$ and

$$-d\Gamma_\lambda(A)f = -\Gamma_\lambda(A)df + \lambda(\Gamma_\lambda(A)f - Af).$$

Note the above equation holds with A replaced by A^*. Then for $f \in \mathfrak{D}(d)$ and $g \in \mathfrak{D}(d^*)$ we have

$$-(g, d\Gamma_\lambda(A^*)f) = -(g, \Gamma_\lambda(A^*)df) + \lambda(g, \Gamma_\lambda(A^*)f) - \lambda(g, A^*f)$$

and rearranging we have

$$(\Gamma_\lambda(A)g, df) = (\Gamma_\lambda(A)d^*g, f) + \lambda(\Gamma_\lambda(A)g, f) - \lambda(Ag, f).$$

It follows that $\Gamma_\lambda(A)g \in \mathfrak{D}(d^*)$ and

$$d^*\Gamma_\lambda(A)g = \Gamma_\lambda(A)d^*g + \lambda(\Gamma_\lambda(A)g - Ag).$$

Hence, it follows from the definition of δ_1 that $\Gamma_\lambda(A) \in \mathfrak{D}(\delta_1)$ and $\delta_1(\Gamma_\lambda(A)) = \lambda(\Gamma_\lambda(A) - A)$. Hence, $\Gamma_\lambda(A) - \lambda^{-1}\delta_1(\Gamma_\lambda(A)) = A$.

Now suppose $A \in \mathfrak{D}(\delta_1)$ and $f, g \in \mathfrak{D}(d^*)$. Since $\delta_1(A)U(t)^*f = d^*AU(t)^*f - Ad^*U(t)^*f$ and $(d/dt)U(t)^*h = -d^*U(t)^*h$ and recalling that $\langle f, g\rangle = (d^*f, g) + (f, d^*g)$ we have

$$\begin{aligned}
(f, \Gamma_\lambda^c(A - \lambda^{-1}\delta_1(A))g) &= \int_0^c \lambda e^{-\lambda t}(U(t)^*f, (A - \lambda^{-1}\delta_1(A))U(t)^*g)\, dt \\
&= -\int_0^c e^{-\lambda t}\langle U(t)^*f, AU(t)^*g\rangle + \frac{d}{dt}e^{-\lambda t}(U(t)^*f, AU(t)^*g)\, dt \\
&= -\int_0^c e^{-\lambda t}(f(t), \pi_o(A)g(t))\, dt + (f, Ag) - e^{-\lambda c}(U(c)^*f, AU(c)^*g) \\
&= (f, (A - \Lambda_\lambda(\pi_o(A)))g) - e^{-\lambda c}(U(c)^*f, (A - \Lambda_\lambda(\pi_o(A)))U(c)^*g).
\end{aligned}$$

Note both sides of the above equation are norm continuous in f and g and since $\mathfrak{D}(d^*)$ is dense in $\mathfrak{H}$ the above equation is valid for all $f, g \in \mathfrak{H}$. Hence, we have

$$\Gamma_\lambda^c(A - \lambda^{-1}\delta_1(A)) = A - \Lambda_\lambda(\pi_o(A)) - e^{-\lambda c}U(c)(A - \Lambda_\lambda(\pi_o(A)))U(c)^*$$

As $c \to \infty$ the second term on the right hand side of the above equation converges strongly to zero and the result of the lemma follows. $\square$

The next lemma characterizes the domain of δ_1 and $\hat{\delta}_1$. We recall that if ϕ is a linear mapping which is σ-weakly closed then $\hat{\phi}$ is the associated mapping on the predual.

Lemma 4.10. *Suppose $\lambda > 0$. We have $A \in \mathfrak{D}(\delta_1)$ if and only if A is of the form $A = \Lambda_\lambda(B) + \Gamma_\lambda(C)$ with $B \in \mathfrak{B}(\mathfrak{K})$ and $C \in \mathfrak{B}(\mathfrak{H})$. We have $\rho \in \mathfrak{D}(\hat{\delta}_1)$ if and only if $\rho = \hat{\Gamma}_\lambda(\omega)$ for some $\omega \in \mathfrak{B}(\mathfrak{H})_*$ with $\hat{\Lambda}_\lambda(\omega) = 0$. Note ρ satisfies $\rho - \lambda^{-1}\hat{\delta}_1(\rho) = \omega$.*

Proof. Suppose $\lambda > 0$. From the previous lemmas it follows that if $A = \Lambda_\lambda(B) + \Gamma_\lambda(C)$ with $B \in \mathfrak{B}(\mathfrak{K})$ and $C \in \mathfrak{B}(\mathfrak{H})$ then $A \in \mathfrak{D}(\delta_1)$. Now suppose $A \in \mathfrak{D}(\delta_1)$. Let $C = A - \lambda^{-1}\delta_1(A)$. Then from Lemma 4.9 we have $\Gamma_\lambda(C) = A - \Lambda_\lambda(\pi_o(A))$ and, hence, $A = \Lambda_\lambda(\pi_o(A)) + \Gamma_\lambda(C)$.

Next suppose $\rho \in \mathfrak{D}(\hat{\delta}_1)$. Let $\omega = \rho - \lambda^{-1}\hat{\delta}_1(\rho)$. Then $\omega(A) = \rho(A) - \lambda^{-1}\rho(\delta_1(A))$ for all $A \in \mathfrak{D}(\delta_1)$. Since $\Lambda_\lambda(B) \in \mathfrak{D}(\delta_1)$ and $\delta_1(\Lambda_\lambda(B)) = \lambda\Lambda_\lambda(B)$ for all $B \in \mathfrak{B}(\mathfrak{K})$ we have $\omega(\Lambda_\lambda(B)) = 0$ for all $B \in \mathfrak{B}(\mathfrak{K})$ so $\hat{\Lambda}_\lambda(\omega) = 0$. Since $\omega(A) = \rho(A) - \lambda^{-1}\rho(\delta_1(A))$ for all $A \in \mathfrak{D}(\delta_1)$ and by the properties of Γ_λ proved in the previous lemma we have $\omega(\Gamma_\lambda(A)) = \rho(A)$ for all $A \in \mathfrak{B}(\mathfrak{H})$. But this means $\rho = \hat{\Gamma}_\lambda(\omega)$.

Now, suppose $\omega \in \mathfrak{B}(\mathfrak{H})_*$ and $\hat{\Lambda}_\lambda(\omega) = 0$. Let $\rho = \hat{\Gamma}_\lambda(\omega)$. Suppose $A \in \mathfrak{D}(\delta_1)$. Then from the previous lemma we have

$$\rho(A - \lambda^{-1}\delta_1(A)) = \omega(\Gamma_\lambda(A - \lambda^{-1}\delta_1(A))) = \omega(A - \Lambda_\lambda(\pi_o(A))) = \omega(A)$$

where the last equality follows from the fact that $\hat{\Lambda}_\lambda(\omega) = 0$. Hence, $\rho \in \mathfrak{D}(\hat{\delta}_1)$ and $\rho - \lambda^{-1}\hat{\delta}_1(\rho) = \omega$. $\square$

In the next definition we introduce notation we will use repeatedly in our analysis of CP-flows.

Definition 4.11. Recall $\mathfrak{H} = \mathfrak{K} \otimes L^2(0,\infty)$ and $U(t)$ is the translation semigroup on $\mathfrak{H}$. Let $E(t) = (I - U(t)U(t)^*)$ and $E(s,t) = E(t) - E(s)$ for $0 < s < t$ and $E(t,\infty) = U(t)U(t)^* = I - E(t)$ for $t \geq 0$. Let

$$\theta_t(A) = U(t)AU(t)^*, \qquad \xi_t(A) = U(t)^*AU(t) \qquad \text{and} \qquad \zeta_t(A) = E(t)AE(t)$$

for all $A \in \mathfrak{B}(\mathfrak{H})$.

For $\lambda > 0$ let Q_λ be the isometry from $\mathfrak{K}$ to $\mathfrak{H}$ given by $(Q_\lambda k)(x) = \sqrt{\lambda} e^{-\frac{1}{2}\lambda x} k$ for $x \geq 0$ and $k \in \mathfrak{K}$. Let Φ_λ be the mapping of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{H})$ given by

$$\Phi_\lambda(A) = Q_\lambda^* A Q_\lambda$$

for $A \in \mathfrak{B}(\mathfrak{H})$. Note if we write Φ without a subscript we mean Φ_λ with $\lambda = 1$.

Note θ_t and ξ_t are semigroups and

$$\xi_t(\theta_t(A)) = A \qquad \text{and} \qquad \theta_t(\xi_t(A)) = E(t,\infty)AE(t,\infty)$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. It follows that $\hat{\theta}_t(\hat{\xi}_t(\eta)) = \eta$ for all $\eta \in \mathfrak{B}(\mathfrak{H})_*$. Note $\xi_t(\Lambda_\lambda(A)) = e^{-\lambda t}\Lambda_\lambda(A)$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t, \lambda \geq 0$. Also, we have

$$\Phi_\lambda(\Lambda_\lambda(A)) = \frac{1}{2}A \qquad \text{and} \qquad \Phi_\lambda(\Gamma_\lambda(A)) = \frac{1}{2}\Phi_\lambda(A)$$

for $A \in \mathfrak{B}(\mathfrak{H})$ and $\lambda > 0$. Using these identities a direct calculation establishes the formulae

$$\hat{\Lambda}_\lambda(\hat{\xi}_t(\eta)) = e^{-\lambda t}\hat{\Lambda}_\lambda(\eta) \tag{4.3a}$$

$$e^{\lambda t}\hat{\Gamma}_\lambda(\hat{\xi}_t(\eta)) = \hat{\Gamma}_\lambda(\eta) + \int_0^t \lambda e^{\lambda s}\hat{\xi}_s(\eta)\, ds \tag{4.3b}$$

$$\hat{\theta}_t(\hat{\Gamma}_\lambda(\hat{\xi}_t(\eta))) = \hat{\Gamma}_\lambda(\eta) \tag{4.3c}$$

$$\hat{\Lambda}_\lambda(\hat{\Phi}_\lambda(\rho)) = \frac{1}{2}\rho \tag{4.3d}$$

$$\hat{\Gamma}_\lambda(\hat{\Phi}_\lambda(\rho)) = \frac{1}{2}\hat{\Phi}_\lambda(\rho) \tag{4.3e}$$

$$\Gamma_\lambda(\Lambda_o(A)) = \Lambda_o(A) - \Lambda_\lambda(A) \tag{4.3f}$$

$$\Gamma(I) = I - \Lambda \tag{4.3g}$$

which are valid for all $t, \lambda > 0$, $\eta \in \mathfrak{B}(\mathfrak{H})_*$, $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $A \in \mathfrak{B}(\mathfrak{H})$.

We establish the last two equations. Suppose $f, g \in \mathfrak{H}$ and so we can represent f and g as $\mathfrak{K}$-valued functions $f(x)$ and $g(x)$ for $x \geq 0$. Then for $A \in \mathfrak{B}(\mathfrak{K})$ and $\lambda > 0$ we have

$$\begin{aligned}(f, \Gamma_\lambda(\Lambda_o(A))g) &= \int_0^\infty \lambda e^{-\lambda t}(f, U(t)\Lambda_o(A)U(t)^*g)\, dt \\ &= \int_0^\infty \lambda e^{-\lambda t}\int_t^\infty (f(x), Ag(x))\, dx\, dt\end{aligned}$$

Integrating by parts we arrive at the formula

$$\begin{aligned}(f, \Gamma_\lambda(\Lambda_o(A))g) &= \int_0^\infty (f(x), Ag(x))\, dx - \int_0^\infty e^{-\lambda t}(f(t), Ag(t))\, dt \\ &= (f, (\Lambda_o(A) - \Lambda_\lambda(A))g).\end{aligned}$$

Hence, we have established (4.3f). Since $\Lambda_o(I) = I$ and $\Lambda_1(I) = \Lambda(I) = \Lambda$ (4.3g) follow from the previous equation when one sets $A = I$ and $\lambda = 1$.

Lemma 4.12. *With ζ_t and θ_t as above we have $\|\eta\| \geq \|\hat{\zeta}_t(\eta)\| + \|\hat{\theta}_t(\eta)\|$ for all $t > 0$ and $\eta \in \mathfrak{B}(\mathfrak{H})_*$.*

Proof. Assume $t > 0$ and $\eta \in \mathfrak{B}(\mathfrak{H})_*$. Suppose A and B are in the unit ball of $\mathfrak{B}(\mathfrak{H})$ and $\hat{\zeta}_t(\eta)(A) = \|\hat{\zeta}_t(\eta)\|$ and $\hat{\theta}_t(\eta)(B) = \|\hat{\theta}_t(\eta)\|$. Let $C = E(t)BE(t) + U(t)AU(t)^*$. Note

$$\begin{aligned} C^*C =& E(t)B^*E(t)BE(t) + U(t)A^*AU(t)^* \\ \leq& E(t)B^*BE(t) + U(t)U(t)^* \leq E(t) + U(t)U(t)^* = I \end{aligned}$$

Hence, $\|C\| \leq 1$. Now we have

$$\|\eta\| \geq |\eta(C)| = |\hat{\zeta}_t(\eta)(B) + \hat{\theta}_t(\eta)(A)| = \|\hat{\zeta}_t(\eta)\| + \|\hat{\theta}_t(\eta)\|$$

which concludes the proof of the lemma. □

Suppose α is a CP_κ-flow over $\mathfrak{K}$. Suppose δ is the generator of α. In the analysis of α an important tool is the resolvent R_λ of δ which is defined for $\lambda > \kappa$ where κ is a growth bound for α by the formula

$$R_\lambda(A) = \int_0^\infty \lambda e^{-\lambda t}\alpha_t(A)\, dt$$

for $A \in \mathfrak{B}(\mathfrak{H})$. If we speak of the resolvent R (with no subscript) we mean the resolvent R_1 where $\lambda = 1$. If α is a CP-semigroup the resolvent is defined for all $\lambda > 0$ (in fact all complex λ with $\mathrm{Re}(\lambda) > 0$) but because $\|\alpha_t(A)\|$ can grow like $e^{\kappa t}$ we see that convergence of above integral is only assured for $\lambda > \kappa$. The resolvent is the inverse of the map $A \to A - \lambda^{-1}\delta(A)$ for $A \in \mathfrak{D}(\delta)$. Precisely, we have for $\lambda > \kappa$ the resolvent maps $\mathfrak{B}(\mathfrak{H})$ onto the domain $\mathfrak{D}(\delta)$ and

$$R_\lambda(A) - \lambda^{-1}\delta(R_\lambda(A)) = A$$

for all $A \in \mathfrak{B}(\mathfrak{H})$. Also, we have

$$R_\lambda(A) - \lambda^{-1}R_\lambda(\delta(A)) = A$$

for all $A \in \mathfrak{D}(\delta)$. The semigroup α can be recovered from the resolvent in a variety of ways. One formula we will use is the formula

$$\alpha_t(A) = \lim_{n\to\infty}(R_{n/t})^n(A)$$

for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$ where the convergence is in the σ-strong topology and is uniform for t in a bounded interval. We use the convention $R_\infty(A) = A$. For a discussion of the resolvent we refer to Chapter 3 of [BR]. Now from equation (4.1) we recall that

$$\alpha_t(A) = E(t)\alpha_t(A)E(t) + U(t)AU(t)^*$$

for $A \in \mathfrak{B}(\mathfrak{H})$. Then we have

$$R_\lambda(A) = \int_0^\infty \lambda e^{-\lambda t}E(t)\alpha_t(A)E(t)\, dt + \Gamma_\lambda(A)$$

so we see that the resolvent is the sum of two terms the second of which is directly computable and the first term contains the information about the particular CP_κ-flow. The next definition allows us to focus on this first term. Our definition is not just the first term above but a what you obtain after applying Φ_λ to it. Our reason for this will become clear with Theorem 4.14.

Definition 4.13. Suppose α is a CP_κ-flow over $\mathfrak{K}$ with a growth bound $\kappa \geq 0$. Suppose $\lambda > \kappa$. The boundary resolvent for α denoted by σ_λ is a completely positive σ-weakly continuous mapping of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$ given by

$$\begin{aligned}\sigma_\lambda(A) &= 2\Phi_\lambda(R_\lambda(A)) - \Phi_\lambda(A) \\ &= 2\int_0^\infty \lambda e^{-\lambda t}\Phi_\lambda(E(t)\alpha_t(A)E(t))\,dt\end{aligned}$$

for $A \in \mathfrak{B}(\mathfrak{H})$ where R_λ is the resolvent for α. In terms of the maps on the predual we have

$$\begin{aligned}\hat{\sigma}_\lambda(\rho) &= 2\hat{R}_\lambda(\hat{\Phi}_\lambda(\rho)) - \hat{\Phi}_\lambda(\rho) \\ &= 2\int_0^\infty \lambda e^{-\lambda t}\hat{\alpha}_t(\hat{\zeta}_t(\hat{\Phi}_\lambda(\rho)))\,dt\end{aligned}$$

for $\rho \in \mathfrak{B}(\mathfrak{K})_*$. If we refer to σ (with no superscript) as the boundary resolvent of a CP-flow we mean σ_λ with $\lambda = 1$.

Theorem 4.14. *Suppose α is a CP_κ-flow over $\mathfrak{K}$ and suppose $\kappa \geq 0$ is a growth bound for α. Suppose $\lambda > \kappa$ and σ_λ is the boundary resolvent of α. Then*

$$\hat{R}_\lambda(\eta) = \hat{\sigma}_\lambda(\hat{\Lambda}_\lambda(\eta)) + \hat{\Gamma}_\lambda(\eta) \tag{4.4}$$

for $\eta \in \mathfrak{B}(\mathfrak{H})_*$ where R_λ is the resolvent of α. We have

$$\hat{\alpha}_t(\hat{\sigma}_\lambda(\hat{\Lambda}_\lambda(\eta)) + e^{\lambda t}\hat{\Gamma}_\lambda(\hat{\xi}_t(\eta))) = e^{\lambda t}(\hat{\sigma}_\lambda(\hat{\Lambda}_\lambda(\eta)) + \hat{\Gamma}_\lambda(\eta)) \tag{4.5}$$

for all $\eta \in \mathfrak{B}(\mathfrak{H})_*$ and $t \geq 0$ and for arbitrary $\nu \in \mathfrak{B}(\mathfrak{H})_*$ we have

$$\hat{\alpha}_t(\nu - \hat{\zeta}_t(\nu)) = \hat{\theta}_t(\nu). \tag{4.6}$$

Proof. Assume the hypothesis and notation of the theorem. We begin with (4.6). Assume $\nu \in \mathfrak{B}(\mathfrak{H})_*$. We have from Lemma 4.1 that

$$\hat{\alpha}_t(\nu - \hat{\zeta}_t(\nu))(A) = \nu(\alpha_t(A) - E(t)\alpha_t(A)E(t)) = \nu(U(t)AU(t)^*) = \hat{\theta}_t(\nu)(A)$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. Hence, equation (4.6) is established.

Next we establish equation (4.4). Let δ be the generator of α and $\hat{\delta}$ its preadjoint which is the generator of $\hat{\alpha}$. Suppose $\eta \in \mathfrak{B}(\mathfrak{H})_*$. Let $\rho = \hat{\Lambda}_\lambda(\eta)$ and let $\eta_1 = \eta - 2\hat{\Phi}_\lambda(\rho)$. Since $2\hat{\Lambda}_\lambda(\hat{\Phi}_\lambda(\rho)) = \rho$ we have $\hat{\Lambda}_\lambda(\eta_1) = 0$. Then we have from Lemma 4.9 that $\hat{\Gamma}_\lambda(\eta_1) \in \mathfrak{D}(\hat{\delta}_1)$ and $\lambda\hat{\Gamma}_\lambda(\eta_1) - \hat{\delta}_1(\hat{\Gamma}_\lambda(\eta_1)) = \lambda\eta_1$. From Lemma 4.4 we have that δ_1 is an extension of δ so $\hat{\delta}$ is an extension of $\hat{\delta}_1$. Hence, $\hat{\Gamma}_\lambda(\eta_1) \in \mathfrak{D}(\hat{\delta})$ and $\hat{\delta}(\hat{\Gamma}_\lambda(\eta_1)) = \lambda\hat{\Gamma}_\lambda(\eta_1) - \lambda\eta_1$. In terms of resolvents this means $\hat{R}_\lambda(\eta_1) = \hat{\Gamma}_\lambda(\eta_1)$. Let σ_λ be the boundary resolvent of α. Since $2\hat{\Phi}_\lambda(\rho) = \eta - \eta_1$ it follows from Definition 4.13 that

$$\hat{R}_\lambda(\eta - \eta_1) = \hat{\sigma}_\lambda(\rho) + \hat{\Phi}_\lambda(\rho)$$

Since $\hat{R}_\lambda(\eta_1) = \hat{\Gamma}_\lambda(\eta_1)$ and $\rho = \hat{\Lambda}_\lambda(\eta)$ we find from the above equation that

$$\hat{R}_\lambda(\eta) = \hat{\sigma}_\lambda(\hat{\Lambda}_\lambda(\eta)) + \hat{\Gamma}_\lambda(\eta) + \hat{\Phi}_\lambda(\hat{\Lambda}_\lambda(\eta)) - 2\hat{\Gamma}_\lambda(\hat{\Phi}_\lambda(\hat{\Lambda}_\lambda(\eta))).$$

From equations (4.3f) the last two terms cancel and we have established equation (4.4) of the theorem.

Finally, we establish equation (4.5). Suppose $t > 0$ and $\eta \in \mathfrak{B}(\mathfrak{H})_*$. From equation (4.4) applied to $e^{\lambda t}\xi_t(\eta)$ instead of η we have

$$\begin{aligned}\hat{R}_\lambda(e^{\lambda t}\hat{\xi}_t(\eta)) &= \hat{\sigma}_\lambda(\hat{\Lambda}_\lambda(\eta)) + \hat{\Gamma}_\lambda(\eta) + \int_0^t \lambda e^{\lambda s}\hat{\xi}_s(\eta)\,ds \\ &= \hat{R}_\lambda(\eta) + \int_0^t \lambda e^{\lambda s}\hat{\xi}_s(\eta)\,ds = e^{\lambda t}\nu_t\end{aligned}$$

where we have used equations (4.3a,b) to compute $\hat{\Lambda}_\lambda(\hat{\xi}_t(\eta))$ and $\hat{\Gamma}_\lambda(\hat{\xi}_t(\eta))$ where the last equality in the second line is just the definition of ν_t. In terms of the generator $\hat{\delta}$ this means $\nu_t \in \mathfrak{D}(\hat{\delta})$

$$\lambda\nu_t - \hat{\delta}(\nu_t) = \lambda\xi_t(\eta) \qquad \text{or} \qquad \hat{\delta}(\nu_t) = \lambda(\nu_t - \xi_t(\eta))$$

for $t \geq 0$. Since

$$\nu_t = e^{-\lambda t}(\hat{R}_\lambda(\eta) + \int_0^t \lambda e^{\lambda s}\hat{\xi}_s(\eta)ds)$$

we see that ν_t is differentiable and

$$\frac{d}{dt}\nu_t = -\lambda(\nu_t - \xi_t(\eta))$$

Suppose $t_o > 0$ and $\vartheta_t = \nu_{t_o - t}$ for $t \in [0, t_o]$. We see that $\vartheta_t \in \mathfrak{D}(\hat{\delta})$ and ϑ_t is differentiable for $t \in [0, t_o]$ and

$$\frac{d}{dt}\vartheta_t = \hat{\delta}(\vartheta_t)$$

so from Theorem 2.8 we have $\vartheta_t = \hat{\alpha}_t(\vartheta_o)$ for $t \in [0, t_o]$ and for $t = t_o$ we have $\vartheta_{t_o} = \hat{\alpha}_{t_o}(\vartheta_o)$ which says

$$e^{-\lambda t_o}\alpha_{t_o}(\hat{R}_\lambda(\eta) + \int_0^{t_o} \lambda e^{\lambda s}\hat{\xi}_s(\eta)ds) = \hat{R}_\lambda(\eta)$$

for $t_o \geq 0$. Multiplying this equation by $e^{\lambda t_o}$ and using equation (4.4) yields equation (4.5). $\square$

We found the next theorem a surprise. It says that CP_κ-flows are CP-flows.

Theorem 4.15. *Suppose α is a CP_κ-flow over $\mathfrak{K}$. Then α is a CP-flow over $\mathfrak{K}$.*

Proof. Suppose α is a CP_κ-flow over $\mathfrak{K}$ and $\kappa \geq 0$ is a growth bound for α. Suppose $\lambda > \kappa$ and $\hat{\sigma}_\lambda$ is the boundary resolvent of α. We will first show that $\sigma_\lambda(\rho)(I) \leq \rho(I)$ for all positive $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Then we will use this to show α is a CP-flow.

Assume $t > 0$ and $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Applying equation (4.5) to $2\hat{\Phi}_\lambda(\rho)$ and using equations (4.3) we find

$$\hat{\alpha}_t(\hat{\sigma}_\lambda(\rho) + 2e^{\lambda t}\hat{\Gamma}_\lambda(\hat{\xi}_t(\hat{\Phi}_\lambda(\rho)))) = e^{\lambda t}(\hat{\sigma}_\lambda(\rho) + 2\hat{\Gamma}_\lambda(\hat{\Phi}_\lambda(\rho))).$$

Applying this to the unit I and noting that $\alpha_t(I) = I + \zeta_t(\alpha_t(I) - I)$ and using equations (4.3) we find

$$\begin{aligned}(e^{\lambda t} - 1)\hat{\sigma}_\lambda(\rho)(I) =& \hat{\sigma}_\lambda(\rho)(\zeta_t(\alpha_t(I) - I)) + (e^{\lambda t} - 1)\rho(I) \\ &+ 2e^{\lambda t}\hat{\Gamma}_\lambda(\hat{\xi}_t(\hat{\Phi}_\lambda(\rho)))(\zeta_t(\alpha_t(I) - I)).\end{aligned}$$

Now we assume $\rho \in \mathfrak{B}(\mathfrak{K})_*$ is positive. We have κ is a growth bound for α so $\alpha_t(I) \leq e^{\kappa t}I$ and

$$\zeta_t(\alpha_t(I) - I) \leq (e^{\kappa t} - 1)\zeta_t(I) = (e^{\kappa t} - 1)E(t).$$

Since σ_λ, Γ_λ, ξ_t and Φ_λ are completely positive and ρ is positive if we substitute $(e^{\kappa t} - 1)E(t)$ for $\zeta_t(\alpha_t(I) - I)$ in the equation above we obtain the inequality

$$\begin{aligned}(e^{\lambda t} - 1)\hat{\sigma}_\lambda(\rho)(I) \leq& (e^{\kappa t} - 1)\hat{\sigma}_\lambda(\rho)(E(t)) + (e^{\lambda t} - 1)\rho(I) \\ &+ 2(e^{\kappa t} - 1)e^{\lambda t}\hat{\Gamma}_\lambda(\hat{\xi}_t(\hat{\Phi}_\lambda(\rho)))(E(t)).\end{aligned}$$

A direct computation shows the last term in the above inequality is $(e^{\kappa t} - 1)(e^{\lambda t} - 1)\rho(I)$ so after dividing by $(e^{\lambda t} - 1)$ we have

$$\hat{\sigma}_\lambda(\rho)(I) \leq \frac{e^{\kappa t} - 1}{e^{\lambda t} - 1}\hat{\sigma}_\lambda(\rho)(E(t)) + e^{\kappa t}\rho(I).$$

Since $\hat{\sigma}_\lambda(\rho)$ is normal we have $\hat{\sigma}_\lambda(\rho)(E(t)) \to 0$ as $t \to 0+$. Then taking the limit as $t \to 0+$ we find

$$\hat{\sigma}_\lambda(\rho)(I) \leq \rho(I)$$

for all positive $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Now suppose $\eta \in \mathfrak{B}(\mathfrak{H})_*$ and η is positive. Then we have from equation (4.4) that

$$\hat{R}_\lambda(\eta)(I) = \hat{\sigma}_\lambda(\hat{\Lambda}_\lambda(\eta))(I) + \hat{\Gamma}_\lambda(\eta)(I) \leq (\hat{\Lambda}_\lambda(\eta))(I) + \eta(I - \Lambda_\lambda(I)) = \eta(I).$$

Hence, $R_\lambda(I) \leq I$. Suppose $t > 0$ and $n = 1, 2, \cdots$. Since $R_{t/n}$ is completely positive we have

$$(R_{t/n})^n(I) \leq (R_{t/n})^{n-1}(I) \leq \cdots \leq R_{t/n}(I) \leq I.$$

Since

$$\alpha_t(I) = \lim_{n\to\infty}(R_{t/n})^n(I)$$

we have $\alpha_t(I) \leq I$ for all $t \geq 0$. Since α_t is completely positive we have α_t is a contraction for all $t \geq 0$ so α is a CP-semigroup. $\square$

The mappings Λ_λ, Γ_λ, σ_λ, Φ_λ and R_λ have a subscript λ. For CP-flows we will only need computations with a fixed $\lambda = 1$. While working with CP_κ-flows it was necessary to be able to choose $\lambda > \kappa$ where κ is a growth bound. Now that we know that CP_κ-flows are CP-flows so that $\kappa = 0$ is a growth bound we are free to fix $\lambda = 1$. So in the sequel we will write Λ, Γ, σ, Φ and R without a subscript and we remind the reader that this means we have set $\lambda = 1$. One exception to this general rule is Λ_o which we will need occasionally.

Now we begin our analysis of the boundary resolvent σ with no subscript so we mean σ_1. We begin with a definition of boundary weights. As we will see the boundary resolvent is the integral of a boundary weight.

Definition 4.16. Suppose $\mathfrak{K}$ is a separable Hilbert space and $\mathfrak{H} = \mathfrak{K} \otimes L^2(0,\infty)$. Suppose $U(t)$ for $t \geq 0$ is translation on $\mathfrak{H}$ and the mappings θ, ζ, Λ, Γ and Φ are as defined in Definitions 4.6, 4.8 and 4.11. We define the null boundary algebra $\mathfrak{A}(\mathfrak{H})$ of $\mathfrak{B}(\mathfrak{H})$ as the algebra of all operators of the form

$$A = (I - \Lambda)^{\frac{1}{2}} B (I - \Lambda)^{\frac{1}{2}}$$

with $B \in \mathfrak{B}(\mathfrak{H})$. We say ω is a boundary weight on $\mathfrak{B}(\mathfrak{H})$ if $\omega \in \mathfrak{A}(\mathfrak{H})_*$ or more explicitly ω a linear functional on $\mathfrak{A}(\mathfrak{H})$ and there is a normal functional $\mu \in \mathfrak{B}(\mathfrak{H})_*$ so that

$$\omega((I - \Lambda)^{\frac{1}{2}} A (I - \Lambda)^{\frac{1}{2}}) = \mu(A)$$

for all $A \in \mathfrak{B}(\mathfrak{H})$. The weight norm of ω is the norm of μ above. When we speak of the norm of a weight ω or say ω is bounded and do not explicitly say the weight norm we mean the usual norm of ω which can be infinite as opposed to weight norm which is always finite. If ω is a boundary weight then the truncated boundary weight ω_t defined for $t > 0$ is the normal functional $\omega_t \in \mathfrak{B}(\mathfrak{H})_*$ so that

$$\omega_t(A) = \omega((I - E(t))A(I - E(t)))$$

for $A \in \mathfrak{B}(\mathfrak{H})$. The mapping $\rho \to \omega(\rho)$ defined for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ is a boundary weight map if this mapping is a linear mapping of $\mathfrak{B}(\mathfrak{K})_*$ into boundary weights on $\mathfrak{B}(\mathfrak{H})$ and this mapping is a completely bounded with the norm on $\mathfrak{B}(\mathfrak{K})_*$ the usual norm and the norm on the boundary weights is the boundary weight norm. A boundary weight map is positive if it is completely positive. A boundary weight map ω is unital if $\omega(\rho)(I - \Lambda) = \rho(I)$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$.

Maintaining the notation of the above definition we note that $U(t)AU(t)^* \in \mathfrak{A}(\mathfrak{H})$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t > 0$. Recall the mapping Γ defined in Definition 4.8. Since Γ is completely positive and $\Gamma(I) = I - \Lambda$ so $\Gamma(I) \in \mathfrak{A}(\mathfrak{H})$ it follows that $\Gamma(A) \in \mathfrak{A}(\mathfrak{H})$ for all $A \in \mathfrak{B}(\mathfrak{H})$. This may be seen as follows. Suppose $A \in \mathfrak{B}(\mathfrak{H})$ and $0 \leq A \leq I$. Then we have $0 \leq \Gamma(A) \leq I - \Lambda$. Then for $f, g \in \mathfrak{D}((I - \Lambda)^{-\frac{1}{2}})$ the bilinear form

$$\langle f, g \rangle = ((I - \Lambda)^{-\frac{1}{2}} f, \Gamma(A)(I - \Lambda)^{-\frac{1}{2}} g)$$

is well defined and $0 \leq \langle f, f\rangle \leq (f, f)$ so there is a bounded operator $B \in \mathfrak{B}(\mathfrak{H})$ so that $(f, Bg) = \langle f, g\rangle$ for all $f, g \in \mathfrak{D}((I-\Lambda)^{-\frac{1}{2}})$. Then we have

$$(f, \Gamma(A)g) = ((I-\Lambda)^{\frac{1}{2}} f, B(I-\Lambda)^{\frac{1}{2}} g)$$

for all $f, g \in \mathfrak{D}((I-\Lambda)^{-\frac{1}{2}})$. If follows that

$$\Gamma(A) = (I-\Lambda)^{\frac{1}{2}} B(I-\Lambda)^{\frac{1}{2}}$$

and $\Gamma(A) \in \mathfrak{A}(\mathfrak{H})$. Since each operator $A \in \mathfrak{B}(\mathfrak{H})$ is the linear combination of four positive operators it follows that Γ maps $\mathfrak{B}(\mathfrak{H})$ into the null boundary algebra $\mathfrak{A}(\mathfrak{H})$. Note that if ω is a boundary weight then $\eta(A) = \omega(\Gamma(A))$ defined for all $A \in \mathfrak{B}(\mathfrak{H})$ defines an element $\eta \in \mathfrak{B}(\mathfrak{H})_*$ so if ω is a boundary weight then $\hat{\Gamma}(\omega)$ is a well defined element of $\mathfrak{B}(\mathfrak{H})_*$.

Theorem 4.17. *Suppose α is a CP-flow over $\mathfrak{K}$ and σ is the boundary resolvent of α. Recall $\theta_t(A) = U(t)AU(t)^*$ for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. Then for each $t > 0$ the mapping*

$$\rho \to \hat{\sigma}(\rho) - e^{-t}\hat{\theta}_t(\hat{\sigma}(\rho)) \tag{4.7}$$

is completely positive mapping of $\mathfrak{B}(\mathfrak{K})_*$ into $\mathfrak{B}(\mathfrak{H})_*$. The boundary resolvent satisfies the normalization inequality $\hat{\sigma}(\rho)(I) \leq \rho(I)$ for all positive $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $\hat{\sigma}(\rho)(I) = \rho(I)$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$ if and only if α is unital.

Suppose σ is a completely positive σ-weakly continuous contraction of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$ so that the mapping (4.7) is completely positive for all $t > 0$. Then there is a completely positive boundary weight map $\rho \to \omega(\rho)$ of $\mathfrak{B}(\mathfrak{K})_*$ into $\mathfrak{A}(\mathfrak{H})_*$ (boundary weights on $\mathfrak{B}(\mathfrak{H})$) so that

$$\hat{\sigma}(\rho)(A) = \int_0^\infty e^{-t}\omega(\rho)(U(t)AU(t)^*)\, dt = \hat{\Gamma}(\omega(\rho))(A) \tag{4.8}$$

for $A \in \mathfrak{B}(\mathfrak{H})$. And ω satisfies the normalization condition

$$\omega(\rho)(I-\Lambda) = \hat{\sigma}(\rho)(I) \leq \rho(I) \tag{4.9}$$

for ρ positive.

Conversely, if $\rho \to \omega(\rho)$ is a completely positive boundary weight map $\mathfrak{B}(\mathfrak{K})_*$ into $\mathfrak{A}(\mathfrak{H})_*$ satisfying the normalization condition (4.8) and $\hat{\sigma}(\rho)$ is defined by (4.7) then the mapping (4.7) is completely positive for all $t > 0$ and this mapping satisfies the normalization condition $\hat{\sigma}(I)(\rho) \leq \rho(I)$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ positive.

Proof. Suppose α is a CP-flow over $\mathfrak{K}$ and σ is the boundary resolvent of α. We will show the mapping (4.7) is completely positive for $t > 0$. From Definition 4.13 we recall

$$\hat{\sigma}(\rho)(A) = \int_0^\infty e^{-t}\hat{\Phi}(\rho)(E(t)\alpha_t(A)E(t))\, dt$$

for all $A \in \mathfrak{B}(\mathfrak{H})$.

Suppose $t > 0$. Let $\hat{\vartheta}_t(\rho) = \hat{\sigma}(\rho) - e^{-t}\hat{\alpha}_t(\hat{\sigma}(\rho))$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$. We show the mapping $\rho \rightarrow \hat{\vartheta}_t$ is completely positive. Suppose $\rho \in \mathfrak{B}(\mathfrak{K})_*$, $A \in \mathfrak{B}(\mathfrak{H})$ and $s > 0$. We have with repeated use of Lemma 4.1 that

$$\begin{aligned}
e^{-s}\hat{\sigma}(\rho)(\alpha_s(A)) =& 2\int_0^\infty e^{-t-s}\hat{\Phi}(\rho)(E(t)\alpha_{t+s}(A)E(t))\,dt \\
=& 2\int_0^\infty e^{-t-s}\hat{\Phi}(\rho)(E(t)\alpha_{t+s}(A))\,dt \\
=& 2\int_0^\infty e^{-t-s}\hat{\Phi}(\rho)(E(t+s)\alpha_{t+s}(A) - E(t,t+s)\alpha_{t+s}(A))\,dt \\
=& 2\int_s^\infty e^{-t}\hat{\Phi}(\rho)(E(t)\alpha_t(A)E(t))\,dt \\
& - 2\int_0^\infty e^{-t-s}\hat{\Phi}(\rho)E(t,t+s)\alpha_{t+s}(A)E(t,t+s)\,dt.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\hat{\vartheta}_s(\rho)(A) =& 2\int_0^s e^{-t}\hat{\Phi}(\rho)(E(t)\alpha_t(A)E(t))\,dt \\
& + 2\int_0^\infty e^{-t-s}\hat{\Phi}(\rho)(E(t,t+s)\alpha_{t+s}(A)E(t,t+s))\,dt.
\end{aligned}$$

Since all the mappings in the above formula for $\hat{\vartheta}_s$ are completely positive in their dependence on ρ the mapping $\rho \rightarrow \hat{\vartheta}_s$ is completely positive. Then from Lemma 4.1 we have

$$\hat{\sigma}(A) - e^{-t}\hat{\sigma}(U(t)AU(t)^*) = \hat{\vartheta}_t(A) + \nu_t(A)$$

where

$$\nu_t(A) = e^{-t}\hat{\sigma}(\rho)(E(t)\alpha_t(A)E(t)).$$

Since α_t and the mapping $\rho \rightarrow \hat{\sigma}(\rho)$ are completely positive we see that the mapping $\rho \rightarrow \nu_t$ is completely positive. Hence, for each $t > 0$ the mapping $\rho \rightarrow \hat{\sigma}(\rho) - e^{-t}\hat{\theta}_t(\hat{\sigma}(\rho))$ is the sum of two completely positive maps and, hence, it is completely positive.

The normalization inequality $\hat{\sigma}(\rho)(I) \leq \rho(I)$ for all positive $\rho \in \mathfrak{B}(\mathfrak{K})_*$ was established in the proof of Theorem 4.15. Recalling equation (4.4) of Theorem 4.14 we have

$$\hat{R}(\eta) = \hat{\sigma}(\hat{\Lambda}(\eta)) + \hat{\Gamma}(\eta) \tag{4.4}$$

for $\eta \in \mathfrak{B}(\mathfrak{H})_*$ where R is the resolvent of α. Suppose α is unital. Then R is unital and setting $\eta = 2\Phi(\rho)$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ in the above equation and using equations (4.3) we have $\hat{\sigma}(\rho)(I) = \rho(I)$. Conversely, if $\hat{\sigma}(\rho)(I) = \rho(I)$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$ then from equation (4.4) above and equations (4.3) we have

$$\eta(R(I)) = \eta(\Lambda) + \eta(I - \Lambda) = \eta(I)$$

for all $\eta \in \mathfrak{B}(\mathfrak{H})_*$. Hence, $R(I) = I$ and we have

$$\int_0^\infty e^{-t}(I - \alpha_t(I))\,dt = 0.$$

Since the integrand above is positive we have $\alpha_t(I) = I$ for all $t \geq 0$ so α is unital.

Now suppose σ satisfies the conclusion of the first paragraph of the theorem so the mapping (4.7) is completely positive. We begin by constructing ω for fixed ρ. Assume $\rho \in \mathfrak{B}(\mathfrak{K})_*$ is positive. Since ρ will be fixed for the first part of our argument we will write expressions like $\sigma(\rho)$ and $\omega(\rho)$ as σ and ω to simplify notation. If I is the interval $[a, b)$ let

$$\eta_I(A) = e^{-a}\hat{\sigma}(U(a)AU(a)^*) - e^{-b}\hat{\sigma}(U(b)AU(b)^*)$$

for all $A \in \mathfrak{B}(\mathfrak{H})$. Since for $A \in \mathfrak{B}(\mathfrak{H})$ we have

$$\eta_I(A) = e^{-a}\hat{\sigma}(U(a)AU(a)^* - e^{-(b-a)}U(b-a)U(a)AU(a)^*U(b-a)^*)$$

it follows that η_I is positive. From the definition of η_I and the properties of $\hat{\sigma}$ we have

$$\eta_{[a+t,b+t)}(A) = e^{-t}\eta_{[a,b)}(U(t)AU(t)^*)$$
$$\eta_{[a+t,b+t)}(I) \leq e^{-t}\eta_{[a,b)}(I) \leq \eta_{[a,b)}(I)$$

for all numbers a, b, c and t satisfying $0 \leq a \leq b \leq c$ and $t \geq 0$ and all $A \in \mathfrak{B}(\mathfrak{H})$. We use the same convention used in the definite integral, namely, $\eta_{[a,b)} = -\eta_{[b,a)}$. Suppose $a > 0$ and n is a positive integer. Then

$$\eta_{[0,a)}(I) = \sum_{k=0}^{n-1}\eta_{[ka/n,(k+1)a/n)}(I) \geq n\eta_{[a,a+a/n)}(I)$$

And if n and m are positive integers we have

$$\eta_{[a,a+ma/n)}(I) = \sum_{k=0}^{m-1}\eta_{[a+ka/n,a+(k+1)a/n)}(I) \leq m\eta_{[a,a+a/n)}(I)$$

Then combining these two inequalities we have

$$\eta_{[a,a+ma/n)}(I) \leq \frac{ma}{n}a^{-1}\eta_{[0,a)}(I)$$

for all $a > 0$ and positive integers n and m. Hence, $\eta_{[a,a+t)}(I) \leq (t/a)\eta_{[0,a)}(I)$ for all positive t so that t/a is rational. Since $\eta_{[a,b)}(I)$ is continuous in b it follows that

$$\eta_{[a,a+t)}(I) \leq (t/a)\eta_{[0,a)}(I) \leq (t/a)\hat{\sigma}(I) \tag{4.10}$$

for all $t, a > 0$. It follows that for every $A \in \mathfrak{B}(\mathfrak{H})$ and $a, t > 0$ the function $q(t) = \eta_{[a,a+t)}(A)$ satisfies a Lipschitz condition of order one. Hence, the derivative dq/dt exists almost everywhere and

$$\eta_{[a,b)}(A) = \int_{[a,b)\cap S(A)} \frac{d}{dt}\eta_{[a,a+t)}(A)\, dt$$

where $S(A)$ is the set of t for which the derivative exists. Suppose $a > 0$. Let C_n be a sequence of hermitian compact operators whose finite linear span is norm dense in the compact operators and let $C_o = I$. Let $S = \cap_{n=0}^{\infty} S(C_n)$. Let $\mathfrak{N}$ be the set of operators which are finite linear combinations of the C_n for $n = 1, 2, \cdots$. For $t \in S$ let ω^t be the linear functional on $\mathfrak{N}$ given by

$$\omega^t(A) = \frac{d}{ds}\eta_{[a,a+s)}(A)|_{s=t}$$

and for $t \notin S$ we define $\omega^t(A) = 0$ for all $A \in \mathfrak{N}$. Note from inequality (4.10) it follows that for $A \in \mathfrak{N}$ we have $|\omega^t(A)| \leq \|A\|\hat{\sigma}(I)/t$ for $t \in S$ and for $t \notin S$ we have $\omega^t.(A) = 0$. Since $\mathfrak{N}$ is norm dense in the compact operators and ω^t is norm continuous ω^t has a unique norm continuous extension to the compact operators which we also denote by ω^t and $\|\omega^t\| \leq \hat{\sigma}(I)/t$.

We note that ω^t is positive. To see this suppose A is a positive compact operator and $t \in S$. Suppose $\{A_n\}$ is a sequence of operators in $\mathfrak{N}$ converging in norm to A. Let $B_n = \frac{1}{2}A_n + \frac{1}{2}A_n^*$ and $D_n = B_n + \|A - B_n\|I$. Since $D_n \geq A \geq 0$ we have

$$\omega^t(B_n) + s_o\|A - B_n\| = \lim_{h\to 0} h^{-1}\eta_{[a+t,a+t+h)}(D_n) \geq 0.$$

where $s_o = (d/dt)\eta_{[a,a+t)}(I) \geq 0$. Recall that we have adopted the convention that $\eta_{[x,y)}(A) = -\eta_{[y,x)}(A)$ which is how we interpret the above expression when $h < 0$. Hence, $\omega^t(B_n) \geq -s_o\|A - B_n\|$. Then we have $\omega^t(A) = \lim_{n\to\infty} \omega^t(B_n) \geq \lim_{n\to\infty} -s_o\|A - B_n\| = 0$. Hence, ω^t is a bounded positive functional on the compact operators for $t \in S$.

Suppose $0 < s < t$ and $s, t \in S$ and $C \in \mathfrak{B}(\mathfrak{H})$ is compact. We show $\omega^t(C) = e^{s-t}\omega^s(U(t-s)CU(t-s)^*)$. Suppose $\epsilon > 0$. Let $\kappa = 2\hat{\sigma}(I)/s$. Then for $[a,b) \subset [s/2,\infty)$ we have from inequality (4.10) and the positivity of $\eta_{[a,b)}$ that $\|\eta_{[a,b)}\| \leq (b-a)\kappa$. Also we have $\|\omega^x\| < \kappa$ for $x \in S$ and $x \geq s$. Since $\mathfrak{N}$ is dense in the compact operators there is an operator $C_1 \in \mathfrak{N}$ with $\|C - C_1\| < (4\kappa)^{-1}\epsilon$. Then we have

$$|\omega^t(C) - \omega^t(C_1)| < \epsilon/4.$$

Now for $h \neq 0$ we have

$$h^{-1}\eta_{[t,t+h)}(C_1) = h^{-1}e^{s-t}\eta_{[s,s+h)}(U(t-s)C_1U(t-s)^*).$$

Since $C_1 \in \mathfrak{N}$ we have the limit of the left hand side of the above equation tends to $\omega^t(C_1)$ and, therefore, the right hand side also tends to $\omega^t(C_1)$ as $h \to 0$. Since $U(t-s)C_1U(t-s)^*$ is compact there is a an operator $C_2 \in \mathfrak{N}$ so that $\|C_2 - U(t-s)C_1U(t-s)\| \leq (4\kappa)^{-1}\epsilon$. Then we have

$$|h^{-1}e^{s-t}\eta_{[s,s+h)}(U(t-s)C_1U(t-s)^*) - h^{-1}e^{s-t}\eta_{[s,s+h)}(C_2)| < \epsilon/4$$

for $h \neq 0$ sufficiently small. Hence, we have in the limit that

$$|\omega^t(C_1) - e^{s-t}\omega^s(C_2)| < \epsilon/4.$$

And we also have

$$|e^{s-t}\omega^s(C_2) - e^{s-t}\omega^s(U(t-s)C_1U(t-s)^*)| < \epsilon/4$$

and

$$|e^{s-t}\omega^s(U(t-s)C_1U(t-s)^*) - e^{s-t}\omega^s(U(t-s)CU(t-s)^*)| < \epsilon/4$$

Combining the four $\epsilon/4$ inequalities above we find

$$|\omega^t(C) - e^{s-t}\omega^s(U(t-s)CU(t-s)^*)| < \epsilon$$

and since $\epsilon > 0$ is arbitrary we have $\omega^t(C) = e^{s-t}\omega^s(U(t-s)CU(t-s)^*)$. Hence, $\omega^t = e^{s-t}\hat{\theta}_{t-s}\omega^s$ for all $0 < s < t$ with $s, t \in S$. Since the complement of S has Lebesgue measure zero there is a decreasing sequence of real numbers $s_n \in S$ tending to zero. Let us define ω^t for all $t > 0$ by the limit

$$\omega^t = \lim_{n\to\infty} e^{s_n - t}\hat{\theta}_t\omega_{s_n}$$

Note that for $t \in S$ this leaves ω^t unchanged and for $t \notin S$ this defines ω^t so that ω^t is norm continuous in t. Note that for the newly defined ω^t we have $\omega^t = e^{s-t}\hat{\theta}_t\omega^s$ for all $0 < s < t$. Since the complement of S has Lebesgue measure zero we have

$$\eta_{[a,b)}(A) = \int_{S\cap[a,b)} \omega^t(A)\,dt = \int_a^b \omega^t(A)\,dt \tag{4.11}$$

for all $A \in \mathfrak{N}$. Since each side of the above equation is σ-weakly continuous the above equation extends to all $A \in \mathfrak{B}(\mathfrak{H})$.

We now define $\omega(A) = \lim_{t\to 0+} e^t\omega^t(U(t)^*AU(t))$ for $A \in \cup_{t>0}U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$. We see that for $s > 0$ and $A \in \mathfrak{B}(\mathfrak{H})$ we have

$$\begin{aligned}\omega(U(s)AU(s)^*) &= \lim_{t\to 0+} e^t\omega_t(U(t)^*U(s)AU(s)^*U(t)) \\ &= \lim_{t\to 0+} e^t\omega^t(U(s-t)AU(s-t)^*) = e^s\omega^s(A)\end{aligned}$$

Combining this with equation (4.11) we have

$$\eta_{[a,b)}(A) = \int_a^b e^{-t}\omega(U(t)AU(t)^*)\,dt$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ and $a, b \in (0, \infty)$ with $a < b$. Since

$$\eta_{[a,b)}(A) = e^{-a}\hat{\sigma}(U(a)AU(a)^*) - e^{-b}\hat{\sigma}(U(b)AU(b)^*)$$

for $A \in \mathfrak{B}(\mathfrak{H})$ and as $a \to 0+$ and $b \to \infty$ this converges to $\hat{\sigma}(A)$ we have

$$\hat{\sigma}(A) = \int_0^\infty e^{-t}\omega(U(t)AU(t)^*)\,dt$$

for all $A \in \mathfrak{B}(\mathfrak{H})$.

To establish ω is a restriction of a boundary weight to $\cup_{t>0}U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$. we will need some estimates. As we saw in establishing equation (4.3f) we have

$$\int_0^\infty e^{-t}U(t)U(t)^*\,dt = I - \Lambda.$$

Then we have

$$\begin{aligned}\hat{\sigma}(U(t)U(t)^*) &= \int_0^\infty e^{-s}\omega(U(t+s)U(t+s)^*)\,ds \\ &= \int_0^\infty e^t e^{-s}\omega^t(U(s)U(s)^*)\,ds = e^t\omega^t(I-\Lambda).\end{aligned}$$

Hence, we have $\omega^t(I-\Lambda) = e^{-t}\hat{\sigma}(U(t)U(t)^*) \leq e^{-t}\hat{\sigma}(I)$. Now for $t > 0$ and $A \in \mathfrak{B}(\mathfrak{H})$ let

$$\begin{aligned}\mu_t(A) =&\omega(U(t)U(t)^*(I-\Lambda)^{\frac{1}{2}}A(I-\Lambda)^{\frac{1}{2}}U(t)U(t)^*) \\ =&e^t\omega^t(U(t)^*(I-\Lambda)^{\frac{1}{2}}A(I-\Lambda)^{\frac{1}{2}}U(t)).\end{aligned}$$

Then recalling that $\omega^t(I) \leq \hat{\sigma}(I)/t$ and $\omega^t(I-\Lambda) \leq e^{-t}\hat{\sigma}(I)$ we have

$$\begin{aligned}\mu_t(I) =&e^t\omega^t(U(t)^*(I-\Lambda)U(t)) = e^t\omega^t(I-e^{-t}\Lambda) \\ =&\omega^t(I-\Lambda) + (e^t-1)\omega^t(I) \leq e^{-t}\hat{\sigma}(I) + (e^t-1)\hat{\sigma}(I)/t\end{aligned}$$

Since $\mu_t(I)$ increases as t deceases toward zero and the limit of the expression on the right of the above inequality converges to $2\hat{\sigma}(I)$ we have $\mu_t(I) \leq 2\hat{\sigma}(I)$ for all $t > 0$.

Now from the definition of μ_t and the fact $U(t)U(t)^*$ commutes with Λ we have

$$\mu_{t+s}(A) = \mu_t(U(t+s)U(t+s)^*AU(t+s)U(t+s)^*) \tag{4.12}$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ and $s,t \in (0,\infty)$. It then follows from Lemma 2.10 that for $t,s \in (0,\infty)$ we have

$$\begin{aligned}\|\mu_t - \mu_{t+s}\|^2 \leq&2\|\mu_t\|^2 - 2\|\mu_{t+s}\|^2 = 2(\|\mu_t\| + \|\mu_{t+s}\|)(\|\mu_t\| - \|\mu_{t+s}\|) \\ \leq&4\|\mu_t\|(\|\mu_t\| - \|\mu_{t+s}\|) \leq 8\hat{\sigma}(I)(\|\mu_t\| - \|\mu_{t+s}\|)\end{aligned}$$

Hence,

$$\|\mu_t - \mu_s\| \leq 2\sqrt{2\hat{\sigma}(I)}\sqrt{|\,\|\mu_t\| - \|\mu_s\|\,|}$$

for $t,s \in (0,\infty)$. Since $\|\mu_t\|$ converges to a limit as $t \to 0+$ it follows that $t \to \mu_t$ is a Cauchy net in norm as $t \to 0+$ and, hence, μ_t converges in norm to a positive element $\mu \in \mathfrak{B}(\mathfrak{H})_*$ and from equation (4.12) it follows that for $t \geq 0$ and $A \in \mathfrak{B}(\mathfrak{H})$ and $t > 0$ we have $\mu_t(A) = \mu(U(t)U(t)^*AU(t)U(t)^*)$. Since $U(t)^*U(t) = I$ and $U(t)U(t)^*$ commutes with Λ we have

$$\begin{aligned}\mu(U(t)AU(t)^*) =&\mu_t(U(t)AU(t))\\ =&e^t\omega^t(U(t)^*(I-\Lambda)^{\frac{1}{2}}U(t)AU(t)^*(I-\Lambda)^{\frac{1}{2}}U(t))\\ =&\omega(U(t)U(t)^*(I-\Lambda)^{\frac{1}{2}}U(t)AU(t)^*(I-\Lambda)^{\frac{1}{2}}U(t)U(t)^*\\ =&\omega((I-\Lambda)^{\frac{1}{2}}U(t)AU(t)^*(I-\Lambda)^{\frac{1}{2}})\end{aligned}$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t > 0$. We extend ω to the whole null boundary algebra $\mathfrak{A}(\mathfrak{H})$ by the relation

$$\omega((I-\Lambda)^{\frac{1}{2}}A(I-\Lambda)^{\frac{1}{2}}) = \mu(A)$$

for all $A \in \mathfrak{B}(\mathfrak{H})$. Hence, this extension of to the whole null boundary algebra $\mathfrak{A}(\mathfrak{H})$ gives us a boundary weight which satisfies equation (4.8). As for normalization condition (4.9) we have only established $\mu(I) = \omega(I-\Lambda) \leq 2\hat{\sigma}(I)$. However, now the existence of μ has been established we have $\mu(I) = \omega(I-\Lambda) = \hat{\sigma}(I) \leq \rho(I)$ by direct calculation.

In our calculations we have suppressed indicating the dependence of σ, ω and μ on ρ. Now we will return to indicating this dependence by writing $\sigma(\rho)$, $\omega(\rho)$ and $\mu(\rho)$. Summarizing our progress up to this point we have shown that for a positive $\rho \in \mathfrak{B}(\mathfrak{K})_*$ the boundary weight $\omega(\rho)$ is positive and satisfies equation (4.8) We have shown the mapping $\rho \to \omega(\rho)$ is positive. To complete the first part of the proof we must show this mapping is completely positive. To show a mapping $\rho \to \omega(\rho)$ is completely positive is equivalent to showing the mapping $\nu \otimes \rho \to \nu \otimes \omega(\rho)$ is positive from $\mathfrak{B}(\mathfrak{K}_1)_*$ to $\mathfrak{B}(\mathfrak{H}_1)_*$ where $\mathfrak{K}_1 = \mathfrak{K}_o \otimes \mathfrak{K}$ and $\mathfrak{H}_1 = \mathfrak{K}_o \otimes \mathfrak{H} = \mathfrak{K}_o \otimes \mathfrak{K} \otimes L^2(0,\infty) = \mathfrak{K}_1 \otimes L^2(0,\infty)$. The argument that the mapping $\rho \to \mu$ is completely positive is obtained by simply repeating our argument above for the tensored map from $\mathfrak{B}(\mathfrak{K}_1)_*$ to $\mathfrak{B}(\mathfrak{H}_1)_*$ and replacing our use of positivity above with complete positivity which is the same as positivity for the tensored maps. Since all this involves is a change in notation we will skip the details.

Conversely, suppose $\rho \to \omega(\rho)$ is a completely positive mapping of $\mathfrak{B}(\mathfrak{K})_*$ into $\mathfrak{A}(\mathfrak{H})_*$ satisfying the normalization condition (4.9). Suppose $t > 0$. Then we have

$$\hat{\sigma}(\rho)(A) - e^{-t}\hat{\theta}_t(\hat{\sigma}(\rho))(A) = \int_0^t e^{-s}\omega(\rho)(U(s)AU(s)^*)\,dt$$

for $A \in \mathfrak{B}(\mathfrak{H})$. Since the mappings $\rho \to \hat{\theta}_s(\omega(\rho))$ is completely positive for each $s > 0$ the mapping $\rho \to \hat{\sigma}(\rho) - e^{-t}\hat{\theta}_t(\hat{\sigma}(\rho))$ is completely positive and the normalization condition follows from direct computation. $\square$

The next two lemmas provide some useful norm estimates.

Lemma 4.18. *Suppose $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Then*

$$\left\| \int_0^t e^s \hat{\zeta}_t(\hat{\xi}_s(\hat{\Phi}(\rho)))\,ds \right\| \leq \|\rho\|(\frac{e^t + e^{-t}}{2} - 1)$$

for all $t > 0$ so the above expression is $O(t^2)$.

Proof. Suppose $\rho \in \mathfrak{B}(\mathfrak{K})_*$. We have

$$\begin{aligned} e^s\hat{\zeta}_t(\hat{\xi}_s(\hat{\Phi}(\rho)))(A) =& e^s\hat{\Phi}(\rho)(U(s)^*E(t)AE(t)U(s)) \\ =& e^{2s}\hat{\Phi}(\rho)(U(s)U(s)^*E(t)AE(t)U(s)U(s)^*) \end{aligned}$$

for all $A \in \mathfrak{B}(\mathfrak{H})$. For $0 \leq s \leq t$ we have $E(t)U(s)U(s)^* = E(s,t)$ so

$$e^s\hat{\zeta}_t(\hat{\xi}_s(\hat{\Phi}(\rho)))(A) = e^{2s}\hat{\Phi}(\rho)(E(s,t)AE(s,t))$$

Next we will estimate the norm of the above expression. Let S be a partial isometry so that $\rho(A) = \phi(AS)$ where ϕ is positive and $\|\phi\| = \phi(I) = \phi(S^*S) = \rho(S^*) = \|\rho\|$. For $A \in \mathfrak{B}(\mathfrak{K})$ we define $\Lambda_o(A)$ the operator on $\mathfrak{B}(\mathfrak{H})$ given by

$$(\Lambda_o(A)f)(x) = Af(x)$$

for $x \geq 0$ so $\Lambda_o(A) = A \otimes I$ acting on $\mathfrak{H} = \mathfrak{K} \otimes L^2(0,\infty)$. Note from the definition of $\hat{\Phi}$ we have $\hat{\Phi}(\rho)(A) = \hat{\Phi}(\phi)(A\Lambda_o(S))$ for all $A \in \mathfrak{B}(\mathfrak{H})$. Then we have

$$\begin{aligned} e^s\hat{\zeta}_t(\hat{\xi}_s(\hat{\Phi}(\rho)))(A) =& e^{2s}\hat{\Phi}(\phi)(E(s,t)AE(s,t)\Lambda_o(S)) \\ =& e^{2s}\hat{\Phi}(\phi)(E(s,t)A\Lambda_o(S)E(s,t)). \end{aligned}$$

Since the functional $A \to \hat{\Phi}(\phi)(E(s,t)AE(s,t))$ is positive the norm of this functional is obtained by evaluating this functional at $A = I$. Hence, we have

$$\begin{aligned} |e^s\hat{\zeta}_t(\hat{\xi}_s(\hat{\Phi}(\rho)))(A)| =& |e^{2s}\hat{\Phi}(\phi)(E(s,t)A\Lambda_o(S)E(s,t))| \\ \leq& e^{2s}\hat{\Phi}(\phi)(E(s,t))\|A\Lambda_o(S)\| \leq (e^s - e^{2s-t})\|\rho\|\,\|A\| \end{aligned}$$

Hence, $\|e^s\hat{\zeta}_t(\hat{\xi}_s(\hat{\Phi}(\rho)))\| \leq (e^s - e^{2s-t})\|\rho\|$. Evaluating the above expression with $A = \Lambda_o(S^*)$ proves the reverse inequality so we have $\|e^s\hat{\zeta}_t(\hat{\xi}_s(\hat{\Phi}(\rho)))\| = (e^s - e^{2s-t})\|\rho\|$. Hence, we have

$$\left\|\int_0^t e^s\hat{\zeta}_t(\hat{\xi}_s(\hat{\Phi}(\rho)))\,ds\right\| \leq \|\rho\|\int_0^t e^s - e^{2s-t}\,ds = \frac{1}{2}\|\rho\|(e^t + e^{-t} - 2).$$

And we have

$$\left\|\int_0^t e^s\hat{\zeta}_t(\hat{\xi}_s(\hat{\Phi}(\rho)))\,ds\right\| = O(t^2)$$

as $t \to 0+$ (i.e., t^{-2} times the above expression is bounded). $\square$

Lemma 4.19. *Suppose $\eta \in \mathfrak{B}(\mathfrak{H})_*$. Then, we have*

$$\|\hat{\zeta}_t(\hat{\Gamma}(\hat{\xi}_t(\eta)) - \hat{\Phi}(\hat{\Lambda}(\eta)))\|/t \to 0$$

and

$$\|\hat{\zeta}_t(\hat{\Gamma}(\eta) - \hat{\Phi}(\hat{\Lambda}(\eta)))\|/t \to 0$$

as $t \to 0+$ so $\|\hat{\zeta}_t(\hat{\Gamma}(\hat{\xi}_t(\eta)) - \hat{\Phi}(\hat{\Lambda}(\eta)))\|$ and $\|\hat{\zeta}_t(\hat{\Gamma}(\eta) - \hat{\Phi}(\hat{\Lambda}(\eta)))\|$ are $o(t)$ as $t \to 0+$.

Proof. Suppose $\eta \in \mathfrak{B}(\mathfrak{H})_*$. We will prove that

$$\|e^t\hat{\zeta}_t(\hat{\Gamma}(\hat{\xi}_t(\eta)) - e^t\hat{\Phi}(\hat{\Lambda}(\eta)))\|/t \to 0$$

as $t \to 0+$. Now from equations (4.3) we have

$$e^t\hat{\Gamma}(\hat{\xi}_t(\eta)) = \hat{\Gamma}(\eta) + \int_0^t e^s\hat{\xi}_s(\eta)\,ds.$$

We will show that the norm of $\hat{\zeta}_t$ applied to the second term is $o(t)$ as $t \to 0+$. Now we have

$$\hat{\zeta}_t(\int_0^t e^s\hat{\xi}_s(\eta)ds)(A) = \int_0^t e^s\eta(U(s)^*E(t)AE(t)U(s))\,ds.$$

Now suppose η is positive then the norm of the above functional is attained for $A = I$ and we have

$$\begin{aligned}\left\|\hat{\zeta}_t(\int_0^t e^s\hat{\xi}_s(\eta)ds)\right\| &= \int_0^t e^s\eta(U(s)^*E(t)U(s))\,ds = \int_0^t e^s\eta(E(t-s))\,ds\\ &\le \int_0^t e^s\eta(E(t))\,ds = (e^t-1)\eta(E(t)).\end{aligned}$$

Since $\eta(E(t)) \to 0$ as $t \to 0+$ we have

$$t^{-1}\left\|\hat{\zeta}_t(\int_0^t e^s\hat{\xi}_s(\eta)ds)\right\| \to 0$$

as $t \to 0+$. Since an arbitrary $\eta \in \mathfrak{B}(\mathfrak{H})_*$ is the linear combination of four positive elements the above results holds for all $\eta \in \mathfrak{B}(\mathfrak{H})_*$. With this established we have

$$\|e^t\hat{\zeta}_t(\hat{\Gamma}(\hat{\xi}_t(\eta)) - e^t\hat{\Phi}(\hat{\Lambda}(\eta)))\| = \|\hat{\zeta}_t(\hat{\Gamma}(\eta) - e^t\hat{\Phi}(\hat{\Lambda}(\eta)))\| + o(t).$$

Let $\nu = \eta - 2\hat{\Phi}(\hat{\Lambda}(\eta))$ so $\eta = 2\hat{\Phi}(\hat{\Lambda}(\eta)) + \nu$ and $\hat{\Lambda}(\nu) = 0$. Then

$$\hat{\zeta}_t(\hat{\Gamma}(\eta) - e^t\hat{\Phi}(\hat{\Lambda}(\eta))) = (1-e^t)\hat{\zeta}_t(\hat{\Phi}(\hat{\Lambda}(\eta))) + \hat{\zeta}_t(\hat{\Gamma}(\nu))$$

and direct calculation shows that $\|\hat{\zeta}_t(\hat{\Phi}(\hat{\Lambda}(\eta)))\| = (1-e^{-t})\|\hat{\Lambda}(\eta)\|$ so

$$\|\hat{\zeta}_t(\hat{\Gamma}(\eta) - e^t\hat{\Phi}(\hat{\Lambda}(\eta)))\| = \|\hat{\zeta}_t(\hat{\Gamma}(\nu))\| + O(t^2)$$

and combining the with the previous estimate we have

$$\|e^t\hat{\zeta}_t(\hat{\Gamma}(\hat{\xi}_t(\eta)) - e^t\hat{\Phi}(\hat{\Lambda}(\eta)))\| = \|\hat{\zeta}_t(\hat{\Gamma}(\nu))\| + o(t).$$

Then the proof of the lemma reduces to showing $\|\hat{\zeta}_t(\hat{\Gamma}(\nu))\|$ is $o(t)$ for all $\nu \in \mathfrak{B}(\mathfrak{H})_*$ with $\hat{\Lambda}(\nu) = 0$. Suppose then that $\nu \in \mathfrak{B}(\mathfrak{H})_*$ and $\hat{\Lambda}(\nu) = 0$. Suppose $t > 0$. We note the mapping $A \to \Gamma(\zeta_t(A))$ is completely positive. Hence, the norm of this mapping is attained at the unit. We have

$$\Gamma(\zeta_t(I)) = \int_0^\infty e^{-s}U(s)E(t)U(s)^*\,ds$$

We recall from equation (4.3f) we established the formula

$$\int_0^\infty e^{-s}U(s)U(s)^*\,ds = I - \Lambda$$

Then we have

$$\begin{aligned}\Gamma(\zeta_t(I)) =& \int_0^\infty e^{-s}U(s)(I - U(t)U(t)^*)U(s)^*\,ds \\ =& I - \Lambda - U(t)(I-\Lambda)U(t)^* \\ =& E(t) - \Lambda + e^t\Lambda U(t)U(t)^* \\ =& (I-\Lambda)E(t) + (e^t - 1)\Lambda U(t)U(t)^*\end{aligned}$$

Note the operator $\Gamma(\zeta_t(I))$ is multiplication by function $q(x) = 1 - e^{-x}$ for $x \in [0,t]$ and $q(x) = e^{t-x} - e^{-x}$ for $x \in [t,\infty)$. Then $\|\Gamma(\zeta_t(I))\| = 1 - e^{-t}$ since $0 < q(x) \leq 1 - e^{-t}$ for all $x \geq 0$ and $q(x) \to 1 - e^{-t}$ as $x \to t$. Since $A \to \Gamma(\zeta_t(A))$ is completely positive we have $\|\Gamma(\zeta_t(A))\| \leq (1 - e^{-t})\|A\|$ for all $A \in \mathfrak{B}(\mathfrak{H})$. Hence, $\|\hat{\zeta}_t(\hat{\Gamma}(\nu))\|$ is $O(t)$. Using the fact that $\hat{\Lambda}(\nu) = 0$ we will show $\|\hat{\zeta}_t(\hat{\Gamma}(\nu))\|$ is $o(t)$. Suppose this is not the case. Let $A(t)$ be an element in the unit ball of $\mathfrak{B}(\mathfrak{H})$ with $\hat{\zeta}_t(\hat{\Gamma}(\nu))(A(t)) = \|\hat{\zeta}_t(\hat{\Gamma}(\nu))\|$ for each $t > 0$. Let

$$B(t) = t^{-1}\Gamma(\zeta_t(A(t))) = t^{-1}\int_0^\infty e^{-s}U(s)E(t)A(t)E(t)U(s)^*\,ds$$

We have $\|B(t)\| \leq (1 - e^{-t})/t \leq 1$ and $\nu(B(t)) = \|\hat{\zeta}_t(\hat{\Gamma}(\nu))\|/t$ and by assumption $\|\hat{\zeta}_t(\hat{\Gamma}(\nu))\|/t$ does not tend to zero as $t \to 0+$ we have $\limsup_{t\to 0+} \nu(B(t)) > 0$. Since $\nu(B(t))$ is bounded there is a sequence $t_n \to 0+$ so that $\nu(B(t_n)) \to c$ as $n \to \infty$ and $c > 0$. Since the unit ball of $\mathfrak{B}(\mathfrak{H})$ is σ-weakly compact and $\mathfrak{H}$ is separable we can by passing to a subsequence (which we also denote by t_n) arrange it so $B_n = B(t_n) \to B_o$ as $n \to \infty$ and $\nu(B_o) = c > 0$. We will show that $B_o = \Lambda(C_o)$ for some $C_o \in \mathfrak{B}(\mathfrak{K})$. We begin by showing that $B_o = E(t)B_oE(t) + e^{-t}U(t)B_oU(t)^*$ for each $t > 0$. As a preliminary to that we show B_o commutes with $E(t)$ for all $t > 0$. Since $U(s)^*E(t) = 0$ for $s \geq t$ we have

$$\begin{aligned}B_nE(t) =& t_n^{-1}\int_0^\infty e^{-s}U(s)E(t_n)A(t_n)E(t_n)U(s)^*E(t)\,ds \\ =& t_n^{-1}\int_0^t e^{-s}U(s)E(t_n)A(t_n)E(t_n)U(s)^*E(t)\,ds\end{aligned}$$

And since $E(t+t_n)U(s)E(t_n) = U(s)E(t_n)$ for $s \in [0,t]$ we have $B_nE(t) = E(t+t_n)B_nE(t)$. Hence, if $f \in (I-E(t))\mathfrak{H}$ and $g \in E(t)\mathfrak{H}$ we have

$$\begin{aligned}|(f,B_og)| &= \lim_{n\to\infty} |(f,B_ng)| = \lim_{n\to\infty} |(E(t,t+t_n)f,B_ng)| \\ &\leq \lim_{n\to\infty} \|E(t,t+t_n)f\|\,\|B_n\|\,\|g\| \\ &\leq \lim_{n\to\infty} \|E(t,t+t_n)f\|\,\|g\| = 0\end{aligned}$$

Hence, $(I-E(t))B_oE(t) = 0$. Calculating $E(t)B_n$ as we did above we find $E(t)B_n = E(t)B_nE(t+t_n)$ and taking the limit as above we find $E(t)B_oE(t,\infty) = 0$. Hence, we have $E(t)B_o = B_oE(t) = E(t)B_oE(t)$ for all $t > 0$.

We now investigate $U(t)B_oU(t)^*$. Now for $t > 0$ we have

$$\begin{aligned}U(t)B_nU(t)^* =& t_n^{-1}\int_0^\infty e^{-s}U(t+s)E(t_n)A(t_n)E(t_n)U(t+s)^*\,ds \\ =& t_n^{-1}e^t\int_0^\infty e^{-t-s}U(t+s)E(t_n)A(t_n)E(t_n)U(t+s)^*\,ds\end{aligned}$$

Then we have

$$B_n - e^{-t}U(t)B_nU(t)^* = t_n^{-1}\int_0^t e^{-s}U(s)E(t_n)A(t_n)E(t_n)U(s)^*\,ds$$

Let $C_n = B_n - e^{-t}U(t)B_nU(t)^* - B_nE(t)$. Combining the above equation with the integral for $B_nE(t)$ derived earlier and using the fact that $E(t_n)U(s)^*E(t) = E(t_n)U(s)^*$ for $s \in [0,t-t_n]$ we have

$$C_n = -t_n^{-1}\int_{t-t_n}^t e^{-s}U(s)E(t_n)A(t_n)E(t_n)U(s)^*(I-E(t))\,ds.$$

Since $E(t_n)U(s)^* = E(t_n)U(s)^*E(t+t_n)$ for $s \in [t-t_n,t]$ we have

$$C_n = -t_n^{-1}\int_{t-t_n}^t e^{-s}U(s)E(t_n)A(t_n)E(t_n)U(s)^*(E(t+t_n)-E(t))\,ds.$$

Hence, we have from the above that $C_n = C_nE(t,t+t_n)$ and $\|C_n\| \leq 1$. Let $C_o = \lim_{n\to\infty} C_n = B_o - e^{-t}U(t)B_oU(t)^* - B_oE(t)$ where we are taking the limit in the sense of weak convergence. Then we have for $f,g \in \mathfrak{H}$ that

$$\begin{aligned}|(f,C_og)| &= \lim_{n\to\infty} |(f,C_ng)| = \lim_{n\to\infty} |(f,C_nE(t,t+t_n)g)| \\ &\leq \lim_{n\to\infty} \|f\|\,\|E(t,t+t_n)g\| = 0\end{aligned}$$

Hence, $C_o = 0$ and since $B_oE(t) = E(t)B_oE(t)$ we have that

$$B_o = E(t)B_oE(t) + e^{-t}U(t)B_oU(t)^*$$

for all $t > 0$. We will now show that the above equation implies $B_o = \Lambda(C_o)$ for some operator $C_o \in \mathfrak{B}(\mathfrak{K})$. Note the above equation implies $E(t)B_o = B_oE(t) =$

$E(t)B_oE(t)$ for all $t > 0$. Multiplying the above expression for B_o by $U(t)^*$ on the left we find $U(t)^*B_o = e^{-t}B_oU(t)^*$ for all $t > 0$. And multiplying the above equation for B_o by $U(t)$ on the right we find $B_oU(t) = e^{-t}U(t)B_o$ for all $t > 0$. It follows from differentiating these equations that $B_o\mathfrak{D}(d) \subset \mathfrak{D}(d)$ and $B_o\mathfrak{D}(d^*) \subset \mathfrak{D}(d^*)$ and $B_of = d^*B_of - B_od^*f$ for all $f \in \mathfrak{D}(d^*)$. It follows that $B_o \in \mathfrak{D}(\delta_1)$ and $\delta_1(B_o) = B_o$. Hence, it follows from Lemma 4.7 that $B_o = \Lambda(C_o)$ for some $C_o \in \mathfrak{B}(\mathfrak{K})$. We recall that $\nu(B_o) = \nu(\Lambda(C_o)) = c > 0$. But this is a contradiction since $\hat{\Lambda}(\nu) = 0$. Hence, $\|\hat{\zeta}_t(\hat{\Gamma}(\nu))\|$ is $o(t)$. $\square$

The next theorem is one of the main results of this section. In the statement of the theorem we use the norm $\|\eta\|_+$ which we now describe. If $\eta \in \mathfrak{B}(\mathfrak{H})_*$ and η is hermitian then η has a canonical decomposition as the difference of two disjoint positive functionals η_+ and η_- so $\eta = \eta_+ - \eta_-$. For a discussion of this decomposition we refer to section 4.3 of [KR] and we present the well known properties of this decomposition. For hermitian $\eta \in \mathfrak{B}(\mathfrak{H})_*$ we have $\|\eta\| = \|\eta_+\| + \|\eta_-\|$ and there are unique hermitian projections $E_+, E_- \in \mathfrak{B}(\mathfrak{H})$ so that $\eta(AE_+) = \eta_+(A)$ and $\eta(AE_-) = -\eta_-(A)$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and E_+ and E_- are the smallest projections with this property and $E_+ + E_- \leq I$. Also $\|\eta_+\| = \sup(\eta(A) : 0 \leq A \leq I)$ and the supremum is actually attained for $A = E_+$. If η is an hermitian functional we define $\|\eta\|_+ = \|\eta_+\| = \sup(\eta(A) : 0 \leq A \leq I)$. Note that for an hermitian functional with $\eta = \eta_+ - \eta_-$ its canonical decomposition into the difference of disjoint positive functional we have $\| - \eta\|_+ = \|\eta_-\|$ and $\|\eta\| = \|\eta\|_+ + \| - \eta\|_+$.

Next we introduce some notation. Suppose ϕ is a σ-weakly continuous linear mapping of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$. Let $\mathfrak{K}_o$ be an infinite dimensional separable Hilbert space and let $\mathfrak{H}_1 = \mathfrak{K}_o \otimes \mathfrak{H}$ and $\mathfrak{K}_1 = \mathfrak{K}_o \otimes \mathfrak{K}$. Let ϕ' be the mapping of $\mathfrak{B}(\mathfrak{H}_1)$ into $\mathfrak{B}(\mathfrak{K}_1)$ given by $\phi'(A \otimes B) = A \otimes \phi(B)$ for all $A \in \mathfrak{B}(\mathfrak{K}_o)$ and $B \in \mathfrak{B}(\mathfrak{H})$. The statement ϕ is completely positive or completely contractive is equivalent to the statement ϕ' is positive or contractive. Suppose α is a CP-flow over $\mathfrak{K}$ so α is a CP-semigroup of $\mathfrak{B}(\mathfrak{H})$ where $\mathfrak{H} = \mathfrak{K} \otimes L^2(0, \infty)$. Let $\mathfrak{K}_o$ be an infinite dimensional separable Hilbert space and let $\mathfrak{H}_1 = \mathfrak{K}_o \otimes \mathfrak{H}$ and $\mathfrak{K}_1 = \mathfrak{K}_o \otimes \mathfrak{K}$. Let α' be the CP-flow over $\mathfrak{K}_1$ given by $\alpha'_t(A \otimes B) = A \otimes \alpha_t(B)$ for $t \geq 0$, $A \in \mathfrak{B}(\mathfrak{K}_o)$ and $B \in \mathfrak{B}(\mathfrak{H})$. To show α is a CP-semigroup is equivalent to showing α' is a semigroup of positive contractions. Note all the operators and mappings $U(t)$, $E(t) = I - U(t)U(t)^*$, θ, ξ, ζ, $,\Lambda$, $,\Phi$, $,\Gamma$ and σ all have obvious primed operators and mappings where we replace $\mathfrak{K}$ with $\mathfrak{K}_1 = \mathfrak{K}_o \otimes \mathfrak{K}$ and $\mathfrak{H}_1 = \mathfrak{K}_o \otimes \mathfrak{H}$. When we put a prime on a mapping (e.g. $U'(t)$, σ' or Φ' and speak of the tensored operators or maps we mean the operators or maps one obtains by tensoring with $\mathfrak{B}(\mathfrak{K}_o)$. So showing that a map ϕ is completely positive is the same as showing ϕ' is positive. Note all the theorems and lemmas we have proved concerning CP-flows remain true if we replace the maps and operators in the theorems and lemmas with the primed maps and operators since all that is needed is to replace $\mathfrak{K}$ with $\mathfrak{K}_1 = \mathfrak{K}_o \otimes \mathfrak{K}$.

Theorem 4.20. *Suppose α is a CP-flow over $\mathfrak{K}$ and σ is the boundary resolvent of α. Recall $E(t) = I - U(t)U(t)^*$, $\theta_t(A) = U(t)AU(t)^*$ and $\zeta(A) = E(t)AE(t)$ for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. Recall from Lemma 4.16 it follows that for each $t > 0$ the mapping $\rho \to \hat{\sigma}(\rho) - e^{-t}\hat{\theta}_t(\hat{\sigma}(\rho))$ is completely positive linear contraction of $\mathfrak{B}(\mathfrak{K})_*$ into $\mathfrak{B}(\mathfrak{H})_*$. Assumed the primed mappings are the tensored mappings just* *described and a subscript one on a Hilbert space mean the Hilbert space without a subscript tensored with the infinite dimensional Hilbert space $\mathfrak{K}_o$. Then for each $\rho \in \mathfrak{B}(\mathfrak{K}_1)_*$ we have*

$$\liminf_{t\to 0+} t^{-1}(\|\hat{\zeta}'_t(e^t\hat{\Phi}'(\rho)+\hat{\sigma}'(\rho))\| - \|e^t\hat{\sigma}'(\rho) - \hat{\theta}'_t(\hat{\sigma}'(\rho))\|) \geq 0 \tag{4.13}$$

and for each hermitian $\rho \in \mathfrak{B}(\mathfrak{K}_1)_*$ we have

$$\liminf_{t\to 0+} t^{-1}(\|\hat{\zeta}'_t(e^t\hat{\Phi}'(\rho)+\hat{\sigma}'(\rho))\|_+ - \|e^t\hat{\sigma}'(\rho) - \hat{\theta}'_t(\hat{\sigma}'(\rho))\|_+) \geq 0 \tag{4.13+}$$

where $\|\eta\|_+$ is the is the norm of η_+ where $\eta = \eta_+ - \eta_-$ is the canonical decomposition η as the difference of disjoint positive functionals.

Conversely, suppose $\rho \to \hat{\sigma}(\rho) - e^{-t}\hat{\theta}_t(\hat{\sigma}(\rho))$ is a completely positive linear contraction of $\mathfrak{B}(\mathfrak{K})_*$ into $\mathfrak{B}(\mathfrak{H})_*$ for each $t \geq 0$ and the primed mappings are defined as described above and for all $\rho \in \mathfrak{B}(\mathfrak{K}_1)_*$ we have

$$\limsup_{t\to 0+} t^{-1}(\|\hat{\zeta}'_t(e^t\hat{\Phi}'(\rho)+\hat{\sigma}'(\rho))\| - \|e^t\hat{\sigma}'(\rho) - \hat{\theta}'_t(\hat{\sigma}'(\rho))\|) \geq 0 \tag{4.14}$$

and for all hermitian $\rho \in \mathfrak{B}(\mathfrak{K}_1)_*$ we have

$$\limsup_{t\to 0+} t^{-1}(\|\hat{\zeta}'_t(e^t\hat{\Phi}'(\rho)+\hat{\sigma}'(\rho))\|_+ - \|e^t\hat{\sigma}'(\rho) - \hat{\theta}_t(\hat{\sigma}'(\rho))\|_+) \geq 0 \tag{4.14+}$$

Then there is a unique CP-flow α over $\mathfrak{K}$ whose boundary resolvent is σ. If in addition the mapping $\rho \to \hat{\sigma}(\rho)$ is unital the same conclusion follows if one only requires condition (4.14) (i.e., in the unital case condition (4.14+) it follows from (4.14)).

Proof. Before we begin the proof we remark that conditions given in (4.13+) and (4.14+) above imply (4.13) and (4.14), respectively. This is seen as follows. Note for an hermitian functional η we have $\|\eta\| = \|\eta\|_+ + \|-\eta\|_+$ and, hence, (4.13) and (4.14) follow from (4.13+) and (4.14+) in the case of hermitian functionals. Because $\mathfrak{K}_o$ is infinite dimensional the truth of (4.13) and (4.14) for hermitian functional implies the truth of the relations for arbitrary functionals. This follows from the following observation. Suppose $\mathfrak{N}$ is a Hilbert space and $\eta \in \mathfrak{B}(\mathfrak{N})_*$ is an arbitrary. Let $\eta_1 \in \mathfrak{B}(\mathfrak{N}\oplus\mathfrak{N})$ the functional given in matrix form as follows

$$\eta_1 = \begin{bmatrix} 0 & \eta \\ \eta^* & 0 \end{bmatrix} \qquad \text{so} \qquad \eta_1\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \eta(B) + \overline{\eta}(C^*)$$

Let S be an element in the unit ball so that $\eta(S) = \|\eta\|$ and let

$$S_1 = \begin{bmatrix} 0 & S \\ S^* & 0 \end{bmatrix}.$$

Note $S_1 = S_1^*$ and $S_1^*S_1$ has positive diagonal entries of norm less than or equal to one so $S_1^*S_1 \leq I$. Hence, we have $\|S_1\| \leq 1$ and $\eta_1(S_1) = 2\|\eta\|$ and, therefore, $\|\eta_1\| \geq 2\|\eta\|$. On the other hand, suppose $T_1 \in \mathfrak{B}(\mathfrak{N}\oplus\mathfrak{N})$ is of the form

$$T_1 = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

and $\|T_1\| \leq 1$. Then we have $\|B\| \leq 1$ and $\|C\| \leq 1$ and

$$|\eta_1(T_1)| = |\eta(B) + \overline{\eta}(C^*)| \leq \|\eta\| + \|\eta\| = 2\|\eta\|$$

and, hence, $\|\eta_1\| \leq 2\|\eta\|$. Combining this with the previous inequality gives $\|\eta_1\| = 2\|\eta\|$. Hence, the norm of an arbitrary functional η can be obtained from the norm of the hermitian functional η_1. Since $\mathfrak{K}_o$ is infinite dimensional $\mathfrak{B}(\mathfrak{K}_o)$ is isomorphic to $\mathfrak{B}(\mathfrak{K}_o \oplus \mathfrak{K}_o)$ the properties of all the primed mappings persist if $\mathfrak{K}_o$ is replaced by $\mathfrak{K}_o \oplus \mathfrak{K}_o$ and by the procedure described above the norm of an arbitrary functional can be determined from the norm of an associated hermitian functional. It follows that if relations (4.13) or (4.14) hold for hermitian functionals they hold for arbitrary functionals. In the statement of the theorem we included both the conditions with and without the $(+)$ because in the unital case the only the versions without the $(+)$ are needed.

We begin the proof of the theorem by establishing condition (4.13) of the theorem. Suppose $\rho \in \mathfrak{B}(\mathfrak{K}_1)_*$ and $\eta, \nu \in \mathfrak{B}(\mathfrak{H}_1)_*$ and $\hat{\Lambda}'(\eta) = \rho$. Then from equations (4.5) and (4.6) of Theorem 4.14 we have

$$\begin{aligned}\hat{\alpha}_t'(\hat{\sigma}'(\rho) + &\hat{\Gamma}'(\eta) + \int_0^t e^s \hat{\xi}_s'(\eta)\, ds - \nu + \hat{\zeta}_t'(\nu)) \\ &= e^t(\hat{\sigma}'(\rho) + \hat{\Gamma}'(\eta)) - \hat{\theta}_t'(\nu)\end{aligned}$$

Let $\eta = 2\hat{\Phi}'(\rho)$ and let

$$\begin{aligned}\nu = \hat{\sigma}'(\rho) + &\hat{\Gamma}'(\eta) + \int_0^t e^s \hat{\xi}_s'(\eta)\, ds \\ &= \hat{\sigma}'(\rho) + \hat{\Phi}'(\rho) + 2\int_0^t e^s \hat{\xi}_s'(\hat{\Phi}'(\rho))\, ds\end{aligned}$$

Then

$$\begin{aligned}\hat{\alpha}_t'(\hat{\zeta}_t'(\nu)) =& e^t\hat{\sigma}'(\rho) - \hat{\theta}_t'(\hat{\sigma}'(\rho)) \\ &+ e^t\hat{\Gamma}'(\eta) - \hat{\theta}_t'(\hat{\Gamma}'(\eta)) - \int_0^t e^s \hat{\theta}_t'(\hat{\xi}_s'(\eta))\, ds \\ \hat{\alpha}_t'(\hat{\zeta}_t'(\nu)) =& e^t\hat{\sigma}'(\rho) - \hat{\theta}_t'(\hat{\sigma}'(\rho)) \\ &+ (e^t - e^{-t})\hat{\Phi}'(\rho) - 2\int_0^t e^s \hat{\theta}_t'(\hat{\xi}_s'(\hat{\Phi}'(\rho)))\, ds\end{aligned}$$

We calculate the last term in the above equation. Since for $t \geq s$ we have $\hat{\theta}_t'(\hat{\xi}_s'(\eta)) = \hat{\theta}_{t-s}'(\hat{\theta}_s'(\hat{\xi}_s'(\eta))) = \hat{\theta}_{t-s}'(\eta)$ and, hence,

$$\begin{aligned}2\int_0^t e^s \hat{\theta}_t'(\hat{\xi}_s'(\hat{\Phi}'(\rho)))\, ds =& 2\int_0^t e^s \hat{\theta}_{t-s}'(\hat{\Phi}'(\rho))\, ds \\ =& 2\int_0^t e^{2s-t}\hat{\Phi}'(\rho)\, ds = (e^t - e^{-t})\hat{\Phi}'(\rho)\end{aligned}$$

Hence, in the expression for $\hat{\alpha}'_t(\hat{\zeta}'_t(\nu))$ the last two terms cancel and we have

$$\hat{\alpha}'_t(\hat{\zeta}'_t(\nu)) = e^t\hat{\sigma}'(\rho) - \hat{\theta}'_t(\hat{\sigma}'(\rho))$$

We have

$$\begin{aligned}\hat{\zeta}'_t(\nu) = \hat{\zeta}'_t(\hat{\sigma}'(\rho)+&e^t\hat{\Phi}'(\rho)) - (e^t-1)\hat{\zeta}'_t(\hat{\Phi}'(\rho))\\ &+2\int_0^t e^s\hat{\zeta}'_t(\hat{\xi}'_s(\hat{\Phi}'(\rho)))\,ds\end{aligned} \tag{4.15}$$

for $A \in \mathfrak{B}(\mathfrak{H})$. From Lemma 4.18 we have

$$\left\|\int_0^t e^s\hat{\zeta}'_t(\hat{\xi}'_s(\hat{\Phi}'(\rho)))\,ds\right\| = O(t^2)$$

as $t \to 0+$ (i.e., t^{-2} times the above expression is bounded).

We note that the norm of the second to last term in equation (4.15) is

$$\|(e^t-1)\hat{\zeta}'_t(\hat{\Phi}'(\rho))\| = (e^t-1)(1-e^{-t})\|\rho\| = (e^t+e^{-t}-2)\|\rho\|$$

which is also $O(t^2)$. Hence, we have

$$\|\hat{\zeta}'_t(\nu)\| = \|\hat{\zeta}'_t(e^t\hat{\Phi}'(\rho)+\hat{\sigma}'(\rho))\| + O(t^2)$$

Since α_t is a complete contraction of $\mathfrak{B}(\mathfrak{H})$ into itself the extended α'_t is a contraction of $\mathfrak{B}(\mathfrak{H}_1)$ into itself. Hence, $\|\hat{\zeta}'_t(\nu)\| \geq \|\hat{\alpha}'_t(\hat{\zeta}'_t(\nu))\| = \|e^t\hat{\sigma}'(\rho) - \hat{\theta}'_t(\hat{\sigma}'(\rho))\|$ for all $t > 0$. From the estimate above for $\|\hat{\zeta}'_t(\nu)\|$ the limit condition (4.13) of the theorem follows.

We now show condition (4.13+) holds. Suppose $\rho \in \mathfrak{B}(\mathfrak{K}_1)*$ and $\eta, \nu \in \mathfrak{B}(\mathfrak{H}_1)_*$ and $\hat{\Lambda}'(\eta) = \rho$ and all the functionals are hermitian. Repeating the calculations above we arrive at the expressions for $\hat{\alpha}'_t(\hat{\zeta}'_t(\nu))$ and $\hat{\zeta}'_t(\nu)$ given above. We note $\|\hat{\alpha}'_t(\hat{\zeta}'_t(\nu))\|_+ \leq \|\hat{\zeta}'_t(\nu)\|_+$. This may be seen as follows. Note that since α_t is completely positive and completely contractive α'_t is positivity preserving and contractive. Since for $0 \leq A \leq I$ we have $0 \leq \alpha'_t(A) \leq \alpha'_t(I) \leq I$ we see that

$$\begin{aligned}\|\hat{\alpha}'_t(\hat{\zeta}'_t(\nu))\|_+ &= \sup(\hat{\alpha}'_t(\hat{\zeta}'_t(\nu))(A) : A \in \mathfrak{B}(\mathfrak{H}_1),\ 0 \leq A \leq I)\\ &= \sup(\hat{\zeta}'_t(\nu)(\alpha'_t(A)) : A \in \mathfrak{B}(\mathfrak{H}_1),\ 0 \leq A \leq I)\\ &\leq \sup(\hat{\zeta}'_t(\nu)(A) : A \in \mathfrak{B}(\mathfrak{H}_1),\ 0 \leq A \leq I) = \|\hat{\zeta}'_t(\nu)\|_+.\end{aligned}$$

Hence, $\|\hat{\zeta}'_t(\nu)\|_+ \geq \|\hat{\alpha}'_t(\hat{\zeta}'_t(\nu))\|_+ = \|e^t\hat{\sigma}'(\rho) - \hat{\theta}'_t(\hat{\sigma}'(\rho))\|_+$ for all $t > 0$. From expression for $\hat{\zeta}'_t(\nu)$ in equation (4.15) and the fact that the norm of the second two terms is $O(t^2)$ it follows that

$$\|\hat{\zeta}'_t(\nu)\|_+ = \|\hat{\zeta}'_t(e^t\hat{\Phi}'(\rho)+\hat{\sigma}'(\rho))\|_+ + O(t^2)$$

This with the upper estimate for $\|e^t\hat{\sigma}'(\rho) - \hat{\theta}_t'(\hat{\sigma}'(\rho))\|_+$ gives condition (4.13+) of the theorem.

Now we prove the reverse implication. Suppose $\rho \to \hat{\sigma}(\rho) - e^{-t}\hat{\theta}_t(\hat{\sigma}(\rho))$ is completely positive linear contraction of $\mathfrak{B}(\mathfrak{K})_*$ into $\mathfrak{B}(\mathfrak{H})_*$ for each $t > 0$ satisfying conditions (4.14) and (4.14+). We define $\hat{\delta}$ by equation (4.4) of Theorem 4.14. Specifically we define the domain of $\hat{\delta}$ to be all ν of the form $\nu = \hat{\sigma}(\hat{\Lambda}(\eta)) + \hat{\Gamma}(\eta)$ for some $\eta \in \mathfrak{B}(\mathfrak{H})_*$ and $\hat{\delta}(\nu) = \nu - \eta$. It is clear that the range of the mapping $\rho \to \rho - \hat{\delta}(\rho)$ from $\mathfrak{D}(\hat{\delta})$ to $\mathfrak{B}(\mathfrak{H})_*$ is all of $\mathfrak{B}(\mathfrak{H})_*$. All we need to establish that $\hat{\delta}$ is the generator of a continuous semigroup of contractions of $\mathfrak{B}(\mathfrak{H})_*$ is show that $\hat{\delta}$ is dissipative. In fact, we will show $\hat{\delta}$ is completely dissipative so we will work with the primed maps. Now each $\nu \in \mathfrak{D}(\hat{\delta}')$ is of the form $\nu_o = \hat{\sigma}'(\hat{\Lambda}'(\eta)) + \hat{\Gamma}'(\eta)$ for some $\eta \in \mathfrak{B}(\mathfrak{H}_1)_*$. We will show there is an element S in the unit ball of $\mathfrak{B}(\mathfrak{H}_1)$ so that $\nu_o(S) = \|\nu_o\|$ and $\mathrm{Re}(\hat{\delta}'(\nu_o(S))) = Re(\nu_o(S) - \eta(S)) = \|\nu_o\| - Re(\eta(S)) \leq 0$. To slightly simplify some of the following formulae let $\rho = \hat{\Lambda}'(\eta)$. Now let $\nu_t = \hat{\sigma}'(\rho) + e^t\hat{\Gamma}'(\hat{\xi}_t'(\eta))$ for $t \geq 0$. Note ν_t for $t = 0$ is ν_o. We will estimate the difference $\|\nu_t\| - e^t\|\nu_o\|$. We have from Lemma 4.12 that $\|\nu_t\| \geq \|\hat{\zeta}_t'(\nu_t)\| + \|\hat{\theta}_t'(\nu_t)\|$. We have

$$\begin{aligned}\|\hat{\zeta}_t'(\nu_t)\| =&\|\hat{\zeta}_t'(e^t\hat{\Phi}'(\rho) + \hat{\sigma}'(\rho) + e^t\hat{\Gamma}'(\hat{\xi}_t'(\eta_o)) - e^t\hat{\Phi}'(\rho))\| \\ \geq&\|\hat{\zeta}_t'(e^t\hat{\Phi}'(\rho) + \hat{\sigma}'(\rho))\| - e^t\|\hat{\zeta}_t'(\hat{\Gamma}'(\hat{\xi}_t'(\eta_o)) - \hat{\Phi}'(\rho))\|.\end{aligned}$$

From Lemma 4.19 the second term above is $o(t)$ so

$$\|\hat{\zeta}_t'(\nu_t)\| = \|\hat{\zeta}_t'(e^t\hat{\Phi}(\rho) + \hat{\sigma}'(\rho))\| + o(t).$$

Now from equations (4.3) we have

$$\hat{\theta}_t'(\nu_t) = \hat{\theta}_t'(\hat{\sigma}(\rho)) + e^t\hat{\Gamma}'(\eta)$$

So we have

$$\begin{aligned}\|\nu_t\| \geq&\|\hat{\zeta}_t'(\nu_t)\| + \|\hat{\theta}_t'(\nu_t)\| \\ =&\|\hat{\zeta}_t'(e^t\hat{\Phi}'(\rho) + \hat{\sigma}'(\rho))\| + \|\hat{\theta}_t'(\hat{\sigma}'(\rho)) + e^t\hat{\Gamma}'(\eta)\| + o(t).\end{aligned}$$

Now

$$\begin{aligned}\|\hat{\theta}_t'(\hat{\sigma}'(\rho)) + e^t\hat{\Gamma}'(\eta)\| =&\|e^t\nu_o - e^t\hat{\sigma}'(\rho) + \hat{\theta}_t'(\hat{\sigma}'(\rho))\| \\ \geq&e^t\|\nu_o\| - \|e^t\hat{\sigma}'(\rho) - \hat{\theta}_t'(\hat{\sigma}'(\rho))\|.\end{aligned}$$

Then combining the two inequalities above we have

$$\|\nu_t\| - e^t\|\nu_o\| \geq \|\hat{\zeta}_t'(e^t\hat{\Phi}'(\rho) + \hat{\sigma}'(\rho))\| - \|e^t\hat{\sigma}'(\rho) - \hat{\theta}_t'(\hat{\sigma}'(\rho))\| + o(t) \tag{4.16}$$

Now, let $S(t)$ be an element of the unit ball of $\mathfrak{B}(\mathfrak{H}_1)$ so that $\nu_t(S_t) = \|\nu_t\|$. Since the superior limit is an accumulation point there is a decreasing sequence t_n' of positive numbers converging to zero so that if the limit (4.14) is taken with the sequence t_n' the limit superior is achieved. Since the unit ball of $\mathfrak{B}(\mathfrak{H})$ is σ-weakly compact there is a subsequence $t_n = t'_{k(n)}$ so that $S(t_n)$ converges σ-weakly to a limit S_o as $n \to \infty$. Note S_o is in the unit ball of $\mathfrak{B}(\mathfrak{H})$ since it is the weak limit of elements in the unit ball. Since $\|\nu_{t_n} - \nu_o\| \to 0$ and, therefore, $|(\nu_o - \nu_{t_n})(S_{t_n})| \to 0$ as $n \to \infty$ we have

$$\begin{aligned}\nu_o(S_o) &= \lim_{n\to\infty} \nu_o(S_{t_n}) \\ &= \lim_{n\to\infty} \nu_{t_n}(S_{t_n}) + (\nu_o - \nu_{t_n})(S_{t_n}) \\ &= \lim_{n\to\infty} \|\nu_{t_n}\| = \|\nu_o\|\end{aligned}$$

Hence, S_o is an element in the unit ball of $\mathfrak{B}(\mathfrak{H})$ with $\nu_o(S_o) = \|\nu_o\|$.

Applying equations (4.3) to the expression for ν_t we have

$$\nu_t = \hat{\sigma}'(\rho) + e^t\hat{\Gamma}'(\hat{\xi}'_t(\eta)) = \hat{\sigma}'(\rho) + \hat{\Gamma}'(\eta) + \int_0^t e^s\hat{\xi}'_s(\eta)\, ds$$

Then, we have

$$\|t^{-1}(\nu_t - \nu_o) - \eta\| \leq t^{-1}\int_0^t \|e^s\hat{\xi}'_s(\eta) - \eta\|\, ds \to 0$$

as $t \to 0+$. Then we have

$$t_n^{-1}(\nu_{t_n}(S(t_n)) - \nu_o(S(t_n))) = \eta(S(t_n)) + (t_n^{-1}(\nu_t - \nu_o) - \eta)(S(t_n))$$

Since the norm of second functional on the right hand side of the above equation converges to zero as $n \to \infty$ and the $S(t_n)$ are in the unit ball of $\mathfrak{B}(\mathfrak{H})$ we have this term converges to zero as $n \to \infty$. Since $S(t_n)$ converges σ-weakly to S_o the first term on the right hand side of the above equations converges to $\eta(S_o)$. Hence, we have

$$\lim_{n\to\infty} t_n^{-1}(\nu_{t_n}(S(t_n)) - \nu_o(S(t_n))) = \eta(S_o)$$

Then we have

$$\begin{aligned}Re(\hat{\delta}'(\nu_o(S_o))) =& Re(\nu_o(S_o) - \eta(S_o)) = \|\nu_o\| - Re(\eta(S_o)) \\ =& \|\nu_o\| - \lim_{n\to\infty} t_n^{-1}Re(\nu_{t_n}(S(t_n)) - \nu_o(S(t_n))) \\ =& \|\nu_o\| - \lim_{n\to\infty} t_n^{-1}Re(\|\nu_{t_n}\| - \nu_o(S(t_n))) \\ \leq& \|\nu_o\| - \lim\sup_{n\to\infty} t^{-1}Re(\|\nu_{t_n}\| - \|\nu_o\|) \\ \leq& \|\nu_o\| - \lim\sup_{n\to\infty} t_n^{-1}(\|\nu_{t_n}\| - e^{t_n}\|\nu_o\| + (e^{t_n} - 1)\|\nu_o\|) \\ =& -\lim\sup_{n\to\infty} t_n^{-1}(\|\nu_{t_n}\| - e^{t_n}\|\nu_o\|) \\ \leq -\lim\sup_{t_n\to\infty} & t_n^{-1}(\|\hat{\zeta}'_{t_n}(e^{t_n}\hat{\Phi}'(\rho) + \hat{\sigma}'(\rho))\| - \|e^{t_n}\hat{\sigma}'(\rho) - \hat{\theta}'_{t_n}(\hat{\sigma}'(\rho))\|)\end{aligned}$$

where the last inequality follows from (4.16). Recall that the sequence $\{t_n\}$ is a subsequence of the sequence $\{t'_n\}$ where for the sequence $\{t'_n\}$ the above limit superior equals the limit superior as $t \to 0+$. Hence, the above limit superior equals the limit superior as $t \to 0+$ and by assumption (limit inequality (4.14)) this limit is greater than or equal to zero. Hence, $\mathrm{Re}(\hat{\delta}'(\nu_o(S_o))) \leq 0$ and $\hat{\delta}'$ is dissipative and since $\hat{\delta}' is$ dissipative $\hat{\delta}$ is completely dissipative.

Recall $\hat{\delta}$ is defined on its domain $\mathfrak{D}(\hat{\delta})$ of $\nu \in \mathfrak{B}(\mathfrak{H})_*$ of the form $\nu = \hat{\sigma}(\hat{\Lambda}(\eta)) + \hat{\Gamma}(\eta)$ and $\hat{\delta}(\nu) = \nu - \eta$. We see that $\hat{\delta}$ is a closed dissipative operator and the range of the mapping $\nu \to \nu - \hat{\delta}(\nu)$ for $\nu \in \mathfrak{D}(\hat{\delta})$ is all of $\mathfrak{B}(\mathfrak{H})_*$. Then from Theorem 2.7 we have $\hat{\delta}$ is the generator of a strongly continuous one parameter semigroup $\hat{\alpha}$ of contractions of $\mathfrak{B}(\mathfrak{H})_*$ and $\hat{\delta}$ and, therefore, α is uniquely determined by the mapping $\rho \to \hat{\sigma}(\rho)$. Since $\hat{\delta}$ is completely dissipative $\hat{\alpha}$ and, therefore, α is completely contractive.

We now prove the last statement of the theorem. We assume then the mapping $\rho \to \hat{\sigma}(\rho)$ is unital. Then for $\nu = \hat{\sigma}(\hat{\Lambda}(\eta)) + \hat{\Gamma}(\eta) \in \mathfrak{D}(\hat{\delta})$ we have

$$\begin{aligned}\hat{\delta}(\nu)(I) &= \hat{\sigma}(\hat{\Lambda}(\eta))(I) + \hat{\Gamma}(\eta)(I) - \eta(I) \\ &= \eta(\Lambda) + \eta(I - \Lambda) - \eta(I) = 0.\end{aligned}$$

Hence,

$$\frac{d}{dt}\nu(\alpha_t(I)) = \hat{\delta}(\hat{\alpha}_t(\nu))(I) = 0$$

for all $\nu \in \mathfrak{D}(\hat{\delta})$ and $t > 0$. Hence, $\nu(I) = \nu(\alpha_t(I))$ for all $\nu \in \mathfrak{D}(\hat{\delta})$ and $t \geq 0$ and since $\mathfrak{D}(\hat{\delta})$ is dense in $\mathfrak{B}(\mathfrak{H})_*$ we have $\alpha_t(I) = I$ for all $t \geq 0$. Hence, α is unital. Since α is unital and completely contractive α is completely positive.

Now that we have proved the last statement of the theorem we now drop the assumption that $\rho \to \hat{\sigma}(\rho)$ is unital. We will show that condition (4.14+) insures that α is completely positive or what is the same thing that α' is positive. As mentioned earlier in the proof condition (4.14+) implies (4.14) so by the argument above we have α is a strongly continuous semigroup of completely contractive mappings of $\mathfrak{B}(\mathfrak{H})$ into itself. As we saw in Theorem 2.9 α' is positivity preserving if and only if for all $\lambda \in (0,1)$ we have $\nu - \lambda\hat{\delta}'(\nu) \geq 0$ implies $\nu \geq 0$ or what is the same thing α' is positivity preserving if and only if for $\lambda \in (0,1)$ and $\nu \in \mathfrak{D}(\hat{\delta}')$ is hermitian and ν is not positive then $\nu - \lambda\hat{\delta}'(\nu)$ is not positive. Suppose then that $\lambda \in (0,1)$ and $\nu \in \mathfrak{D}(\hat{\delta}')$ is hermitian and ν is not positive. Let $\nu = \nu_+ - \nu_-$ be the canonical decomposition of ν as the difference of two disjoint positive functionals and let E_+ and E_- be the support projections of ν_+ and ν_-, respectively. Since ν is not positive $\nu(E_-) = -\|\nu_-\| < 0$. Since $\nu \in \mathfrak{D}(\hat{\delta}')$ we have $\nu = \hat{\sigma}'(\hat{\Lambda}'(\eta)) + \hat{\Gamma}'(\eta)$. Let $P_t = E'(t)A(t)E'(t) + \theta_t'(E_-)$ for $t \geq 0$ where $A(t)$ is a positive operator in the unit ball of $\mathfrak{B}(\mathfrak{H}_1)$. We see that P_t is positive and in the unit ball so

$$(\hat{\sigma}'(\hat{\Lambda}'(\eta)) + \hat{\Gamma}'(\eta))(P_t - E_-) \geq 0$$

for $t > 0$. So for $t > 0$ we have

$$\begin{aligned}&\hat{\sigma}'(\hat{\Lambda}'(\eta))(\zeta_t'(A(t))) + \hat{\Gamma}'(\eta)(\zeta_t'(A(t))) + \hat{\Gamma}'(\eta)(\theta_t'(E_-) - E_-) \\ &\quad + (e^t - 1)\hat{\sigma}'(\hat{\Lambda}'(\eta))(E_-) - \hat{\sigma}'(\hat{\Lambda}'(\eta))(e^t E_- - \theta_t'(E_-)) \geq 0\end{aligned}$$

Since

$$\lim_{t\to 0+} t^{-1}\hat{\Gamma}'(\eta)(\theta_t'(E_-) - E_-) = \hat{\Gamma}'(\eta)(E_-) - \eta(E_-)$$

and from Lemma 4.19 we have $\|\hat{\zeta}_t'(\hat{\Gamma}'(\eta) - \hat{\Phi}'(\hat{\Lambda}'(\eta)))\|/t \to 0$ as $t \to 0+$ and since $\|(e^t - 1)\hat{\zeta}_t'(\hat{\Phi}'(\hat{\Lambda}'(\eta)))\|$ is $O(t^2)$ and, hence, is $o(t)$ we have the above expression is equal to the expression below

$$\begin{aligned}\hat{\zeta}_t'(\hat{\sigma}'(\hat{\Lambda}'(\eta)) + &e^t\hat{\Phi}'(\hat{\Lambda}'(\eta)))(A(t)) + t(\hat{\sigma}'(\hat{\Lambda}'(\eta)) \\ &+\hat{\Gamma}'(\eta))(E_-) - t\eta(E_-) \\ &-(e^t\hat{\sigma}'(\hat{\Lambda}'(\eta)) - \hat{\theta}_t'(\hat{\sigma}'(\hat{\Lambda}'(\eta)))))(E_-) + o(t)\end{aligned}$$

So we have

$$\begin{aligned}t\eta(E_-) \leq \hat{\zeta}_t'(\hat{\sigma}'(\hat{\Lambda}'(\eta)) + e^t\hat{\Phi}'(\eta))(A(t)) + t(\hat{\sigma}'(\hat{\Lambda}'(\eta)) + \hat{\Gamma}'(\eta))(E_-) \\ - (e^t\hat{\sigma}'(\hat{\Lambda}'(\eta)) - \hat{\theta}_t'(\hat{\sigma}'(\hat{\Lambda}'(\eta)))))(E_-) + o(t)\end{aligned}$$

Recall that the only assumption on $A(t) \in \mathfrak{B}(\mathfrak{H}_1)$ is $0 \leq A(t) \leq I$. Now let us choose $A(t)$ so that

$$\hat{\zeta}_t'(\hat{\sigma}'(\hat{\Lambda}'(\eta)) + e^t\hat{\Phi}'(\eta))(A(t)) = -\|\hat{\zeta}_t'(\hat{\sigma}'(\hat{\Lambda}'(\eta)) + e^t\hat{\Phi}'(\eta))\|_-$$

where $\|\mu\|_- = \|\mu_-\|$ with $\mu = \mu_+ - \mu_-$ is the canonical decomposition of μ into the difference of disjoint positive functionals. Since E_- is a hermitian projection we have

$$(e^t\hat{\sigma}'(\hat{\Lambda}'(\eta)) - \hat{\theta}_t'(\hat{\sigma}'(\hat{\Lambda}'(\eta)))))(E_-) \geq -\|e^t\hat{\sigma}'(\hat{\Lambda}'(\eta)) - \hat{\theta}_t'(\hat{\sigma}'(\hat{\Lambda}'(\eta)))\|_-.$$

Hence, we have

$$\begin{aligned}\eta(E_-) \leq &(\hat{\sigma}'(\hat{\Lambda}'(\eta)) + \hat{\Gamma}'(\eta))(E_-) - t^{-1}\|\hat{\zeta}_t'(\hat{\sigma}'(\hat{\Lambda}'(\eta)) + e^t\hat{\Phi}'(\eta))\|_- \\ &+ t^{-1}\|e^t\hat{\sigma}'(\hat{\Lambda}'(\eta)) - \hat{\theta}_t'(\hat{\sigma}'(\hat{\Lambda}'(\eta)))\|_- + o(t)/t\end{aligned}$$

Note $(\hat{\sigma}'(\hat{\Lambda}'(\eta)) + \hat{\Gamma}'(\eta))(E_-) = \nu(E_-) = -\|\nu\|_- < 0$. Hence, we have

$$\eta(E_-) \leq -\|\nu\|_- \leq -\limsup_{t\to 0+} D(t)$$

with

$$D(t) = t^{-1}(\|\hat{\zeta}_t'(\hat{\sigma}'(\hat{\Lambda}'(\eta)) + e^t\hat{\Phi}'(\eta))\|_- - \|e^t\hat{\sigma}'(\hat{\Lambda}'(\eta)) - \hat{\theta}_t'(\hat{\sigma}'(\hat{\Lambda}'(\eta)))\|_-)$$

Since $\|-\mu\|_+ = \|\mu\|_-$ for any hermitian functional μ it follows that relation (4.14+) holds with the $\|\cdot\|_+$ norms replaced by the $\|\cdot\|_-$ norms. Since we have assumed (4.14+) holds and $-\|\nu\|_-$ is strictly negative we have $\eta(E_-) < 0$. Hence, we have

$$\begin{aligned}(\nu - \lambda\hat{\delta}'(\nu))(E_-) =&(1-\lambda)(\hat{\sigma}'(\hat{\Lambda}'(\eta)) + \hat{\Gamma}'(\eta))(E_-) + \lambda\eta(E_-) \\ =&(1-\lambda)\nu(E_-) + \lambda\eta(E_-) \\ <& -(1-\lambda)\|\nu\|_- < 0\end{aligned}$$

and, hence, $(\nu - \lambda\hat{\delta}'(\nu))$ is not positive. It then follows from Theorem 2.9 that $\hat{\alpha}'$ is positivity preserving and, hence, α is completely positive.

We show that $U(t)$ intertwines α. We recall each $\nu \in \mathfrak{D}(\hat{\delta})$ is of the form $\nu = \hat{\sigma}(\hat{\Lambda}(\eta)) + \hat{\Gamma}(\eta)$ for some $\eta \in \mathfrak{B}(\mathfrak{H})_*$. If follows that if $\eta \in \mathfrak{B}(\mathfrak{H})_*$ and $\hat{\Lambda}(\eta) = 0$ then $\hat{\Gamma}(\eta) \in \mathfrak{D}(\hat{\delta})$ and $\hat{\delta}(\hat{\Gamma}(\eta)) = \hat{\Gamma}(\eta) - \eta$. It follows from Lemma 4.10 that $\hat{\delta}$ is an extension of $\hat{\delta}_1$ (i.e., $\hat{\delta}(\nu) = \delta_1(\nu)$ for all $\nu \in \mathfrak{D}(\hat{\delta}_1)$). Hence, it follows that δ_1 is an extension of δ. Suppose that $f \in \mathfrak{D}(d)$ and $A \in \mathfrak{D}(\delta)$. Then we have

$$\begin{aligned} h^{-1}(\alpha_{t+h}(A)U(t+h)f - \alpha_t(A)U(t)f) =& h^{-1}(\alpha_{t+h}(A)(U(h)-I)U(t)f) \\ &+ h^{-1}(\alpha_{t+h}(A) - \alpha_t(A))U(t)f \end{aligned}$$

for $t > 0$ and $t + h > 0$. Taking the limit as $h \to 0$ we find

$$\begin{aligned} \frac{d}{dt}\alpha_t(A)U(t)f =& \alpha_t(A)dU(t)f - \delta(\alpha_t(A))U(t)f = \\ =& -\alpha_t(A)dU(t)f - \delta_1(\alpha_t(A))U(t)f = -d\alpha_t(A)U(t)f. \end{aligned}$$

Since $f_t = \alpha_t(A)U(t)f \in \mathfrak{D}(d)$ and $(d/dt)f_t = -df_t$ it follows that $f_t = U(t)f_o = U(t)f$. Hence, we have $U(t)Af = \alpha_t(A)U(t)f$ for all $f \in \mathfrak{D}(d)$ and $A \in \mathfrak{D}(\delta)$. Since for fixed t each side of this equation is norm continuous in f this equation extends to all $f \in \mathfrak{H}$. Since each side of this equation is σ-strongly continuous in A and $\mathfrak{D}(\delta)$ is σ-strongly dense in $\mathfrak{B}(\mathfrak{H})$ it follows that $U(t)Af = \alpha_t(A)U(t)f$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $f \in \mathfrak{H}$. Hence, $U(t)$ intertwines α. $\square$

We see from the previous theorem that for an understanding of CP-flows it is essential that we understand the limits (4.13+) and (4.14+). The next lemma shows us that the superior limit in (4.14+) is always finite.

Lemma 4.21. *Suppose $\rho \in \mathfrak{B}(\mathfrak{K})_*$ is hermitian and $\sigma \in \mathfrak{B}(\mathfrak{H})_*$ is hermitian then*

$$\|\hat{\zeta}_t(e^t\hat{\Phi}(\rho) + \sigma)\|_+ - \|e^t\sigma - \hat{\theta}_t(\sigma)\|_+ \le (e^t - 1)(\|\sigma\|_- + \|\rho\|_+)$$

for all $t > 0$. The same result holds for an hermitian $\rho \in \mathfrak{B}(\mathfrak{K}_1)_*$ and an hermitian $\sigma' \in \mathfrak{B}(\mathfrak{H}_1)_*$ with all the maps above replaced by the primed maps as described before Theorem 4.20.

Proof. Assume the hypothesis of the lemma and $t > 0$. Let $\eta = \hat{\zeta}_t(e^t\hat{\Phi}(\rho) + \sigma)$ and let $\eta = \eta_+ + \eta_-$ be the unique decomposition of η into the difference of disjoint positive functionals and let E_+ be the support projection for η_+ so $\eta(E_+) = \|\eta\|_+$ and E_+ is the smallest projection with this property. It follows that $E(t)E_+E(t) = E_+$. Let

$$B = \sum_{n=0}^{\infty} U(nt)E_+U(nt)^*$$

Since $E_+ \le E(t)$ we have

$$B = \sum_{n=0}^{\infty} U(nt)E_+U(nt)^* \le \sum_{n=0}^{\infty} U(nt)E(t)U(nt)^* = I$$

Hence, $0 \leq B \leq I$. We have

$$\begin{aligned} e^t B - \theta_t(B) =& (e^t - 1)B + \sum_{n=0}^{\infty} U(nt)E_+U(nt)^* - \sum_{n=1}^{\infty} U(nt)E_+U(nt)^* \\ =& (e^t - 1)B + E_+ \end{aligned}$$

Since $0 \leq B \leq I$ we have

$$\begin{aligned} \|e^t\sigma - \hat{\theta}_t(\sigma)\|_+ \geq& \sigma(e^t B - \theta_t(B)) = (e^t - 1)\sigma(B) + \sigma(E_+) \\ =& (e^t - 1)\sigma(B) + \eta(E_+) - \hat{\zeta}_t(e^t\hat{\Phi}(\rho))(E_+) \\ \geq& (e^t - 1)\sigma(B) + \|\eta\|_+ - \|\hat{\zeta}_t(e^t\hat{\Phi}(\rho))\|_+ \\ \geq& -(e^t - 1)\|\sigma\|_- + \|\eta\|_+ - (e^t - 1)\|\rho\|_+ \end{aligned}$$

Note in the last line we used the fact that $\|\hat{\zeta}_t(e^t\hat{\Phi}(\rho))\|_+ = (e^t - 1)\|\rho\|_+$ which follow from direct computation. Recalling $\eta = \hat{\zeta}_t(e^t\hat{\Phi}(\rho) + \sigma)$ the estimate of the lemma follows. The proof for the primed maps is identical. $\square$

Lemma 4.22. *Suppose $\rho \to \omega(\rho)$ is a completely positive boundary weight map of $\mathfrak{B}(\mathfrak{K})_*$ into $\mathfrak{A}(\mathfrak{H})_*$ as described in Definition 4.16 and suppose $\rho \to \hat{\sigma}(\rho)$ is the mapping of $\mathfrak{B}(\mathfrak{K})_*$ into $\mathfrak{B}(\mathfrak{H})_*$ given by*

$$\hat{\sigma}(\rho)(A) = \int_0^\infty e^{-t}\omega(\rho)(U(t)AU(t)^*)\,dt = \hat{\Gamma}(\omega(\rho))(A) \tag{4.8}$$

for all $A \in \mathfrak{B}(\mathfrak{H})$. Let $\mathfrak{K}_o$ be an infinite dimensional separable Hilbert space and let $\mathfrak{K}_1 = \mathfrak{K}_o \otimes \mathfrak{K}$ and $\mathfrak{H}_1 = \mathfrak{K}_1 \otimes \mathfrak{H} = \mathfrak{K}_o \otimes \mathfrak{K} \otimes L^2(0,\infty)$ and let the primed operators and mappings be the tensored mapping as described before Theorem 4.20. Let $\mathfrak{A}(\mathfrak{H}_1)$ be then null boundary algebra of all operators of the form

$$A = (I - \Lambda'(I))^{\frac{1}{2}}B(I - \Lambda'(I))^{\frac{1}{2}}$$

with $B \in \mathfrak{B}(\mathfrak{H}_1)$. Then the mapping $\rho \to \hat{\sigma}(\rho)$ is the boundary resolvent of a CP-flow over $\mathfrak{K}$ if and only if for each hermitian $\rho \in \mathfrak{B}(\mathfrak{K}_1)_*$ there is an operator $T \in \mathfrak{B}(\mathfrak{K}_1)$ with $0 \leq T \leq I$ so that if $A = A^* \in \mathfrak{A}(\mathfrak{H}_1)$ and

$$0 \leq A + \Lambda'(T) \leq I \qquad \textit{then} \qquad \rho(T) \geq \omega'(\rho)(A). \tag{4.17}$$

Proof. Assume the hypothesis and notation of the lemma. Suppose for each hermitian $\rho \in \mathfrak{B}(\mathfrak{K}_1)_*$ there is an operator $T \in \mathfrak{B}(\mathfrak{K})$ with $0 \leq T \leq I$ so that for $A = A^* \in \mathfrak{A}(\mathfrak{H}_1)$ inequality (4.17) above is satisfied. From Theorems 4.17 and 4.20 we see that the mapping $\rho \to \hat{\sigma}(\rho)$ defines a CP-semigroup provided limit inequality (4.14+) holds. Suppose then that $\rho \in \mathfrak{B}(\mathfrak{K}_1)_*$ is hermitian and $T \in \mathfrak{B}(\mathfrak{K}_1)$ with $0 \leq T \leq I$ so that (4.17) is satisfied. Suppose $t > 0$ and $C \in \mathfrak{B}(\mathfrak{H}_1)$ with $0 \leq C \leq I$ and

$$(e^t\hat{\sigma}'(\rho) - \hat{\theta}_t(\hat{\sigma}'(\rho)))(C) = \|e^t\hat{\sigma}'(\rho) - \hat{\theta}_t(\hat{\sigma}'(\rho))\|_+$$

Note we can let C be the support projection of the positive part of $e^t\hat{\sigma}'(\rho)-\hat{\theta}_t(\hat{\sigma}'(\rho))$. Let $B = \Lambda_o'(T)E'(t)$ where Λ_o was defined in Definition 4.6. Since $0 \leq T \leq I$ we have $0 \leq B \leq I$ and $\zeta_t'(B) = B$ it follows that

$$\|\hat{\zeta}_t'(e^t\hat{\Phi}'(\rho) + \hat{\sigma}'(\rho))\|_+ \geq (e^t\hat{\Phi}'(\rho) + \hat{\sigma}'(\rho))(B)$$

Then we have

$$\begin{aligned} \|\hat{\zeta}_t'(e^t\hat{\Phi}'(\rho) + \hat{\sigma}'(\rho))\|_+ - \|e^t\hat{\sigma}'(\rho) - \hat{\theta}_t(\hat{\sigma}'(\rho))\|_+ &\geq Q(t) \\ = (e^t\hat{\Phi}'(\rho) + \hat{\sigma}'(\rho))(B) - (e^t\hat{\sigma}'(\rho) - \hat{\theta}_t(\hat{\sigma}'(\rho)))(C) \end{aligned} \tag{4.18}$$

We examine $Q(t)$. We have

$$\begin{aligned} Q(t) = (e^t-1)\rho(T) + \int_0^\infty e^{-s}\omega'(\rho)(U'(s)\Lambda_o'(T)E'(t)U'(s)^*)\,ds \\ - e^t\int_0^t e^{-s}\omega'(\rho)(U'(s)CU'(s)^*)\,ds \end{aligned}$$

We can write the above formula in the form

$$Q(t) = (e^t - 1)\rho(T) - \omega'(\rho)(B_1 - B_2)$$

where

$$B_1 = e^t\int_0^t e^{-s}U'(s)CU'(s)^*\,ds$$

and

$$\begin{aligned} B_2 = &\int_0^\infty e^{-s}U'(s)\Lambda_o'(T)E'(t)U'(s)^*\,ds \\ =&\Lambda_o'(T) - \Lambda'(T) - U'(t)(\Lambda_o'(T) - \Lambda'(T))U'(t)^*. \\ =&\Lambda_o'(T)E'(t) - \Lambda'(T) + e^t\Lambda'(T)(I - E'(t)) \end{aligned}$$

where the second equality comes from applying equation (4.3f) and the fact that $E'(t) = I - U'(t)U'(t)^*$ and the second equality comes from the commutation properties of $U'(t)$ with $\Lambda_\lambda'(T)$ as stated after Definition 4.6. We calculate

$$\begin{aligned} (e^t - 1)\Lambda'(T) - B_2 =&e^t\Lambda'(T) - \Lambda_o'(T)E'(t) - e^t\Lambda'(T)(I - E'(t)) \\ =&(e^t\Lambda'(T) - \Lambda_o'(T))E'(t). \end{aligned}$$

For f a $\mathfrak{K}_1$ valued function $f(x)$ we have $((\mathrm{e}^t\Lambda'(T) - \Lambda_o'(T))E'(t)f)(x) = (e^te^{-x} - 1)Tf(x)$ for $x \in [0, t]$ and the function is the zero vector for $x > t$. Since $T \geq 0$ we see the above operator is positive. Since B_1 is positive we have

$$B_1 - B_2 + (e^t - 1)\Lambda'(T) \geq 0$$

If in the expression for B_1 above we replace C by the unit I we will obtain a larger operator. Hence, we have

$$\begin{aligned} B_1 \leq D =& e^t \int_0^t e^{-s} U'(s) U'(s)^* \, ds \\ =& e^t (I - \Lambda'(I)) - U'(t)(I - \Lambda'(I)) U'(t)^* \\ =& (e^t - 1)(I - E'(t)) + e^t (I - \Lambda'(I)) E'(t). \end{aligned}$$

Hence, we have

$$\begin{aligned} &(e^t - 1)I - B_1 + B_2 - (e^t - 1)\Lambda'(T) \\ &\quad \geq (e^t - 1)I - D + B_2 - (e^t - 1)\Lambda'(T) \\ &\quad = ((e^t - 1)I - (e^t - 1)(I - E'(t))) - e^t (I - \Lambda'(I)) E'(t) \\ &\quad\quad - (e^t \Lambda'(T) - \Lambda'_o(T)) E'(t). \\ &\quad = (e^t I - I - e^t I + e^t \Lambda'(I) - e^t \Lambda'(T) + \Lambda'_o(T)) E'(t) \\ &\quad = (-\Lambda'_o(I) + e^t \Lambda'(I) - e^t \Lambda'(T) + \Lambda'_o(T)) E'(t) \\ &\quad = (e^t \Lambda'(I - T) - \Lambda'_o(I - T)) E'(t) \end{aligned}$$

For $f \in \mathfrak{H}_1$ represented by a $\mathfrak{K}_1$ valued function $f(x)$ we have $((e^t \Lambda'(I-T) - \Lambda'_o(I-T))E'(t)f)(x) = (e^t e^{-x} - 1)(I - T)f(x)$ for $x \in [0, t]$ and the function is the zero vector for $x > t$. Recalling that $T \leq I$ we see the above operator is positive. Hence, we have

$$0 \leq B_1 - B_2 + (e^t - 1)\Lambda'(T) \leq (e^t - 1)I$$

Let $A = (e^t - 1)^{-1}(B_1 - B_2)$. Since B_1 and B_2 are in $\mathfrak{A}(\mathfrak{H}_1)$ we have $A \in \mathfrak{A}(\mathfrak{H}_1)$ and we have $0 \leq A + \Lambda'(T) \leq I$ so we have $\rho(T) \geq \omega'(\rho)(A)$ and we have

$$Q(t) = (e^t - 1)\rho(T) - \omega'(\rho)(B_1 - B_2) \geq 0.$$

Hence, $Q(t)/t \geq 0$ so from (4.18) we see the limit (4.14+) of Theorem 4.20 is non negative. Hence, from Theorem 4.20 it follows that the mapping $\rho \to \hat{\sigma}(\rho)$ defines a unique CP-flow α.

Conversely, suppose σ is the boundary resolvent of a CP-flow α. Suppose $\rho \in \mathfrak{B}(\mathfrak{K}_1)_*$ is hermitian. Let $A(t) \in \mathfrak{B}(\mathfrak{H}_1)$ be hermitian with $0 \leq A(t) \leq E'(t)$ for each $t > 0$ so that

$$\hat{\zeta}'_t(e^t \hat{\Phi}'(\rho) + \hat{\sigma}'(\rho))(A(t)) = \|\hat{\zeta}'_t(e^t \hat{\Phi}'(\rho) + \hat{\sigma}'(\rho))\|_+$$

Let

$$B(t) = \sum_{n=0}^{\infty} e^{-nt} U'(nt) A(t) U'(nt)^*$$

Let $X(t) = t^{-1}\Gamma'(\zeta'_t(A(t))) = t^{-1}\Gamma'(A(t))$. Note that $X(t) - B(t)$ converges σ-weakly to zero as $t \to 0+$. The fact that for $f, g \in \mathfrak{H}_1$ $(f, (X(t) - B(t))g) \to 0$ as $t \to 0+$ is just the argument that the Riemann integral can be replace by Riemann sums for a continuous function. And then since $X(t) - B(t)$ is uniformly bounded one obtain σ-weak convergence. Repeating the argument in the proof of Lemma 4.19 we have there exists a decreasing sequence of positive t_n so that $t_n \to 0$ as $n \to \infty$ and $X(t_n) \to \Lambda'(T)$ σ-weakly as $n \to \infty$. In constructing the sequence t_n we can arrange it so it is a subsequence of any given sequence converging to zero so we will make the further assumption that $t_n = 2^{-k(n)}$ (i.e., t_n is the reciprocal of a power of two). Since the $X(t)$ are positive and $\|X(t)\| \leq 1$ (see the argument in the proof of Lemma 4.19) we have $0 \leq T \leq I$. Since $X(t) - B(t) \to 0$ σ-weakly as $t \to 0+$ we have $B(t_n) \to \Lambda'(T)$ as $n \to \infty$. We claim condition (4.17) is satisfied for this operator T. Suppose this is not the case so there an hermitian $A_1 \in \mathfrak{A}(\mathfrak{H}_1)$ so that $0 \leq A_1 + \Lambda'(T) \leq I$ and $\rho(T) < \omega'(\rho)(A_1)$. Note $\omega'(E'(t,\infty)A_1E'(t,\infty)) \to \omega'(A_1)$ as $t \to 0+$ so there is a $t_o > 0$ so that if $A_o = E'(t_o,\infty)A_1E'(t_0,\infty)$ then $\rho(T) < \omega'(\rho)(A_o)$. Furthermore, shrinking t_o if necessary we can assume t_o is the reciprocal of a power of two. One checks that since $0 \leq A_1 + \Lambda'(T) \leq I$ we have $0 \leq A_o + \Lambda'(T) \leq I$

We will need to introduce some notation. Let

$$\omega^t(A) = t^{-1}\int_0^t e^{-s}\omega'(\rho)(U'(s)AU'(s)^*)\,ds$$

and

$$\nu_t(A) = \omega^t(E'(t_o,\infty)AE'(t_o,\infty))$$

for all $A \in \mathfrak{B}(\mathfrak{H}_1)$ and $t > 0$. As we saw in the proof of Theorem 4.17 the expression for ω^t is well defined and is given by

$$\omega^t = t^{-1}(\hat{\sigma}'(\rho) - e^{-t}\hat{\theta}'_t(\hat{\sigma}'(\rho)))$$

As for ν_t, since $\omega'(\rho)$ restricted to $U'(t_o)\mathfrak{B}(\mathfrak{H}_1)U'(t_o)^*$ is a normal functional we have $\nu_t \in \mathfrak{B}(\mathfrak{H}_1)_*$ and ν_t converges in norm to the limit ν_o as $t \to 0+$ where $\nu_o(A) = \omega'(\rho)(E'(t_o,\infty)AE'(t_o,\infty))$ for all $A \in \mathfrak{B}(\mathfrak{H}_1)$. In terms of ω^t we have for $t > 0$ that

$$\begin{aligned}
t^{-1}\|\hat{\zeta}'_t(e^t\hat{\Phi}'(\rho) + \hat{\sigma}'(\rho))\|_+ &= t^{-1}\hat{\zeta}'_t(e^t\hat{\Phi}'(\rho) + \hat{\sigma}'(\rho))(A(t)) \\
&= t^{-1}(e^t\hat{\Phi}'(\rho) + \hat{\sigma}'(\rho))(A(t)) \\
&= t^{-1}e^t\hat{\Phi}'(\rho)(A(t)) + t^{-1}\int_0^\infty e^{-s}\omega'(\rho)(U'(s)A(t)U'(s)^*)\,ds \\
&= t^{-1}(e^t - e^{-t})\hat{\Phi}'(\rho)(B(t)) + \omega^t(B(t))
\end{aligned}$$

And

$$t^{-1}(e^t\hat{\sigma}'(\rho) - \hat{\theta}_t(\hat{\sigma}'(\rho)))(A) = e^t\omega^t(A)$$

for all $A \in \mathfrak{B}(\mathfrak{H}_1)$. We define

$$q(t) = t^{-1}(\|\hat{\zeta}'_t(e^t\hat{\Phi}'(\rho) + \hat{\sigma}'(\rho))\|_+ - \|e^t\hat{\sigma}'(\rho) - \hat{\theta}'_t(\hat{\sigma}'(\rho))\|_+)$$

Since σ is the boundary resolvent of a CP-flow over $\mathfrak{K}$ we have $\liminf_{t\to 0+} q(t) \geq 0$ from Theorem 4.20. Now for $C(t) \in \mathfrak{B}(\mathfrak{H}_1)$ hermitian and $0 \leq C(t) \leq I$ for $t > 0$ we have

$$t^{-1}(e^t - e^{-t})\hat{\Phi}'(\rho)(B(t)) + \omega^t(B(t)) \geq e^t\omega^t(C(t)) + q(t).$$

Dividing by e^t and with a slight rearrangement we have

$$t^{-1}(1-e^{-2t})\hat{\Phi}'(\rho)(B(t)) - e^{-t}q(t) \geq \omega^t(C(t) - e^{-t}B(t))$$

for $t > 0$. Then we have for all hermitian $A \in U'(t_o)\mathfrak{B}(\mathfrak{H}_1)U'(t_o)^*$

$$t^{-1}(1-e^{-2t})\hat{\Phi}'(\rho)(B(t)) - e^{-t}q(t) \geq \omega^t(A) \qquad \text{if} \qquad 0 \leq A + e^{-t}B(t) \leq I$$

Since t_o and t_n for $n = 1, 2, \cdots$ are the reciprocals of a power of two we have $E'(t_o)$ commutes with $B(t_n)$ for $t_n \leq t_o$ so we have if A satisfies the inequality $0 \leq A + e^{-t_n}B(t_n) \leq I$ then the operator $A' = E'(t_o,\infty)AE'(t_o,\infty)$ satisfies the same inequality. Hence, we have for all hermitian $A \in \mathfrak{B}(\mathfrak{H}_1)$ with $0 \leq A + e^{-t_n}B(t_n) \leq I$

$$t_n^{-1}(1-e^{-2t_n})\hat{\Phi}'(\rho)(B(t_n)) - q(t_n) \geq \nu_{t_n}(A)$$

Note ν_{t_n} converges in norm to ν_o as $n \to \infty$ and

$$t_n^{-1}(1-e^{-2t_n})\hat{\Phi}'(\rho)(B(t_n)) \to 2\hat{\Phi}'(\rho)(\Lambda'(T)) = \rho(T)$$

and the inferior limit of the $e^{-t_n}q(t_n)$ is non negative. From these facts it follows that for every $\epsilon > 0$ there is an integer N so that for each $n \geq N$ if $A = A^* \in \mathfrak{B}(\mathfrak{H}_1)$ with $0 \leq A + e^{-t_n}B(t_n) \leq I$ we have $\rho(T) + \epsilon > \nu_o(A)$. Note $\rho(T) < \omega'(\rho)(A_o) = \nu_o(A_o)$ and $0 \leq A_o + \Lambda'(T) \leq I$. We choose $\epsilon = \epsilon_o = \frac{1}{2}(\nu_o(A_o) - \rho(T)) > 0$ so for $n \geq N = N(\epsilon_o)$ if $A = A^* \in \mathfrak{B}(\mathfrak{H}_1)$ and $0 \leq A + e^{-t_n}B(t_n) \leq I$ we have

$$\rho(T) + \epsilon_o > \nu_o(A)$$

We will show that this inequality leads to the conclusion that $\rho(T) \geq \nu_o(A_o)$ which is a contradiction.

Suppose $D_i = \exp(-t_{n_i})B(t_{n_i})$, $\lambda_i > 0$ and $n_i \geq N$ for $i = 1, \cdots, p$ and

$$\sum_{i=1}^{p} \lambda_i = 1 \qquad \text{and} \qquad C = \sum_{i=1}^{p} \lambda_i D_i$$

Suppose $A = A^* \in \mathfrak{B}(\mathfrak{H}_1)$ and $0 \leq A + C \leq I$. We show $\rho(T) + \epsilon_o > \nu_o(A)$. Let $A_i = A + C - D_i$ for $i = 1, \cdots, p$. Then $0 \leq A_i + D_i \leq I$ so we have $\rho(T) + \epsilon_o > \nu_o(A_i)$ for $i = 1, \cdots, p$. Then we have

$$\rho(T) + \epsilon_o = \sum_{i=1}^{p} \lambda_i(\rho(T) + \epsilon_o) > \sum_{i=1}^{p} \nu_o(\lambda_i A_i) = \nu_o(A)$$

Since the set of C of the above form is a convex set and the σ-weak and σ-strong closure of a convex set are equal and since $e^{-t_n}B(t_n) \to \Lambda'(T)$ σ-weakly as $n \to \infty$ we have $\Lambda'(T)$ can be approximated arbitrarily well by operators C in the above form in the σ-strong topology. Hence, there is a sequence C_n of operators of the above form so that $C_n \to \Lambda'(T)$ as $n \to \infty$ in the σ-strong topology. Let ϕ be the real valued function given by $\phi(x) = 0$ for $x \leq 0$, $\phi(x) = x$ for $x \in [0,1]$, $\phi(x) = 2 - x$ for $x \in [1,2]$ and $\phi(x) = 0$ for $x \geq 2$. Note ϕ is a continuous function of compact support. As shown in (Theorem 5.3.4 *p.* 328 of [KR]) the mapping $A \to \phi(A)$ is strongly continuous on the hermitian operators. Recall A_o was the hermitian operator satisfying $0 \leq A_o + \Lambda'(T) \leq I$. Let $A_n = \phi(A_o + C_n) - C_n$ for $n = 1, 2, \cdots$. Since $0 \leq \phi(A_o + C_n) \leq I$ we have $0 \leq A_n + C_n \leq I$ and, hence, $\rho(T) + \epsilon_o > \nu_o(A_n)$ for $n = 1, 2, \cdots$. Since $C_n \to \Lambda'(T)$ and $\phi(A_o + C_n) \to \phi(A_o + \Lambda'(T)) = A_o + \Lambda'(T)$ as $n \to \infty$ in the strong operator topology we have

$$A_n \to A_o + \Lambda'(T) - \Lambda'(T) = A_o$$

strongly as $n \to \infty$ and since the A_n are uniformly bounded we have convergence in the σ-strong topology. Hence, $\nu_o(A_n) \to \nu_o(A_o)$ as $n \to \infty$ and since $\rho(T) + \epsilon_o > \nu_o(A_n)$ for all $n = 1, 2, \cdots$ we have $\rho(T) + \epsilon_o \geq \nu_o(A_o)$. We recall $\epsilon_o = \frac{1}{2}(\nu_o(A_o) - \rho(T))$ so we have $\frac{1}{2}\rho(T) \geq \frac{1}{2}\nu_o(A_o)$. But this is a contradiction since $\rho(T) < \nu_o(A_o)$. Hence, $\rho(T) \geq \omega'(\rho)(A)$ for all hermitian $A \in \mathfrak{A}(\mathfrak{H}_1)$ with $0 \leq A + \Lambda'(T) \leq I$. $\square$

Theorem 4.23. *Suppose $\rho \to \omega(\rho)$ is a completely positive boundary weight map $\mathfrak{B}(\mathfrak{K})_*$ into $\mathfrak{A}(\mathfrak{H})_*$. For $s > 0$ suppose ω_s is the truncated boundary weight map so $\omega_s(\rho)(A) = \omega(\rho)(E(s,\infty)AE(s,\infty))$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $s > 0$. Suppose $\rho \to \hat{\sigma}(\rho)$ is the mapping of $\mathfrak{B}(\mathfrak{K})_*$ into $\mathfrak{B}(\mathfrak{H})_*$ given by*

$$\hat{\sigma}(\rho)(A) = \int_0^\infty e^{-t}\omega(\rho)(U(t)AU(t)^*)\,dt \tag{4.8}$$

for all $A \in \mathfrak{B}(\mathfrak{H})$. Let $\mathfrak{K}_o$ be an infinite dimensional separable Hilbert space and let $\mathfrak{K}_1 = \mathfrak{K}_o \otimes \mathfrak{K}$ and $\mathfrak{H}_1 = \mathfrak{K}_1 \otimes \mathfrak{H} = \mathfrak{K}_o \otimes \mathfrak{K} \otimes L^2(0,\infty)$ and let the primed operators and maps be the tensored operators and maps as describe before Theorem 4.20. Suppose the mapping σ is the boundary resolvent of a CP-flow over $\mathfrak{K}$. Then for each hermitian $\rho \in \mathfrak{B}(\mathfrak{K}_1)_*$ and each $s > 0$ we have

$$\|\rho + \hat{\Lambda}'(\omega_s'(\rho))\|_+ \geq \|\omega_s'(\rho)\|_+ \tag{4.19}$$

Conversely, suppose for each hermitian $\rho \in \mathfrak{B}(\mathfrak{K}_1)_*$ we have

$$\limsup_{s \to 0+} \|\rho + \hat{\Lambda}'(\omega_s'(\rho))\|_+ - \|\omega_s'(\rho)\|_+ \geq 0. \tag{4.20}$$

Then the mapping $\rho \to \hat{\sigma}(\rho)$ defines a CP-flow over $\mathfrak{K}$.

Proof. Assume the hypothesis and notation of the theorem. First let us assume the mapping $\rho \to \hat{\sigma}(\rho)$ defines an CP-flow over $\mathfrak{K}$. Suppose $\rho \in \mathfrak{B}(\mathfrak{K}_1)$ is hermitian and $s > 0$. Then from the previous lemma there is an operator $T = T^* \in \mathfrak{B}(\mathfrak{K}_1)$ with $0 \leq T \leq I$ so that $\rho(T) \geq \omega'(\rho)(A)$ for $A = A^* \in \mathfrak{A}(\mathfrak{H}_1)$ (the null boundary algebra) with $0 \leq A + \Lambda'(T) \leq I$. Note if A satisfies $0 \leq A + \Lambda'(T) \leq I$ so does $A' = E'(s,\infty)AE'(s,\infty)$. Hence, we have $\rho(T) \geq \omega_s'(\rho)(A)$ for all $A = A^* \in \mathfrak{B}(\mathfrak{H}_1)$ with $0 \leq A + \Lambda'(T) \leq I$. Hence, we have $\rho(T) + \omega_s'(\rho)(\Lambda'(T)) \geq \omega_s'(\rho)(A + \Lambda'(T))$ for all $A = A^* \in \mathfrak{B}(\mathfrak{H}_1)$ with $0 \leq A + \Lambda'(T) \leq I$. Since $\|\omega_s'(\rho)\|_+ = \sup(\omega_s'(\rho)(C) :$ $C \in \mathfrak{B}(\mathfrak{H}_1)$ with $0 \leq C \leq I)$ we have $(\rho + \hat{\Lambda}'(\omega_s'(\rho)))(T) \geq \|\omega_s'(\rho)\|_+$ and since $0 \leq T \leq I$ we have $\|\rho + \hat{\Lambda}'(\omega_s'(\rho))\|_+ \geq (\rho + \hat{\Lambda}'(\omega_s'(\rho)))(T)$ and inequality (4.19) follows.

Conversely, suppose inequality (4.20) holds for all hermitian $\rho \in \mathfrak{B}(\mathfrak{K}_1)_*$. Suppose $\rho \in \mathfrak{B}(\mathfrak{K}_1)_*$ is hermitian. Then there is a decreasing sequence $s_n \to 0$ so that if

$$q_n = \|\rho + \hat{\Lambda}'(\omega'_{s_n}(\rho))\|_+ - \|\omega'_{s_n}(\rho)\|_+$$

then $\lim_{n\to\infty} q_n \geq 0$. Let P_n be the support projection of the positive part of $\rho + \hat{\Lambda}'(\omega'_{s_n}(\rho))$ so

$$(\rho + \hat{\Lambda}'(\omega'_{s_n}(\rho)))(P_n) = \|\rho + \hat{\Lambda}'(\omega'_{s_n}(\rho))\|_+.$$

Then for each $n = 1, 2, \cdots$ we have

$$\rho(P_n) + \omega'_{s_n}(\rho)(\Lambda'(P_n)) \geq \omega'_{s_n}(\rho)(B) + q_n$$

for all hermitian $B \in \mathfrak{B}(\mathfrak{H}_1)$ with $0 \leq B \leq I$ and, therefore, we have $\rho(P_n) \geq \omega'_{s_n}(\rho)(A) + q_n$ for $A = A^* \in \mathfrak{B}(\mathfrak{H}_1)$ with $0 \leq A + \Lambda'(P_n) \leq I$. And from the definition of $\omega'_{s_n}(\rho)$ we have $\rho(P_n) \geq \omega'(A) + q_n$ for all $A = A^* \in U'(s_n)\mathfrak{B}(\mathfrak{H}_1)U'(s_n)$ with $0 \leq A + \Lambda'(P_n) \leq I$. Since the unit ball of $\mathfrak{B}(\mathfrak{H}_1)$ is σ-weakly compact and $\mathfrak{H}_1$ is separable there is a subsequence $s_{n(k)}$ so that $P_{n(k)}$ converges σ-weakly to an operator T as $k \to \infty$. We relabel the subsequence as s_k and P_k so s_k is a decreasing sequence converging to zero and $P_k \to T$ σ-weakly as $k \to \infty$. We claim inequality (4.17) holds with T the operator just constructed. Suppose this is not the case. Then there is a $c > 0$ and a hermitian operator $A_o \in U'(c)\mathfrak{B}(\mathfrak{H}_1)U'(c)^*$ and $0 \leq A_o + \Lambda'(T) \leq I$ so that $\rho(T) < \omega'(\rho)(A_o) = \omega'_c(A_o)$. Note that if $A = A^* \in \mathfrak{B}(\mathfrak{H}_1)$ and $0 \leq A + \Lambda'(P_n) \leq I$ or $0 \leq A + \Lambda'(T) \leq I$ then the same inequalities hold with A replaced by $A' = E'(c,\infty)AE'(c,\infty)$. Hence, we have $\rho(P_n) \geq \omega'_c(A) + q_n$ for hermitian $A \in \mathfrak{B}(\mathfrak{H}_1)$ with $0 \leq A + \Lambda'(P_n) \leq I$. Since $P_n \to T$ σ-weakly as $n \to \infty$ and $\lim_{n\to\infty} q_n \geq 0$ for each $\epsilon > 0$ there is an integer N so that for each $n \geq N$ if $A = A^* \in \mathfrak{B}(\mathfrak{H}_1)$ with $0 \leq A + \Lambda'(P_n) \leq I$ we have

$$\rho(T) + \epsilon > \omega'_c(A)$$

We are now in precisely the same situation we had in the proof of the second part of Lemma 4.22 and repeating the argument there produces the contradiction $\rho(T) \geq \omega'_c(A_o)$. Hence, inequality (4.17) holds and from Theorem 4.22 we have $\rho \to \hat{\sigma}(\rho)$ defines a CP-flow. $\square$

A natural way to construct E_o-semigroups or CP-flows is through the boundary representation π_o as given in Definition 4.5. One may simply require that the boundary representation π_o of $\mathfrak{D}(\delta)$ be σ-weakly continuous and, therefore, have a σ-weakly continuous extension π to all of $\mathfrak{B}(\mathfrak{H})$. In earlier work it was natural to focus on the boundary representation. For example, it was shown in [P3] (Theorem 4.6) if α satisfies the conclusion of Theorem 4.20 then α is a completely spatial E_o-semigroup of $\mathfrak{B}(\mathfrak{H})$ if and only if the mapping $A \to \pi_o(A)$ from the domain of the generator δ of α to $\mathfrak{B}(\mathfrak{H})$ extends to a σ-weakly continuous $*$-representation π of $\mathfrak{B}(\mathfrak{H})$ on $\mathfrak{B}(\mathfrak{K})$ and with the further property that $A = \pi(\Lambda(A))$ only if $A = 0$. Here are ways to connect a normal boundary representation with a CP-flow.

Theorem 4.24. *Suppose σ is the boundary resolvent of a CP-flow over $\mathfrak{K}$ and δ is the generator of α. Suppose π is a completely positive normal contraction of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$. Then the following are equivalent.*

(i) *$\hat{\Phi}(\rho) \in \mathfrak{D}(\hat{\delta})$ and $\hat{\delta}(\hat{\Phi}(\rho)) = \hat{\pi}(\rho) - \hat{\Phi}(\rho)$ for each $\rho \in \mathfrak{B}(\mathfrak{K})_*$.*

(ii) *$\hat{\sigma}(\rho - \hat{\Lambda}(\hat{\pi}(\rho))) = \hat{\Gamma}(\hat{\pi}(\rho))$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$.*

(iii) *$\pi(A) = \pi_o(A)$ for all $A \in \mathfrak{D}(\delta)$ where π_o is the boundary representation introduced in Definition 4.5.*

Proof. Assume the hypothesis and notation of the theorem apply. Assume condition (ii). Since α is defined from $\hat{\sigma}$ we have for all $\eta \in \mathfrak{B}(\mathfrak{H})_*$ that $\hat{\sigma}(\hat{\Lambda}(\eta))+\hat{\Gamma}(\eta) \in \mathfrak{D}(\hat{\delta})$ and $\hat{\delta}(\hat{\sigma}(\hat{\Lambda}(\eta)) + \hat{\Gamma}(\eta)) = \hat{\sigma}(\hat{\Lambda}(\eta)) + \hat{\Gamma}(\eta) - \eta$ where δ is the generator of α. For $\eta = 2\hat{\Phi}(\rho) - \hat{\pi}(\rho)$ we have $\hat{\Lambda}(\eta) = \rho - \hat{\Lambda}(\hat{\pi}(\rho))$ and $\hat{\Gamma}(\eta) = \hat{\Phi}(\rho) - \hat{\Gamma}(\hat{\pi}(\rho))$ and, therefore, from the equation above we have $\hat{\Phi}(\rho) \in \mathfrak{D}(\hat{\delta})$ and $\hat{\delta}(\hat{\Phi}(\rho)) = \hat{\pi}(\rho) - \hat{\Phi}(\rho)$. Hence, (ii) $\Rightarrow$ (i).

Conversely if α satisfies (i) we have $\hat{\Phi}(\rho) \in \mathfrak{D}(\hat{\delta})$ and $\hat{\delta}(\hat{\Phi}(\rho)) = \hat{\pi}(\rho) - \hat{\Phi}(\rho)$ where $\hat{\delta}$ is the generator of $\hat{\alpha}$. Then for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ there is an $\eta \in \mathfrak{B}(\mathfrak{H})_*$ so that $\hat{\sigma}(\hat{\Lambda}(\eta))+\hat{\Gamma}(\eta) = \hat{\Phi}(\rho)$ and then $\hat{\delta}(\hat{\sigma}(\hat{\Lambda}(\eta))+\hat{\Gamma}(\eta)) = \hat{\sigma}(\hat{\Lambda}(\eta))+\hat{\Gamma}(\eta)-\eta = \hat{\Phi}(\rho)-\eta = \hat{\pi}(\rho) - \hat{\Phi}(\rho)$. Hence, $\eta = 2\hat{\Phi}(\rho) - \hat{\pi}(\rho)$. Since $\hat{\Gamma}(\hat{\Phi}(\rho)) = \frac{1}{2}\hat{\Phi}(\rho)$ and $\hat{\Lambda}(\hat{\Phi}(\rho)) = \frac{1}{2}\rho$ we have $\hat{\sigma}(\hat{\Lambda}(\eta)) = \hat{\sigma}(\rho - \hat{\Lambda}(\hat{\pi}(\rho))) = \hat{\Gamma}(\hat{\pi}(\rho))$. Hence, (i) $\Rightarrow$ (ii).

Next we show (ii) and (iii) are equivalent. Since α is defined from the map $\rho \to \hat{\sigma}(\rho)$ means that each element of $\mathfrak{D}(\hat{\delta})$ is of the form $\hat{\sigma}(\hat{\Lambda}(\eta)) + \hat{\Gamma}(\eta)$ and $\hat{\delta}(\hat{\sigma}(\hat{\Lambda}(\eta))+\hat{\Gamma}(\eta)) = \hat{\sigma}(\hat{\Lambda}(\eta))+\hat{\Gamma}(\eta)-\eta$ for some $\eta \in \mathfrak{B}(\mathfrak{H})_*$. This translates over to the dual space $\mathfrak{B}(\mathfrak{H})$ to give that each element of $\mathfrak{D}(\delta)$ is of the form $\Lambda(\sigma(A))+\Gamma(A)$ and $\delta(\Lambda(\sigma(A)) + \Gamma(A)) = \Lambda(\sigma(A)) + \Gamma(A) - A$. Recalling the properties of the boundary representation we note that $\pi_o(\Gamma(A)) = 0$ and $\pi_o(\Lambda(B)) = B$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $B \in \mathfrak{B}(\mathfrak{K})$. Hence, we have $\pi_o(\Lambda(\sigma(A))+\Gamma(A)) = \sigma(A)$ for all $A \in \mathfrak{B}(\mathfrak{H})$. It follows that condition (iii) is then equivalent to the equation $\sigma(A) = \pi(\Lambda(\sigma(A)) +\Gamma(A))$ for all $A \in \mathfrak{B}(\mathfrak{H})$. Note all the mapping in this equation are σ-weakly continuous so translating this equation to the predual gives condition (ii). Hence, (ii) and (iii) are equivalent. $\square$

Definition 4.25. We say a CP-flow α over $\mathfrak{K}$ is derived from the completely positive normal contraction π of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$ if it satisfies one and, therefore, all the conditions of Theorem 4.24.

The next theorem shows that for each such π there is a CP-flow α derived from π.

Theorem 4.26. *Suppose π is a completely positive σ-weakly continuous linear contraction of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$. Then for each $\rho \in \mathfrak{B}(\mathfrak{K})_*$ the sum*

$$\hat{\sigma}(\rho) = \hat{\Gamma}(\hat{\pi}(\rho) + \hat{\pi}(\hat{\Lambda}(\hat{\pi}(\rho))) + \hat{\pi}(\hat{\Lambda}(\hat{\pi}(\hat{\Lambda}(\hat{\pi}(\rho))))) + \cdots)$$

converges in norm and σ is the boundary resolvent of a CP-flow α which is derived from π. Furthermore, this α is the minimal CP-flow derived from π in that if $\rho \to \hat{\sigma}_2(\rho)$ defines a second CP-semigroup derived from π then $\hat{\sigma}(\rho) \leq \hat{\sigma}_2(\rho)$ for *all positive $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Furthermore, if $(\pi \circ \Lambda)^n(I) \to 0$ weakly as $n \to \infty$ then α defined above is the unique (i.e., α is the only CP-flow derived from π).*

Proof. Suppose π is a completely positive σ-weakly continuous linear contraction of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$. For $\rho \in \mathfrak{B}(\mathfrak{K})_*$ let

$$\hat{\sigma}_n(\rho) = \hat{\Gamma}(\hat{\pi}(\rho)) + \hat{\Gamma}(\sum_{k=1}^{n} \hat{\pi}((\hat{\Lambda} \circ \hat{\pi})^k(\rho)))$$

Note the mapping $\rho \to \hat{\sigma}_n(\rho)$ is completely positive and each of the terms in the sum for $\hat{\sigma}_n$ are completely positive. Suppose $\rho \in \mathfrak{B}(\mathfrak{H})_*$ and ρ is positive. Using the fact that $\pi(I) \leq I$ we find

$$\begin{aligned}\hat{\sigma}_n(\rho)(I) =& \rho(\pi(I - \Lambda)) + \sum_{k=1}^{n} \rho((\pi \circ \Lambda)^k \pi(I - \Lambda)) \\ \leq& \rho(I - \pi(\Lambda)) + \sum_{k=1}^{n} \rho((\pi \circ \Lambda)^k (I - \pi(\Lambda))) \\ =& \rho(I) - \rho((\pi \circ \Lambda)^{n+1}(I)) \leq \rho(I)\end{aligned}$$

for all $n \geq 1$. Since for $n \geq m$ we have $\|\hat{\sigma}_n - \hat{\sigma}_m\| = \hat{\sigma}_n(I) - \hat{\sigma}_m(I)$ and this tends to zero as $n, m \to \infty$ it follows that $\hat{\sigma}_n$ converges in norm to a limit which we denote by $\hat{\sigma}(\rho)$ as $n \to \infty$. Since each $\rho \in \mathfrak{B}(\mathfrak{K})_*$ is the complex linear combination of four positive elements it follows that $\hat{\sigma}_n(\rho)$ converges in norm to a limit as $n \to \infty$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Note for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $A \in \mathfrak{B}(\mathfrak{H})$ we have

$$(\hat{\sigma}_n(\rho) - e^{-t}\hat{\theta}_t(\hat{\sigma}_n(\rho)))(A) = \int_0^t e^{-s} \sum_{k=0}^{n} \rho((\pi \circ \Lambda)^k \pi(U(s)AU(s)^*))\, ds$$

Since each of the terms in the above sum is completely positive the mapping $\rho \to \hat{\sigma}_n(\rho) - e^{-t}\hat{\theta}_t(\hat{\sigma}_n(\rho))$ is completely positive. Since $\hat{\sigma}_n$ converges in norm to $\hat{\sigma}$ as $n \to \infty$ the mapping $\rho \to \hat{\sigma}(\rho) - e^{-t}\theta_t(\hat{\sigma}(\rho))$ is completely positive. To show that $\hat{\sigma}$ defines a CP-flow we need to establish the limit inequality (4.14+) of Theorem 4.20. We do not know how to do this directly because although the expression for $\hat{\sigma}(\rho)$ converges in norm as $n \to \infty$ the expression on which $\hat{\Gamma}$ acts in the definition of $\hat{\sigma}(\rho)$ need not converge. (In fact, we know of examples where it fails to converge.) To fix this problem we will replace π by $\lambda\pi$ with $0 < \lambda < 1$ which makes the typical sums which occur convergent. Then we will take the limit as $\lambda \to 1-$.

For $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $0 \leq \lambda < 1$ we define

$$\rho_\lambda = \rho + \lambda\hat{\Lambda}(\hat{\pi}(\rho)) + \lambda^2\hat{\Lambda}(\hat{\pi}(\hat{\Lambda}(\hat{\pi}(\rho)))) + \cdots$$

and

$$\hat{\sigma}^\lambda(\rho) = \lambda\hat{\Gamma}(\hat{\pi}(\rho_\lambda)).$$

Note that since $\lambda < 1$ the series for ρ_λ converges in norm. We show the mapping $\rho \to \hat{\sigma}^\lambda(\rho)$ defines a CP-flow over $\mathfrak{K}$. This mapping is completely positive by the same argument that the mapping $\rho \to \hat{\sigma}(\rho)$ is completely positive and $\hat{\sigma}^\lambda(\rho)(I) \leq \rho(I)$ for $\rho \geq 0$ by the same computation that showed this for $\hat{\sigma}(\rho)$. We show $\hat{\sigma}\lambda(\rho)$ satisfies the limit inequality (4.14+). As describe before Theorem 4.20 we use the primed maps to indicate the extension of the unprimed map to tensor product space $\mathfrak{K}_1 = \mathfrak{K}_o \otimes \mathfrak{K}$ by rule $\gamma'(A \otimes B) = A \otimes \gamma(B)$. Suppose $\rho \in \mathfrak{B}(\mathfrak{K}_1)_*$ is hermitian and $t > 0$. Then we have

$$\begin{aligned}\hat{\zeta}_t'(e^t\hat{\Phi}'(\rho) + \hat{\sigma}^{\lambda\prime}(\rho)) =&\hat{\zeta}_t'(e^t\hat{\Phi}'(\rho) + \lambda\hat{\Phi}'(\hat{\Lambda}'(\hat{\pi}'(\rho_\lambda)))) \\ &+ \lambda\hat{\zeta}_t'(\hat{\Gamma}'(\hat{\pi}'(\rho_\lambda)) - \hat{\Phi}'(\hat{\Lambda}'(\hat{\pi}'(\rho_\lambda))))\end{aligned}$$

By Lemma 4.19 the norm of the second term on the left hand of the above equation is $o(t)$. Hence, we have

$$\begin{aligned}&\lim_{t\to 0+} t^{-1}\|\hat{\zeta}_t'(e^t\hat{\Phi}'(\rho) + \hat{\sigma}^{\lambda\prime}(\rho))\|_+ \\ &\qquad = \lim_{t\to 0+} t^{-1}\|\hat{\zeta}_t'(e^t\hat{\Phi}'(\rho) + \lambda\hat{\Phi}'(\hat{\Lambda}'(\hat{\pi}'(\rho_\lambda))))\|_+ \\ &\qquad = \lim_{t\to 0+} t^{-1}(1 - e^{-t})\|e^t\rho + \lambda\hat{\Lambda}'(\hat{\pi}'(\rho_\lambda))\|_+ \\ &\qquad = \|\rho + \lambda\hat{\Lambda}'(\hat{\pi}'(\rho_\lambda))\|_+ = \|\rho_\lambda\|_+\end{aligned}$$

And we have

$$\begin{aligned}t^{-1}(e^t\hat{\sigma}^{\lambda\prime}(\rho) - \hat{\theta}_t(\hat{\sigma}^{\lambda\prime}(\rho))) &= t^{-1}\lambda e^t\int_0^t e^{-s}\hat{\theta}_s'(\hat{\pi}'(\rho_\lambda))\,ds \\ &\to \lambda\hat{\pi}'(\rho_\lambda)\end{aligned}$$

as $t \to 0+$. Hence, we have

$$\lim_{t\to 0+} t^{-1}\|e^t\hat{\sigma}^{\lambda\prime}(\rho) - \hat{\theta}_t(\hat{\sigma}^{\lambda\prime}(\rho))\|_+ = \lambda\|\hat{\pi}'(\rho_\lambda)\|_+$$

Hence, inequality (4.14+) is satisfied if and only if

$$\lambda\|\hat{\pi}'(\rho_\lambda)\|_+ \leq \|\rho_\lambda\|_+$$

for all hermitian $\rho \in \mathfrak{B}(\mathfrak{K}_1)_*$. But this follows immediately from the fact that $\lambda < 1$ and π is a completely positive contraction so $\pi' is$ a positive contraction. Hence, by Theorem 4.20 there is a CP-flow α^λ of $\mathfrak{B}(\mathfrak{H})$ whose boundary resolvent is σ^λ. The properties of the semigroup α^λ are essential to the remainder of our argument.

Suppose $\rho \in \mathfrak{B}(\mathfrak{K}_1)$ is hermitian. Suppose $t > 0$. Then from equations (4.15) in the proof of Theorem 4.20 we have

$$\hat{\alpha}_t^{\lambda\prime}(\hat{\zeta}_t'(\nu)) = e^t\hat{\sigma}^{\lambda\prime}(\rho) - \hat{\theta}_t'(\hat{\sigma}^{\lambda\prime}(\rho))$$

and

$$\begin{aligned}\hat{\zeta}_t'(\nu) = \hat{\zeta}_t'(\hat{\sigma}^{\lambda\prime}(\rho) + e^t\hat{\Phi}'(\rho)) - (e^t - 1)\hat{\zeta}_t'(\hat{\Phi}'(\rho)) \\ + 2\int_0^t e^s\hat{\zeta}_t'(\hat{\xi}_s'(\hat{\Phi}'(\rho)))\,ds\end{aligned}$$

where the somewhat complicated expression for ν (which fortunately we do not need) is given in the proof of Theorem 4.20. Since α_t^λ is completely positive and, therefore, $\alpha_t^{\lambda\prime}$ is positive for $t \geq 0$ we have $\|\hat{\alpha}_t^{\lambda\prime}(\hat{\zeta}_t'(\nu))\|_+ \leq \|\hat{\zeta}_t'(\nu)\|_+$. We note that both $\hat{\alpha}_t^{\lambda\prime}(\hat{\zeta}_t'(\nu))$ and $\hat{\zeta}_t'(\nu)$ converge in norm to limits as $\lambda \to 1-$ and, hence, the inequality $\|\hat{\alpha}_t^{\lambda\prime}(\hat{\zeta}_t'(\nu))\|_+ \leq \|\hat{\zeta}_t'(\nu)\|_+$ holds in the limit obtained by setting $\lambda = 1$. Hence, we have

$$t^{-1}(\|\hat{\zeta}_t'(\nu)\|_+ - \|e^t\hat{\sigma}'(\rho) - \hat{\theta}_t'(\hat{\sigma}'(\rho))\|_+) \geq 0$$

where $\hat{\zeta}_t'(\nu)$ is the expression given above with $\hat{\sigma}^{\lambda\prime}(\rho)$ replaced by $\hat{\sigma}'(\rho)$. As shown in the proof of Theorem 4.20 (after equation 4.15) the norm of each of the second two terms in the expression for $\hat{\zeta}_t'(\nu)$ are $O(t^2)$. Combining this with the inequality above we have

$$\liminf_{t\to 0+} t^{-1}(\|\hat{\zeta}_t'(e^t\hat{\Phi}'(\rho) + \hat{\sigma}'(\rho))\|_+ - \|e^t\hat{\sigma}'(\rho) - \hat{\theta}_t'(\hat{\sigma}'(\rho))\|_+) \geq 0$$

which is the limit inequality (4.13+) which implies (is stronger than) that the limit inequality (4.14+). Hence, by Theorem 4.20 σ is the boundary resolvent of a CP-flow over $\mathfrak{K}$. We show α is derived from π. Suppose $\rho \in \mathfrak{B}(\mathfrak{K})_*$. A direct computation from the definition of $\hat{\sigma}$ shows that

$$\hat{\sigma}(\rho - \hat{\Lambda}(\hat{\pi}(\rho))) = \hat{\Gamma}(\hat{\pi}(\rho)).$$

for $\rho \in \mathfrak{B}(\mathfrak{K})_*$. From Theorem 4.24 it follows that α is derived from π.

Next suppose then that β is a CP-flow over $\mathfrak{K}$ derived from π and let σ^2 be the boundary resolvent of β. Since β is derived from π we have from Theorem 4.24 that

$$\hat{\sigma}^2(\rho - \hat{\Lambda}(\hat{\pi}(\rho))) = \hat{\Gamma}(\hat{\pi}(\rho))$$

for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$. For $\rho \in \mathfrak{B}(\mathfrak{K})_*$ let

$$\rho_n = \sum_{k=0}^{n}(\hat{\Lambda}\cdot\hat{\pi})^k(\rho).$$

Then $\rho_n - \hat{\Lambda}(\hat{\pi}(\rho_n)) = \rho - (\hat{\Lambda}\cdot\hat{\pi})^{n+1}(\rho)$ and, hence,

$$\hat{\sigma}^2(\rho - (\hat{\Lambda}\cdot\hat{\pi})^{n+1}(\rho)) = \hat{\Gamma}(\hat{\pi}(\rho_n))$$

Early in this proof when we constructed σ we showed that $\hat{\Gamma}(\hat{\pi}(\rho_n))$ converges in norm to $\hat{\sigma}(\rho)$ as $n \to \infty$. Hence, we have

$$\hat{\sigma}^2(\rho) = \hat{\sigma}(\rho) + \lim_{n\to\infty}\hat{\sigma}^2((\hat{\Lambda}\cdot\hat{\pi})^n(\rho))$$

where the above limit exists. Since the mappings $\rho \to \hat{\sigma}^2(\rho)$ and $(\hat{\Lambda}\cdot\hat{\pi})^n$ are completely positive for all $n = 1, 2, \cdots$ is follows the mapping $\rho \to \hat{\sigma}^2(\rho) - \hat{\sigma}(\rho)$ is completely positive.

Now suppose $\rho \geq 0$. Then $\|(\hat{\Lambda} \cdot \hat{\pi})^n(\rho)\| = \rho((\pi \circ \Lambda)^n(I))$. Now let us make the assumption $(\pi \circ \Lambda)^n(I) \to 0$ weakly as $n \to \infty$. It then follows that $\|(\hat{\Lambda} \cdot \hat{\pi})^n(\rho)\| \to 0$ as $n \to \infty$ and since each $\rho \in \mathfrak{B}(\mathfrak{K})_*$ is a linear combination of four positive elements this limit holds for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$. From the normalization condition for $\hat{\sigma}_2$ it follows that $\hat{\sigma}_2((\hat{\Lambda} \cdot \hat{\pi})^n) \to 0$ in norm. Hence, we have

$$\hat{\sigma}_2(\rho) = \hat{\sigma}(\rho) + \lim_{n \to \infty} \hat{\sigma}_2((\hat{\Lambda} \cdot \hat{\pi})^n(\rho)) = \hat{\sigma}(\rho)$$

Hence, $\beta = \alpha$ and α is the unique CP-flow derived from π if $(\pi \circ \Lambda)^n(I) \to 0$ strongly as $n \to \infty$. $\square$

Next we show that if the weights $\omega(\rho)$ are bounded for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$ then the CP-flow defined from ω is derived from a completely positive normal contraction π of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$.

Theorem 4.27. *Suppose α is a CP-flow over $\mathfrak{K}$ and σ is the boundary resolvent of α. Suppose*

$$\hat{\sigma}(\rho)(A) = \int_0^\infty e^{-t}\omega(\rho)(U(t)AU(t)^*)\, dt$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ where $\omega(\rho)$ is the weight defined in Theorem 4.17 and suppose the weight $\omega(\rho)$ is bounded for each $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Then there is a unique completely positive σ-weakly continuous contraction π of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$ so that α is the unique CP-flow derived from π and the relation between π and ω is given by

$$\omega = \hat{\pi}(I - \hat{\Lambda} \cdot \hat{\pi})^{-1} \qquad \text{and} \qquad \pi = \omega(I + \hat{\Lambda} \cdot \omega)^{-1}$$

In particular, it follows that if α is a CP-flow over $\mathfrak{K}$ and $\rho \to \omega_t(\rho)$ are the associated maps for $t > 0$ as described in Theorem 4.23 then for each $t > 0$ there is a unique completely positive normal contraction π_t of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$ so that there is a unique CP-flow $\alpha^{(t)}$ derived from π_t and the associated map $\rho \to \sigma^t(\rho)$ is given by

$$\hat{\sigma}^t(\rho)(A) = \int_0^\infty e^{-s}\omega_t(\rho)(U(s)AU(s)^*)\, ds$$

for $A \in \mathfrak{B}(\mathfrak{H})$. The relation between π_t and ω_t is given by the relations

$$\omega_t = \hat{\pi}_t(I - \hat{\Lambda} \cdot \hat{\pi}_t)^{-1} \qquad \text{and} \qquad \pi_t = \omega_t(I + \hat{\Lambda} \cdot \omega_t)^{-1}$$

for each $t > 0$.

Proof. Assume the hypothesis and notation of the first paragraph of the theorem holds. We first show that the mapping $\rho \to \omega(\rho)$ is closed. We must show that if $\|\rho_n\| \to 0$ and $\|\omega(\rho_n) - \eta\| \to 0$ as $n \to \infty$ then $\eta = 0$. Let $\omega_s(\rho)(A) = \omega(\rho)(E(s,\infty)AE(s,\infty))$ for $s > 0$ and $A \in \mathfrak{B}(\mathfrak{H})$. Suppose $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and ρ is positive. Since $\omega(\rho)(I - \Lambda) \leq \rho(I)$ we have for $s > 0$ that

$$\begin{aligned} \omega_s(\rho)(I) =& \omega(\rho)(I - E(s)) \leq (1 - e^{-s})^{-1}\omega(\rho)(I - \Lambda) \\ \leq& (1 - e^{-s})^{-1}\rho(I). \end{aligned}$$

Hence, $\|\omega_s(\rho)\| \leq (1-e^{-s})^{-1}\|\rho\|$ for ρ positive and since the mapping $\rho \to \omega_s(\rho)$ is completely positive we have this inequality holds for all ρ and the mapping $\rho \to \omega_s(\rho)$ is bounded with bound less than or equal to $(1-e^{-s})^{-1}$ for all $s > 0$. Suppose then that $\|\rho_n\| \to 0$ and $\|\omega(\rho_n) - \eta\| \to 0$ as $n \to \infty$. Then we have $\omega_s(\rho_n)(A) \to 0$ as $n \to \infty$ for each $s > 0$. Hence, $\eta(E(s,\infty)AE(s,\infty)) = 0$ for all $s > 0$ and $A \in \mathfrak{B}(\mathfrak{H})$. Hence $\eta = 0$ and the mapping $\rho \to \omega(\rho)$ is closed and, hence, by the closed graph theorem the mapping is bounded so there is a constant K so that $\|\omega(\rho)\| \leq K\|\rho\|$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$.

Then next step is to show that mapping $\rho \to \rho + \hat{\Lambda}(\omega(\rho))$ is invertible. We have $\hat{\sigma}(\rho) = \hat{\Gamma}(\omega(\rho))$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$. We use the primed maps as described before Theorem 4.20. Then for all hermitian $\rho \in \mathfrak{B}(\mathfrak{K}_1)_*$ we have

$$\begin{aligned}\hat{\zeta}'_t(e^t\hat{\Phi}'(\rho) + \hat{\sigma}'(\rho)) =& \hat{\zeta}'_t(e^t\hat{\Phi}'(\rho) + \hat{\Phi}'(\hat{\Lambda}'(\omega'(\rho)))) \\ &+ \hat{\zeta}'_t(\hat{\Gamma}'(\omega(\rho)) - \hat{\Phi}'(\hat{\Lambda}'(\omega'(\rho))))\end{aligned}$$

By Lemma 4.19 the norm of the second term in the above equation is $o(t)$ and, hence,

$$\begin{aligned}&\lim_{t\to 0+} t^{-1}\|\hat{\zeta}'_t(e^t\hat{\Phi}'(\rho) + \hat{\sigma}'(\rho))\|_+ \\ &\qquad = \lim_{t\to 0+} t^{-1}\|\hat{\zeta}'_t(\hat{\Phi}'(e^t\rho + \hat{\Lambda}'(\omega'(\rho))))\|_+ = \|\rho + \hat{\Lambda}'(\omega'(\rho))\|_+\end{aligned}$$

And we have

$$(e^t\sigma'(\rho) - \hat{\theta}'_t(\sigma'(\rho)))(A) = e^t \int_0^t e^{-s}\omega'(\rho)(U'(s)AU'(s)^*)\,ds$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t > 0$. It follows from the above that

$$\lim_{t\to 0+} t^{-1}(e^t\sigma'(\rho) - \hat{\theta}'_t(\sigma'(\rho))) = \omega'(\rho)$$

and, therefore,

$$\lim_{t\to 0+} t^{-1}\|e^t\hat{\sigma}'(\rho) - \hat{\theta}_t(\hat{\sigma}'(\rho))\|_+ = \|\omega'(\rho)\|_+$$

Then it follows from Theorem 4.20 (inequality 4.13+) that

$$\|\rho + \hat{\Lambda}'(\omega'(\rho))\|_+ \geq \|\omega'(\rho)\|_+$$

for all hermitian $\rho \in \mathfrak{B}(\mathfrak{K}_1)_*$. As described in the proof of Theorem 4.20 this implies $\|\rho + \hat{\Lambda}'(\omega'(\rho))\| \geq \|\omega'(\rho)\|$ for all $\rho \in \mathfrak{B}(\mathfrak{K}_1)_*$ and this trivially implies $\|\rho + \hat{\Lambda}(\omega(\rho))\| \geq \|\omega(\rho)\|$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Note $\|\rho + \hat{\Lambda}(\omega(\rho))\| \geq \|\omega(\rho)\|$ implies

$$\|\rho + \hat{\Lambda}(\omega(\rho))\| \geq \|\rho\| - \|\hat{\Lambda}(\omega(\rho))\| \geq \|\rho\| - \|\omega(\rho)\| \geq \|\rho\| - \|\rho + \hat{\Lambda}(\omega(\rho))\|.$$

So we have $\|\rho + \hat{\Lambda}(\omega(\rho))\| \geq \frac{1}{2}\|\rho\|$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$. It follows that the map $\rho \to \rho + \hat{\Lambda}(\omega(\rho))$ is one to one and this map has a bounded left inverse.

We will show that this mapping has range $\mathfrak{B}(\mathfrak{K})_*$ and so this left inverse is also a right inverse and the mapping is invertible. Suppose $0 \leq y \leq 1$. Consider the mapping $\rho \to y\hat{\sigma}(\rho)$ and then $\rho \to y\omega(\rho)$ is the corresponding differentiated map. Note from Lemma 4.22 that the mapping $\rho \to y\omega(\rho)$ satisfies inequality (4.17) since the mapping $\rho \to \omega(\rho)$ does. Hence, the mapping $\rho \to y\hat{\sigma}(\rho)$ corresponds to a CP-flow and by the argument above (with $y\omega(\rho)$ replacing $\omega(\rho)$) we have $\|\rho + y\hat{\Lambda}(\omega(\rho))\| \geq \frac{1}{2}\|\rho\|$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $y \in [0,1]$. Let T_y be the mapping $\rho \to \rho + y\hat{\Lambda}(\omega(\rho))$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and let Θ_y be the left inverse of T_y for $y \in [0,1]$. Recall $\|\omega(\rho)\| \leq K\|\rho\|$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Now for $y \in [0,1/K)$ we have

$$\Theta_y(\rho) = \rho - y\hat{\Lambda}(\omega(\rho)) + y^2\hat{\Lambda}(\omega(\hat{\Lambda}(\omega(\rho)))) - \cdots$$

and the geometric series converges. Note for $y \in [0,1/K)$ we have Θ_y is both a right and left inverse of T_y so the range of T_y is $\mathfrak{B}(\mathfrak{K})_*$ for $y \in [0,1/K)$. Suppose $y \in [0,1]$ and for this value of y we have T_y has range $\mathfrak{B}(\mathfrak{K})_*$ so Θ_y is both a right and left inverse of T_y. Then for $x \in [0,1]$ we have

$$T_{x+y} = T_y + x\hat{\Lambda} \cdot \omega = (T_y + x\hat{\Lambda} \cdot \omega)\Theta_y T_y = (I + x(\hat{\Lambda} \cdot \omega)\Theta_y)T_y.$$

Note since $\|T_y\rho\| \geq \frac{1}{2}\|\rho\|$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ we have $\|\Theta_y\| \leq 2$ and, hence, $\|x(\hat{\Lambda} \cdot \omega)\Theta_y\| \leq 2xK$. Hence for $x \in [0, \frac{1}{2}K^{-1})$ we have $(I + x(\hat{\Lambda} \cdot \omega)\Theta_y)$ is invertible (since the geometric series for it converges). Since T_y has range $\mathfrak{B}(\mathfrak{K})_*$ it follows that T_{x+y} has range all of $\mathfrak{B}(\mathfrak{K})_*$ for $x \in [0, \frac{1}{2}K^{-1})$. Then we can extend the interval $[0,(2/3)K^{-1}]$ on which we know T_y has range $\mathfrak{B}(\mathfrak{K})_*$ to $[0,K^{-1}]$ on which T_y has range $\mathfrak{B}(\mathfrak{K})_*$ then to $[0,(4/3)K^{-1}]$ on which T_y has range $\mathfrak{B}(\mathfrak{K})_*$ and in a finite number of steps we can extend the interval for which we know T_y has range $\mathfrak{B}(\mathfrak{K})_*$ to an interval containing $[0,1]$. Hence, T_1 has range $\mathfrak{B}(\mathfrak{K})_*$ and Θ_1 is both a right and left inverse for T_1.

It follows that $\Theta_1(\rho) + \hat{\Lambda}(\omega(\Theta_1(\rho))) = \rho$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Then for the primed maps we have

$$\|\rho\|_+ = \|\Theta_1'(\rho) + \hat{\Lambda}'(\omega_1'(\Theta_1'(\rho)))\|_+ \geq \|\omega_1'(\Theta_1'(\rho))\|_+$$

for all hermitian $\rho \in \mathfrak{B}(\mathfrak{K}_1)_*$. Let $\hat{\pi}(\rho) = \omega(\Theta_1(\rho))$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Then the above inequality says $\|\rho\|_+ \geq \|\hat{\pi}'(\rho)\|_+$ for all hermitian $\rho \in \mathfrak{B}(\mathfrak{K}_1)_*$ which is equivalent to saying π is a completely positive normal contraction of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$. Since $\Theta_1(\rho) + \hat{\Lambda}(\omega(\Theta_1(\rho))) = \rho$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$ we have $\omega(\Theta_1(\rho)) + \omega(\hat{\Lambda}(\omega(\Theta_1(\rho)))) = \omega(\rho)$ or

$$\omega(\rho - \hat{\Lambda}(\hat{\pi}(\rho))) = \hat{\pi}(\rho)$$

for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Hence, we have $\hat{\sigma}(\rho - \hat{\Lambda}(\hat{\pi}(\rho))) = \hat{\Gamma}(\hat{\pi}(\rho))$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and α is derived from π.

As in the proof of Theorem 4.26 for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $n \geq 0$ let

$$\rho_n = \rho + \hat{\Lambda}(\hat{\pi}(\rho)) + \cdots + (\hat{\Lambda} \cdot \hat{\pi})^n(\rho).$$

Then we have $\omega(\rho-(\hat{\Lambda}\cdot\hat{\pi})^{n+1}(\rho)) = \hat{\pi}(\rho_n)$. Then assuming further that ρ is positive we have

$$\hat{\pi}(\rho_n)(I) = \omega(\rho)(I) - \omega((\hat{\Lambda}\cdot\hat{\pi})^{n+1}(\rho))(I) \leq \omega(\rho)(I).$$

for all $n = 1, 2, \cdots$ and this implies $\hat{\pi}((\hat{\Lambda}\cdot\hat{\pi})^n\rho)(I) = ((\hat{\Lambda}\cdot\hat{\pi})^n\rho)(\pi(I)) \to 0$ as $n \to \infty$ for all positive $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Since $\Lambda \leq I$ we have $((\hat{\Lambda}\cdot\hat{\pi})^{n+1}\rho)(I) \leq ((\hat{\Lambda}\cdot\hat{\pi})^n\rho)(\pi(I))$ for ρ positive so $\rho((\pi\cdot\Lambda)^n(I)) \to 0$ for all positive $\rho \in \mathfrak{B}(\mathfrak{K})_*$ as $n \to \infty$. Since each element in $\mathfrak{B}(\mathfrak{K})_*$ is the linear combination of four positive elements we have $(\pi\cdot\Lambda)^n(I)$ tends weakly to zero so from Theorem 4.26 we have α is the unique CP-flow derived from π.

Note we have shown that $\|\omega_s(\rho)\| \leq (1-e^{-s})^{-1}\|\rho\|$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $s > 0$ so the theorem's last paragraph is an immediate consequence of the theorem's first paragraph and the proof is complete. $\square$

We consider the following exercise. Suppose π is a completely positive normal contraction of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$ so that $\|\hat{\Lambda}\cdot\hat{\pi}\| < 1$ and α is the unique CP-flow derived from π. Suppose σ is the boundary resolvent of α. As we saw in the above proof for $\lambda \in [0,1]$ the mapping $\rho \to \lambda\hat{\sigma}(\rho)$ gives rise to a CP-flow $\alpha^{(\lambda)}$ and since the mapping $\rho \to \lambda\omega(\rho)$ is bounded it follows from the above theorem that $\alpha^{(\lambda)}$ is derived from a completely positive normal contraction π_λ of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$. We leave as an exercise determining the relation between π and π_λ which is given by $\hat{\pi}_\lambda = \lambda\hat{\pi}(I-(1-\lambda)\hat{\Lambda}\cdot\hat{\pi})^{-1}$ for the predual maps and $\pi_\lambda = \lambda(I-(1-\lambda)\pi\cdot\Lambda)^{-1}\pi$ for the operator maps.

At this point we have reached the most important result of this section. We see how a CP-flow is characterized by the family of completely positive normal contraction π_t of the previous theorem and these contractions completely determine the CP-semigroup α. Because of their importance we give this family a name.

Definition 4.28. If α is a CP-semigroup over $\mathfrak{K}$ we say $\rho \to \omega(\rho)$ is the boundary weight map of α if the boundary resolvent σ of α is given by equation (4.8). We denote by $\pi^\#$ called the generalized boundary representation of α (or ω) the family of mappings $\pi_t^\# = \pi_t$ (where π_t are mappings defined in Theorem 4.27) for $t > 0$. A boundary weight map $\rho \to \omega(\rho)$ is said to be q-positive if the generalized boundary representation maps $\pi_t^\#$ are completely positive contractions of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$ for all $t > 0$.

Theorem 4.27 shows that the problem of constructing CP-flows is equivalent to constructing q-positive boundary weight maps. We consider this to be the most important result of this paper. Since every spatial E_o-semigroup is cocycle conjugate to one dilated from a unital CP-flow this gives us a way to construct E_o-semigroups. There are a number of strategies for constructing q-positive boundary weight maps. When the Hilbert space $\mathfrak{K}$ is one dimensional they are simply given by a simple boundary weight on $L^2(0,\infty)$. When $\mathfrak{K}$ is finite dimensional we have not classified the q-positive boundary weight maps but the problem seems tractable. And in the case where $\mathfrak{K}$ is infinite dimensional we can construct new E_o-semigroups with surprising properties as we will see at the end of this section. Note the generalized boundary representation of α completely determines α. Also note that if $\pi^\#$ is a generalized boundary representation of a CP-semigroup then $\pi_t^\#$ is determined by $\pi_s^\#$ for all $t > s$ so it is the properties of $\pi_s^\#$ as $s \to 0+$ that are important. The generalized boundary representation is of importance in determining the order structure for CP-semigroups.

Theorem 4.29. *Suppose α and β are CP-flows over $\mathfrak{K}$ and $\pi^\#$ and $\phi^\#$ are the generalized boundary representations of α and β, respectively. If β is a subordinate of α then $\pi_s^\# \geq \phi_s^\#$ (i.e., the map $A \to \pi_s^\#(A) - \phi_s^\#(A)$ from $\mathfrak{B}(\mathfrak{H})$ to $\mathfrak{B}(\mathfrak{K})$ is completely positive) for all $s > 0$. Conversely, if $\pi_{s_n}^\# \geq \phi_{s_n}^\#$ for all $n = 1, 2, \cdots$ where $s_n \to 0+$ as $n \to \infty$ then β is a subordinate of α.*

Proof. Suppose α and β are CP-flows over $\mathfrak{K}$ and $\pi^\#$ and $\phi^\#$ are the generalized boundary representations of α and β, respectively. Suppose α dominates β. Let Θ be the semigroup of $\mathfrak{B}(\mathfrak{H} \oplus \mathfrak{H})$ constructed from α and β as described in the Lemma 3.6. Since α dominates β we have Θ is a CP-semigroup and, hence, its generalized boundary representation which is given below

$$\begin{bmatrix} \pi_s^\# & \phi_s^\# \\ \phi_s^\# & \phi_s^\# \end{bmatrix}$$

is by Theorem 4.27 completely positive for all $s > 0$. Then $\pi_s^\# \geq \phi_s^\#$ for all $s > 0$.

Conversely, suppose $\pi_{s_n}^\# \geq \phi_{s_n}^\#$ for all n where $s_n \to 0+$ as $n \to \infty$. Let Ω be the boundary weight given by the matrix of weights

$$\begin{bmatrix} \omega & \eta \\ \eta & \eta \end{bmatrix}$$

where ω is the boundary weight associated with α and η is the boundary weight associated with β. Consider the matrix of truncated weights

$$\begin{bmatrix} \omega_s & \eta_s \\ \eta_s & \eta_s \end{bmatrix}$$

where $\omega_s(A) = \omega(E(s,\infty)AE(s,\infty))$ for $A \in \mathfrak{B}(\mathfrak{H})$ and the same for η_s. Since $\pi_{s_n}^\# \geq \phi_{s_n}^\#$ we have the above weight is the weight of a CP-flow over $\mathfrak{K} \oplus \mathfrak{K}$ for $s = s_n$. Since $s_n \to 0+$ it follows from Theorem 4.23 that the above matrix of weights is the boundary weight of a CP-flow over $\mathfrak{K} \oplus \mathfrak{K}$ and this is clearly a CP-flow of the form of Θ_t given in Lemma 3.6. Since Θ_t is a CP-flow we have $\alpha \geq \beta$ from Lemma 3.6. $\square$

Lemma 4.30. *Suppose α and β are CP-flows over $\mathfrak{K}$ and $\pi^\#$ and $\phi^\#$ are their generalized boundary representations, respectively. Suppose for some $t > 0$ we have $\pi_t^\# \geq \phi_t^\#$. Then $\pi_s^\# \geq \phi_s^\#$ for all $s \geq t$.*

Proof. Assume the hypothesis and notation of the theorem. Let ω and η be the boundary weights of α and β, respectively and let ω_t and η_t the truncated weights at t, so

$$\omega_t(\rho)(A) = \omega(\rho)(E(t,\infty)AE(t,\infty)) \quad \text{and} \quad \eta_t(\rho)(A) = \eta(\rho)(E(t,\infty)AE(t,\infty))$$

for all A in the null boundary algebra $\mathfrak{A}(\mathfrak{H})$ and $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Theorem 4.27 shows there are CP-flows α^1 and β^1 associated with ω_t and η_t, respectively. Let γ the mapping of $\mathfrak{B}(\mathfrak{H} \oplus \mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K} \oplus \mathfrak{K})$ given by

$$\gamma(\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}) = \begin{bmatrix} \pi_t^{\#}(X_{11}) & \phi_t^{\#}(X_{12}) \\ \phi_t^{\#}(X_{21}) & \phi_t^{\#}(X_{22}) \end{bmatrix}$$

for $X_{ij} \in \mathfrak{B}(\mathfrak{H})$. One checks that γ is completely positive and there is a unique CP-semigroup Θ derived from γ which is given for each $t \geq 0$ by the matrix

$$\Theta_t(\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}) = \begin{bmatrix} \alpha_t^1(X_{11}) & \beta_t^1(X_{12}) \\ \beta_t^1(X_{21}) & \beta_t^1(X_{22}) \end{bmatrix}.$$

Note the fact that Θ is unique follow from the fact that $\|\gamma \cdot \Lambda\| \leq e^{-t} < 1$. From Lemma 3.6 it follows that $\alpha^1 \geq \beta^1$ and from Theorem 4.29 that $\pi_s^1 \geq \phi_s^1$ for all $s > 0$ where $\pi^{1\#}$ and $\phi^{1\#}$ are the generalized boundary representations of α^1 and β^1, respectively. Since $\pi_s^{1\#} = \pi_s^{\#}$ and $\phi_s^{1\#} = \phi_s^{\#}$ for $s \geq t$ the conclusion of the lemma follows. $\square$

The difficulty in computing the generalized boundary representation from the boundary weight for a CP-flow is computing the inverse of the map $T\rho = \rho + \hat{\Lambda}\omega(\rho)$. Even when $\mathfrak{K}$ is two dimensional this is a complicated problem in linear algebra. The situation is tractable in the case of Schur maps which we now describe. A mapping ϕ of $\mathfrak{B}(\mathfrak{H})$ into itself is called Schur product with respect to an orthonormal basis $\{f_i : i = 1, 2, \cdots\}$ of $\mathfrak{H}$ if there are complex numbers ϕ_{ij} so that $(f_i, \phi(A)f_j) = \phi_{ij}(f_i, Af_j)$ for $i, j = 1,2,\cdots$. This means ϕ acts on A by multiply the matrix coefficients of A by ϕ_{ij}. This product has been called the Schur product, the Hadamard product or Kronecker product. We will call this the Schur product. In the case where $\mathfrak{H}$ is finite dimensional one easily sees that if ϕ is diagonal with coefficients ϕ_{ij} then ϕ is completely positive if and only if the coefficients ϕ_{ij} are those of a positive operator. A similar result holds for infinite dimensional Hilbert spaces. Note the spectrum of ϕ as a mapping are the numbers ϕ_{ij}. We see then that a completely positive mapping can have negative spectrum and even complex spectrum.

Definition 4.31. The mapping $\rho \to \omega(\rho)$ from $\mathfrak{B}(\mathfrak{K})_*$ to weights defined on the null boundary algebra $\mathfrak{A}(\mathfrak{H})$ is said to be Schur diagonal with respect to an orthonormal basis $\{f_i : i = 1, 2, \cdots\}$ of $\mathfrak{K}$ if $\rho_{ij}(A) = (f_i, Af_j)$ for $A \in \mathfrak{B}(\mathfrak{K})$ and $e_i f = (f_i, f)f_i$ then

$$\omega(\rho_{ij})(A) = \omega(\rho_{ij})((e_i \otimes I)A(e_j \otimes I))$$

for all A in the null boundary algebra $\mathfrak{A}(\mathfrak{H})$ for all $i, j = 1, 2, \cdots$. In this case the matrix elements of the mapping $\rho \to \omega(\rho)$ are the weights

$$\omega_{ij}(A) = \omega(\rho_{ij})(e_{ij} \otimes A)$$

defined for A in the null boundary algebra $\mathfrak{A}(L^2(0, \infty)$ where $\{e_{ij}\}$ are the set of matrix units defined by $e_{ij}f = (f_j, f)f_i$ for all $f \in \mathfrak{K}$ and i, j =1,2,$\cdots$.

The next lemma shows that if the mapping $\rho \to \omega(\rho)$ is completely positive to show the mapping is Schur diagonal we need only check the diagonal entries.

Lemma 4.32. *Suppose the mapping $\rho \to \omega(\rho)$ from $\mathfrak{B}(\mathfrak{K})_*$ to weights defined on the null boundary algebra $\mathfrak{A}(\mathfrak{H})$ is completely positive. Suppose $\{f_i : i = 1, 2, \cdots\}$ is an orthonormal basis for $\mathfrak{K}$ and $\rho_{ij}(A) = (f_i, Af_j)$ for all $A \in \mathfrak{B}(\mathfrak{K})$ and all i and j and $e_{ij}f = (f_j, f)f_i$ for $f \in \mathfrak{K}$. Suppose $\omega_{ij} = \omega(\rho_{ij})$ and suppose the diagonal weights ω_{ii} are Schur diagonal, so $\omega_{ii}(A) = \omega_{ii}((e_{ii} \otimes I)A(e_{ii} \otimes I))$ for A in the null boundary algebra $\mathfrak{A}(L^2(0, \infty))$ for each i and j. Then ω is Schur diagonal with respect to $\{f_i : i = 1, 2, \cdots\}$.*

Proof. Assume the hypothesis and notation of the lemma are satisfied. Suppose f_i and f_j are distinct vectors in the orthonormal basis for $\mathfrak{K}$. Since $\rho \to \omega(\rho)$ is completely positive we have

$$\omega_{ii}(A^*A) + \omega_{ij}(A^*B) + \omega_{ji}(B^*A) + \omega_{jj}(B^*B) \geq 0$$

for A, B in the null boundary algebra $\mathfrak{A}(L^2(0, \infty))$ Multiply B by z with z a complex number and minimizing the above expression we find the above inequality is equivalent to the inequality

$$|\omega_{ij}(A^*B)|^2 \leq \omega_{ii}(A^*A)\omega_{jj}(B^*B)$$

for $A, B \in \mathfrak{A}(L^2(0, \infty))$ Replacing B by $B((I - e_{jj}) \otimes I)$ we have $\omega_{ij}(A^*B((I - e_{jj}) \otimes I)) = 0$ or $\omega_{ij}(A^*B) = \omega_{ij}(A^*B(e_{jj} \otimes I))$ for all $A, B \in \mathfrak{A}(L^2(0, \infty))$. Applying this argument to A, and replacing A by $A((I - e_{ii}) \otimes I)$ we recalculate that $\omega_{ij}(A^*B) = \omega_{ij}((e_{ii} \otimes I)A^*B)$ for all $A, B \in \mathfrak{A}(L^2(0, \infty))U(t)$. Since A and B are arbitrary we combine these results to obtain $\omega_{ij}(A) = \omega_{ij}((e_{ii} \otimes I)A(e_{jj} \otimes I))$ for all $A \in \mathfrak{A}(L^2(0, \infty))$. Hence ω is Schur diagonal with respect to the basis $\{f_i : i = 1, 2, \cdots\}$. □

The following theorem gives a reasonably computable condition that the Schur diagonal mapping $\rho \to \omega(\rho)$ gives rise to a CP-flow.

Theorem 4.33. *Suppose $\mathfrak{K}$ is finite dimensional and $\rho \to \omega(\rho)$ is a linear mapping of $\mathfrak{B}(\mathfrak{K})_*$ into weights $\omega(\rho)$ on the null boundary algebra $\mathfrak{A}(\mathfrak{H})$ which is Schur diagonal with respect to an orthonormal basis $\{f_i : i = 1,2,\cdots, n\}$ and $\rho_{ij}(A) = (f_i, Af_j)$ for each i and j and for $A \in \mathfrak{B}(\mathfrak{K})$. For $t > 0$ and $\rho \in \mathfrak{B}(\mathfrak{K})_*$ let*

$$\omega_t(\rho)(A) = \omega(\rho)(E(t, \infty)AE(t, \infty))$$

for all $A \in \mathfrak{A}(\mathfrak{H})$. Note $\rho \to \omega_t(\rho)$ is Schur diagonal with the same basis. For $\rho \in \mathfrak{B}(\mathfrak{K})_*$ let

$$\hat{\sigma}(\rho)(A) = \int_0^\infty e^{-t}\omega(\rho)(U(t)AU(t)^*)\, dt$$

for all $A \in \mathfrak{B}(\mathfrak{H})$. Then the mapping σ is the boundary resolvent of a CP-flow if and only if for each $t > 0$ the matrix with entries given by

$$\eta_{ij} = \frac{\omega_t(\rho_{ij})}{1 + \omega_t(\rho_{ij})(\Lambda)}$$

for $i,j = 1,2,\cdots,n$ *are the matrix elements of a completely positive contraction of* $\mathfrak{B}(\mathfrak{K})_*$ *into* $\mathfrak{B}(\mathfrak{H})_*$.

Proof. Assume the hypothesis and notation given in the statement of the theorem apply. Suppose the mapping $\rho \to \omega(\rho)$ is the boundary weight map of a CP-flow. Suppose $t > 0$. From Theorem 4.23 it follows the mapping $\rho \to \omega_t(\rho)$ defines a CP-flow. From Theorem 4.27 it follows that this semigroup is derived from a completely positive normal contraction π of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$ and from the details of the proof of Theorem 4.27 it follows that $\hat{\pi}(\rho) = \omega_t(\Theta(\rho))$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$ where Θ is the inverse of the map $\rho \to \rho + \hat{\Lambda}(\omega_t(\rho))$. Let $\rho_{ij}(A) = (f_i, Af_j)$ for $A \in \mathfrak{B}(\mathfrak{K})$ and $e_{ij}f = (f_j,f)f_i$ for $i,j = 1,\cdots,n$. Let Θ' be the linear mapping of $\mathfrak{B}(\mathfrak{K})_*$ into $\mathfrak{B}(\mathfrak{K})_*$ given by $\Theta'(\rho_{ij}) = (1+\omega_t(\rho_{ij})(\Lambda))^{-1}\rho_{ij}$ for $i,j = 1,\cdots,n$. Since $\rho \to \omega_t(\rho)$ is Schur diagonal with the basis $\{f_i : i = 1,\cdots,n\}$ a direct calculations shows that Θ' is the inverse of the map $\rho \to \rho + \hat{\Lambda}(\omega_t(\rho))$ and since the inverse is unique $\Theta' = \Theta$. Hence, we have

$$\hat{\pi}(\rho_{ij}) = \frac{\omega_t(\rho_{ij})}{1+\omega_t(\rho_{ij})(\Lambda)}$$

and since $\rho \to \omega_t(\rho)$ is Schur diagonal and $\hat{\pi}$ is a completely positive contraction of $\mathfrak{B}(\mathfrak{K})_*$ into $\mathfrak{B}(\mathfrak{H})_*$ the conclusion of the theorem follows for the η_{ij}.

Conversely, suppose for each $t > 0$ the matrix entries η_{ij} given in the statement of the theorem are define a completely positive contraction $\hat{\pi}_t$ of $\mathfrak{B}(\mathfrak{K})_*$ into $\mathfrak{B}(\mathfrak{H})_*$. Then it follows from Theorem 4.26 that $\rho \to \omega_t(\rho)$ is the boundary weight map of a CP-flow which is derived from π_t. Since $\rho \to \omega_t(\rho)$ gives rise to a CP-flow for each $t > 0$ it follows from Theorem 4.23 that inequality (4.19) of Theorem 4.23 is satisfied and this implies weaker limit inequality (4.20) which implies $\rho \to \omega(\rho)$ defines a CP-flow. $\square$

We begin our investigation of the limit π_o of $\pi_s^{\#}$ as $s \to 0+$ where $\pi^{\#}$ is a generalized boundary representation.

Lemma 4.34. *Suppose* α *is a* CP*-flow over* $\mathfrak{K}$ *and* $\pi^{\#}$ *is the associated generalized boundary representation. If* $t_o > 0$ *and* $0 < s \le t \le t_o$ *then the mapping* $A \to \pi_t^{\#}(E(t_o,\infty)AE(t_o,\infty)) - \pi_s^{\#}(E(t_o,\infty)AE(t_o,\infty))$ *is completely positive (i.e., the mapping* $\phi_s(A) = \pi_s^{\#}(E(t_o,\infty)AE(t_o,\infty))$ *is an increasing (in sense of complete positivity) function for* $s \in (0,t_o]$*).*

Proof. Suppose $\pi^{\#}$ is a generalized boundary representation described above. As we have done before when we will put a prime on mapping to indicate the associated map where $\mathfrak{K}$ is replaced by $\mathfrak{K}_1 = \mathfrak{K} \otimes \mathfrak{K}_o$ and $\mathfrak{H}$ is replaced by $\mathfrak{H}_1 = \mathfrak{H} \otimes \mathfrak{K}_o$ where $\mathfrak{K}_o$ is a separable infinite dimensional Hilbert space and the primed mapping is the usual tensor extensions. To prove the lemma all we need do is prove the mapping described in the lemma is positive for the primed maps. Suppose then that $A \in \mathfrak{B}(\mathfrak{H}_1)$ is positive. Suppose $t_o > 0$ and $0 < s < t < t_o$. We show $\pi_s'^{\#}(E'(t,\infty)AE(t,\infty)) \le \pi_t'^{\#}(E'(t,\infty)AE'(t,\infty))$. Suppose $\rho \in \mathfrak{B}(\mathfrak{K}_1)_*$ and $\rho \ge 0$. Let

$$Q(t) = \rho(\pi_t'^{\#}(E'(t,\infty)AE'(t,\infty)) - \pi_s'^{\#}(E'(t,\infty)AE'(t,\infty))).$$

Then we have

$$\begin{aligned} Q(t) =& (\hat{\pi}_t'^{\#} - \hat{\pi}_s'^{\#})(\rho)(E'(t,\infty)AE'(t,\infty)) \\ =& (\omega_t'(I + \hat{\Lambda}'\omega_t')^{-1} - \omega_s'(I + \hat{\Lambda}'\omega_s')^{-1})(\rho)(E'(t,\infty)AE'(t,\infty)). \end{aligned}$$

Because $\omega_s'(E'(t,\infty)AE'(t,\infty)) = \omega_t'(A)$ we have

$$\begin{aligned} Q(t) =& (\omega_t'(I + \hat{\Lambda}'\omega_t')^{-1} - \omega_t'(I + \hat{\Lambda}'\omega_s')^{-1})(\rho)(A) \\ =& \omega_t'((I + \hat{\Lambda}'\omega_t')^{-1}((I + \hat{\Lambda}'\omega_s') - (I + \hat{\Lambda}'\omega_t'))(I + \hat{\Lambda}'\omega_s')^{-1})(\rho)(A) \\ =& \omega_t'((I + \hat{\Lambda}'\omega_t')^{-1}\hat{\Lambda}'(\omega_s' - \omega_t')(I + \hat{\Lambda}'\omega_s')^{-1})(\rho)(A) \\ =& \hat{\pi}_t'^{\#}(\hat{\Lambda}'(\omega_s - \omega_t)((I + \hat{\Lambda}'\omega_s)^{-1})(\rho))(A). \end{aligned}$$

Since $A \geq 0$ we see $Q(t) \geq 0$ if the mapping in the brackets following $\pi_t'^{\#}$ is positive. To give this mapping a name we call it Ψ. Suppose $B \in \mathfrak{B}(\mathfrak{K}_1)$ and $B \geq 0$ and $\eta \in \mathfrak{B}(\mathfrak{K}_1)_*$ is positive, then we have

$$\begin{aligned} \Psi(\eta)(B) =& \hat{\Lambda}'(\omega_s' - \omega_t')((I + \hat{\Lambda}'\omega_s')^{-1}\eta)(B) \\ =& \omega'((I + \hat{\Lambda}'\omega_s')^{-1}\eta)(E'(s,\infty)\Lambda'(B)E'(s,\infty) - E'(t,\infty)\Lambda'(B)E'(t,\infty)). \end{aligned}$$

Since $E'(x,\infty)$ commutes with $\Lambda'(B)$ for all $x > 0$ we have

$$\Psi(\eta)(B) = \omega'((I + \hat{\Lambda}'\omega_s')^{-1}\eta)(E'(s,t)\Lambda'(B)E'(s,t)).$$

Since $\omega'(E'(s,t)CE'(s,t)) = \omega_s'(E'(s,t)CE'(s,t))$ for all $C \in \mathfrak{B}(\mathfrak{H})$ we have

$$\begin{aligned} \Psi(\eta)(B) =& \omega_s'((I + \hat{\Lambda}'\omega_s')^{-1}\eta)(E'(s,t)\Lambda'(B)E'(s,t)). \\ =& \hat{\pi}_s'^{\#}(\eta)(E'(s,t)\Lambda'(B)E'(s,t)). \end{aligned}$$

Since $\pi_s'^{\#}$ is positivity preserving Ψ is positivity preserving and $Q(t) = \hat{\pi}_t^{\#}(\Psi(\rho))(A)$ is positive. Replacing A by $E'(t_o,\infty)AE'(t_o,A)$ in the expression for $Q(t)$ completes the proof of the lemma. $\square$

Theorem 4.35. *Suppose α is a CP-flow over $\mathfrak{K}$ and $\pi^{\#}$ is the generalized boundary representation of α. Then $\pi_s^{\#}(A) \to \pi_o^{\#}(A)$ for as $s \to 0+$ in the σ-strong topology for each $A \in \cup_{t>0}U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$ where $\pi_o^{\#}$ is a σ-weakly continuous completely positive contraction of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$.*

Proof. Suppose α is a CP-flow over $\mathfrak{K}$ with generalized boundary representation $\pi^{\#}$. Let $\phi_s(A) = \pi_s^{\#}(E(t,\infty)AE(t,\infty))$ for $0 < s \leq t$ and $A \in \mathfrak{B}(\mathfrak{H})$. From the previous lemma we have ϕ_s is an increasing function of s in the sense of complete positivity. From the Stinespring Theorem we have $\phi_t(A) = V^*\gamma(A)V$ for $A \in \mathfrak{B}(\mathfrak{H})$ where γ is a $*$-representation of $\mathfrak{B}(\mathfrak{H})$ on a Hilbert space $\mathfrak{H}_o$ and V is a linear contraction from $\mathfrak{K}$ to $\mathfrak{H}_o$. Since $\phi_s \leq \phi_t$ for $s \leq t$ we have $\phi_s(A) = V^*C_s\gamma(A)V$ for $A \in \mathfrak{B}(\mathfrak{H})$ where $C_s \in \gamma(\mathfrak{B}(\mathfrak{H}))'$ and $0 \leq C_s \leq I$. Since the ϕ_s are increasing we have $0 \leq C_x \leq C_y \leq I$ for $0 < x \leq y \leq t$. Since the C_s are decreasing as

$s \to 0+$ we have the C_s converge strongly to a limit C_o as $t \to 0+$. Hence, $\phi_s(A)$ converges σ-strongly to $\phi_o(A) = V^*C_o\gamma(A)V$ as $s \to 0+$. For $A \in U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$ then we define $\pi_o(A) = \phi_o(A)$. The mapping ϕ_o depends on t but we note that for two $\phi's$ defined for two $t's$ the ϕ_o from the smaller t_1 agrees with the ϕ_o from the larger t_2 on $U(t_2)\mathfrak{B}(\mathfrak{H})U(t_2)^*$. Then for $A \in U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$ we define $\pi_o^\#(A)$ defined from any ϕ_o constructed from a $t_1 \leq t$. This defines the mapping $\pi_o^\#$ on $\cup_{t>0}U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$ and we have $\pi_s^\#(A) \to \pi_o^\#(A)$ in the σ-strong topology as $s \to 0+$ for $A \in \cup_{t>0}U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$ We now show $\pi_o^\#$ is σ-weakly continuous. Note that $\phi_o(A) = V^*C_o\gamma(A)V$ for $A \in \mathfrak{B}(\mathfrak{H})$ is σ-weakly continuous since γ is a $*$-representation of $\mathfrak{B}(\mathfrak{H})$ and, hence, we have $A \to \pi_o^\#(E(t,\infty)AE(t,\infty))$ is σ-weakly continuous for all $t > 0$. Suppose $\eta \in \mathfrak{B}(\mathfrak{K})_*$ and $\eta \geq 0$ and $\rho_t(A) = \eta(\pi_o(E(t,\infty)AE(t,\infty)))$ for $t > 0$ and $A \in \mathfrak{B}(\mathfrak{H})$. Then from Lemma 2.10 we have

$$\|\rho_t - \rho_s\|^2 \leq 2\eta(\pi_o^\#(E(s,\infty)))^2 - 2\eta(\pi_o^\#(E(t,\infty)))^2 \leq 4\|\eta\|\eta(\pi_o^\#(E(s,t)))$$

Since $\lim_{s\to 0+} \eta(\pi_o^\#(s,\infty)) \leq \eta(I)$ we have if s_n is a sequence of positive numbers decreasing to zero we see from the above estimate that the ρ_{s_n} form a Cauchy sequence in norm. Since each element of $\mathfrak{B}(\mathfrak{K})_*$ can be written as a sum of four positive elements we see that functionals $A \to \eta(\pi_o^\#(A))$ are norm limits of normal functionals and, hence, these functionals are normal so $\pi_o^\#$ is normal which implies $\pi_o^\#$ is σ-weakly continuous. $\square$

Definition 4.36. If α is a CP-flow over $\mathfrak{K}$ and $\pi^\#$ is the generalized boundary representation of α then $\pi_o^\#$ as defined in the previous theorem is called the normal spine of α.

Lemma 4.37. *Suppose ϕ is a σ-weakly continuous completely positive contraction of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$ and α is the minimal CP-flow derived from ϕ. Suppose $\pi^\#$ is the generalized boundary representation for α and $\pi_o^\#$ is the normal spine of α. Then $\pi_o^\# = \phi$.*

Proof. Assume the hypothesis and notation of the lemma. For $t > 0$ let $\phi_t(A) = \phi(E(t,\infty)AE(t,\infty))$ for $A \in \mathfrak{B}(\mathfrak{H})$. We establish a formula for $\pi_t^\#$. From Theorem 4.26 we have the boundary weight $\omega(\rho)$ for α is given by

$$\omega(\rho) = \hat{\phi}(\rho) + \hat{\phi}(\hat{\Lambda}(\hat{\phi}(\rho))) + \hat{\phi}(\hat{\Lambda}(\hat{\phi}(\hat{\Lambda}(\hat{\phi}(\rho))))) + \cdots$$

where the sum converges on the null boundary algebra $\mathfrak{A}(\mathfrak{H})$ and for ρ positive the sum satisfies $\omega(\rho)(I - \Lambda) \leq \rho(I)$. Suppose $t > 0$. The truncated weight ω_t is given by

$$\omega_t(\rho) = \hat{\phi}_t(\rho) + \hat{\phi}_t(\hat{\Lambda}(\hat{\phi}(\rho))) + \hat{\phi}_t(\hat{\Lambda}(\hat{\phi}(\hat{\Lambda}(\hat{\phi}(\rho))))) + \cdots$$

where now the sum converges in norm. Then $\pi_t^\#$ is given by $\hat{\pi}_t^\# = \omega_t(I + \hat{\Lambda}\omega_t)^{-1}$ or $\hat{\pi}_t^\#(I + \hat{\Lambda}\omega_t) = \omega_t$. Then applying this equation to $I - \hat{\Lambda}\hat{\phi}$ and canceling terms which is permissible since the sums converge we find $\hat{\pi}_t^\#(I - \hat{\Lambda}(\hat{\phi} - \hat{\phi}_t)) = \hat{\phi}_t$. Then applying this equation to the finite geometric sum of powers of $\hat{\Lambda}(\phi - \phi_t)$ we find

$$\hat{\pi}_t^\#(I - (\hat{\Lambda}(\hat{\phi} - \hat{\phi}_t))^{n+1}) = \hat{\phi}_t(I + \hat{\Lambda}(\hat{\phi} - \hat{\phi}_t) + \cdots + (\hat{\Lambda}(\hat{\phi} - \hat{\phi}_t))^n)$$

Since $E(t,\infty)$ commutes with $\Lambda(A)$ for $A \in \mathfrak{B}(\mathfrak{H})$ it follows that

$$\begin{aligned}(\phi - \phi_t)(\Lambda(A)) =& \phi(\Lambda(A) - E(t,\infty)\Lambda(A)E(t,\infty)) \\ =& \phi((I - E(t,\infty))\Lambda(A)) \\ =& \phi(E(t)\Lambda(A)) = \phi(E(t)\Lambda(A)E(t))\end{aligned}$$

Hence, $\hat{\Lambda}(\hat{\phi} - \hat{\phi}_t)$ is a completely positive map. Since the terms in the above equation are positive it follows that the series converges in norm. We will show that the $\hat{\pi}_t^{\#}((\hat{\Lambda}(\hat{\phi} - \hat{\phi}_t))^{n+1}$ term converges to zero as $n \to \infty$. Recall the equation $\hat{\pi}_t^{\#}(I + \hat{\Lambda}\omega_t) = \omega_t$. Applying this to $(\hat{\Lambda} \cdot \hat{\phi})^n$ we find

$$\hat{\pi}_t^{\#}(\hat{\Lambda} \cdot \hat{\phi})^n = (I - \hat{\pi}_t^{\#}\hat{\Lambda})\omega_t(\hat{\Lambda} \cdot \hat{\phi})^n$$

Since the series for ω_t converges it follows that $\omega_t(\hat{\Lambda} \cdot \hat{\phi})^n$ converges pointwise to zero in norm. Since $\pi_t^{\#}$ is a contraction $\hat{\pi}_t^{\#}(\hat{\Lambda} \cdot \hat{\phi})^n$ converges pointwise to zero in norm as $n \to \infty$. A bit of computation shows that $(\hat{\Lambda} \cdot \hat{\phi})^n \geq (\hat{\Lambda}(\hat{\phi} - \hat{\phi}_t))^n$ so for positive $\rho \in \mathfrak{B}(\mathfrak{K})_*$ we have

$$\begin{aligned}\|\hat{\pi}_t^{\#}(\hat{\Lambda} \cdot \hat{\phi})^n\rho\| =& (\hat{\pi}_t^{\#}(\hat{\Lambda} \cdot \hat{\phi})^n\rho)(I) \geq (\hat{\pi}_t^{\#}(\hat{\Lambda}(\hat{\phi} - \hat{\phi}_t))^n\rho)(I) \\ =& \|\hat{\pi}_t^{\#}(\hat{\Lambda}(\hat{\phi} - \hat{\phi}_t))^n\rho\|\end{aligned}$$

Since each $\rho \in \mathfrak{B}(\mathfrak{K})_*$ is the linear combination of four positive elements we have $\|\hat{\pi}_t^{\#}(\hat{\Lambda}(\hat{\phi} - \hat{\phi}_t))^n\rho\| \to 0+$ as $n \to \infty$. Using this in the equation for $\hat{\pi}_t^{\#}$ we find

$$\hat{\pi}_t^{\#} = \hat{\phi}_t(I + \hat{\Lambda}(\hat{\phi} - \hat{\phi}_t) + (\hat{\Lambda}(\hat{\phi} - \hat{\phi}_t))^2 + \cdots) \tag{4.21}$$

where the series converges in norm for each $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Assume $s > 0$. Applying this to a positive $\rho \in \mathfrak{B}(\mathfrak{K})_*$ we have for $t \in (0, s]$ and a positive $A \in \mathfrak{B}(\mathfrak{H})$ that

$$\pi_t^{\#}(\rho)(E(s,\infty)AE(s,\infty)) = (\hat{\phi}_s(I + \hat{\Lambda}(\hat{\phi} - \hat{\phi}_t) + (\hat{\Lambda}(\hat{\phi} - \hat{\phi}_t))^2 + \cdots))(A)$$

for $A \in \mathfrak{B}(\mathfrak{H})$. Since the terms above are positive and decreasing as t decreases it follows that $\pi_t^{\#}(\rho)(E(s,\infty)AE(s,\infty)) \to \phi(E(s,\infty)AE(s,\infty))$ as $t \to 0+$. By linearity this result extends to all $\rho \in \mathfrak{B}(\mathfrak{K})_*$ and $A \in \mathfrak{B}(\mathfrak{H})$. Since $s > 0$ is arbitrary and from the definition of $\pi_o^{\#}$ we have $\pi_o^{\#} = \phi$. $\square$

We suspect that in the previous lemma with more work one could show $\|\hat{\pi}_t^{\#}(\rho) - \hat{\phi}(\rho)\| \to 0$ as $t \to 0+$ for each $\rho \in \mathfrak{B}(\mathfrak{K})_*$.

Lemma 4.38. *Let α be a CP-flow over $\mathfrak{K}$ and let $\pi^{\#}$ be the generalized boundary representation for α. Let α^s be the minimal CP-flow derived from $\pi_s^{\#}$ for $s > 0$. Then $\alpha_t^s(A) \to \alpha_t(A)$ σ-weakly as $s \to 0+$ for $t \geq 0$ and $A \in \mathfrak{B}(\mathfrak{H})$ and the convergence is uniform for t in a finite interval.*

Proof. Assume the hypothesis and notation of the lemma. We will use the Trotter convergence theorem for resolvents. Let R^s and R be the resolvent of α^s and α and let ω^s and ω be the boundary weights of α^s and α. Then we have from Theorem 4.17 and the Definition (4.13) of the boundary resolvent that

$$\hat{R}^s(\eta) = \hat{\Gamma}(\omega^s(\hat{\Lambda}(\eta))) + \hat{\Gamma}(\eta)$$

and

$$\hat{R}(\eta) = \hat{\Gamma}(\omega(\hat{\Lambda}(\eta))) + \hat{\Gamma}(\eta)$$

for $\eta \in \mathfrak{B}(\mathfrak{H})_*$. Suppose further that η is positive and $\|\eta\| \leq 1$. We have

$$\begin{aligned} \|\hat{R}(\eta)\| =& \Gamma(\omega(\hat{\Lambda}(\eta)) + \eta)(I) \\ \leq& \hat{\Gamma}(\omega(\hat{\Lambda}(\eta)))(I) = \int_0^\infty h(t)\,dt \leq 1 \end{aligned}$$

where $h(t) = e^{-t}\omega(\hat{\Lambda}(\eta))(E(t,\infty))$. Now suppose $A \in \mathfrak{B}(\mathfrak{H})$ with $\|A\| \leq 1$. Then we have

$$\begin{aligned} |(\hat{R} - & \hat{R}^s)(\eta)(A)| = |\hat{\Gamma}(\omega(\hat{\Lambda}(\eta)) - \omega^s(\hat{\Lambda}(\eta)))(A)| \\ &= \left| \int_0^\infty e^{-t}\omega(\hat{\Lambda}(\eta))(U(t)AU(t)^* - E(s,\infty)U(t)AU(t)^*E(s,\infty))dt \right| \end{aligned}$$

Now $U(t)^*E(s,\infty) = E(s-t,\infty)U(t)^*$ for $t \in [0,s]$ and $U(t)^*E(s,\infty) = U(t)^*$ for $t \geq s$. Hence, we have

$$\begin{aligned} |((\hat{R} - \hat{R}^s)(\eta)(A)| =& \left| \int_0^s e^{-t}\omega(\hat{\Lambda}(\eta))(U(t)(A - E(s-t,\infty)AE(s-t,\infty))U(t)^*)dt \right| \\ =& \left| \int_0^s e^{-t}\hat{\theta}_t\omega(\hat{\Lambda}(\eta))(A - E(s-t,\infty)AE(s-t,\infty))dt \right| \\ \leq& \left| \int_0^s e^{-t}2\|\hat{\theta}_t\omega(\hat{\Lambda}(\eta))\|dt \right| \\ =& \left| \int_0^s e^{-t}2\omega(\hat{\Lambda}(\eta))(E(t,\infty)dt \right| = 2\int_0^s h(t)\,dt \end{aligned}$$

Since this estimate is true for all $A \in \mathfrak{B}(\mathfrak{H})$ with $\|A\| \leq 1$ and $h \in L^1(0,\infty)$ we have

$$\|(\hat{R} - \hat{R}^s)(\eta)\| \leq 2\int_0^s h(t)\,dt \to 0$$

as $s \to 0+$. Since each $\eta \in \mathfrak{B}(\mathfrak{K})_*$ is the linear combination of four positive elements we have $\|(\hat{R} - \hat{R}^s)(\eta)\| \to 0$ as $s \to 0+$ for all $\eta \in \mathfrak{B}(\mathfrak{K})_*$. Then by the Trotter resolvent convergence theorem [BR] (Theorem 3.1.26) we have $\|\hat{\alpha}_t^s(\eta) - \hat{\alpha}_t(\eta)\| \to 0$ as $s \to 0+$ for all $\eta \in \mathfrak{B}(\mathfrak{K})_*$ and $t \geq 0$ where the convergence is uniform for t in a bounded interval. This result for the predual maps implies the conclusion of the lemma for the maps α^s and α. $\square$

Lemma 4.39. *Suppose ϕ is a σ-weakly continuous completely positive contraction of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$. Let $\phi_s(A) = \phi(E(s,\infty)AE(s,\infty))$ for $s > 0$ and $A \in \mathfrak{B}(\mathfrak{H})$. Let α be the minimal CP-flow derived from ϕ and let α^s be the minimal CP-flow derived from ϕ_s for $s > 0$. Then $\alpha_t^s(A) \to \alpha_t(A)$ σ-weakly as $s \to 0+$ for $t \geq 0$ and $A \in \mathfrak{B}(\mathfrak{H})$ and the convergence is uniform for t in a finite interval.*

Proof. Assume the hypothesis and notation of the lemma. Again we will use the Trotter convergence theorem for resolvents. Let R^s and R be resolvents of α^s and α and let ω^s and ω be the boundary weights of α^s and α. Then from Theorem 4.27 and Definition 4.28 we have

$$\hat{R}^s(\eta) = \hat{\Gamma}(\omega^s(\hat{\Lambda}(\eta))) + \hat{\Gamma}(\eta)$$

and

$$\hat{R}(\eta) = \hat{\Gamma}(\omega(\hat{\Lambda}(\eta))) + \hat{\Gamma}(\eta).$$

for $s > 0$ and $\eta \in \mathfrak{B}(\mathfrak{K})_*$. Assume $\eta \in \mathfrak{B}(\mathfrak{H})_*$ is positive and let $\rho = \hat{\Lambda}\eta$. Then from Theorem 4.26 we have

$$\hat{\Gamma}(\omega(\rho)) = \hat{\Gamma}(\hat{\phi}(\rho) + \hat{\phi}(\hat{\Lambda}(\hat{\phi}(\rho))) + \hat{\phi}(\hat{\Lambda}(\hat{\phi}(\hat{\Lambda}(\hat{\phi}(\rho))))) + \cdots)$$

and

$$\hat{\Gamma}(\omega^s(\rho)) = \hat{\Gamma}(\hat{\phi}_s(\rho) + \hat{\phi}_s(\hat{\Lambda}(\hat{\phi}_s(\rho))) + \hat{\phi}_s(\hat{\Lambda}(\hat{\phi}_s(\hat{\Lambda}(\hat{\phi}_s(\rho))))) + \cdots)$$

As we saw in the proof of Theorem 4.26 the series above converge with the $\hat{\Gamma}$ term included. Note the series above is uniformly bounded since we can compute the norms by evaluating on the unit I and we obtain the estimates

$$\|\hat{\Gamma}(\hat{\phi}_s(\hat{\Lambda}(\cdots(\hat{\phi}_s(\rho))\cdots)))\| \leq \|\hat{\Gamma}(\hat{\phi}(\hat{\Lambda}(\cdots(\hat{\phi}(\rho))\cdots)))\|$$

Since each of the terms with the ϕ_s converge to the corresponding term with the ϕ and since we have uniform bounds on the sum of the norms of the terms we have $\|\hat{\Gamma}(\omega^s(\rho)) - \hat{\Gamma}(\omega(\rho))\| \to 0$ as $s \to 0+$. Hence, $\|(\hat{R} - \hat{R}^s)(\eta)\| \to 0$ as $s \to 0+$. Again since each $\eta \in \mathfrak{B}(\mathfrak{H})_*$ is the linear combination of four positive elements this results holds for all $\eta \in \mathfrak{B}(\mathfrak{H})_*$. Then using the Trotter convergence theorem as we did in the previous lemma the result of the lemma follows. □

Lemma 4.40. *Suppose π and ϕ are two σ-weakly continuous completely positive contractions of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$. Suppose α and β are the minimal CP-flows derived from π and ϕ, respectively. Then $\alpha \geq \beta$ if and only if $\pi \geq \beta$.*

Proof. Assume the hypothesis and notation of the lemma. Assume further that $\pi \geq \phi$. Let $\pi_s(A) = \pi(E(s,\infty)AE(s,\infty))$ and $\phi_s(A) = \phi(E(s,\infty)AE(s,\infty))$ for $s > 0$ and $A \in \mathfrak{B}(\mathfrak{H})$. For each $s > 0$ let α^s and β^s be the minimal CP-flows derived from π_s and ϕ_s, respectively. Suppose $\pi_t^{s\#}$ and $\phi_t^{s\#}$ and are the generalized boundary representations of α^s and β^s. Note $\pi_t^{s\#} = \pi_s^{s\#} = \pi_s$ and $\phi_t^{s\#} = \phi_s^{s\#} = \phi_s$ for $t \in (0,s]$. Since $\pi_s \geq \phi_s$ we have $\alpha^s \geq \beta^s$ from Theorem 4.29. From Lemma 4.39 we have $\alpha_t^s(A) \to \alpha_t(A)$ and $\beta_t^s(A) \to \beta_t(A)$ σ-weakly as $s \to 0+$ for all $t \geq 0$ and $A \in \mathfrak{B}(\mathfrak{H})$. Since $\alpha^s \geq \beta^s$ is follows that $\alpha \geq \beta$ in the limit of $s \to 0+$.

Conversely suppose $\alpha \geq \beta$. Then $\pi_s^\# \geq \phi_s^\#$ for all $s > 0$ where $\pi^\#$ and $\phi^\#$ are the generalized boundary representations of α and β. Since the normal spines of α and β ($\pi_o^\#$ and $\phi_o^\#$, respectively) are limits of the $\pi_s^\#$ and $\phi_s^\#$ we have $\pi_o^\# \geq \phi_o^\#$. From Lemma 4.37 we have $\pi_o^\# = \pi$ and $\phi_o^\# = \phi$ so $\pi \geq \phi$. □

Lemma 4.41. *Suppose α is a CP-flow over $\mathfrak{K}$ and $\pi_o^\#$ is the normal spine of α. Suppose β is the minimal CP-flow derived from $\pi_o^\#$. Then $\alpha \geq \beta$.*

Proof. Assume the hypothesis and notation of the lemma. Let $\phi = \pi_o^\#$ and let ϕ_s be defined as in Lemma 4.39. For $s > 0$ let β^s be the minimal CP-flow derived from ϕ_s and let α^s be the minimal CP-flow derived from $\pi_s^\#$ where the family $\pi^\#$ is the generalized boundary representation of α. From Lemma 4.34 and the definition of the normal spine $\pi_o^\#$ we have $\pi_s^\# \geq \phi_s$ for each $s > 0$. Then from Lemma 4.40 we have $\alpha^s \geq \beta^s$. From Lemmas 4.38 and 4.39 we have $\alpha_t^s(A) \to \alpha_t(A)$ and $\beta_t^s(A) \to \beta_t(A)$ σ-weakly as $s \to 0+$ for all $t \geq 0$ and $A \in \mathfrak{B}(\mathfrak{H})$. Since $\alpha^s \geq \beta^s$ it follows that $\alpha \geq \beta$ in the limit as $s \to 0+$. $\square$

Theorem 4.42. *Suppose α is a CP-flow over $\mathfrak{K}$ and $\pi_o^\#$ is the normal spine of α. Suppose ϕ is a σ-weakly continuous completely positive contraction of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$ and β is the minimal CP-flow derived from ϕ. Then $\alpha \geq \beta$ if and only if $\pi_o^\# \geq \phi$.*

Proof. Assume the hypothesis and notation of the theorem. Suppose $\alpha \geq \beta$. Let $\pi_t^\#$ and $\phi_t^\#$ be the generalize boundary representations of α and β, respectively. From Theorem 4.29 we have $\pi_t^\# \geq \phi_t^\#$ for all $t > 0$. Let $\pi_o^\#$ and $\phi_o^\#$ be normal spines of α and β, respectively. Since $\pi_o^\#$ and $\phi_o^\#$ are defined in terms of limits of the $\pi_t^\#$ and $\phi_t^\#$ we have $\pi_o^\# \geq \phi_o^\#$. From Lemma 4.37 we have $\phi_o^\# = \phi$ so $\pi_o^\# \geq \phi$.

Next suppose $\pi_o^\# \geq \phi$. Let γ be the minimal CP-flow over $\mathfrak{K}$ derived from $\pi_o^\#$. From Lemma 4.41 we have $\alpha \geq \gamma$ and from Lemma 4.40 we have $\gamma \geq \beta$. Hence, $\alpha \geq \beta$. $\square$

Arveson defines the index of a unital CP-semigroup α in terms of semigroups $S(t)$ of contractions so that if $\Omega_t(A) = S(t)AS(t)^*$ for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$ then $e^{kt}\alpha_t \geq \Omega_t$. This index is of great importance since if γ is the minimal dilation of α, so γ is an E_o-semigroup then the index of γ is the Arveson index of α. The factor of e^{kt} which Arveson allowed we will eliminate by rescaling $S(t)$ with a factor of $e^{-\frac{1}{2}kt}$. The following lemmas lead up to a Theorem 4.46 which enables us to determine when a CP-semigroup of the form $\Omega_t(A)$ just given is a subordinate of α. This will enable us to show the Arveson index of a CP-flow is just the rank of the normal spine.

Lemma 4.43. *Suppose α is a CP-flow over $\mathfrak{K}$ and $S(t)$ is a strongly continuous one parameter semigroup of contractions of $\mathfrak{H}$ and $\Omega_t(A) = S(t)AS(t)^*$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$ and $\alpha_t - \Omega_t$ is positive for all $t \geq 0$. Then if $-d$ is the generator of $U(t)$ (so $U(t) = \exp(-td)$) and $-D$ is the generator of $S(t)$ there is a complex number c with non-negative real part and a linear operator V from $\mathfrak{H}$ to $\mathfrak{K}$ with norm satisfying $\|V\| \leq \sqrt{2Re(c)}$ so the domain of D is $\mathfrak{D}(D) = \{f \in \mathfrak{D}(d) : f(0) = Vf\}$ and $Df = -d^*f + cf$. (Note as we saw in the discussion of the boundary representation that each element of $\mathfrak{D}(d^*)$ has a unique representation as a continuous $\mathfrak{K}$-valued function $f(x)$ so in particular $f(0)$ is well defined.)*

Proof. Assume the hypothesis of the lemma. It follows that for all $A \in \mathfrak{B}(\mathfrak{H})$ with $A \geq 0$ and $t \geq 0$ we have from Lemma 4.1 that

$$U(t)^*S(t)AS(t)^*U(t) \leq U(t)^*\alpha_t(A)U(t) = A.$$

If A is a rank one projection and f is a units vector in the range of A if follows that $U(t)^*S(t)f = xf$ for some complex number x. Now if g is a second unit vector orthogonal to f then $U(t)^*S(t)g = yg$ and $U(t)^*S(t)(f+g) = z(f+g) = xf+yg$ with y and z complex numbers. Since f and g are orthogonal we have $x = y = z = a(t)$ where $a(t)$ is constant independent of the vector f so $U(t)^*S(t) = a(t)I$. Since both $U(t)$ and $S(t)$ are semigroups we have

$$a(t_1 + t_2)I = U(t_2)^*U(t_1)^*S(t_1)S(t_2) = a(t_1)U(t_2)^*S(t_2) = a(t_1)a(t_2)I$$

Since $a(t)$ is continuous we have $a(t) = e^{-ct}$ for all $t \geq 0$ where c is a complex number and since $U(t)$ and $S(t)$ are contractions the real part of c is non-negative. Let $W(t) = e^{ct}S(t)$. Then $U(t)^*W(t) = I$ for all $t \geq 0$. Since $S(t)AS(t)^* \leq \alpha_t(A)$ for $A \geq 0$ and $t \geq 0$ we have

$$W(t)W(t)^* = e^{2Re(c)t}S(t)S(t)^* \leq e^{2Re(c)t}\alpha_t(I)$$

and since $\|W(t)\|^2 = \|W(t)W(t)^*\|$ we have $\|W(t)\| \leq e^{Re(c)t}$ for all $t \geq 0$. Let $-T$ be the generator of $W(t)$ so $W(t) = e^{-tT}$. Suppose $f \in \mathfrak{D}(d)$ and $g \in \mathfrak{D}(T)$. Then we have

$$\frac{d}{dt}(U(t)f, W(t)g)|_{t=0} = -(df, g) - (f, Tg) = 0$$

It follows that $g \in \mathfrak{D}(d^*)$ and $Tg = -d^*g$. Hence, T is a restriction of $-d^*$. Note that for $f \in \mathfrak{D}(T)$ we have

$$\begin{aligned}\frac{d}{dt}\|W(t)f\|^2 =&(W(t)d^*f, W(t)f) + (W(t)f, W(t)d^*f)\\ =&(d^*W(t)f, W(t)f) + (W(t)f, d^*W(t)f) = \|(W(t)f)(0)\|^2\end{aligned}$$

So $s(t) = \|W(t)f\|^2$ is a function with a continuous positive derivative and since $s(0) = \|f\|^2$ and $s(t) \leq e^{2Re(c)t}\|f\|^2$ for $t \geq 0$ we have

$$\frac{d}{dt}\|W(t)f\|^2|_{t=0} = \|f(0)\|^2 \leq 2Re(c)\|f\|^2$$

For $f \in \mathfrak{D}(T) \subset \mathfrak{D}(d^*)$ the mapping $f \to Vf = f(0)$ is clearly linear and from the above inequality we have $\|V\| \leq (2Re(c))^{\frac{1}{2}}$. We show $\mathfrak{D}(T)$ consists of all $f \in \mathfrak{D}(d^*)$ so that $f(0) = Vf$. Suppose $\lambda > Re(c)$. Since $\|W(t)\| \leq e^{Re(c)t}$ for $t \geq 0$ it follows that the integral of $e^{-\lambda t}W(t)$ from zero to infinity exists and gives the inverse of $(T + \lambda I)$. Hence, we have

$$(T + \lambda I)^{-1} = \int_0^\infty e^{-\lambda t}W(t)\,dt$$

Now suppose $f \in \mathfrak{D}(d^*)$ and $f(0) = Vf$. Let $g = -d^*f + \lambda f$. Let $f_1 = (T + \lambda I)^{-1}g$. Then

$$-d^*f_1 + \lambda I f_1 = (T + \lambda I)f_1 = g = -d^*f + \lambda f.$$

Hence, $d^*(f - f_1) = \lambda(f - f_1)$. But this implies $(f - f_1)(x) = e^{-\lambda x}(f - f_1)(0)$ so we have

$$\|f(0) - f_1(0)\|^2 = 2\lambda\|f - f_1\|^2$$

Now we have

$$\|f(0) - f_1(0)\|^2 = \|V(f - f_1)\|^2 \leq 2Re(c)\|f - f_1\|^2$$

Combining these inequalities we have

$$2(\lambda - Re(c))\|f - f_1\|^2 \leq 0.$$

Since $\lambda > Re(c)$ we have $f = f_1$. Hence, we have shown that $\mathfrak{D}(T)$ consists of all $f \in \mathfrak{D}(d^*)$ so that $f(0) = Vf$. Since $S(t) = e^{-ct}W(t)$ for $t \geq 0$ the conclusion of the lemma follows. $\square$

Lemma 4.44. *Suppose D satisfies the conclusion of the previous lemma so D is defined on $\mathfrak{D}(D) = \{f \in \mathfrak{D}(d^*) : f(0) = Vf\}$ by $Df = -d^*f + cf$ where V is a linear operator from $\mathfrak{H}$ to $\mathfrak{K}$ with norm satisfying $\|V\| \leq \sqrt{2Re(c)}$ and $Re(c) \geq 0$. Then $-D$ is the generator of strongly continuous semigroup $S(t)$ of contractions and if $f \in \mathfrak{H}$ is of the form $f(x) = e^{-sx}h$ for $x \geq 0$ with $s > 0$ and $h \in \mathfrak{K}$ then $t^{-1}S(t)^*E(t)f \to V^*h$ as $t \to 0+$ and we have the uniform estimate $t^{-1}\|S(t)^*E(t)f\| \leq \|V\|\,\|h\|$ for all $h \in \mathfrak{K}$.*

Proof. The proof of the lemma can be extracted from [PP]. Since the situation is different we give a complete proof.

Suppose D satisfies the hypothesis of the lemma. Now for $f \in \mathfrak{D}(D)$ we have

$$\begin{aligned} Re(f, -Df) =& Re((f, d^*f) - c(f, f)) = \frac{1}{2}\|f(0)\|^2 - Re(c)\|f\|^2 \\ =& \frac{1}{2}\|Vf\|^2 - Re(c)\|f\|^2 \leq \frac{1}{2}(\|V\|^2 - 2Re(c))\|f\|^2 \leq 0. \end{aligned}$$

Hence, $-D$ is dissipative. We show $\mathfrak{D}(D)$ is dense in $\mathfrak{H}$. For $s > 0$ let Q_s be the isometry of from $\mathfrak{K}$ to $\mathfrak{H}$ given by $(Q_sk)(x) = \sqrt{s}ke^{-\frac{1}{2}sx}$ for $k \in \mathfrak{K}$. Then $\|VQ_s\| \leq \|V\|$ and, hence, $(I - s^{-\frac{1}{2}}VQ_s)$ is invertible for $s > \|V\|^2$. Suppose $f \in \mathfrak{D}(d)$ and $s > \|V\|^2$. Let $g = f + s^{-\frac{1}{2}}Q_s(I - s^{-\frac{1}{2}}VQ_s)^{-1}Vf$. We have $g \in \mathfrak{D}(d^*)$ and

$$\begin{aligned} Vg =& Vf - s^{\frac{1}{2}}(I - s^{-\frac{1}{2}}VQ_s)s^{-\frac{1}{2}}(I - s^{-\frac{1}{2}}VQ_s)^{-1}Vf + s^{\frac{1}{2}}s^{-\frac{1}{2}}(I - s^{-\frac{1}{2}}VQ_s)^{-1}Vf \\ =& Vf - Vf + (I - s^{-\frac{1}{2}}VQ_s)^{-1}f = g(0). \end{aligned}$$

Hence, $g \in \mathfrak{D}(D)$. Now we have

$$\|s^{-\frac{1}{2}}Q_s(I - s^{-\frac{1}{2}}VQ_s)^{-1}Vf\| \leq s^{-\frac{1}{2}}\|(I - s^{-\frac{1}{2}}VQ_s)^{-1}Vf\| \leq \frac{s^{-\frac{1}{2}}\|Vf\|}{1 - s^{-\frac{1}{2}}\|VQ_s\|}$$

and as $s \to \infty$ the above tends to zero. Hence, for each $f \in \mathfrak{D}(d)$ and each $\epsilon > 0$ there is a element $g \in \mathfrak{D}(D)$ with $\|f - g\| < \epsilon$. Since, $\mathfrak{D}(d)$ is dense in $\mathfrak{H}$ we have $\mathfrak{D}(D)$ is dense in $\mathfrak{H}$. Next we show that the range of $D + I$ is $\mathfrak{H}$. Suppose $g \in \mathfrak{H}$. If $(D + I)f = g$ then f satisfies the differential equation

$$\frac{df}{dx}(x) + (c + 1)f(x) = g(x)$$

and solving this equation we find

$$f(x) = f(0)e^{-(c+1)x} + e^{-(c+1)x}\int_0^x e^{(c+1)t}g(t)\,dt$$

or in operator form $f = Wf(0) + Bg$ where W is the operator from $\mathfrak{K}$ to $\mathfrak{H}$ given above and B is the operator from $\mathfrak{H}$ to $\mathfrak{H}$ given above. Note $B = (-d+(c+1)I)^{-1}$. Since $f \in \mathfrak{D}(D)$ we must have

$$f(0) = Vf = VWf(0) + VBg \qquad \text{or} \qquad (I - VW)f(0) = VBg$$

Now $\|W\| = (2Re(c)+2)^{-\frac{1}{2}}$ which implies

$$\|VW\| \leq (\frac{2Re(c)}{2+2Re(c)})^{\frac{1}{2}} < 1$$

so $(I - VW)$ is invertible and we find $f(0) = (I - VW)^{-1}VBg$ and the range of $D+I$ is all of $\mathfrak{H}$. Hence, $-D$ is a densely defined dissipative operator with the range of $I + D$ is $\mathfrak{H}$ and by the standard tools described in section II we have $-D$ is the generator of a strongly continuous one parameter semigroup of contractions $S(t)$.

We show $U(t)^*S(t) = e^{-ct}I$ for $t \geq 0$. Suppose $f \in \mathfrak{D}(d)$ and $g \in \mathfrak{D}(D)$. Then

$$\begin{aligned}\frac{d}{dt}(f, U(t)^*S(t)g) &= -\,(dU(t)f, S(t)g) - (U(t)f, DS(t)g)\\ &= -\,(U(t)f, d^*S(t)g) - (U(t)f, (-d^* + cI)S(t)g)\\ &= -\,c(f, U(t)^*S(t)g)\end{aligned}$$

If then follows that $(f, U(t)^*S(t)g) = e^{-ct}(f,g)$ for all $f \in \mathfrak{D}(d)$, $g \in \mathfrak{D}(D)$ and $t \geq 0$. This equation extends by continuity to all $f, g \in \mathfrak{H}$ and we find $U(t)^*S(t) = e^{-ct}I$ for all $t \geq 0$. Then we have $S(t) = E(t)S(t) + U(t)U(t)^*S(t) = E(t)S(t) + e^{-ct}U(t)$ for all $t \geq 0$. Using this we can establish the uniform estimate of the lemma. Since the range of $E(t)$ and $U(t)$ are orthogonal compliments we have $\|S(t)f\|^2 = \|E(t)S(t)f\|^2 + e^{-2Re(c)t}\|U(t)f\|^2$. Since $S(t)$ is a contraction if follows that

$$\|E(t)S(t)f\|^2 \leq \|f\|^2 - e^{-2Re(c)t}\|f\|^2$$

for all $t \geq 0$. Hence,

$$\|E(t)S(t)\| \leq (1 - e^{-2Re(c)t})^{\frac{1}{2}} \leq \sqrt{2tRe(c)}$$

for all $t \geq 0$. Now if $f(x) = e^{-sx}h$ for $x \geq 0$ with $s > 0$ we have

$$\begin{aligned}\|S(t)^*E(t)f\| \leq& \|S(t)E(t)E(t)f\| \leq (2tRe(c))^{\frac{1}{2}}\|E(t)f\|\\ \leq& \quad (2\mathrm{tRe(c)})^{\frac{1}{2}}(\frac{1-e^{-2st}}{2s})^{\frac{1}{2}}\|h\| \leq t(2Re(c))^{\frac{1}{2}}\|h\| \leq t\|V\|\,\|h\|\end{aligned}$$

for $t \geq 0$.

As in [PP] compute the action of D^* on $\mathfrak{D}(d^*)$. Suppose $f \in \mathfrak{D}(d^*)$ and $g \in \mathfrak{D}(D)$. Then we have

$$\begin{aligned}(f, Dg) =& -(f, d^*g) + c(f, g)\\ =& -(f, d^*g) - (d^*f, g) + ((d^* + \overline{c}I)f, g)\\ =& -(f(0), g(0)) + ((d^* + \overline{c}I)f, g)\\ =& -(f(0), Vg) + ((d^* + \overline{c}I)f, g)\\ =& (((d^* + \overline{c}I)f - V^*f(0)), g)\end{aligned}$$

Hence, $f \in \mathfrak{D}(D^*)$ and $D^*f = (d^* + \overline{c}I)f - V^*f(0)$. Then for $f \in \mathfrak{D}(d^*)$ we have $t^{-1}(S(t)^* - I)f \to -D^*f$ as $t \to 0+$. Hence, for $f \in \mathfrak{D}(d^*)$ we have

$$\begin{aligned}t^{-1}S(t)^*E(t)f =& t^{-1}S(t)^*(I - U(t)U(t)^*)f\\ =& t^{-1}(S(t)^*f - e^{-\overline{c}t}U(t)^*f)\\ =& t^{-1}(S(t)^*f - f - (e^{-\overline{c}t}U(t)^*f - f))\\ &\to -D^*f - (-d^* - \overline{c}I)f = V^*f(0)\end{aligned}$$

As $t \to 0+$. Now if $f(x) = e^{-sx}h$ for $x \geq 0$ with $s > 0$ and $h \in \mathfrak{K}$ we have $f \in \mathfrak{D}(d^*)$ and $f(0) = h$. $\square$

Lemma 4.45. *Suppose $\{S(t) : t \geq 0\}$ is a strongly continuous semigroup of contractions of $\mathfrak{H} = \mathfrak{K} \otimes L^2(0, \infty)$ satisfying the conclusion of the Lemma 4.43 so $S(t) = e^{-tD}$ where the domain of D is given by $\mathfrak{D}(D) = \{f \in \mathfrak{D}(d^*) : f(0) = Vf\}$ and $Df = -d^*f + cf$ and V is a linear operator from $\mathfrak{H}$ to $\mathfrak{K}$ with norm satisfying $\|V\| \leq \sqrt{2Re(c)}$. We assume further that $Re(c) > 0$ and $\|V\| < \sqrt{2Re(c)}$. For $t > 0$ let*

$$\beta_t(A) = (1 - e^{-2Re(c)t})^{-1}E(t)S(t)AS(t)^*E(t) + U(t)AU(t)^*$$

for $A \in \mathfrak{B}(\mathfrak{H})$. Then for $A \in \mathfrak{B}(\mathfrak{H})$ we have $(\beta_{t/n})^n(A) \to \gamma_t(A)$ in the strong operator topology as $n \to \infty$ for each $t > 0$ where γ is the minimal CP-semigroup of $\mathfrak{B}(\mathfrak{H})$ derived from the completely positive normal map $\pi(A) = (2Re(c))^{-1}VAV^*$ as defined in Definition 4.25 and constructed in Theorem 4.26.

Proof. Assume the hypothesis and notation of the lemma. We have made the further hypothesis that $\|V\| < \sqrt{2Re(c)}$. With this additional assumption $\|\pi(A)\| \leq (1 - \epsilon)\|A\|$ for $A \in \mathfrak{B}(\mathfrak{H})$ with $\epsilon > 0$ and by Theorem 4.26 there is only one CP-semigroup γ derived from π and, furthermore, the geometric series occurring in the calculations we need converge.

We use the ingenious inequalities of Chernoff [Ch] and let $D_n(A) = n(\beta_{s/n}(A) - A)$. Note D_n is the generator of a semigroup given by

$$\beta_t^{(n)}(A) = \exp(tD_n)(A) = e^{-tn}\sum_{k=0}^{\infty}\frac{t^k n^k (\beta_{s/n})^n(A)}{k!}$$

We note each of the term in the above series is completely positive and, hence, $\beta_t^{(n)}$ is completely positive. Evaluating $\beta_t^{(n)}(I)$ we see that $0 \leq \beta_t^{(n)}(I) \leq I$ so $\beta_t^{(n)}$ is a contraction. Using Chernoff's inequality (see Lemma 3.1.11 of [BR]) we have

$$\|\beta_1^{(n)}(A) - (\beta_{s/n})^n(A)\| \leq \sqrt{n}\|\beta_{s/n}(A) - A\|$$

for $A \in \mathfrak{B}(\mathfrak{H})$. For a typical operator $\|\beta_{s/n}(A) - A\|$ is the order of one as $n \to \infty$ so the above inequality is not very helpful. However, at this point it is profitable to work on the predual and the same inequality holds there, namely,

$$\|\hat{\beta}_1^{(n)}(\eta) - (\hat{\beta}_{s/n})^n(\eta)\| \leq \sqrt{n}\|\hat{\beta}_{s/n}(\eta) - \eta\| \tag{4.22}$$

for all $\eta \in \mathfrak{B}(\mathfrak{H})_*$. Let γ be the minimal CP-semigroup which is intertwined by $U(t)$ derived from π. Since $\|\pi\| < 1$ it follows from Theorem 4.26 that γ is the unique CP-semigroup derived from π. Let δ be the generator of γ and $\hat{\delta}$ be the generator $\hat{\gamma}$ (the action of γ on the predual). We establish the key estimate of the lemma which says that $\hat{D}_n\eta \to s\hat{\delta}(\eta)$ as $n \to \infty$ for all $\eta \in \mathfrak{D}(\hat{\delta})$. Our arguments draw heavily on Theorem 4.26 and we assume the notation used in that theorem is in effect. Let $\rho \to \hat{\sigma}(\rho)$ be the integrated boundary map which generates γ. Since γ is derived from π we have $\hat{\sigma}(\rho - \hat{\Lambda}(\hat{\pi}(\rho))) = \hat{\Gamma}(\hat{\pi}(\rho))$ for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$. Since $\|\pi\| < 1$ the mapping $\rho \to \rho - \hat{\Lambda}(\hat{\pi}(\rho))$ is invertible and we have

$$\hat{\sigma}(\rho) = \hat{\Gamma}(\hat{\pi}(\rho) + \hat{\pi}(\hat{\Lambda}(\hat{\pi}(\rho))) + \cdots)$$

where the geometric series converges for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$. From the definition of $\hat{\sigma}$ we have each element of $\mathfrak{D}(\hat{\delta})$ is of the form $\hat{\sigma}(\hat{\Lambda}(\eta)) + \hat{\Gamma}(\eta)$ for some $\eta \in \mathfrak{B}(\mathfrak{H})_*$ and

$$\hat{\delta}(\hat{\sigma}(\hat{\Lambda}(\eta)) + \hat{\Gamma}(\eta)) = \hat{\sigma}(\hat{\Lambda}(\eta)) + \hat{\Gamma}(\eta) - \eta.$$

It follow that each element of $\mathfrak{D}(\hat{\delta})$ is of the form $\hat{\Gamma}(\nu)$ and

$$\hat{\delta}(\hat{\Gamma}(\nu)) = \hat{\Gamma}(\nu) - \eta$$

where

$$\nu = \eta + \hat{\pi}(\hat{\Lambda}(\eta)) + \hat{\pi}(\hat{\Lambda}(\hat{\pi}(\hat{\Lambda}(\eta)))) + \cdots$$

for some $\eta \in \mathfrak{B}(\mathfrak{H})_*$. The above equation for ν is equivalent to the equation $\eta = \nu - \hat{\pi}(\hat{\Lambda}(\nu))$. We compute $\hat{D}_n(\hat{\Gamma}(\nu))$. From direct calculation for $A \in \mathfrak{B}(\mathfrak{H})$ we have

$$\begin{aligned}\hat{D}_n(\hat{\Gamma}(\nu))(A) =& n(1 - e^{-2Re(c)s/n})^{-1}\nu(\Gamma(E(s/n)S(s/n)AS(s/n)^*E(s/n))) \\ &- ne^{s/n}\int_0^{s/n} e^{-t}\nu(U(t)AU(t)^*)\,dt + n(e^{s/n} - 1)\nu(\Gamma(A)).\end{aligned}$$

Then rewriting this purely in terms elements of $\mathfrak{B}(\mathfrak{H})_*$ we have

$$\begin{aligned}\hat{D}_n(\hat{\Gamma}(\nu)) =& n(1 - e^{-2Re(c)s/n})^{-1}\hat{\psi}_{s/n}(\hat{\zeta}_{s/n}(\hat{\Gamma}(\nu))) \\ &- ne^{s/n}\int_0^{s/n} e^{-t}\hat{\theta}_t(\nu)\,dt + n(e^{s/n} - 1)\hat{\Gamma}(\nu)\end{aligned}$$

where $\psi(A) = S(t)AS(t)^*$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. It is clear that the second and third term above converge to $s\nu$ and $s\hat{\Gamma}(\nu)$ in norm as $n \to \infty$. From Lemma 4.19 we have $\|\hat{\zeta}_t(\hat{\Gamma}(\nu) - \hat{\Phi}(\hat{\Lambda}(\nu)))\|/t \to 0$ as $t \to 0+$ so we have

$$\begin{aligned}\hat{D}_n(\hat{\Gamma}(\nu)) =& n(1 - e^{-2Re(c)s/n})^{-1}\hat{\psi}_{s/n}(\hat{\zeta}_{s/n}(\hat{\Phi}(\hat{\Lambda}(\nu)))) \\ &- s\nu + s\hat{\Gamma}(\nu) + o(n)\end{aligned}$$

as $n \to \infty$. Now $\hat{\Lambda}(\nu)$ has a decomposition so that for $A \in \mathfrak{B}(\mathfrak{K})$

$$\hat{\Lambda}(\nu)(A) = \sum_{i=1} \lambda_i(h_i, Ak_i)$$

where $h_i, k_i \in \mathfrak{K}$ are unit vectors and $\lambda_i > 0$ for $i = 1, 2, \cdots$ and the sum of the λ_i is bounded. Then

$$\hat{\Phi}(\hat{\Lambda}(\nu))(A) = \sum_{i=1} \lambda_i(f_i, Ag_i)$$

for $A \in \mathfrak{B}(\mathfrak{H})$ where $f_i(x) = e^{-\frac{1}{2}x}h_i$ and $g_i(x) = e^{-\frac{1}{2}x}k_i$ for all $x \in [0, \infty)$ for $i = 1, 2, \cdots$. Then we have

$$n(1 - e^{-2Re(c)s/n})^{-1}\hat{\psi}_{s/n}(\hat{\zeta}_{s/n}(\hat{\Phi}(\hat{\Lambda}(\nu)))) = \sum_{i=1} \eta_i^n$$

where

$$\eta_i^n(A) = n(1 - e^{-2Re(c)s/n})^{-1}\lambda_i(S(s/n)^*E(s/n)f_i, AS(s/n)E(s/n)g_i)$$

for $A \in \mathfrak{B}(\mathfrak{H})$ and $i = 1, 2, \cdots$. From the previous Lemma 4.44 we have

$$(n/s)\|S(s/n)^*E(s/n)f_i\| \leq \|V\|\,\|h_i\| \leq \|V\| < \sqrt{2Re(c)}$$

for $i = 1, 2, \cdots$ and the same estimate applies with the f_i replaced by g_i. Also, from Lemma 4.44 we have

$$(n/s)S(s/n)^*E(s/n)f_i \to V^*h_i \qquad \text{and} \qquad (n/s)S(s/n)^*E(s/n)g_i \to V^*k_i$$

as $n \to \infty$ for $i = 1, 2, \cdots$. Hence, $\eta_i^n \to \eta_i^\infty$ as $n \to \infty$ where

$$\eta_i^\infty(A) = \frac{s}{2Re(c)}\lambda_i(V^*h_i, AV^*k_i)$$

for $A \in \mathfrak{B}(\mathfrak{H})$ and we have the uniform estimate that $\|\eta_i^n\| < s$ independent of n for $i = 1, 2, \cdots$. Since the sum of the λ_i converges and with our uniform estimate and the convergence for each i = 1,2,$\cdots$ and the definition of π we have

$$n(1 - e^{-2Re(c)s/n})^{-1}\hat{\psi}_{s/n}(\hat{\zeta}_{s/n}(\hat{\Phi}(\hat{\Lambda}(\nu)))) \to s\hat{\pi}(\hat{\Lambda}(\nu))$$

as $n \to \infty$. Hence, we have $\hat{D}_n(\hat{\Gamma}(\nu)) \to s\hat{\pi}(\hat{\Lambda}(\nu)) - s\nu + s\hat{\Gamma}(\nu)$ as $n \to \infty$. Recalling $\eta = \nu - \hat{\pi}(\hat{\Lambda}(\nu))$ we have that

$$\hat{D}_n(\hat{\Gamma}(\nu)) \to -s\eta + s\hat{\Gamma}(\nu) = s\hat{\delta}(\hat{\Gamma}(\nu))$$

as $n \to \infty$. Hence, $\hat{D}_n(\eta) \to s\hat{\delta}(\eta)$ for all $\eta \in \mathfrak{D}(\hat{\delta})$. Then from Chernoff's inequality (4.22) we have

$$\|\hat{\beta}_1^{(n)}(\eta) - (\hat{\beta}_{s/n})^n(\eta)\| \leq \sqrt{n}\|\hat{\beta}_{s/n}(\eta) - \eta\| = \|D_n(\eta)\|/\sqrt{n} \to 0$$

as $n \to \infty$ for all $\eta \in \mathfrak{D}(\hat{\delta})$. Now by standard convergence arguments we have for $\eta \in \mathfrak{D}(\hat{\delta})$ that

$$\hat{\beta}_1^{(n)}(\eta) - \hat{\gamma}_s(\eta) = \int_0^1 \hat{\beta}_t^{(n)}((D_n - s\hat{\delta})(\hat{\gamma}_{st}(\eta)))\,dt.$$

Since the integrand is uniformly bounded and converges pointwise to zero in norm we have $\|\hat{\beta}_1^{(n)}(\eta) - \hat{\gamma}_s(\eta)\| \to 0$ as $n \to \infty$. Combining this with our previous inequality we have $\|(\hat{\beta}_{s/n})^n(\eta) - \hat{\gamma}_s(\eta)\| \to 0$ as $n \to \infty$ for all $\eta \in \mathfrak{D}(\hat{\delta})$. Since $\mathfrak{D}(\hat{\delta})$ is dense in $\mathfrak{B}(\mathfrak{H})_*$ and the mappings are uniformly bounded we have $\|(\hat{\beta}_{s/n})^n(\eta) - \hat{\gamma}_s(\eta)\| \to 0$ as $t \to 0$ for all $\eta \in \mathfrak{B}(\mathfrak{H})_*$ and this immediately gives us σ-strong convergence on $\mathfrak{B}(\mathfrak{H})$. $\square$

The next theorem gives a relatively computable condition that a CP-semigroup α of $\mathfrak{B}(\mathfrak{H})$ intertwined by $U(t)$ dominates Ω_t with $\Omega_t(A) = S(t)AS(t)^*$ for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$.

Theorem 4.46. *Suppose α is a CP-flow over $\mathfrak{K}$ and $S(t)$ is a strongly continuous one parameter semigroup and $\Omega_t(A) = S(t)AS(t)^*$ for $t \geq 0$ and $A \in \mathfrak{B}(\mathfrak{H})$ is a subordinate of α. Then $S(t)$ is a strongly continuous one parameter semigroup of contractions with generator $-D$ where $\mathfrak{D}(D) = \{f \in \mathfrak{D}(d^*) : f(0) = Vf\}$ and $Df = -d^*f + cf$ where c is a complex number with non-negative real part and V is a linear operator from $\mathfrak{H}$ to $\mathfrak{K}$ with norm satisfying $\|V\|^2 \leq 2Re(c)$. Furthermore, if $\pi(A) = (2Re(c))^{-1}VAV^*$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and γ is the minimal CP-semigroup derived from π then α dominates γ. In the case $Re(c) = 0$ we take define $\pi = 0$.*

Conversely, if c is a complex number with $Re(c) > 0$ and V is a linear operator from $\mathfrak{H}$ to $\mathfrak{K}$ with norm satisfying $\|V\|^2 \leq 2Re(c)$ and if $\pi(A) = (2Re(c))^{-1}VAV^*$ for $A \in \mathfrak{B}(\mathfrak{H})$ and γ is the minimal CP-semigroup derived from π and α dominates γ then if D is an operator with domain $\mathfrak{D}(D) = \{f \in \mathfrak{D}(d^*) : f(0) = Vf\}$ and $Df = -d^*f + cf$. Then $-D$ is the generator of a contraction semigroup $S(t)$ and if $\Omega_t(A) = S(t)AS(t)^*$ for $t \geq 0$ and $A \in \mathfrak{B}(\mathfrak{H})$ and α dominates Ω.

Proof. Suppose the hypothesis and notation of the first paragraph of the theorem is satisfied. Then it follow from Lemma 4.43 that $S(t)$ is a strongly continuous one parameter semigroup of contractions with generator $-D$ where $\mathfrak{D}(D) = \{f \in \mathfrak{D}(d^*) : f(0) = Vf\}$ and $Df = -d^*f + cf$ where c is a complex number with non-negative real part and V is a linear operator from $\mathfrak{H}$ to $\mathfrak{K}$ with norm satisfying $\|V\|^2 \leq 2Re(c)$. Let $\pi(A) = (2Re(c))^{-1}VAV^*$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and let γ be the minimal CP-semigroup derived from π. (In case Re(c) $= 0$ we define $\pi = 0$.) Here we make a slight change. Note if we replace c with $c + \epsilon$ with $\epsilon > 0$ we replace $S(t)$ with $e^{-\epsilon t}S(t)$ and all the hypothesis concerning Ω_t remains true. Note with this change we have $\|\pi\| < 1$. In what follows we will assume this replacement of c with $c + \epsilon$ has been made.

Let $A \in \mathfrak{B}(\mathfrak{H})$ be a positive. Then we have

$$S(t)AS(t)^* \leq \alpha_t(A) = E(t)\alpha_t(A)E(t) + U(t)AU(t)^*$$

Since $U(t)^*S(t) = e^{-ct}I$ we have

$$S(t) = U(t)U(t)^*S(t) + E(t)S(t) = e^{-ct}U(t) + E(t)S(t)$$

Combining this with the previous inequality we have

$$E(t)(\alpha_t(A) - S(t)AS(t)^*)E(t) - e^{-ct}E(t)S(t)AU(t)^* \\ - e^{-ct}U(t)AS(t)^*E(t) + (1 - e^{-2Re(c)t})U(t)AU(t)^* \geq 0$$

If X denotes the operator above and $h = U(t)g + E(t)f$ then we have

$$(h, Xh) =(E(t)f, (\alpha_t(A) - S(t)AS(t)^*)E(t)f) - 2Re(e^{-ct}(f, E(t)S(t)Ag)) \\ + (1 - e^{-2Re(c)t})(g, Ag) \geq 0.$$

for all $f, g \in \mathfrak{H}$. Then by the Schwarz inequality the above inequality is satisfied if and only if

$$|(f, E(t)S(t)Ag)|^2 \leq (e^{2Re(c)t} - 1)(E(t)f, (\alpha_t(A) - S(t)AS(t)^*)E(t)f)(g, Ag)$$

for all $f, g \in \mathfrak{H}$. Specializing this inequality to the case when $A = E$ with E an hermitian rank one projection and g is a unit vector in the range of E (so $Eg = g$ and $Ef = (g, f)f$ for $f \in \mathfrak{H}$) we find

$$e^{2Re(c)t}|(f, E(t)S(t)g)|^2 \leq (e^{2Re(c)t} - 1)(E(t)f, \alpha_t(E)E(t)f)$$

for all $f \in \mathfrak{H}$. Then we find

$$\|ES(t)^*E(t)f\|^2 \leq (1 - e^{-2Re(c)t})(E(t)f, \alpha_t(E)E(t)f)$$

for all $f \in \mathfrak{H}$ and this is equivalent to the operator inequality

$$E(t)((1 - e^{-2Re(c)t})\alpha_t(E) - S(t)ES(t)^*)E(t) \geq 0.$$

Now if $A \in \mathfrak{B}(\mathfrak{H})$ is of the form $A = \sum_{i=1}^n \lambda_i E_i$ where the E_i are hermitian rank one projections and the $\lambda_i > 0$ for $i = 1, \cdots, n$ then it follows from the above inequality that

$$E(t)((1 - e^{-2Re(c)t})\alpha_t(A) - S(t)AS(t)^*)E(t) \geq 0.$$

And since every positive $A \in \mathfrak{B}(\mathfrak{H})$ can be approximated in the σ-strong topology by expressions of the above form it follows that the above inequality is valid for all positive $A \in \mathfrak{B}(\mathfrak{H})$. For $t \geq 0$ let β_t be the map

$$\beta_t(A) = (1 - e^{-2Re(c)t})^{-1}E(t)S(t)AS(t)^*E(t) + U(t)AU(t)^*$$

for $A \in \mathfrak{B}(\mathfrak{H})$. Note that

$$\alpha_t(A) - \beta_t(A) = E(t)(\alpha_t(A) - (1 - e^{-2Re(c)t})^{-1}S(t)AS(t)^*)E(t)$$

for $A \in \mathfrak{B}(\mathfrak{H})$. We have shown that $\alpha_t - \beta_t$ is positive for each $t \geq 0$. As was done in the last section we can introduce the primed maps which are obtained by replacing $\mathfrak{H}$ with $\mathfrak{H} \otimes \mathfrak{K}_o$ with $\mathfrak{K}_o$ in infinite dimensional separable Hilbert space and making the obvious definitions. Since for each $t \geq 0$ we have $A \to \alpha_t(A) - \Omega_t(A)$ is completely positive the mapping $A \to \alpha'_t(A) - \Omega'_t(A)$ is positive and the argument that $\alpha_t - \beta_t$ is positive extends directly to the primed maps and we find that $\alpha'_t - \beta'_t$ is positive and this is equivalent to the fact that $\alpha_t - \beta_t$ is completely positive. Now for each $t > 0$ we have

$$\alpha_t(A) - (\beta_{t/n})^n(A) = \sum_{k=1}^{n} (\alpha_{t/n})^{k-1}((\alpha_{t/n} - \beta_{t/n})(\beta_{t/n})^{n-k}(A))$$

for $A \in \mathfrak{B}(\mathfrak{H})$. Since the mappings above are all completely positive we have $A \to \alpha_t(A) - (\beta_{t/n})^n(A)$ is completely positive. Now from Lemma 4.5 we have $(\beta_{t/n})^n(A) \to \gamma_t(A)$ σ-strongly as $n \to \infty$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. Here, is where we needed the fact that we had replaced c with $c+\epsilon$ since we needed $\|\pi\| < 1$. Taking the limit as $n \to \infty$ we have the mapping $A \to \alpha_t(A) - \gamma_t(A)$ is completely positive for all $t \geq 0$ and, hence, α dominates γ.

Now we deal with the fact that we replaced c with $c + \epsilon$. Let us now denote the dependence of π and γ on ϵ by writing π^ϵ and γ^ϵ. We have shown that α dominates γ^ϵ for all $\epsilon > 0$ and γ^ϵ is the unique CP-semigroup derived from π^ϵ. As was shown in the proof of Theorem 4.26 we have $\gamma^\epsilon_t(A)$ converges σ-weakly $\gamma^t_o(A)$ for each $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$ where γ^o is the minimal CP-semigroup γ^o derived from π^o. Hence, we have α dominates γ where γ is the minimal CP-semigroup derived from π (now $\epsilon = 0$). This completes the proof of the implication of the theorem in one direction.

Now suppose the hypothesis and notation of the second paragraph of the statement of the theorem is satisfied. As we did in the first part of the proof of this theorem we replace c by $c + \epsilon$ with $\epsilon > 0$ in the definition of D (and, therefore, $S(t)$) and π. Again the hypothesis remain true after this replacement. For $t > 0$ let

$$\beta_t(A) = (1 - e^{-2Re(c)t})^{-1} E(t)S(t)AS(t)^*E(t) + U(t)AU(t)^*$$

and $\beta_o(A) = A$ for $A \in \mathfrak{B}(\mathfrak{H})$. Since $S(t) = E(t)S(t) + e^{-ct}U(t)$ we have

$$(\beta_t(A) - \Omega_t(A)) = (e^{2Re(c)t} - 1)^{-1} B(t)AB(t)^*$$

with

$$B(t) = E(t)S(t) - (e^{2Re(c)t} - 1)e^{-ct}U(t)$$

from which it follows that the mapping $A \to \beta_t(A) - \Omega_t(A)$ is completely positive for all $t \geq 0$. Note $\Omega_t(\Omega_s(A)) = \Omega_{t+s}(A)$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t, s \geq 0$. Then we have

$$\begin{aligned}(\beta_{t/n})^n(A) - \Omega_t(A) =&(\beta_{t/n})^n(A) - (\Omega_{t/n})^n(A)\\ =&\sum_{k=1}^{n} (\beta_{t/n})^{k-1}((\beta_{t/n} - \Omega_{t/n})((\Omega_{t/n})^{n-k}(A)))\end{aligned}$$

for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$ and n a positive integer. Since all of the mappings in the above sum are completely positive we have the mapping $A \to (\beta_{t/n})^n(A) - \Omega_t(A)$

is completely positive. From Lemma 4.5 we have $(\beta_{t/n})^n(A) \rightarrow \gamma_t(A)$ σ-strongly as $n \rightarrow \infty$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. Hence, $\gamma \geq \Omega$.

Now we deal with the fact that we replaced c by $c + \epsilon$. Again we denote the dependence of γ_t, Ω_t and π on ϵ by writing γ_t^ϵ, Ω_t^ϵ and $\pi^\in$. Then we have shown that $\gamma^\epsilon \geq \Omega^\epsilon$ where γ^ϵ is the unique CP-semigroup derived from π^ϵ. As $\epsilon \rightarrow 0+$ we have from the proof of Theorem 4.26 that $\gamma_t^\epsilon(A) \rightarrow \gamma_t^o(A)$ in the σ-strong topology for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$ where γ^o is the minimal CP-semigroup derived from π^o. Since $\Omega_t^\epsilon(A) \rightarrow \Omega_t^o(A)$ as $\epsilon \rightarrow 0+$ for each $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$ in the σ-strong topology we have $\gamma^o \geq \Omega^o$. Or with $\epsilon = 0$ we have $\gamma \geq \Omega$. Since $\alpha \geq \gamma$ we have $\alpha \geq \Omega$. $\square$

Before we show how to compute the Arveson index of a CP-flow we make a simple definition and prove a useful lemma.

Definition 4.47. Suppose α is a CP-semigroup and β is a subordinate of α. We say β is trivially maximal if $\beta'_t = e^{st}\beta_t$ for $t \geq 0$ with $s > 0$ then β' is not a subordinate of α.

Note that if β is a subordinate of α there is a unique subordinate β' of the form $\beta'_t = e^{st}\beta_t$ for $t \geq 0$ with $s \geq 0$ so that β' is a trivially maximal subordinate of α. In discussing subordinates it is often useful to consider trivially maximal subordinates.

Lemma 4.48. *Suppose α is a spatial E_o-semigroup of $\mathfrak{B}(\mathfrak{H})$ and β is an extremal subordinate of α which is trivially maximal (where by extremal we mean every subordinate γ of β is of the form $\gamma_t = e^{-st}\beta_t$ for all $t \geq 0$ where $s \geq 0$). Then there is a strongly continuous one parameter semigroup of isometries $S(t)$ which intertwine α_t for each $t \geq 0$ so that*

$$\beta_t(A) = S(t)S(t)^*\alpha_t(A) = \alpha_t(A)S(t)S(t)^* = S(t)AS(t)^*$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$.

Proof. Assume the hypothesis of the lemma. Since α is spatial there is a strongly continuous one parameter semigroup of isometries $U(t)$ which intertwine α_t for each $t \geq 0$. As we saw in Theorem 3.4 there is a local cocycle C so that $\beta_t(A) = C(t)\alpha_t(A)$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. Let $\gamma_t(A) = C(t)^2\alpha_t(A)$ for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. Since $0 \leq C(t) \leq I$ we have $0 \leq C(t)^2 \leq C(t)$ for all $t \geq 0$ so γ is a subordinate of β and since β is extremal we have $\gamma_t = e^{-st}\beta_t$ for all $t \geq 0$ with $s \geq 0$. Hence, $C(t)^2 = e^{-st}C(t)$ for all $t \geq 0$. Hence, $Q(t) = e^{st}C(t)$ is a projection valued local cocycle so if $\eta_t(A) = Q(t)\alpha_t(A)$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$ then η is a subordinate of α and β is a subordinate of η. Since β is trivially maximal if follows that $\eta = \beta$ and $C(t)$ is a projection for all $t \geq 0$.

Next consider $R(t) = C(t)U(t)$. We have

$$\begin{aligned} R(t)R(s) =& C(t)U(t)C(s)U(s) = C(t)\alpha_t(C(s))U(t)U(s) \\ =& C(t+s)U(t+s) = R(t+s) \end{aligned}$$

for $t, s \geq 0$. Note that $R(t)$ intertwines α_t for each $t \geq 0$ since

$$R(t)A = C(t)U(t)A = C(t)\alpha_t(A)U(t) = \alpha_t(A)C(t)U(t) = \alpha_t(A)R(t).$$

We note $R(t)^*R(t)$ commutes with A for all $A \in \mathfrak{B}(\mathfrak{H})$ since

$$R(t)^*R(t)A = R(t)^*\alpha_t(A)R(t) = AR(t)^*R(t).$$

Hence, $R(t)^*R(t)$ is a multiple of the identity and from the semigroup property of R we have $R(t)^*R(t) = e^{-st}I$ for $t \geq 0$ where $s \geq 0$. Let $S(t) = e^{\frac{1}{2}st}R(t)$ for $t \geq 0$. We see $S(t)$ is a strongly continuous one parameter semigroup of isometries which intertwines α_t for $t \geq 0$. Note $t \to F(t) = S(t)S(t)^*$ is local cocycle so ν given by $\nu_t(A) = F(t)\alpha_t(A)$ for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$ is a subordinate of α. Since

$$F(t) = S(t)S(t)^* = e^{st}C(t)U(t)U(t)^*C(t)$$

and $C(t)$ for $t \geq 0$ are projections we see that $F(t) \leq C(t)$ and, hence, ν is a subordinate of β. Since β is extremal we have $\nu_t = e^{-at}\beta_t$ for $t \geq 0$ with $a \geq 0$. Since $C(t)$ and $F(t)$ are projections we have $a = 0$ and $\beta = \nu$. Hence,

$$\beta_t(A) = S(t)S(t)^*\alpha_t(A) = \alpha_t(A)S(t)S(t)^* = S(t)AS(t)^*$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. $\square$

Theorem 4.49. *The Arveson index of a unital CP-flow α which is equal to the index of the minimal dilation α^d of α to an E_o-semigroup is the rank (given in Definition 3.2) of the normal spine of α.*

Proof. Suppose α is a CP-flow over $\mathfrak{K}$. Arveson's definition of the index of a CP-semigroup involves identifying the semigroups $S(t)$ so that $\Omega_t(A) = S(t)AS(t)^*$ for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$ is subordinate of α and computing the covariance of two such semigroups. From Theorem 4.46 we can easily identify such semigroups but computing the covariance is not something we know how to do easily. The Arveson index of α is equal to the index of the minimal dilation α^d of α to an E_o-semigroup. (This was the point of Arveson's definition.) So to prove the corollary we will show the index of the minimal dilation α^d of α is the rank of the normal spine of α.

Suppose α^d is the minimal dilation of α to an E_o-semigroup and π_o is the normal spine of α. Recalling the relation between α and α^d as described in the last section we have α^d is an E_o-semigroup of $\mathfrak{B}(\mathfrak{H}_1)$ and W is a isometry of $\mathfrak{H}$ into $\mathfrak{H}_1$ so that WW^* is an increasing projection for α^d and α^d is minimal over the range of W and $\alpha_t(A) = W^*\alpha_t^d(WAW^*)W$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. Now it was shown in [P4] (see section 4) that E_o-semigroup α^d is of index p if and only if there are $p+1$ minimal projective local cocycles F_i for $i = 0, 1, \cdots, p$ which are lattice independent and maximal in that one can not add another minimal projective local cocycle and maintain lattice independence. In greater detail F is a projective local cocycle if $F(t)$ is a projection valued local cocycle (i.e. $F(t)\alpha_t^d(F(s)) = F(t+s)$ and $F(t) \in \alpha_t^d(\mathfrak{B}(\mathfrak{H}_1))'$ for $t, s \geq 0$). And F is minimal if G is a projective local cocycle so that $0 \leq G(t) \leq F(t)$ for $t \geq 0$ then $G(t) = F(t)$ for all $t \geq 0$. The projective local cocycles F_i are lattice independent if the suprema of the F_i for $i \neq j$ is not greater than F_j.

In the language of subordinates a minimal projective local cocycle F for α^d corresponds to an extremal subordinate γ of α^d which are trivially maximal. This means F is a minimal projective local cocycle for α^d if and only if the mapping $\gamma_t(A) = F(t)\alpha_t^d(A)$ for $A \in \mathfrak{B}(\mathfrak{H}_1)$ and $t \geq 0$ is an extremal subordinate of α^d. Note that extremal subordinates of α^d which are trivially maximal correspond to minimal projective local cocycle as was shown in Lemma 4.48.

Now we use Theorem 4.46 and the order isomorphism of Theorem 3.5 which gives us an order isomorphism from the extremal subordinates of α^d to the extremal subordinates of α. Suppose the normal spine π_o of α is of finite rank p. This means π_o is of the form

$$\pi_o(A) = \sum_{i=1}^{p} C_i A C_i^*$$

for $A \in \mathfrak{B}(\mathfrak{H})$ where the C_i are linearly independent operators from $\mathfrak{H}$ to $\mathfrak{K}$ for $i = 1, \cdots, p$. Let $\phi_i(A) = C_i A C_i^*$ for $i = 1, \cdots, p$ and let $\phi_o(A) = 0$ for $A \in \mathfrak{B}(\mathfrak{H})$. Let D_i be the operator with domain $\mathfrak{D}(D_i) = \{f \in \mathfrak{D}(d^*) : f(0) = C_i f\}$ and $Df = -d^* f + \frac{1}{2} f$ for $i = 1, \cdots, p$ and let $D_o = d$. Let β_i be the minimal CP-flow over $\mathfrak{K}$ derived from ϕ_i for $i = 0,1,\cdots, r$. Since $\pi_o \geq \phi_i$ we have from Theorem 4.42 that $\alpha \geq \beta_i$ for each $i = 0, 1, \cdots, p$. And from Theorem 4.46 we have D_i is the generator of a contraction semigroup $S_i(t)$ and $\Omega_{it}(A) = S_i(t) A S_i(t)^*$ for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$ is a subordinate of α for $i = 0, 1, \cdots, p$. It is clear that the Ω_i are extremal subordinates of α which are trivially maximal. The fact that the Ω_i are lattice independent may be seen as follows. Let η_i be the suprema of the Ω_j with $j \leq i$ for $i = 0, 1, \cdots, p$. Note η_i is the minimal CP-flow over $\mathfrak{K}$ derived from $\phi_o + \phi_1 + \cdots + \phi_i$. We see then for each $i = 1, \cdots, p$ we have η_i is strictly greater than η_{i-1}. If the Ω_i were not lattice independent we would have $\eta_i = \eta_{i-1}$ for some $i = 1, \cdots, p$. Note η_p is the CP-flow over $\mathfrak{K}$ derived from π_o. Now suppose β is an extremal subordinate of α which is trivially maximal. From the order isomorphism of Theorem 3.4 there is an extremal subordinate γ of α^d

$$\beta_t(A) = W^* \gamma_t(W A W^*) W$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. Since γ is extremal we have from Lemma 4.48 there is a strongly continuous semigroup of isometries $S_1(t)$ which intertwine α_t^d for each $t \geq 0$ so that $\gamma_t(A) = S_1(t) A S_1(t)^*$. Hence, we have $\beta_t(A) = W^* S_1(t) W A W^* S_1(t)^* W$ for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. Since WW^* is an increasing projection for α^d it follows that $S_1(t)^*$ maps the range of W into itself. This is seen as follows. Suppose $t \geq 0$. Then $S_1(t) W W^* = \alpha_t^d(WW^*) S_1(t)$ and taking adjoints and multiplying by WW^* on the right we find

$$WW^* S_1(t)^* WW^* = S_1(t)^* \alpha_t^d(WW^*) WW^* = S_1(t)^* WW^*$$

so $S_1(t)^*$ maps the range of W into itself. It follows that $S(t)^* = W^* S_1(t)^* W$ is a strongly continuous semigroup of contractions since

$$\begin{aligned} S(t)^* S(s)^* =& W^* S_1(t)^* W W^* S_1(s)^* W = W^* S_1(t)^* S_1(s)^* W \\ =& W^* S_1(t+s)^* W = S(t+s)^* \end{aligned}$$

for $s, t \geq 0$. Hence, $\beta_t(A) = S(t) A S(t)^*$ for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. Since β is a subordinate of α Theorem 4.46 applies to the semigroup $S(t)$ and the generator of

$S(t)$ has to satisfy certain conditions regarding the normal spine π_o of α. But π_o is also the normal spine of η_d and therefore $S(t)$ satisfies these same continuous for the normal spine of η_d and, hence, by Theorem 4.46 we have that β is a subordinate of η_d. Hence, we have shown that every non zero extremal subordinate of α is a subordinate of η_d so we have proved the subordinates β_i for $i = 0, 1, \cdots, p$ are a lattice independent set and maximal in the sense that any other extremal subordinate β of α is a subordinate of η_d the suprema of the β_i for $i = 0, 1, \cdots, p$. Hence, the index of α^d is p and the Arveson index of α is p. In the case where the normal spine π_o is of infinite rank the above argument shows that the index of α^d is greater than any positive integer so the index of α^d is infinite. $\square$

In the next lemma we show that if α is a unital CP-flow over $\mathfrak{K}$ and α^d is the minimal dilation of α to an E_o-semigroup then α^d is a CP-flow over $\mathfrak{K}_1$.

Lemma 4.50. *Suppose α is a unital CP-flow over $\mathfrak{K}$ and α^d is the minimal dilation of α to an E_o-semigroup and suppose the relation between α and α^d is given by*

$$\alpha_t(A) = W^*\alpha_t^d(WAW^*)W$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ (with $\mathfrak{H} = \mathfrak{K} \otimes L^2(0,\infty)$) and $t \geq 0$ where W is an isometry from $\mathfrak{H}$ to $\mathfrak{H}_1$ and WW^* is an increasing projection for α^d and α^d is minimal over the range of W. Then $\mathfrak{H}_1$ can be expressed as $\mathfrak{H}_1 = \mathfrak{K}_1 \otimes L^2(0,\infty)$ and α^d is a CP-flow over $\mathfrak{K}_1$ so that if $U(t)$ and $U_1(t)$ are right translation on $\mathfrak{H}$ and $\mathfrak{H}_1$ for α and α^d, respectively, then $U_1(t)W = WU(t)$ and $U_1(t)^*W = WU(t)^*$ for all $t \geq 0$. This means W as a mapping of $\mathfrak{H} = \mathfrak{K} \otimes L^2(0,\infty)$ into $\mathfrak{H}_1 = \mathfrak{K}_1 \otimes L^2(0,\infty)$ can be expressed the form $W = W_1 \otimes I$ where W_1 is an isometry from $\mathfrak{K}$ into $\mathfrak{K}_1$.

Proof. Assume the hypothesis and notation of the lemma are satisfied. Since α^d is minimal over the range of W the linear combination of vectors of the form

$$\alpha_{t_1}^d(WA_1W^*)\cdots\alpha_{t_n}^d(WA_nW^*)Wf$$

with $f \in \mathfrak{K}$ and $A_i \in \mathfrak{B}(\mathfrak{H})$ and $t_i \geq 0$ for $i = 1, \cdots, n$ and $n = 1,2,\cdots$ are dense in $\mathfrak{H}_1$. For $t \geq 0$ we define $U_1(t)$ on a vectors of the above form by the equation

$$\begin{aligned} U_1(t)\alpha_{t_1}^d(WA_1W^*)&\cdots\alpha_{t_n}^d(WA_nW^*)Wf \\ &= \alpha_{t_1+t}^d(WA_1W^*)\cdots\alpha_{t_n+t}^d(WA_nW^*)WU(t)f. \end{aligned}$$

By expressing the inner product of such vectors in terms of α and using the fact that $U(t)$ intertwines α_t one first checks that $(U_1(t)F, U_1(t)G) = (F,G)$ for F and G vectors of the above form. Then it follows that these relations uniquely define an isometry $U_1(t)$ of $\mathfrak{H}_1$. Recalling how α^d is defined as explained in the last section we can show that $U_1(t)$ intertwines α_t^d so $U_1(t)A = \alpha_t^d(A)U_1(t)$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. Next it follows from the above equation that $U_1(t)W = WU(t)$ for all $t \geq 0$. We note that

$$U_1(t)^*WU(t) = U_1(t)^*U_1(t)W = W = WU(t)^*U(t)$$

for all $t \geq 0$. It follow that $U_1(t)^*W = WU(t)^*$ for all $t \geq 0$ if and only if $B(t) = U_1(t)^*W(I - U(t)U(t)^*) = 0$ for all $t \geq 0$. We find

$$\begin{aligned} B(t)^*B(t) =&(I - U(t)U(t)^*)W^*U_1(t)U_1(t)^*W(I - U(t)U(t)^*) \\ =&W^*U_1(t)U_1(t)^*W - U(t)U(t)^* \end{aligned}$$

We show $B(t)^*B(t) = 0$. Since $U_1(t)^*\alpha_t^d(A) = AU_t(t)^*$ we have

$$W^*U_1(t)U_1(t)^*\alpha_t^d(WAW^*)W = W^*U_1(t)WAW^*U_1(t)^*W = U(t)AU(t)^*$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. Then setting $A = I$ in this equation and noting that since WW^* is an increasing projection for α^d we have $\alpha_t^d(WW^*)W = W$ so the above equation gives

$$W^*U_1(t)U_1(t)^*W = U(t)U(t)^*$$

for all $t \geq 0$. Hence, $B(t) = 0$ for all $t \geq 0$ and $U_1(t)^*W = WU(t)^*$ for all $t \geq 0$.

Next we show $U_1(t)^* \to 0$ strongly as $t \to \infty$. Since $U_1(t)$ intertwines α_t^d we have for $t_i \geq 0,\ A_i \in \mathfrak{B}(\mathfrak{H})$ for $i = 1, \cdots, n$ and $t \geq t_1$ that

$$\begin{aligned} &U_1(t)^*\alpha_{t_1}^d(WA_1W^*)\cdots\alpha_{t_n}^d(WA_nW^*)Wf \\ &\qquad = U_1(t-t_1)^*WA_1W^*U_1(t_1)^*\alpha_{t_2}(A_2)\cdots\alpha_{t_n+t}^d(WA_nW^*)WU(t)f \\ &\qquad = WU(t-t_1)^*A_1W^*U_1(t_1)^*\alpha_{t_2}(A_2)\cdots\alpha_{t_n+t}^d(WA_nW^*)WU(t)f. \end{aligned}$$

Since $U(t)^* \to 0$ strongly as $t \to \infty$ we have the above expression tends to zero in norm as $t \to \infty$. Since the linear span of vectors of the above form is dense in $\mathfrak{H}_1$ it follows that $U_1(t)^* \to 0$ strongly as $t \to \infty$. Hence, $U_1(t)$ is a pure shift. Since $U_1(t)$ is a pure shift we can realize $\mathfrak{H}_1$ in the form $\mathfrak{K}_1 \otimes L^2(0,\infty)$ where $U_1(t)$ is right translation by t for $t \geq 0$. The details of this realization are as follows. Let M_1 be the von Neumann algebra generated by $U_1(t)$ for $t \geq 0$. Since the action of the right shift operators $S(t)$ are irreducible on $L^2(0,\infty)$ we have M_1 can be identified with $\mathfrak{B}(L^2(0,\infty))$ and $U(t)$ corresponds to the right shift $S(t)$ for all $t \geq 0$. Since M_1 is a type I factor its commutant M_1' is a type I factor which we identify as $\mathfrak{B}(\mathfrak{K}_1)$. In this way we realize $\mathfrak{H}_1 = \mathfrak{K}_1 \otimes L^2(0,\infty)$ and $U_1(t)$ as the right shift by t on $\mathfrak{H}_1$ for all $t \geq 0$. Similarly in the realization of $\mathfrak{H}$ as $\mathfrak{H} = \mathfrak{K} \otimes L^2(0,\infty)$ we let M be the von Neumann algebra generated by $U(t)$. Since W intertwines the action of $U_1(t)$ and $U_1(t)^*$ with $U(t)$ and $U(t)^*$ we see that if ϕ is the natural isomorphism of M with M_1 induced by identifying $U(t)$ with $U_1(t)$ we see that $(I \otimes \phi(A))W = W(I \otimes A)$ for all $A \in M$. Note WW^* is in the commutant of M_1 so WW^* corresponds to a projection in $\mathfrak{B}(\mathfrak{K}_1)$. Let $\mathfrak{N}$ be the subspace of $\mathfrak{K}_1$ corresponding to the range of this projection. Now if we simply think of ϕ as the identity map by which we mean we identify $U_1(t) = I \otimes S(t)$ and $U(t) = I \otimes S(t)$ where in the first case I is the unit of $\mathfrak{B}(\mathfrak{K}_1)$ and in the second case I is the unit of $\mathfrak{B}(\mathfrak{K})$ for $t \geq 0$ then W is of the form $W = W_1 \otimes I$ where W_1 is an isometry of $\mathfrak{K}$ into $\mathfrak{K}_1$ with range $\mathfrak{N}$. This completes the proof of the lemma. $\square$

Theorem 4.51. *Suppose α is a unital CP-flow over $\mathfrak{K}$ and α^d is the minimal dilation of α to an E_o-semigroup and suppose the relation between α and α^d is given by*

$$\alpha_t(A) = W^*\alpha_t^d(WAW^*)W$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ (with $\mathfrak{H} = \mathfrak{K} \otimes L^2(0,\infty)$) and $t \geq 0$ where W is an isometry from $\mathfrak{H}$ to $\mathfrak{H}_1$ and WW^* is an increasing projection for α^d and α^d is minimal over the range of W. Suppose $S(t)$ is a strongly continuous semigroup of contractions of $\mathfrak{H}$ and Ω given by $\Omega_t(A) = S(t)AS(t)^*$ for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$ is a subordinate of α. Further assume Ω is trivially maximal. Then there is a unique strongly continuous one parameter semigroup of isometries $S_1(t)$ which intertwine α_t^d for each $t \geq 0$ and

$$S(t) = W^*S_1(t)W$$

for all $t \geq 0$.

Conversely, if $S_1(t)$ is a strongly continuous one parameter semigroup of isometries which intertwine α_t^d for each $t \geq 0$ then if $S(t)$ is as defined in the equation above we have $S(t)$ is a strongly continuous one parameter semigroup of contractions so that Ω defined by $\Omega_t(A) = S(t)AS(t)^*$ for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$ is a subordinate of α which is trivially maximal.

Proof. Assume the hypothesis and notation for α and α^d is in effect. The second paragraph in the statement of the theorem was established in the proof of Theorem 4.49.

Suppose the hypothesis of the first paragraph of the lemma is satisfied. Since Ω is an extremal subordinate of α which is trivially maximal it follows from the order isomorphism of Theorem 3.5 that there is an extremal subordinate γ of α^d which is trivially maximal and $\Omega_t(A) = W^*\gamma_t(WAW^*)W$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. From Lemma 4.48 we have γ is of the form

$$\gamma_t(A) = S_1(t)S_1(t)^*\alpha_t^d(A) = S_1(t)AS_1(t)^*$$

for all $A \in \mathfrak{B}(\mathfrak{H}_1)$ and $t \geq 0$ where $S_1(t)$ is a strongly continuous one parameter semigroup of isometries which intertwine α_t^d for each $t \geq 0$. Note since γ is uniquely determined by Ω we have the semigroup $S_1(t)$ is uniquely determined except for a unitary phase factor (i.e., the semigroup $S_1'(t) = e^{ist}S_1(t)$ for $t \geq 0$ with s real gives the same γ). We have

$$S(t)AS(t)^* = \Omega_t(A) = W^*S_1(t)WAW^*S_1(t)W$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. It follows that $S(t) = e^{ist}W^*S_1(t)W$. for all $t \geq 0$ where s is real. Now replacing $S_1(t)$ with $e^{ist}S_1(t)$ we have $S(t) = W^*S_1(t)W$ and we have established the connection between $S(t)$ and $S_1(t)$ as stated in the theorem. □

Theorem 4.52. *Suppose α is a unital CP-flow over $\mathfrak{K}$ and α^d is the minimal dilation of α to an E_o-semigroup. Then α^d is completely spatial if and only if α is the minimal CP-flow derived from $\pi_o^\#$ the normal spine of α.*

Proof. Assume the notation of the theorem. As shown in the last paragraph of section 4 of [P3] a spatial E_o-semigroup is completely spatial if an only if it is least upper bound of its extremal subordinates (in [P3] these extremal subordinates were called minimal compressions). From Theorem 3.5 we know there is an order isomorphism from the subordinates of α^d with the subordinates of α. Hence, α^d is completely spatial if and only if α is the least upper bound of its extremal subordinates.

Let γ be the least upper bound of the extremal subordinates of α and let ϕ be the normal spine of γ. Since the extremal subordinates of α are of the form $\beta_t(A) = S(t)AS(t)^*$ for $t \geq 0$ (see Lemma 4.48 and Theorem 4.51) with $S(t)$ a strongly continuous one parameter semigroup and if β is such a CP-semigroup then $\gamma \geq \beta$ is determined only by ϕ (see Theorems 4.42 and 4.46) it follows that γ must be the minimal CP-flow derived from ϕ. To see this simply replace γ by the CP-flow derived from ϕ and we have a CP-flow γ' with $\gamma \geq \gamma'$ and γ' is still an upper bound for the extremal subordinates of α. Then it follows that ϕ is the least upper bound of all subordinates of $\pi_o^\#$ of the form $\pi(A) = CAC^*$ for $A \in \mathfrak{B}(\mathfrak{H})$ with C an operator from $\mathfrak{H}$ to $\mathfrak{K}$. Since ϕ is an upper bound we have $\phi \geq \pi_o^\#$ and since ϕ is a least upper bound we have $\pi_o^\# \geq \phi$ so $\phi = \pi_o^\#$. Hence, we have shown that the least upper bound of the extremal subordinates of α is the minimal CP-flow derived from $\pi_o^\#$. Hence, α is the least upper bound of its extremal subordinate if and only if α is the minimal CP-flow derived from $\pi_o^\#$. $\square$

As in Theorem 3.14 of the last section we characterize corners for CP-flows. We begin with a definition.

Definition 4.53. Suppose α and β are CP-flows over $\mathfrak{K}_1$ and $\mathfrak{K}_2$, respectively. We say γ is a flow corner from α to β if γ is a one parameter family of σ-weakly continuous maps γ_t of $\mathfrak{B}(\mathfrak{H}_2)$ to $\mathfrak{B}(\mathfrak{H}_1)$ (with $\mathfrak{H}_i = \mathfrak{K}_i \otimes L^2(0,\infty)$ for $i = 1, 2$) so that

$$\Theta_t(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}) = \begin{bmatrix} \alpha_t(A_{11}) & \gamma_t(A_{12}) \\ \gamma_t^*(A_{21}) & \beta_t(A_{22}) \end{bmatrix}$$

for $t \geq 0$ and A_{ij} is a bounded linear operator from $\mathfrak{H}_j$ to $\mathfrak{H}_i$ is a CP-flow over $\mathfrak{K}_1 \oplus \mathfrak{K}_2$ where the translation operator $U(t)$ on $(\mathfrak{K}_1 \oplus \mathfrak{K}_2) \otimes L^2(0,\infty)$ is given by

$$U(t) = \begin{bmatrix} U_1(t) & 0 \\ 0 & U_2(t) \end{bmatrix}$$

for $t \geq 0$ where U_i is the translation operator on $\mathfrak{H}_i = \mathfrak{K}_i \otimes L^2(0,\infty)$ for $i = 1, 2$.

Theorem 4.54. *Suppose α is a unital CP-flow over $\mathfrak{K}$ and α^d is the minimal dilation of α to an E_o-semigroup and suppose the relation between α and α^d is a stated in Lemma 4.50. We assume the notation of the Lemma 4.50. Suppose C is an α^d contractive local cocycle so that $C(t)U_1(t) = U_1(t)$ for $t \geq 0$. Then*

$$\gamma_t(A) = W^*C(t)\alpha_t^d(WAW^*)W$$

for $t \geq 0$ and $A \in \mathfrak{B}(\mathfrak{H})$ is a flow corner from α to α. Conversely, if γ is a flow corner from α to α then there is a unique α^d contractive local cocycle C so that $C(t)U_1(t) = U_1(t)$ and

$$\gamma_t(A) = W^*C(t)\alpha_t^d(WAW^*)W$$

for $A \in \mathfrak{B}(\mathfrak{H})$ for all $t \geq 0$.

Proof. Assume the notation and set up of the theorem. Suppose C is an α^d contractive local cocycle and $C(t)U_1(t) = U_1(t)$ for $t \geq 0$ and γ is given in terms of C as stated in the theorem. Then we have

$$\begin{aligned}\gamma_t(A)U(t) &= W^*C(t)\alpha_t^d(WAW^*)WU(t) = W^*C(t)\alpha_t^d(WAW^*)U_1(t)W \\ &= W^*C(t)U_1(t)WAW^*W = W^*U_1(t)WAW^*W \\ &= U(t)W^*WAW^*W = U(t)A\end{aligned}$$

for $t \geq 0$ and $A \in \mathfrak{B}(\mathfrak{H})$. Hence, $U(t)$ intertwines γ_t so we see that Θ as defined in terms of γ in Definition 4.53 is a CP-flow.

Conversely, suppose γ is a flow corner from α to α. Since γ is a corner from α to α we have from Theorem 3.14 that there is a α^d cocycle C so that

$$\gamma_t(A) = W^*C(t)\alpha_t^d(WAW^*)W$$

for $t \geq 0$ and $A \in \mathfrak{B}(\mathfrak{H})$. Since γ is flow corner from α to α we have $U(t)A = \gamma_t(A)U(t)$ for all $t \geq 0$ and $A \in \mathfrak{B}(\mathfrak{H})$. And setting $A = I$ in the equation yields the result that

$$\begin{aligned}\gamma_t(I)U(t) &= W^*C(t)\alpha_t^d(WW^*)WU(t) = W^*C(t)\alpha_t^d(WW^*)U_1(t)W \\ &= W^*C(t)U_1(t)WW^*W = W^*C(t)U_1(t)W = U(t)\end{aligned}$$

for $t \geq 0$. Note $S_1(t) = C(t)U_1(t)$ is a strongly continuous semigroup which intertwines α^d and $W^*S_1(t)W = S(t) = U(t)$ for $t \geq 0$, where we introduce $S(t) = U(t)$ to recall the notation of Theorem 4.51. Note if $\Omega_t(A) = S(t)AS(t)^*$ for $t \geq 0$ and $A \in \mathfrak{B}(\mathfrak{H})$ then Ω is a subordinate of α and applying Theorem 4.51 we see that S_1 is uniquely determined from $S = U$ so $S_1(t) = U_1(t)$ for $t \geq 0$. Hence, $C(t)U_1(t) = U_1(t)$ for $t \geq 0$. $\square$

One ambiguity that occurs with flow corner comes with the definition of maximal and hyper maximal flow corners. In the definition of maximal and hyper maximal we speak of the subordinates Θ' of Θ (see Definition 3.7). The question is do we means subordinates of Θ or do we mean flow subordinates of Θ which are subordinates which are also CP-flows. The next lemma shows that the subordinates Θ' are necessarily CP-flows. This means that for flow corners the two notions of maximal or hyper maximal are equivalent.

Lemma 4.55. *Suppose α and β are CP-semigroups over $\mathfrak{H}_1 = \mathfrak{K}_1 \otimes L^2(0,\infty)$ and $\mathfrak{H}_2 = \mathfrak{K}_2 \otimes L^2(0,\infty)$, respectively. Let $U_i(t)$ be translation on $\mathfrak{H}_i$ for $t \geq 0$ and $i = 1,2$. Suppose γ is a corner from α to β. with the property that $U_1(t)A = \gamma_t(A)U_2(t)$ for all $A \in \mathfrak{B}(\mathfrak{H}_1,\mathfrak{H}_2)$ and $t \geq 0$ (so γ is a flow corner). Then α and β are CP-flows.*

Proof. Assume the hypothesis and notation of the lemma. Let Θ and U be defined as in the above Definition 4.53 and let $\Psi_t(A) = U(t)^*\Theta_t(A)U(t)$ for $t \geq 0$ and $A \in \mathfrak{B}(\mathfrak{H}_1 \oplus \mathfrak{H}_2)$. Then we find

$$\Psi_t(\begin{bmatrix} XX^* & X \\ X^* & X^*X \end{bmatrix}) = \begin{bmatrix} U_1(t)^*\alpha_t(XX^*)U_1(t) & X \\ X^* & U_2(t)^*\beta_t(X^*X)U_2(t) \end{bmatrix}$$

for all partial isometries X from $\mathfrak{H}_2$ to $\mathfrak{H}_1$. Note the diagonal entries in the above matrix are positive contractions since Θ is a CP-semigroup. Since Θ_t is completely positive the matrix on the right hand side of the above equation must also be positive. One checks that this implies $U_1(t)^*\alpha_t(E)U_1(t) \geq E$ for all projections $E \in \mathfrak{B}(\mathfrak{H}_1)$ and for all $t \geq 0$. Since α_t is a contraction we have $U_1(t)^*\alpha_t(I)U_1(t) = I$ and using additivity we find $U_1(t)^*\alpha_t(A)U_1(t) = A$ for all projections $A \in \mathfrak{B}(\mathfrak{H}_1)$ and for all $t \geq 0$. By linearity this extends to all $A \in \mathfrak{B}(\mathfrak{H}_1)$. Now fix $t > 0$ and let $\phi(A) = \alpha_t(A)$ for $A \in \mathfrak{B}(\mathfrak{H}_1)$ and let $V = U_1(t)$. Note $V^*\phi(A)V = A$ for all $A \in \mathfrak{B}(\mathfrak{H}_1)$. Since ϕ is completely positive we have

$$\phi(A) = \sum_{i \in I} S_i A S_i^*$$

for $A \in \mathfrak{B}(\mathfrak{H}_1)$ and the S_i are linearly independent over $\ell^2(\mathbb{N})$. Since $V^*\phi(A)V = A$ for $A \in \mathfrak{B}(\mathfrak{H}_1)$ we have V^*S_i is a multiple of the unit operator for all $i \in I$. Then with a change of basis we can rewrite the sum for ϕ with a new set of S_i where $V^*S_i = 0$ except for $i = 1$ and $V^*S_1 = I$. Since V is an isometry and S_1 is a contraction it follows that $S_1 = V$. Then we have

$$\phi(A) = VAV^* + \sum_{i \in J} S_i A S_i^*$$

for $A \in \mathfrak{B}(\mathfrak{H}_1)$ where J is the index set I with the index $i = 1$ removed. Since $V^*S_j = 0$ for $j \in J$ we have $VA = \phi(A)V$ for $A \in \mathfrak{B}(\mathfrak{H}_1)$. Hence, α is a CP-flow. The same argument shows β is a CP-flow. □

Theorem 4.56. *Suppose α and β are unital CP-flows over $\mathfrak{K}_1$ and $\mathfrak{K}_2$ and α^d and β^d are the minimal dilations of α and β to E_o-semigroups. Suppose γ is a hyper maximal flow corner from α to β. Then α^d and β^d are cocycle conjugate. Conversely, if α^d is a type II_o and α^d and β^d are cocycle conjugate then there is a hyper maximal flow corner from α to β.*

Proof. The first statement of the theorem is just an application of Theorem 3.13. Assume the hypothesis and notation of the last statement of the theorem. We know from Lemma 4.50 that the relation between the CP-flows and the dilated E_o-semigroups is given by

$$\alpha_t(A) = W_1^*\alpha_t^d(W_1AW_1^*)W_1 \qquad \text{and} \qquad \beta_t(B) = W_2^*\beta_t^d(W_2BW_2^*)W_2$$

for $t \geq 0$ and $W_1, W_2, , A$ and B are operators on the appropriate Hilbert spaces with the properties described in Lemma 4.50. Now α^d and β^d are cocycle conjugate and mapping that establishes the cocycle conjugacy maps one parameter semigroups of intertwining isometries for α^d onto one parameter semigroups of intertwining isometries for β^d. Since α^d and β^d are type II_o there is only one semigroup of intertwining isometries up to multiplication by a phase factor. This means that the corner which establishes the cocycle conjugacy for α^d and β^d is after multiplication by a phase factor e^{ist} a flow corner and taking things back to the original CP-semigroups this gives us a hyper maximal flow corner from α to β. □

An important question in the theory of CP-flows is whether two CP-flows dilate to cocycle conjugate E_o-semigroups if and only if there is a hyper maximal flow corner from one to the other. The previous theorem shows the implication one way and both ways in the type II_o case. It would be very nice to know if the implication goes both ways in the type II_n case with $n \geq 0$. If follows from the papers of Alevras ([Al1],[Al2]) this is a question of whether there a unitary local cocycles for and E_o-semigroup maps that maps one semigroup of intertwining isometries onto any other semigroup of intertwining isometries.

In the last section we defined $(n \times n)$-matrices of corners. There is the corresponding notion of flow corners.

Definition 4.57. Suppose α is a CP-flow over $\mathfrak{K}$ and n is positive integer. We say Θ is a positive $(n \times n)$-matrix of flow corners from α to α if Θ is a matrix with coefficients $\theta^{(ij)}$ where the $\theta^{(ij)}$ are strongly continuous semigroups of $\mathfrak{B}(\mathfrak{H})$ for $i, j = 1, \cdots, n$ so that Θ is a CP-flow over $(\oplus_{i=1}^{n} \mathfrak{K})$ and the diagonal entries of Θ are subordinates of α.

Definition 4.58. Suppose α^d is CP-flow over $\mathfrak{K}$ which is also a E_o-semigroup of $\mathfrak{B}(\mathfrak{H})$ with $\mathfrak{H} = \mathfrak{K} \otimes L^2(0, \infty)$ and n is a positive integer. We say C is a positive $(n \times n)$-matrix of α^d contractive local flow cocycles if the coefficients C_{ij} of C are contractive local cocycles for α^d for $i, j = 1, \cdots, n$ which fix the translations $U(t)$ meaning $C_{ij}(t)U(t) = U(t)$ and the matrix $C(t)$ whose entries are $C_{ij}(t)$ is positive for all $t \geq 0$.

We remark if C is a contractive local flow cocycle then C^* is also a contractive local flow cocycle. This is seen as follows. Suppose C is a contractive local cocycle for α^d and $C(t)U(t) = U(t)$ for $t \geq 0$. Then we have

$$
\begin{aligned}
&(C(t)^*U(t) - U(t))^*(C(t)^*U(t) - U(t)) \\
&\qquad = U(t)^*C(t)C(t)^*U(t) - U(t)^*C(t)U(t) - U(t)^*C(t)^*U(t) + I \\
&\qquad = U(t)^*C(t)C(t)^*U(t) - I \leq U(t)^*U(t) - I = 0
\end{aligned}
$$

Since the above expression is positive it must be zero so $C(t)^*U(t) = U(t)$ for $t \geq 0$.

Theorem 4.59. *Suppose α is a unital CP-flow over $\mathfrak{K}$ and α^d is its dilation to an E_o-semigroup α^d of $\mathfrak{B}(\mathfrak{H}_1)$. The relation between α and α^d is given by*

$$\alpha_t(A) = W^*\alpha_t^d(WAW^*)W$$

as described in Lemma 4.50.

Suppose n is a positive integer and Θ is positive $(n \times n)$-matrix of flow corners from α to α. Then there is a unique positive $(n \times n)$-matrix C of contractive local flow cocycles C_{ij} for α^d for $i, j = 1, \cdots, n$ so that

$$\theta_t^{(ij)}(A) = W^*C_{ij}(t)\alpha_t^d(WAW^*)W$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. Conversely, if C is a positive $(n \times n)$-matrix of contractive local flow cocycles for α^d then the matrix Θ whose coefficients $\theta^{(ij)}$ are give above is a positive $(n \times n)$-matrix of flow corners from α to α.

Proof. Once one sees that there is a one to one mapping from flow corners from α to α and flow cocycles for α^d as was established in Theorem 4.54 the theorem follows from Theorem 3.16. $\square$

Theorem 4.60. *Suppose α is a unital CP-flow over $\mathfrak{K}$ and α^d is its dilation to an E_o-semigroup and the relation between α and α^d is as given in the previous theorem. Suppose θ is a flow corner from α to α and C is the local contractive flow cocycle for α^d associated with θ. Then $C(t)$ is an isometry for all $t \geq 0$ if and only if θ is maximal and $C(t)$ is unitary for all $t \geq 0$ if and only if θ is hyper maximal.*

Proof. The proof is the same as the proof of Corollary 3.17 taking into Theorem 4.54 □

We think the next theorem is a surprising result. It is a basically a corollary of Theorem 4.15.

Theorem 4.61. *Suppose α is a CP-flow over $\mathfrak{K}$ and $U(t)$ is translation on $\mathfrak{H} = \mathfrak{K} \otimes L^2(0, \infty)$. Suppose γ is a corner from α to α so that*

$$U(t)A = e^{zt}\gamma_t(A)U(t)$$

for $t \geq 0$ and $A \in \mathfrak{B}(\mathfrak{H})$ where z is a complex number with non negative real part. Let $\beta_t = e^{zt}\gamma_t$ for $t \geq 0$. Then β is a flow corner from α to α.

Proof. Assume the hypothesis and notation of the theorem. Suppose α^d the dilation of α to an E_o-semigroup and the relation between α and α^d is as stated in Lemma 4.50. We assume all the notation of the statement of Lemma 4.50. From Theorem 3.16 there is a local contractive α^d cocycle so that

$$\gamma_t(A) = W^*C(t)\alpha_t^d(WAW^*)W$$

for all $t \geq 0$ and $A \in \mathfrak{B}(\mathfrak{H})$. From Lemma 4.50 we have

$$\begin{aligned} U(t) =& e^{zt}\gamma_t(I)U(t) \\ =& e^{zt}W^*C(t)\alpha_t^d(WW^*)WU(t) \\ =& e^{zt}W^*C(t)\alpha_t^d(WW^*)U_1(t)W \\ =& e^{zt}W^*C(t)U_1(t)W \end{aligned}$$

for all $t \geq 0$. One checks that $S(t) = e^{zt}C(t)U_1(t)$ is a one parameter semigroup that intertwines α_t^d for $t \geq 0$. Since $S(t)$ intertwines α_t^d we have $S(t)^*S(t)$ commutes with $\mathfrak{B}(\mathfrak{H}_1)$ so $S(t)^*S(t)$ is a multiple of the unit for $t \geq 0$. Then $S(t) = e^{st}V(t)$ for $t \geq 0$ where V is semigroup of intertwining isometries for α^d. Since $U(t) = W^*S(t)W$ it follows from Theorem 4.51 that $S(t) = U_1(t)$ for $t \geq 0$. Hence, $C(t)U_1(t) = e^{-zt}U_1(t)$ for $t \geq 0$. Now let $D(t) = e^{2xt}C(t)^*C(t)$ for $t \geq 0$ where x is the real part of z. and let $\Theta_t(A) = D(t)\alpha_t^d(A)$ for $A \in \mathfrak{B}(\mathfrak{H}_1)$. One sees Θ is a CP_κ-flow with growth bound $\kappa = 2x$. Then by Theorem 4.15 we have Θ is a CP-flow so $\|D(t)\| \leq 1$ for all $t \geq 0$. Hence, $e^{zt}C(t)$ is a contractive flow cocycle and β is a flow corner from α to α. □

For the case of type II$_o$ CP-flows this theorem is very useful in calculating the local cocycles for the dilated E_o-semigroup. It says they can all be obtained by analyzing flow corners.

Next we will present some results which show that the relation between a CP-flow and its normal spine $\pi_o^\#$ is not as simple as one would expect. Since $\pi_s^\#$ determines $\pi_t^\#$ for $t > s$ for a generalized boundary representation one gets the impression that the limit $\pi_o^\#$ of a boundary representation $\pi_s^\#$ as $s \to 0+$ determines the CP-flow. The next theorem shows the situation is quite delicate. This theorem shows that if π is a completely positive contraction of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$ and $\Delta = \lim_{n\to\infty}(\pi\cdot\Lambda)^n(I)$ is not zero then there are CP-flows over $\mathfrak{K}$ derived from π other than the minimal one constructed in Theorem 4.26. This theorem is of importance because it shows that the boundary representation does not completely specify the CP-flow. The generalized boundary representation contains more information than the boundary representation.

Theorem 4.62. *Suppose π is a completely positive contraction from $\mathfrak{B}(\mathfrak{H})$ to $\mathfrak{B}(\mathfrak{K})$. Note $(\pi\cdot\Lambda)^{n+1}(I) = (\pi\cdot\Lambda)^n(\pi(\Lambda)) \leq (\pi\cdot\Lambda)^n(I)$ so $(\pi\cdot\Lambda)^n(I)$ is a decreasing sequence of positive operators which then must converge strongly to a limit Δ as $n\to\infty$. Suppose Δ is not zero. Suppose ν is an positive element of $\mathfrak{B}(\mathfrak{H})_*$ with $\nu(I)\leq 1$ and*

$$\omega_o = \nu + \hat{\pi}(\hat{\Lambda}(\nu)) + \hat{\pi}(\hat{\Lambda}(\hat{\pi}(\hat{\Lambda}(\nu)))) + \cdots$$

where the sum converges as a weight on $(I-\Lambda)^{\frac{1}{2}}\mathfrak{B}(\mathfrak{H})(I-\Lambda)^{\frac{1}{2}}$. Let $\rho\to\omega(\rho)$ be the mapping given by

$$\omega(\rho) = \frac{\rho(\Delta)}{(1-\nu(\Lambda(\Delta)))}\omega_o + \hat{\pi}(\rho) + \hat{\pi}(\hat{\Lambda}(\hat{\pi}(\rho))) + \hat{\pi}(\hat{\Lambda}(\hat{\pi}(\hat{\Lambda}(\hat{\pi}(\rho))))) + \cdots$$

for all $\rho\in\mathfrak{B}(\mathfrak{K})_*$. Then the mapping $\rho\to\omega(\rho)$ is a boundary weight mapping of a CP-flow α and α is derived from π. Furthermore, if $\nu(I)=1$ then α is unital.

Proof. Assume the hypothesis and notation of the theorem apply. Since Δ is not zero we have $\|\Delta\|>0$. Note $\pi(\Lambda(\Delta))=\Delta$. We have $(\pi\cdot\Lambda)^n(\|\Delta\|I-\Delta) = \|\Delta\|(\pi\cdot\Lambda)^n(I)-\Delta \to (\|\Delta\|-1)\Delta$ and since the limit is positive we have $\|\Delta\|\geq 1$ and since π and Λ are contractions we have $\|\Delta\|\leq 1$ so we have $\|\Delta\|=1$. The arguments of Theorem 4.26 show the series for $\omega(\rho)$ and ω_o converge as weights. Suppose $\lambda\in(0,1)$ and let ϕ^λ be the mapping of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$ given by

$$\hat{\phi}^\lambda(\rho) = \lambda\hat{\pi}(\rho) + (1-\lambda)\rho(\Delta)\nu$$

for $\rho\in\mathfrak{B}(\mathfrak{K})_*$. It is clear that ϕ^λ is completely positive so to check that ϕ^λ is a contraction we need only check ϕ^λ on the unit. One easily checks that for positive $\rho\in\mathfrak{B}(\mathfrak{K})_*$ we have $\hat{\phi}^\lambda(\rho)(I)\leq\rho(I)$ and since $\nu(I)\leq 1$ we have $\nu(\Lambda(\Delta))<1$ so $\|\hat{\Lambda}\hat{\phi}^\lambda\|<1$. Hence, the CP-flow derived from ϕ^λ is unique and its boundary weight map is given by

$$\omega^\lambda(\rho) = \hat{\phi}^\lambda(\rho) + \hat{\phi}^\lambda\hat{\Lambda}\hat{\phi}^\lambda(\rho) + \hat{\phi}^\lambda\hat{\Lambda}\hat{\phi}^\lambda\hat{\Lambda}\hat{\phi}^\lambda(\rho) + \cdots$$

for $\rho\in\mathfrak{B}(\mathfrak{K})_*$. Computing the series which converges in norm we find

$$\omega^\lambda(\rho) = \frac{\rho(\Delta)}{(1-\nu(\Lambda(\Delta)))}\omega_o^\lambda + \lambda\hat{\pi}(\rho) + \lambda^2\hat{\pi}(\hat{\Lambda}(\hat{\pi}(\rho))) + \lambda^3\hat{\pi}(\hat{\Lambda}(\hat{\pi}(\hat{\Lambda}(\hat{\pi}(\rho))))) + \cdots$$

for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ where

$$\omega_o^\lambda = \nu + \lambda\hat{\pi}(\hat{\Lambda}(\nu)) + \lambda^2\hat{\pi}(\hat{\Lambda}(\hat{\pi}(\hat{\Lambda}(\nu)))) + \cdots .$$

Following the argument of Theorem 4.26 we can take the limit as $\lambda \to 1-$ obtain the mapping $\rho \to \omega(\rho)$ given in the statement of the theorem and we find the limit inequality (4.13+) of Theorem 4.20 is satisfied. Hence, the mapping $\rho \to \omega(\rho)$ is the boundary weight map of a CP-flow α where now we have set $\lambda = 1$. Since $\Delta = \pi(\Lambda(\Delta))$ we have

$$\omega(\rho - \hat{\Lambda}(\hat{\pi}(\rho))) = \rho(\Delta - \pi(\Lambda(\Delta)))\omega_o + \hat{\pi}(\rho) = \hat{\pi}(\rho)$$

so from Theorem 4.24 we have that α is derived from π.

Note that if $\nu(I) = 1$ and $\rho \in \mathfrak{B}(\mathfrak{K})_*$ is positive we have

$$\omega(\rho)(I - \Lambda) = \rho(\Delta)\frac{\nu(I - \Lambda(\Delta))}{1 - \nu(\Lambda(\Delta))} + \rho(I - \Delta) = \rho(I)$$

so in this case α is unital. $\square$

We show that the previous theorem is not vacuous in that there are examples of representations π where Δ is not zero. Let $\mathfrak{K}$ be the infinite tensor product of $L^2(0,\infty)$ so $\mathfrak{K} = \otimes_{k=1}^\infty L^2(0,\infty)$ with the reference vector (see [vN] for details of infinite tensor products of Hilbert spaces)

$$F_o = \lambda_1 e^{-\frac{1}{2}\lambda_1^2 x} \otimes \lambda_2 e^{-\frac{1}{2}\lambda_2^2 x} \otimes \cdots$$

and where $\lambda_i > 0$ for $i \geq 0$ and

$$\sum_{n=1}^\infty \lambda_n^{-2} < \infty \qquad \text{and} \qquad \sum_{n=1}^\infty \frac{|\lambda_n - \lambda_{n+1}|^2}{\lambda_n^2 + \lambda_{n+1}^2} < \infty$$

We note both these conditions are satisfied for $\lambda_n = n$ and the second condition is not satisfied for $\lambda_n = 2^n$. Let S be the unitary mapping of $\mathfrak{H} = \mathfrak{K} \otimes L^2(0,\infty)$ into $\mathfrak{K}$ given by

$$S((f_1 \otimes f_2 \otimes \cdots) \otimes h) = h \otimes f_1 \otimes f_2 \otimes \cdots$$

and let $\pi(A) = SAS^*$ and $\Delta = e^{-x} \otimes e^{-x} \otimes \cdots$ where e^{-x} is shorthand for the operation of multiplication by e^{-x} on $L^2(0,\infty)$. The first sum condition insures that Δ is not zero and the second condition insures that S is well defined. One checks that

$$(\pi \cdot \Lambda)^n(I) = e^{-x} \otimes e^{-x} \otimes \cdots \otimes e^{-x} \otimes I \otimes I \otimes \cdots$$

where there are n factors of e^{-x} and $(\pi \cdot \Lambda)^n(I) \to \Delta$ as $n \to \infty$.

References

[Al1] A. Alevras, *A note of the boundary representation of a continuous spatial semigroup of *-endomorphisms of* $\mathfrak{B}(\mathfrak{H})$, Proc. AMS **123** (1995), no. 10, 3129–3133.

[Al2] ———, *The standard form of an E_o-semigroup*, J. Funct. Anal. **182** (2001), 227–242.

[A1] W.B. Arveson, *Continuous analogues of Fock Space*, Memoirs A.M.S. **80** (1989), no. 409.

[A2] ———, *An addition formula for the index of semigroups of endomorphisms of* $\mathfrak{B}(\mathfrak{H})$, Pac. J. Math. **137** (1989), no. 1, 19–36.

[A3] ———, *Continuous analogues of Fock space II: the spectral C^*-algebra*, J. Funct. Anal. **90** (1990), no. 1, 165-205.

[A4] ———, *Continuous analogues of Fock space III: Singular states*, J. Oper. Th. **22** (1989), 165–205.

[A5] ———, *Continuous analogues of Fock space IV: Essential states*, Acta Math. **164**, 265-300.

[A6] ———, *Minimal E_o-Semigroups:*, Fields Institute Communications **13** (1997), 1-12.

[A7] ———, *The Index of a Quantum Dynamical Semigroup:*, J. of Funct. Anal. **146** (1997), 557-588.

[A8] ———, *The Domain Algebra of a CP-semigroup:*, Pac. J. Math. **203** (2002), no. 1, 67-77.

[Bh] B.V.R. Bhat, *Minimal dilations of quantum dynamical semigroups to semigroups of endomorphisms of C^*-algebras*, J. Ramanujan Math. Soc. **14** (1999), no. 2, 109-124.

[BR] O. Bratteli and D. Robinson, *Operator Algebras and Quantum Statistical Mechanics I*, Springer-Verlag, 1979.

[Ch] Paul R. Chernoff, *Note on product formulas for operator semigroups*, J. Funct. Anal. **2** (1968), 238-242.

[Co] Alain Connes, *Outer Conjugacy Classes of Automorphisms of Factors*, Ann. Scient. Ec. Norm Sup. **4** (1975 pages 383-419), no. 8.

[DS] N. Dunford and J.T. Schwartz, *Linear Operators*, Interscience, New York, 1958-1971.

[KR] Richard V. Kadison and John R. Ringrose, *Fundamentals of the Theory of Operator Algebras, Vol I*, Academic Press, New York, 1983.

[P1] R.T. Powers, *An index theory for semigroups of $*$-endomorphisms of $\mathfrak{B}(\mathfrak{H})$ and type II_1 factors*, Can. Jour. Math. **40** (l988), 86–114.

[P2] ———, *A non-spatial continuous semigroup of $*$-endomorphisms of* $\mathfrak{B}(\mathfrak{H})$, Publ. R.I.M.S. Kyoto Univ. **23** (1987), 1053–1069.

[P3] ———, *On the structure of continuous spatial semigroups of $*$-endomorphisms of* $\mathfrak{B}(\mathfrak{H})$, International Journal of Math. **2**, 323–360.

[P4] ———, *New Examples of Continuous Spatial Semigroups of $*$-endomorphisms of* $\mathfrak{B}(\mathfrak{H})$, International J. Math. **10** (1999), no. 2, 215-288.

[PP] R.T. Powers and G. Price, *Continuous spatial semigroups of $*$-endomorphisms of* $\mathfrak{B}(\mathfrak{H})$, Transactions A. M. S. **321** (1990), 347–361.

[St] W. F. Stinespring, Proc. Amer. Math. Soc. **6** (1955), 211-216.

[T1] Boris Tsirelson, *From random sets to continuous tensor products: answers to three questions of W. Arveson*, preprint arXiv:math FA/0001070, 12 Jan (2000).

[T2] ———, *From slightly coloured noises to unitless product systems*, preprint arXiv:math FA/0006165 v1, 22 June (2000).

[vN] J. von Neumann, *On Infinite Direct Products*, Composito Mathematica **6** (1938), 1.

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