CONSTRUCTION OF E_o-SEMIGROUPS OF $\mathfrak{B}(\mathfrak{H})$ FROM CP-FLOWS

Robert T. Powers

Abstract. This paper constructs new examples of spatial E_o-semigroup of $\mathfrak{B}(\mathfrak{H})$ using CP-flows. A CP-flow is a strongly continuous one parameter semigroup of completely positive contractions of $\mathfrak{B}(\mathfrak{H}) = \mathfrak{B}(\mathfrak{K}) \otimes L^2(0,\infty)$ which are intertwined by translation. Using Bhat's dilation result each unital CP-flow over $\mathfrak{K}$ dilates to an E_o-semigroup of $\mathfrak{B}(\mathfrak{H}_1)$ where $\mathfrak{H}_1$ can be considered to contain $\mathfrak{K} \otimes L^2(0,\infty)$. Every spatial E_o-semigroup is cocycle conjugate to one dilated from a CP-flow. Each CP-flow is determined by its associated boundary weight which determines the generalized boundary representation. Using the machinery for determining whether two CP-flows dilate to cocycle conjugate E_o-semigroups new examples of spatial E_o-semigroups of $\mathfrak{B}(\mathfrak{H})$ are constructed.

I. Introduction to E_o-semigroups.

In 1936 Wigner proved if $A \to \alpha_t(A)$ is a group of $*$-automorphism of $\mathfrak{B}(\mathfrak{H})$ then $\alpha_t(A) = U(t)^*AU(t)$ with $U(t) = e^{-itH}$ and H is a self adjoint operator. To get to E_o-semigroups of $\mathfrak{B}(\mathfrak{H})$ replace group by semigroup and automorphism by endomorphism.

Definition 1.1. We say α is an E_o-semigroup of a $\mathfrak{B}(\mathfrak{H})$ if the following conditions are satisfied.

(i) α_t is a $*$-endomorphism of $\mathfrak{B}(\mathfrak{H})$ for each $t \geq 0$.
(ii) α_o is the identity endomorphism and $\alpha_t \circ \alpha_s = \alpha_{t+s}$ for all $s, t \geq 0$.
(iii) For each $\rho \in \mathfrak{B}(\mathfrak{H})_*$ (the predual of $\mathfrak{B}(\mathfrak{H})$) and $A \in M$ the function $\rho(\alpha_t(A))$ is a continuous function of t.
(iv) $\alpha_t(I) = I$ for each $t \geq 0(\alpha_t$ preserves the unit).

The appropriate notions of when two E_o-semigroups are similar are conjugacy and cocycle conjugacy (which comes from Alain Connes definition of outer conjugacy).

Definition 1.2. Suppose α and β are E_o-semigroups $\mathfrak{B}(\mathfrak{H}_1)$ and $\mathfrak{B}(\mathfrak{H}_2)$. We say α and β are conjugate denoted $\alpha \approx \beta$ if there is $*$-isomorphism ϕ of $\mathfrak{B}(\mathfrak{H}_1)$ onto $\mathfrak{B}(\mathfrak{H}_2)$ so that $\phi \circ \alpha_t = \beta_t \circ \phi$ for all $t \geq 0$. We say α and β are cocycle conjugate denoted $\alpha_t \sim \beta_t$ if α' and β are conjugate where α and α' differ by a unitary cocycle (i.e., there is a strongly continuous one parameter family of unitaries $U(t)$

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on $\mathfrak{B}(\mathfrak{H}_1)$ for $t \geq 0$ satisfying the cocycle condition $U(t)\alpha_t(U(s)) = U(t+s)$ for all $t, s \geq 0$ so that $\alpha'_t(A) = U(t)\alpha_t(A)U(t)^{-1}$ for all $A \in \mathfrak{B}(\mathfrak{H}_1)$ and $t \geq 0$).

We note that the notion of cocycle conjugacy is invariant under bounded perturbations. Suppose $\alpha_t = e^{t\delta}$ where $\delta = *$-derivation and $H = H^*$ is a bounded operator. Let

$$\delta_1(A) = \delta(A) + i(HA - AH)$$

for $A \in \mathfrak{B}(\mathfrak{H})$. Then $\beta_t = \exp(t\delta_1)$ is an E_o-semigroup which is cocycle conjugate to α (see [P3] theorem 2.8).

Next consider the case when there is an invariant projection. Suppose α_t is an E_o-semigroup and E is an invariant projection. Let β be the restriction of α_t to $E\mathfrak{B}(\mathfrak{H})E$ so

$$\beta_t(EAE) = \alpha_t(EAE) = E\alpha_t(EAE)E$$

for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. We see that β_t is an E_o-semigroup of $\mathfrak{B}(\mathfrak{K})$ where $\mathfrak{K} = E\mathfrak{H}$. One notes that if α is a semigroup of proper endomorphism then the invariant projection E must be of infinite dimension so there is an isometry W of $\mathfrak{H}$ onto $\mathfrak{K}$. Let $U(t) = W^*\alpha_t(W)$. One checks U is a unitary cocycle and

$$WU(t)\alpha_t(W^*AW)U(t)^*W^* = E\alpha_t(EAE)E = \beta_t(A)$$

Hence, α and β are cocycle conjugate.

Next we note cocycle conjugacy is invariant under tensoring with a group of automorphisms. Suppose σ is a E_o-semigroup which is actually a group so $\sigma_t(A) = S(t)AS(t)^*$ for $A \in \mathfrak{B}(\mathfrak{K})$ and $t > 0$ where $S(t)$ is a one parameter unitary group. Suppose α is an E_o-semigroup of $\mathfrak{B}(\mathfrak{H})$ and we tensor α and σ. Let β act on the tensor product $\mathfrak{B}(\mathfrak{H}) \otimes \mathfrak{B}(\mathfrak{K})$ by the formula

$$\beta_t(A \otimes B) = \alpha_t(A) \otimes \sigma_t(B)$$

for $A \in \mathfrak{B}(\mathfrak{H})$, $B \in \mathfrak{B}(\mathfrak{K})$ and $t \geq 0$. One can show (see [P4] theorem 2.10) that α and β are cocycle conjugate.

We summarize these results in the following theorem.

Theorem 1.3. *The cocycle conjugacy class of an E_o-semigroup is invariant under*

(i) *Bounded perturbations, i.e. replacing the generator δ by $\delta + iAd(H)$ with $H = H^* \in \mathfrak{B}(\mathfrak{H})$.*
(ii) *Cutting down with an invariant projection.*
(iii) *Tensoring with a automorphism group.*

Next we discuss spatial E_o-semigroups of $\mathfrak{B}(\mathfrak{H})$. An E_o-semigroup α_t is spatial if there is semigroup of isometries $U(t)$ which intertwine so $U(t)A = \alpha_t(A)U(t)$ for $A \in \mathfrak{B}(\mathfrak{H})$ and $t > 0$. The property of being spatial is a cocycle conjugacy invariant. Suppose α is a spatial E_o-semigroup and $U(t)$ is an intertwining semigroup of isometries. Suppose β is cocycle conjugate to α. We show β is spatial. We can assume α and β act on the same Hilbert space and $\beta_t(A) = S(t)\alpha_t(A)S(t)^*$ for $A \in \mathfrak{B}(\mathfrak{H})$ and $t > 0$ where $S(t)$ is a unitary cocycle for α. Let $V(t) = S(t)U(t)$ for $t > 0$. Recall that $U(t)A = \alpha_t(A)U(t)$ for $t > 0$ and $A \in \mathfrak{B}(\mathfrak{H})$. So we have

$$V(t)V(s) = S(t)U(t)S(s)U(s) = S(t)\alpha_t(S(s))U(t)U(s) = S(t+s)U(t+s)$$

So $V(t)$ is a one parameter semigroup of isometries. Now V intertwines β since for $A \in \mathfrak{B}(\mathfrak{H})$ and $t > 0$ we have

$$\begin{aligned} V(t)A &= S(t)U(t)A = S(t)\alpha_t(A)U(t) \\ &= S(t)\alpha_t(A)S(t)^*S(t)U(t) = \beta_t(A)V(t) \end{aligned}$$

Hence, β is spatial. Note the mapping above gives an isomorphism of the intertwining semigroups for α with those that intertwine β.

You can construct spatial E_o-semigroups using the CAR algebra or the CCR algebra. Derek Robinson and I [PR] showed these constructions give the same E_o-semigroups. Now there is a better way to show this. Every spatial E_o-semigroup is cocycle conjugate to an E_o-semigroup in standard form. A spatial E_o-semigroup of $\mathfrak{B}(\mathfrak{H})$ is in standard form if there is an invariant pure state ω_o which is absorbing, meaning if ω is any normal state of $\mathfrak{B}(\mathfrak{H})$ then $\|\omega_o - \alpha_t \circ \omega\| \to 0$ as $t \to \infty$. The state ω_o corresponds to the vacuum state in the CAR and CCR constructions. To see this suppose α is a spatial E_o-semigroup of $\mathfrak{B}(\mathfrak{H})$ and $U(t)$ is an intertwining semigroup of isometries. Let $-d$ be the generator of $U(t)$. In some cases we have $\mathfrak{H} = \mathfrak{K} \otimes L^2(0,\infty)$ and d is just differentiation. Choose $f_o \in \mathfrak{D}(d)$ a unit vector and let

$$-iHf = (f_o, f)df_o - (df_o, f)f_o + (df_o, f_o)f$$

for $f \in \mathfrak{H}$. Note H is a bounded hermitian operator. Let δ be the generator of α and define

$$\delta_1(A) = \delta(A) + i(HA - AH)$$

for $A \in \mathfrak{D}(\delta)$. One finds $\beta_t = \exp(t\delta_1)$ is an E_o-semigroup which is cocycle conjugate to α and $V(t) = \exp(-t(d - iH))$ for $t \geq 0$ is a one parameter semigroup of isometries that intertwine β_t. Note $(d - iH)f_o = 0$ so $V(t)f_o = f_o$ for all $t > 0$. Let $\omega_o(A) = (f_o, Af_o)$ for $A \in \mathfrak{B}(\mathfrak{H})$. We show ω_o is β_t invariant. We have

$$\begin{aligned} \omega_o(\beta_t(A)) &= (f_o, \beta_t(A)f_o) = (f_o, \beta_t(A)V(t)f_o) \\ &= (f_o, V(t)Af_o) = (V(t)^*f_o, Af_o) = \omega_o(A) \end{aligned}$$

for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t > 0$. Let e_o be the rank one projection onto f_o and let $e_t = \beta_t(e_o)$. Since $(f_o, e_t f_o) = 1$ we have $e_t \geq e_o$ for all $t > 0$. Since β_t is a semigroup of $*$-endomorphisms we have

$$e_{t+s} = \beta_t(\beta_s(e_o)) \geq \beta_t(e_o) = e_t$$

for $s, t \geq 0$. So e_t is an increasing family of projections. Let

$$E = \lim_{t \to \infty} e_t$$

Then E is β_t invariant. Now let γ be the restriction of β_t to $E\mathfrak{B}(\mathfrak{H})E$ and as we have seen γ is cocycle conjugate with β. One sees that γ is a spatial E_o-semigroup in standard form since it has an invariant pure normal state ω_o and if e_o is support projection for ω_o then $\gamma_t(e_o) \to I$ as $t \to \infty$.

One of the important questions regarding spatial E_o-semigroups discussed by A. Alevras in [Al2] is if two spatial E_o-semigroups are in standard form then does cocycle conjugacy imply conjugacy? Alevras shows this question is equivalent to a question about equivalence of intertwining semigroups.

In [P3] we defined the boundary representation. Suppose α is a spatial E_o-semigroup and δ is the generator of α so $\alpha_t = e^{t\delta}$ and suppose $U(t) = e^{-td}$ intertwines α so $U(t)A = \alpha_t(A)U(t)$ for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. Differentiating this at $t = 0$ we find that if $A \in \mathfrak{D}(\delta)$ then $A\mathfrak{D}(d) \subset \mathfrak{D}(d)$ and $A\mathfrak{D}(d^*) \subset \mathfrak{D}(d^*)$ so A maps $\mathfrak{D}(d^*)$ mod $\mathfrak{D}(d)$ into itself. By the Wold decomposition it is well known that for every semigroup of isometries $U(t)$ there is a decomposition of the Hilbert space $\mathfrak{H}$ as the sum of two orthogonal subspaces $\mathfrak{H}_o$ and $\mathfrak{H}_1$ and $U(t)$ is unitary on $\mathfrak{H}_o$ and $U(t)$ is a pure shift on $\mathfrak{H}_1$. We can write $\mathfrak{H}_1 = \mathfrak{K} \otimes L^2(0,\infty)$ and $U(t)$ is just right translation by t. The generator of $U(t)$ on $\mathfrak{H}_1$ is $-d$ where $d = d/dx$ with boundary condition $F(0) = 0$ and d^* is $-d/dx$ with no boundary condition at $x = 0$. So if $\mathfrak{H}_1 = \mathfrak{K} \otimes L^2(0,\infty)$ we have $\mathfrak{K} = \mathfrak{D}(d^*)$ mod $\mathfrak{D}(d)$. The boundary representation π is a $*$-representation of $\mathfrak{D}(\delta)$ on $\mathfrak{K}$.

In [P1] we first defined the index to be the multiplicity of π. The problem with this is that it seems to depend on the intertwining semigroup $U(t)$. Arveson defined the index of an E_o-semigroup to be the dimension of the space of intertwining semigroups minus one. Arveson's definition of index is a cocycle conjugacy invariant. In [A2] Arveson proved the addition formula

$$\text{Index } \alpha_t \otimes \beta_t = \text{Index } \alpha_t + \text{Index } \beta_t$$

that was hinted at in [P1]. Later G. Price and I proved in [PP] that the index equals the multiplicity of the normal part of the boundary representation. Later in [Al1] A. Alevras proved the boundary representations constructed from two different intertwining semigroups are unitarily equivalent.

There is a type classification of E_o-semigroups similar to that for factors. An E_o-semigroup is completely spatial or of type I if there are enough intertwining semigroups to reconstruct the E_o-semigroup. Arveson defined and completely classified the completely spatial E_o-semigroups. These E_o-semigroups are determined up to cocycle conjugacy by the index.

In [P4] we constructed an example of a spatial E_o-semigroup which is not completely spatial. Recently Tsirelson [T2] has constructed a one parameter family of non isomorphic product systems of type II and by Arveson's theory of product systems this implies the existence of a one parameter family of non cocycle conjugate spatial E_o-semigroups of $\mathfrak{B}(\mathfrak{H})$ which are not completely spatial. E_o-semigroups which are spatial but not completely spatial are said to be of type II.

There are non spatial E_o-semigroups. In [P2] we constructed an example of a non spatial E_o-semigroup and later Tsirelson [T1] constructed a one parameter family of non isomorphic product systems of type III in the context of Arveson's theory of continuous tensor products of Hilbert spaces and from Arveson's representation theorem this implies the existence of a one parameter family of non cocycle conjugate non spatial E_o-semigroups. Non spatial E_o-semigroups are said to be of type III. In this paper we will restrict our attention to the spatial E_o-semigroups. We summarize the type classification of E_o-semigroups in the following definition.

Definition 1.4. An E_o-semigroup of $\mathfrak{B}(\mathfrak{H})$ is of type I if it is completely spatial, type II if it is spatial but not completely spatial and type III if it is not spatial. The E_o-semigroups of type I and II can be further subdivided into those of type I_n and type II_n where $n = \infty, 0, 1, 2, \ldots$ and n is the index.

Under tensoring we have that if $\{\alpha_t\}$ is an E_o-semigroup of type A_n and $\{\beta_t\}$ is an E_o-semigroup of type B_m where A and B are I,II or III then $\alpha_t \otimes \beta_t$ is an E_o-semigroup of type C_{n+m} where C is the maximum of A and B and for type III the index is superfluous.

II. Bhat's dilation of CP-semigroups to E_o-semigroups.

An extremely useful and well known result in the theory of C^*-algebras is the Gelfand Segal construction of a cyclic $*$-representation of a C^*-algebra associated with a state of the C^*-algebra. In the study of E_o-semigroups there is a result in the same spirit which says that every semigroup of unital completely positive maps of $\mathfrak{B}(\mathfrak{K})$ can be dilated to an E_o-semigroup of $\mathfrak{B}(\mathfrak{H})$ where $\mathfrak{H}$ can be thought of as a larger Hilbert space containing $\mathfrak{K}$. We begin with a review of the properties of completely positive maps.

A linear map ϕ from a C^*-algebra $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{H})$ is completely positive if

$$\sum_{i,j=1}^{n} (f_i, \phi(A_i^* A_j) f_j) \geq 0$$

for $A_i \in \mathfrak{A},\ f_i \in \mathfrak{H}$ for $i = 1, 2, \cdots, n$ and $n = 1, 2, \cdots$. Stinespring's central result [St]is that if $\mathfrak{A}$ has a unit and ϕ is a completely positive map from $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{H})$ then there is a $*$-representation π of $\mathfrak{A}$ on $\mathfrak{B}(\mathfrak{K})$ and an operator V from $\mathfrak{H}$ to $\mathfrak{K}$ so that $\phi(A) = V^*\pi(A)V$ for $A \in \mathfrak{A}$. And π is determined by ϕ up to unitary equivalence if the linear span of $\{\pi(A)Vf\}$ for $A \in \mathfrak{A}$ and $f \in \mathfrak{K}$ is dense in $\mathfrak{H}$. Arveson has shown [A7] that if ϕ is a σ-weakly continuous completely positive map from $\mathfrak{B}(\mathfrak{H})$ to $\mathfrak{B}(\mathfrak{K})$ then ϕ is of the form

$$\phi(A) = \sum_{i=1}^{r} C_i A C_i^*$$

for $A \in \mathfrak{B}(\mathfrak{H})$ where r is a positive integer or $+\infty$ and C_i is a bounded operator from $\mathfrak{H}$ to $\mathfrak{K}$ and the C_i are linearly independent over $\ell^2(\mathbb{N})$ which means if $z_i \in \mathbb{C}$ is square summable and

$$C = \sum_{i=1}^{r} z_i C_i$$

then $C = 0$ if and only if $z_i = 0$ for all $i \in [1, r+1)$. Note the above sum converges in norm! Stinesprings' well known result is that if ϕ and η are completely positive maps of $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{H})$ and $\phi - \eta$ is completely positive then if π is the $*$-representation of $\mathfrak{A}$ induced by ϕ then there is a unique positive operator $C \in \pi(\mathfrak{A})'$ so that

$$\eta(A) = V^*\pi(A)CV.$$

Often we speak of one functional or map dominating another. We would like to introduce a word for the functional or map that is dominated. The word is subordinate. If A is an object which is positive with respect to some order structure we say B is a subordinate of A if B is the same kind of thing A is and B is positive and B is less than A. For example if we are speaking of the positive integers the subordinates of 4 are 4,3,2,1. If A is a positive operator then the subordinates of A are operators B with $A \geq B \geq 0$. Suppose E is a projection. Are the subordinates of a projection E projections under E or the operators under E? The answer depends on the context.

In terms of subordinates Stinespring's theorem becomes the following. Suppose ϕ is a completely positive map from a C^*-algebra $\mathfrak{A}$ into $\mathfrak{B}(\mathfrak{H})$ and π is the induced $*$-representation of $\mathfrak{A}$. Then the subordinates of ϕ are order isomorphic to the subordinates of I in $\pi(\mathfrak{A})'$. We say a subordinate is pure if the space of its subordinates is one dimensional.

Suppose ϕ is a completely positive map from $\mathfrak{B}(\mathfrak{H})$ to $\mathfrak{B}(\mathfrak{K})$ of the form

$$\phi(A) = \sum_{i=1}^{r} C_i A C_i^*$$

with the C_i linearly independent over $\ell^2(\mathbb{N})$. Then η is a pure subordinate of ϕ if and only if η is of the form $\eta(A) = CAC^*$ with

$$C = \sum_{i=1}^{r} z_i C_i \qquad \text{and} \qquad \sum_{i=1}^{r} |z_i|^2 \leq 1$$

Note the pure subordinates of ϕ are isomorphic to the unit ball in a Hilbert space of dimension r. We call r the rank of ϕ.

A CP-semigroup of $\mathfrak{B}(\mathfrak{H})$ is a strongly continuous one parameter semigroup of completely positive maps of $\mathfrak{B}(\mathfrak{H})$ into itself. We now state Bhat's theorem [Bh] for $\mathfrak{B}(\mathfrak{H})$.

Theorem 2.1. *Suppose α is a unital CP-semigroup of $\mathfrak{B}(\mathfrak{H})$ then there is an E_o-semigroup α^d of $\mathfrak{B}(\mathfrak{K})$ and an isometry W from $\mathfrak{H}$ to $\mathfrak{K}$ so that*

$$\alpha_t(A) = W^* \alpha_t^d(WAW^*)W$$

and $\alpha_t(WW^*) \geq WW^*$ for $t > 0$ and if the projection $E = WW^*$ is minimal which means the span of the vectors

$$\alpha_{t_1}^d(EA_1E)\alpha_{t_2}^d(EA_2E)\cdots\alpha_{t_n}^d(EA_nE)Ef$$

for $f \in \mathfrak{K}$, $A_i \in \mathfrak{B}(\mathfrak{H})$, $t_i \geq 0$ for $i = 1, 2, \cdots$ and $n = 1, 2, \cdots$ are dense in $\mathfrak{H}$ then α^d is determined up to conjugacy.

We use the Arveson definition of minimal which is easier to state and equivalent to Bhat's. Arveson has worked out how to calculate the index of a minimal dilation of a unital CP-semigroup α of $\mathfrak{B}(\mathfrak{H})$. You look at the subordinates of α of the form

$$\Omega_t(A) = S(t)AS(t)^*$$

where $S(t)$ is a strongly continuous semigroup of contractions of $\mathfrak{H}$. Arveson tells us the index of α^d is the dimension of the space of semigroups $S(t)$ minus one.

Suppose α is an E_o-semigroup of $\mathfrak{B}(\mathfrak{H})$. What are the subordinates of α? The answer can be described using positive local cocycles. A cocycle is a σ-weakly continuous one parameter family of operator $C(t)$ satisfying the cocycle relation

$$C(t+s) = C(t)\alpha_t(C(s))$$

for all $s, t \geq 0$. The cocycle $C(t)$ is local if $C(t) \in \alpha_t(\mathfrak{B}(\mathfrak{H}))'$ for all $t > 0$. The local cocycles are a cocycle conjugacy invariant.

If α is a CP-semigroup of $\mathfrak{B}(\mathfrak{H})$ then the subordinates of α are CP-semigroups β of $\mathfrak{B}(\mathfrak{H})$ so that $\alpha_t - \beta_t$ is completely positive for each $t \geq 0$. In [P6,P7] we showed the following.

Theorem 2.2. *Suppose α is a unital CP-semigroup of $\mathfrak{B}(\mathfrak{H})$ and α^d is the minimal dilation of α to an E_o-semigroup of $\mathfrak{B}(\mathfrak{K})$ and W is an isometry from $\mathfrak{H}$ to $\mathfrak{K}$ so that WW^* is a minimal projection for α^d and*

$$\alpha_t(A) = W^*\alpha_t^d(WAW^*)W$$

for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$. Then there is an order isomorphism from the subordinates of α to the subordinates of α^d given as follows. Suppose γ is a subordinate of α^d and $C(t) = \gamma_t(I)$ for $t \geq 0$ is the local cocycle associated with γ then the subordinate β of α under this isomorphism is given by

$$\beta_t(A) = W^*\alpha_t(WAW^*)C(t)W$$

for $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$.

As an application we compute the subordinates of an E_o-semigroup of type I_n. Arveson and I independently proved (see [A8] and [P5]) that if α_t is a unital CP-semigroup of $\mathfrak{B}(\mathfrak{H})$ with $\mathfrak{H}$ of finite dimension then the Bhat induced minimal dilation α^d is completely spatial. Define Θ_t to act on an $(n \times n)$-matrix by multiplying the off diagonal entries by e^{-t} and Θ_t leaves the diagonal entries invariant. One computes Θ induces the completely spatial E_o-semigroup of index n (type I_n). We compute the subordinates of Θ. Let Q be the $(n \times n)$-matrix with all ones off the diagonal and zeros down the diagonal. Note $-Q$ is associated with the generator of Θ. The set of subordinates of the type I_n E_o-semigroup is order isomorphic to the space of conditionally positive matrices C with $C \leq Q$. (A matrix C is conditionally positive if $(x, Cx) \geq 0$ for all vectors $x \in \mathbb{C}^n$ with $x_1 + x_2 + \cdots + x_n = 0$.)

Next we discuss cocycle conjugacy. We use the well known trick of A. Connes [Co].

Definition 2.3. Suppose α and β are CP-semigroups of $\mathfrak{B}(\mathfrak{H})$ and $\mathfrak{B}(\mathfrak{K})$. Then γ is a corner from α to β if Θ given by

$$\Theta_t(\begin{bmatrix} A & B \\ C & D \end{bmatrix}) = \begin{bmatrix} \alpha_t(A) & \gamma_t(B) \\ \gamma_t^*(C) & \beta_t(D) \end{bmatrix}$$

for $A \in \mathfrak{B}(\mathfrak{H})$, $D \in \mathfrak{B}(\mathfrak{K})$, B a linear operator from $\mathfrak{K}$ to $\mathfrak{H}$ and C a linear operator from $\mathfrak{H}$ to $\mathfrak{K}$ is a CP-semigroup of $\mathfrak{B}(\mathfrak{H} \oplus \mathfrak{K})$.

Applying Connes' trick to E_o-semigroups one can show that two E_o-semigroups α and β are cocycle conjugate if and only if there is a corner γ from α to β so that Θ above is an E_o-semigroup of $\mathfrak{B}(\mathfrak{H} \oplus \mathfrak{K})$. In order to determine when E_o-semigroups dilated from CP-semigroups are cocycle conjugate we need a more general result (see Theorem 3.13 of [P6,P7]) which we state below.

Theorem 2.4. *Suppose α and β are unital CP-semigroups of $\mathfrak{B}(\mathfrak{H})$ and $\mathfrak{B}(\mathfrak{K})$ and α^d and β^d are the minimal dilations of α and β to E_o-semigroups. Then α^d and β^d are cocycle conjugate if and only if there is a corner γ from α to β so that if*

$$\Theta_t(\begin{bmatrix} A & B \\ C & D \end{bmatrix}) = \begin{bmatrix} \alpha_t(A) & \gamma_t(B) \\ \gamma_t^*(C) & \beta_t(D) \end{bmatrix}$$

and

$$\Theta'_t(\begin{bmatrix} A & B \\ C & D \end{bmatrix}) = \begin{bmatrix} \alpha'_t(A) & \gamma_t(B) \\ \gamma_t^*(C) & \beta'_t(D) \end{bmatrix}$$

for $t \geq 0$ and A, B, C and D operators between the appropriate Hilbert spaces and Θ' is a subordinate of $\Theta(\Theta \geq \Theta' \geq 0)$ then $\Theta' = \Theta$ so $\alpha' = \alpha$ and $\beta' = \beta$.

III. CP-Flows.

In this section we define CP-flows. We believe these are the simplest objects which can be dilated to produce all spatial E_o-semigroups.

Definition 3.1. Suppose $\mathfrak{K}$ is a separable Hilbert space and $\mathfrak{H} = \mathfrak{K} \otimes L^2(0,\infty)$ and $U(t)$ is right translations of $\mathfrak{H}$ by $t \geq 0$. Specifically, we may realize $\mathfrak{H}$ as the space of $\mathfrak{K}$-valued Lebesgue measurable functions with inner product

$$(f,g) = \int_0^\infty (f(x), g(x))\, dx$$

for $f, g \in \mathfrak{H}$. The action of $U(t)$ on an element $f \in \mathfrak{H}$ is given by $(U(t)f)(x) = f(x-t)$ for $x \in [t,\infty)$ and $(U(t)f)(x) = 0$ for $x \in [0,t)$. A semigroup α is a CP-flow over $\mathfrak{K}$ if α is a CP-semigroup of $\mathfrak{B}(\mathfrak{H})$ which is intertwined by the translation semigroup $U(t)$, i.e., $U(t)A = \alpha_t(A)U(t)$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $t \geq 0$.

Every spatial E_o-semigroup can be induced from a CP-flow because every spatial E_o-semigroup is cocycle conjugate to an E_o-semigroup which is also a CP-flow.

We introduce notation for describing CP-flows. Let $\mathfrak{H} = \mathfrak{K} \otimes L^2(0,\infty)$ and $U(t)$ be translation by t. Consider the projections

$$E(t) = I - U(t)U(t)^* \qquad \text{and} \qquad E(a,b) = U(a)U(a)^* - U(b)U(b)^*$$

for $t \in [0,\infty)$ and $0 \leq a < b < \infty$. Let $d = d/dx$ be the differential operator of differentiation with the boundary condition $f(0) = 0$. Precisely, the domain $\mathfrak{D}(d)$ is all $f \in \mathfrak{H}$ of the form

$$f(x) = \int_0^x g(t)\, dt$$

with $g \in \mathfrak{H}$. The hermitian adjoint d^* is $-d/dx$ with no boundary condition at $x = 0$. Precisely, the domain $\mathfrak{D}(d^*)$ consists of the linear span of $\mathfrak{D}(d)$ and the functions $g(x) = e^{-x}k$ with $k \in \mathfrak{K}$.

Suppose α is a CP-flow over $\mathfrak{K}$ and $A \in \mathfrak{B}(\mathfrak{H})$ then for $t > 0$ one finds

$$\alpha_t(A) = U(t)AU(t)^* + E(t)\alpha_t(A)E(t) = U(t)AU(t)^* + B$$

for all $t \geq 0$. Then B commutes with $E(s)$ for all s so B is of the form

$$(Bf)(x) = b(x)f(x)$$

for $x \in (0, t)$ where $b(x) \in \mathfrak{B}(\mathfrak{K})$ depends σ-strongly on A for $x \in (0, t)$. Again we define the boundary representation, π_o. Let δ be the generator of α then for $A \in \mathfrak{D}(\delta)$ we have $A\mathfrak{D}(d) \subset \mathfrak{D}(d)$ and $A\mathfrak{D}(d^*) \subset \mathfrak{D}(d^*)$ so A acts on $\mathfrak{D}(d^*)$ mod $\mathfrak{D}(d) = \mathfrak{K}$. We call this mapping from $\mathfrak{D}(\delta)$ into $\mathfrak{B}(\mathfrak{K})$ the boundary representation π_o. If one thinks of a CP-flow as the operator shifting to the right in $\mathfrak{B}(\mathfrak{K} \otimes L^2(0, \infty))$ then π_o determines what flows in from the origin. The boundary representation need not be σ-weakly continuous. Also the boundary representation does not always completely determine the CP-flow. If π is a σ-weakly continuous completely positive contraction of $\mathfrak{B}(\mathfrak{K} \otimes L^2(0, \infty))$ into $\mathfrak{B}(\mathfrak{K})$ then there is a minimal CP-flow with that boundary representation and if that flow is unital then the E_o-semigroup induced by the flow is completely spatial (type I_n) where n is the rank of π.

We now define the generalized boundary representation. The resolvent R_α for α is given by

$$R_\alpha(A) = \int_0^\infty e^{-t}\alpha_t(A)\,dt$$

Next we introduce some notation. If ϕ is a σ-weakly continuous mapping from $\mathfrak{B}(\mathfrak{H})$ to $\mathfrak{B}(\mathfrak{K})$ we define $\hat{\phi}$ as the predual map from $\mathfrak{B}(\mathfrak{K})_*$ to $\mathfrak{B}(\mathfrak{H})_*$ so we have $\rho(\phi(A)) = (\hat{\phi}\rho)(A)$ for all $A \in \mathfrak{B}(\mathfrak{H})$ and $\rho \in \mathfrak{B}(\mathfrak{K})_*$. We define the mapping Γ as

$$\Gamma(A) = \int_0^\infty e^{-t}U(t)AU(t)^*dt$$

for $A \in \mathfrak{B}(\mathfrak{H})$. Note $R_\alpha - \Gamma$ is completely positive which we denote by writing $R_\alpha - \Gamma \geq 0$. Note Γ is the resolvent of a CP-flow with boundary representation $\pi_o = 0$.

We need one more bit of notation. We define $\Lambda : \mathfrak{B}(\mathfrak{K}) \to \mathfrak{B}(\mathfrak{H})$ for $A \in \mathfrak{B}(\mathfrak{K})$ where $\Lambda(A)$ is given by

$$(\Lambda(A)f)(x) = e^{-x}Af(x)$$

for $x \geq 0$ and $f \in \mathfrak{H}$. We define $\Lambda = \Lambda(I)$. Note $\Gamma(I) = I - \Lambda$.

Now we present our main formula

$$\hat{R}_\alpha(\rho) = \hat{\Gamma}(\omega(\hat{\Lambda}\rho) + \rho)$$

for $\rho \in \mathfrak{B}(\mathfrak{H})_*$ and the mapping $\eta \to \omega(\eta)$ defined to be the boundary weight map. We call $\omega(\eta)$ the boundary weight associated with η. In the unital case we have for $\rho \in \mathfrak{B}(\mathfrak{K})_*$

$$\hat{R}_\alpha(\rho)(I) = \omega(\hat{\Lambda}(\rho))(I - \Lambda) + \rho(I - \Lambda) = \rho(I)$$

so

$$\omega(\hat{\Lambda}(\rho))(I-\Lambda) = \rho(\Lambda) = (\hat{\Lambda}\rho)(I)$$

The mapping $\rho \to \omega(\rho)$ is a completely positive map from $\mathfrak{B}(\mathfrak{K})_*$ into weights on $\mathfrak{B}(\mathfrak{H})$ so that $\mu(\rho)$ given by $\mu(\rho)(A) = \omega(\rho)((I-\Lambda)^{\frac{1}{2}}A(I-\Lambda)^{\frac{1}{2}})$ for $A \in \cup_{t>0} U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$ is normal (i.e. $\mu(\rho) \in \mathfrak{B}(\mathfrak{H})_*$) and $\mu(\rho)(I) \leq \rho(I)$ for ρ positive.

Every CP-flow is given by a boundary weight map $\rho \to \omega(\rho)$. What are the properties? As we have mentioned the map is completely positive. There is a further complicated positivity condition. The condition says if you construct an approximation to the boundary representation π_t, then π_t is completely positive.

Let us describe the connection between boundary weight and boundary representation. We can construct a boundary weight map so that the boundary representation is a given σ-weakly continuous completely positive contraction of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$. Suppose π is a σ-weakly continuous completely positive contraction of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$. Let

$$\omega = \hat{\pi} + \hat{\pi}\hat{\Lambda}\hat{\pi} + \hat{\pi}\hat{\Lambda}\hat{\pi}\hat{\Lambda}\hat{\pi} + \hat{\pi}\hat{\Lambda}\hat{\pi}\hat{\Lambda}\hat{\pi}\hat{\Lambda}\hat{\pi} + \cdots$$

This converges as a weight (i.e. $\omega(\rho)(I-\Lambda)$ converges for all ρ) and this is the boundary weight of a CP-flow. We call this the minimal CP-flow derived from π. Formally $\omega = \hat{\pi}(I - \hat{\Lambda}\hat{\pi})^{-1}$ and solving for π we have

$$\hat{\pi} = \omega(I + \hat{\Lambda}\omega)^{-1}$$

If a boundary weight associated with a CP-flow is bounded then the boundary representation is well defined as stated in the next theorem (see theorem 4.27 of [P6,P7]).

Theorem 3.2. *Suppose α is a CP-flow over $\mathfrak{K}$ and $\rho \to \omega(\rho)$ is the boundary weight map. Suppose $\|\omega(\rho)\| < \infty$ for $\rho \in \mathfrak{B}(\mathfrak{K})_*$ so $\omega(\rho) \in \mathfrak{B}(\mathfrak{H})_*$ for all $\rho \epsilon \mathfrak{B}(\mathfrak{K})_*$. Then the map $\rho \to \rho + \hat{\Lambda}\omega(\rho)$ is invertible (i.e., $(I+\hat{\Lambda}\omega)^{-1}$ exists) and $\hat{\pi}$ given by*

$$\hat{\pi} = \omega(I + \hat{\Lambda}\omega)^{-1}$$

is a completely positive contraction from $\mathfrak{B}(\mathfrak{K})_*$ to $\mathfrak{B}(\mathfrak{H})_*$. There is a unique CP-flow derived from π and its boundary weight map is given by

$$\omega = \hat{\pi} + \hat{\pi}\hat{\Lambda}\hat{\pi} + \hat{\pi}\hat{\Lambda}\hat{\pi}\hat{\Lambda}\hat{\pi} + \hat{\pi}\hat{\Lambda}\hat{\pi}\hat{\Lambda}\hat{\pi}\hat{\Lambda}\hat{\pi} + \cdots$$

So when $\omega(\rho)$ is bounded for all $\rho \in \mathfrak{B}(\mathfrak{K})_*$ we have

$$\omega = \hat{\pi}(I - \hat{\Lambda}\hat{\pi})^{-1} \qquad \text{and} \qquad \hat{\pi} = \omega(I + \hat{\Lambda}\omega)^{-1}$$

Now we introduce a bit of notation. Suppose ω is a boundary weight. We denote by $\omega_t(\rho)$ the functional $\omega_t(\rho)(A) = \omega(\rho)(E(t,\infty)AE(t,\infty))$. Note $\omega_t(\rho) \in \mathfrak{B}(\mathfrak{H})_*$, i.e. $\omega_t(\rho)$ is a bounded σ-weakly continuous functional. Our main result is the following theorem (see theorem 4.23 and 4.27 of [P6,P7]).

Theorem 3.3. *Suppose $\rho \to \omega(\rho)$ is the boundary weight map of a CP-flow over $\mathfrak{K}$. Then for each $t > 0$ we have $\rho \to \omega_t(\rho)$ is the boundary weight map of a CP-flow over $\mathfrak{K}$. Suppose $\rho \to \omega(\rho)$ is a completely positive mapping of $\mathfrak{B}(\mathfrak{K})_*$ into weights on $\mathfrak{B}(\mathfrak{H})$ satisfying $\omega(\rho)(I - \Lambda) \leq \rho(I)$ for ρ positive. Suppose*

$$\hat{\pi}_t = \omega_t(I + \hat{\Lambda}\omega_t)^{-1}$$

is a completely positive contraction of $\mathfrak{B}(\mathfrak{K})_*$ into $\mathfrak{B}(\mathfrak{H})_*$ for each $t > 0$. Then $\rho \to \omega(\rho)$ is boundary weight map of a CP-flow over $\mathfrak{K}$.

If $\rho \to \omega(\rho)$ is a mapping of $\mathfrak{B}(\mathfrak{K})_*$ into weights on $\mathfrak{B}(\mathfrak{H})$ so that $\hat{\pi}_t$ defined above is completely positive for each $t > 0$ we say this map is q-positive. The family $\pi_t^\#$ of completely positive σ-weakly continuous contractions of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$ is called the generalized boundary representation.

We remark that in checking that the $\pi_t^\#$ are completely positive it is only necessary to check for small t. If the mapping $\pi_t^\#$ is completely positive then $\pi_s^\#$ is completely positive for all $s \geq t$. Next we give the order relation for generalized boundary representations.

Theorem 3.4. *If α and β are CP-flows over $\mathfrak{K}$ then β is a subordinate of $\alpha (\alpha \geq \beta)$ if and only if $\pi_t^\# \geq \phi_t^\#$ for all $t > 0$ where $\pi_t^\#$ and $\phi_t^\#$ are the generalized boundary representations of α and β. Also we have if $\pi_t^\# \geq \phi_t^\#$ then $\pi_s^\# \geq \phi_s^\#$ for all $s \geq t$ so one only has to check for a sequence $\{t_n\}$ tending to zero.*

Theorem 3.5. *Suppose α is a CP-flow over $\mathfrak{K}$ and $\pi^\#$ is the generalized boundary representation of α. Then $\pi_s^\#(A) \to \pi_o^\#(A)$ as $s \to 0+$ in the σ-strong topology on $U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$ for each $t > 0$ where $\pi_o^\#$ is a σ-weakly continuous completely positive contraction of $\mathfrak{B}(\mathfrak{H})$ into $\mathfrak{B}(\mathfrak{K})$. The index of α is the rank of $\pi_o^\#$.*

Note $\pi_o^\#$ is not the boundary representation, rather it is the normal part of the boundary representation. So the result of G. Price and myself (see [PP]) generalizes to CP-flows.

The inverse of $(I + \hat{\Lambda}\omega)$ is in general hard to compute. We make a simplifying assumption. We introduce the Schur product of matrices. Sometimes this product is call the Hadamard product or the Kronecker product. The Schur product $C = A \circ B$ of two matrices A and B denoted by the dot between A and B is obtained by multiplying the entries of A and B so $c_{ij} = a_{ij}b_{ij}$ where a_{ij}, b_{ij} and c_{ij} are the coefficients of A, B and C, respectively. Note the Schur product is commutative so $X \circ A = A \circ X$ for matrices A and X. Note the map $A \to X \circ A$ is completely positive if and only if X is positive. We define Schur diagonal maps which is a generalization of the Schur product.

Definition 3.6. The map $\rho \to \omega(\rho)$ from $\mathfrak{B}(\mathfrak{K})_*$ to weights on $\cup_{t>0} U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$ is said to be Schur diagonal with respect to an orthonormal basis $\{f_i : i = 1, 2, \cdots\}$ of $\mathfrak{K}$ if $\rho_{ij}(A) = (f_i, Af_j)$ for $A \in \mathfrak{B}(\mathfrak{K})$ and $e_i f = (f_i, f)f_i$ then

$$\omega(\rho_{ij})(A) = \omega(\rho_{ij})((e_i \otimes I)A(e_j \otimes I))$$

for all $A \in \cup_{t>0} U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$ for all $i, j = 1, 2, \cdots$. In this case the matrix elements of the mapping $\rho \to \omega(\rho)$ are the weights

$$\omega_{ij}(A) = \omega(\rho_{ij})(e_{ij} \otimes A)$$

for $A \in \cup_{t>0} U(t)\mathfrak{B}(L^2(0,\infty))U(t)^*$ where $\{e_{ij}\}$ are the set of matrix units defined by $e_{ij}f = (f_j, f)f_i$ for all $f \in \mathfrak{K}$ and $i, j = 1, 2, \cdots$.

It is an easy exercise to show that if the mapping $\rho \to \omega(\rho)$ is completely positive then to show the mapping is Schur diagonal we need only check the diagonal entries. The next theorem tells us when a Schur diagonal boundary weight map is q-positive. The proof of the next theorem can be found in [P7].

Theorem 3.7. *Suppose $\mathfrak{K}$ is finite dimensional and $\rho \to \omega(\rho)$ is a linear mapping of $\mathfrak{B}(\mathfrak{K})_*$ into weights $\omega(\rho)$ on $\cup_{t>0} U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$ which is Schur diagonal with respect to an orthonormal basis $\{f_i : i = 1,2,\cdots, n\}$ and $\rho_{ij}(A) = (f_i, Af_j)$ for each i and j and for $A \in \mathfrak{B}(\mathfrak{K})$. For $t > 0$ and $\rho \in \mathfrak{B}(\mathfrak{K})_*$ let*

$$\omega_t(\rho)(A) = \omega(\rho)(E(t,\infty)AE(t,\infty))$$

for all $A \in \mathfrak{B}(\mathfrak{H})$. Then the mapping $\rho \to \omega(\rho)$ defines a CP-flow if and only if for each $t > 0$ the matrix with entries given by

$$\eta_{ij} = \frac{\omega_t(\rho_{ij})}{1 + \omega_t(\rho_{ij})(\Lambda)}$$

for $i, j = 1, 2, \cdots, n$ is the matrix of a completely positive contraction of $\mathfrak{B}(\mathfrak{K})_*$ into $\mathfrak{B}(\mathfrak{H})_*$.

For the remainder of this section we will restrict ourselves to the case when $\mathfrak{K}$ is one dimensional. In this case every boundary weight of a CP-flow is of the form $\omega(A) = \mu((I-\Lambda)^{-\frac{1}{2}}A(I-\Lambda)^{-\frac{1}{2}})$ for $A \in \cup_{t>0} U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$ with μ a positive element of $\mathfrak{B}(\mathfrak{H})_*$ with $\mu(I) \leq 1$. If ω_1 and ω_2 are boundary weights giving rise to CP-semigroups α_t and β_t we say ω_1 q-dominates ω_2 written $\omega_1 \geq_q \omega_2$ if $\alpha_t - \beta_t$ is completely positive for each $t \geq 0$. In the case when $\mathfrak{K}$ is one dimensional the generalized boundary representation is given by

$$\pi_t^\#(A) = \frac{\omega_t(A)}{1 + \omega_t(\Lambda)}$$

for all $t > 0$ and $A \in \mathfrak{B}(\mathfrak{H})$ where $\omega_t(A) = \omega((I - E(t))A(I - E(t))$. Then from theorem 3.4 and 3.7 it follows that $\omega_1 \geq_q \omega_2$ if and only if

$$\frac{\omega_{1t}}{1 + \omega_{1t}(\Lambda)} \geq \frac{\omega_{2t}}{1 + \omega_{2t}(\Lambda)}$$

for each $t > 0$. If the above inequality is satisfied it follows that $\omega_{1t}(\Lambda) \geq \omega_{2t}(\Lambda)$ and, therefore,

$$\frac{\omega_{1t}}{1 + \omega_{1t}(\Lambda)} \geq \frac{\omega_{2t}}{1 + \omega_{2t}(\Lambda)} \geq \frac{\omega_{2t}}{1 + \omega_{1t}(\Lambda)}$$

and, hence, $\omega_{1t} \geq \omega_{2t}$ for all $t > 0$ and this is equivalent to $\omega_1 \geq \omega_2$. Hence, $\omega_1 \geq_q \omega_2$ implies $\omega_1 \geq \omega_2$.

A weight ω is q-pure if $\omega \geq_q \rho \geq_q 0$ implies ρ is a multiple of ω. From now on we will use a subscript t or s on a functional for that functional cut off to the left of t so $\omega_t(A) = \omega((I - E(t))A(I - E(t)))$ for $A \in \mathfrak{B}(\mathfrak{H})$ and $t > 0$.

Theorem 3.8. *Suppose ω and η are boundary weights and $\omega \geq_q \eta \geq 0$. Then $\omega \geq \eta$. Furthermore, if $\omega(I) = \infty$ then $\eta(I)$ is infinite or zero.*

Proof. Suppose ω and η are boundary weights and $\omega \geq_q \eta$. We have seen that $\omega \geq \eta$. Next suppose $\omega(I) = \infty$. Since $\omega(I - \Lambda) \leq 1$ we have $\omega_t(\Lambda) \to \infty$ as $t \to 0+$. Since $\omega \geq_q \eta$ we have

$$\frac{\eta_t}{1 + \eta_t(\Lambda)} \leq \frac{\omega_t}{1 + \omega_t(\Lambda)}$$

for all $t > 0$ and, hence, for $0 < t \leq s$ we have

$$\eta(I - E(s)) = \eta_t(I - E(s)) \leq \frac{1 + \eta_t(\Lambda)}{1 + \omega_t(\Lambda)}\omega(I - E(s))$$

Now suppose $\eta(I) < \infty$. Since $\eta(\Lambda) \leq \eta(I) < \infty$ and since $\omega(\Lambda) = \infty$ the above ratio tends to zero as $t \to 0+$ and we have $\eta(I - E(s)) = 0$ for all $s > 0$ but this implies $\eta = 0$ so either $\eta(I) = 0$ or $\eta(I) = \infty$.

Theorem 3.9. *Suppose ω is a boundary weight and $\rho \in \mathfrak{B}(\mathfrak{H})_*$ is positive (so $0 \leq \rho(I) < \infty$) and $\omega \geq \rho$ and $\eta = \lambda(1 + \rho(\Lambda))^{-1}(\omega - \rho)$ with $0 \leq \lambda \leq 1$. Then $\omega \geq_q \eta$. Conversely suppose ω and η are boundary weights and $\omega \geq_q \eta \geq 0$ and $\eta \neq 0$. Then there is a positive $\rho \in \mathfrak{B}(\mathfrak{H})_*$ (so $\rho(I) < \infty$) and a real number $\lambda \in (0, 1]$ so that $\omega \geq \rho$ and $\eta = \lambda(1 + \rho(\Lambda))^{-1}(\omega - \rho)$. Furthermore, if $\omega(I) = \infty$ then ρ and λ are unique.*

Proof. Assume the first sentence in the statement of the theorem is satisfied with $0 < \lambda \leq 1$. Note $\rho(\Lambda) - \rho_t(\Lambda) = \rho(E(t)\Lambda E(t)) \geq 0$ for $t \geq 0$. We have for $t > 0$

$$\begin{aligned}\frac{\eta_t}{1 + \eta_t(\Lambda)} &= \frac{\omega_t - \rho_t}{\lambda^{-1}(1 + \rho(\Lambda)) + \omega_t(\Lambda) - \rho_t(\Lambda)} \\ &\leq \frac{\omega_t - \rho_t}{1 + \rho(\Lambda) + \omega_t(\Lambda) - \rho_t(\Lambda)} \leq \frac{\omega_t}{1 + \omega_t(\Lambda)}\end{aligned}$$

Hence, $\omega \geq_q \eta$. If $\lambda = 0$ then $\omega \geq_q \eta$ so we have proved the implication in one direction.

Next assume ω and η are boundary weights and $\omega \geq_q \eta \geq_q 0$ and $0 < t < s$. Let $h(t) = (1 + \eta_t(\Lambda))/(1 + \omega_t(\Lambda))$ for $t > 0$. Since $\omega \geq_q \eta$ we have

$$\begin{aligned}\frac{\eta_t(\Lambda) - \eta_s(\Lambda)}{1 + \eta_t(\Lambda)} &= \frac{\eta_t((E(s) - E(t))\Lambda)}{1 + \eta_t(\Lambda)} \\ &\leq \frac{\omega_t((E(s) - E(t))\Lambda)}{1 + \omega_t(\Lambda)} = \frac{\omega_t(\Lambda) - \omega_s(\Lambda)}{1 + \omega_t(\Lambda)}\end{aligned}$$

Multiplying by the common denominator and rearranging we have

$$(1 + \eta_t(\Lambda))(1 + \omega_s(\Lambda)) \leq (1 + \eta_s(\Lambda))(1 + \omega_t(\Lambda))$$

and dividing by $(1 + \omega_s(\Lambda))(1 + \omega_t(\Lambda))$ we find $h(t) \leq h(s)$. Hence, h is increasing and the limit as $t \to 0+$ exists. We denote this limit by κ so $h(t) \to \kappa$ as $t \to 0+$.

Since $\omega \geq_q \eta$ we have $h(t)\omega_t \geq \eta_t$ for all $t > 0$ and, hence, $\kappa\omega \geq \eta$. We see that if $\kappa = 0$ then $\eta = 0$ and since $\eta \neq 0$ we have $\kappa > 0$. Let $\rho = \omega - \kappa^{-1}\eta$. Since $\kappa\omega \geq \eta$ we have ρ is positive. Note $\omega \geq \rho$ and $\eta = \kappa(\omega - \rho)$. Since $h(t)$ decreases to κ as $t \to 0+$ we have $h(t) \geq \kappa$ for all $t > 0$ and, hence,

$$\frac{1 + \kappa(\omega_t(\Lambda) - \rho_t(\Lambda))}{1 + \omega_t(\Lambda)} \geq \kappa$$

for $t > 0$. Hence, $\rho_t(\Lambda) \leq \kappa^{-1} - 1$ for $t > 0$ and so $\rho(\Lambda) \leq \kappa^{-1} - 1$. Since $\omega \geq \rho$ and $\omega(I - \Lambda) \leq 1$ we have $\rho(I) = \rho(I - \Lambda) + \rho(\Lambda) \leq \kappa^{-1}$ so $\rho(I) < \infty$ and ρ is normal. Since $\kappa \leq (1 + \rho(\Lambda))^{-1}$ we have $\kappa = \lambda(1 + \rho(\Lambda))^{-1}$ with $\lambda \in (0, 1]$ and $\eta = \lambda(1 + \rho(\Lambda))^{-1}(\omega - \rho)$.

Finally, suppose $\omega(I) = \infty$. Suppose λ and λ' are in $(0, 1]$ and ρ and ρ' are positive elements of $\mathfrak{B}(\mathfrak{H})_*$ and $\eta = \lambda(1 + \rho(\Lambda))^{-1}(\omega - \rho)$ and $\eta = \lambda'(1 + \rho'(\Lambda))^{-1}(\omega - \rho')$. Then we have

$$(\lambda(1 + \rho(\Lambda))^{-1} - \lambda'(1 + \rho'(\Lambda))^{-1})\omega = \lambda(1 + \rho(\Lambda))^{-1}\rho - \lambda'(1 + \rho'(\Lambda))^{-1}\rho'$$

Since ω is unbounded and ρ and ρ' are bounded the coefficient of ω is zero so $\lambda(1 + \rho(\Lambda))^{-1} = \lambda'(1 + \rho'(\Lambda))^{-1}$. Then the right hand side must be zero so $\rho = \rho'$. From what we have just shown this implies $\lambda = \lambda'$. $\square$

Theorem 3.10. *Suppose ω is a boundary weight. Then ω is of the form*

$$\omega(A) = \sum_{k \in I} (f_k, (I - \Lambda)^{-\frac{1}{2}} A (I - \Lambda)^{-\frac{1}{2}} f_k)$$

for $A \in \cup_{t>0} U(t)^* \mathfrak{B}(\mathfrak{H}) U(t)$ where $f_k \in \mathfrak{H} = L^2(0, \infty)$ for each $k \in I$ and the f_k are mutually orthogonal and the sum of the $\|f_k\|^2$ for $k \in I$ is not greater than one. The boundary weight ω is q-pure if and only if the index set I contains one element or each vector g of the form

$$g = \sum_{k \in I} z_k f_k \qquad \text{with} \qquad 0 < \sum_{k \in I} |z_k|^2 \leq 1$$

is not in the domain of $(I - \Lambda)^{-\frac{1}{2}}$ which is the range of $(I - \Lambda)^{\frac{1}{2}}$.

Proof. Suppose ω is a boundary weight. For each $t > 0$ let

$$\nu_t(A) = \omega_t((I - \Lambda)^{\frac{1}{2}} A (I - \Lambda)^{\frac{1}{2}})$$

for each $A \in \mathfrak{B}(\mathfrak{H})$. From the properties of a weight if follows that $\nu_t \in \mathfrak{B}(\mathfrak{H})_*$ is positive and ν_t converges in norm to a positive element $\nu_o \in \mathfrak{B}(\mathfrak{H})_*$ as $t \to 0+$ and $\|\nu_o\| = \nu_o(I) \leq 1$. It is well know that each positive $\nu_o \in \mathfrak{B}(\mathfrak{H})_*$ with $\nu_o(I) \leq 1$ can be expressed in the form

$$\nu_o(A) = \sum_{k \in I} (f_k, A f_k)$$

where the f_k have the properties stated in the theorem. It now follows that ω has the form stated in the theorem.

If the index set I has only one term the conclusion of the theorem follows immediately. Suppose then that I has more than one element. Suppose ω is not q-pure. Then from the previous theorem there is a positive normal functional η with $\omega \geq \eta$. And for each positive normal functional η there is a pure positive normal functional ρ with $\eta \geq \rho$. Then we have $\omega \geq \rho$. Now ρ is of the form $\rho(A) = (f_o, Af_o)$ for all $A \in \mathfrak{B}(\mathfrak{H})$ with $f_o \in \mathfrak{H}$. Then we have

$$\nu_o(A) = \omega((I - \Lambda)^{\frac{1}{2}} A (I - \Lambda)^{\frac{1}{2}}) \geq \rho((I - \Lambda)^{\frac{1}{2}} A (I - \Lambda)^{\frac{1}{2}})$$

for all positive $A \in \cup_{t>0} U(t)^* \mathfrak{B}(\mathfrak{H}) U(t)$ and this extends to all positive $A \in \mathfrak{B}(\mathfrak{H})$. Setting $Af = (h, f)h$ for all $f \in \mathfrak{H}$ where $h \in \mathfrak{H}$ is an arbitrary vector we have

$$\sum_{k \in I} |(f_k, h)|^2 \geq |(g, h)|^2$$

where $g = (I - \Lambda)^{\frac{1}{2}} f_o$. Let $z_k = (f_k, g)/\|f_k\|$ for $k \in I$. Setting $h = f_k$ in the above inequality we see $|z_k|^2 \leq \|f_k\|^2$ so the sum of $|z_k|^2$ over $k \in I$ is not greater than one. Let

$$g_o = g - \sum_{k \in I} z_k f_k$$

Note g_o is orthogonal to the f_k. Setting $h = g_o$ in the inequality above we find $(g, g_o) = 0$. Since $(g, g_o) = (g_o, g_o)$ we have $g_o = 0$. Hence,

$$g = \sum_{k \in I} z_k f_k$$

and since $g = (I - \Lambda)^{\frac{1}{2}} f_o$ we have g is in the domain of $(I - \Lambda)^{-\frac{1}{2}}$.

Conversely, suppose there is a vector g of the form given in the statement of the theorem and g is in the domain of $(I - \Lambda)^{-\frac{1}{2}}$ so $g = (I - \Lambda)^{\frac{1}{2}} f_o$. Let $\rho(A) = (f_o, Af_o)$. Let ν_o be defined from ω as above. Now we have $\nu_o(A) \geq (g, Ag)$ for all positive $A \in \mathfrak{B}(\mathfrak{H})$. Hence,

$$\omega((I - \Lambda)^{\frac{1}{2}} A (I - \Lambda)^{\frac{1}{2}}) = \nu_o(A) \geq (g, Ag) = \rho((I - \Lambda)^{\frac{1}{2}} A (I - \Lambda)^{\frac{1}{2}})$$

for all positive $A \in \cup_{t>0} U(t)^* \mathfrak{B}(\mathfrak{H}) U(t)$. For $A \in \cup_{t>0} U(t)^* \mathfrak{B}(\mathfrak{H}) U(t)$ we can replace A with $(I - \Lambda)^{-\frac{1}{2}} A (I - \Lambda)^{-\frac{1}{2}}$ and we find $\omega(A) \geq \rho(A)$ for all positive $A \in \cup_{t>0} U(t)^* \mathfrak{B}(\mathfrak{H}) U(t)$. Hence, by the previous theorem ω is not q-pure. $\square$

Next we will derive a computable condition for determining whether two pure weights give rise to cocycle conjugate E_o-semigroups (see theorems 3.22 and 3.23). Before we begin we make a comment on matrices of linear functionals. Suppose $\mathfrak{H}$ is a separable Hilbert space and n is a positive integer. We denote by $M_n(\mathfrak{B}(\mathfrak{H}))$ the algebra of $(n \times n)$-matrices with entries in $\mathfrak{B}(\mathfrak{H})$. Suppose $\Omega = [\omega_{ij}]$ is an $(n \times n)$ matrix of functionals $\omega_{ij} \in \mathfrak{B}(\mathfrak{H})_*)$ for $i, j = 1, \cdots, n$. We say Ω is positive if

$$\Omega(A) = \sum_{i,j=1}^{n} \omega_{ij}(A_{ij}) \geq 0$$

for $A = [A_{ij}]$ a positive element in $M_n(\mathfrak{B}(\mathfrak{H}))$. One can check that a matrix Ω is positive if and only if

$$\sum_{i,j=1}^{n} \omega_{ij}(A_i^* A_j) \geq 0$$

for $A_i \in \mathfrak{B}(\mathfrak{H})$ for $i = 1, \cdots, n$. This is seen as follows. Since every positive $(n \times n)$ matrix T with entries in $\mathfrak{B}(\mathfrak{H})$ is of the form $T = X^*X = X^*E_1X + \cdots + X^*E_nX$ with E_i the matrix with all zero entries except for the diagonal (i,i) entry $(E_i)_{ii} = I$ it follows that T is the sum of at most n positive matrices of the form $[A_{ij}] = [A_i^*A_j]$. Note that if $\Omega \in M_n(\mathfrak{B}(\mathfrak{H})_*)$ is positive if an only if there are a countable set of vectors $F_k = (f_{1k}, \cdots, f_{nk}) \in \oplus_{i=1}^n \mathfrak{H})$ for $k \in I$ so that

$$\omega_{ij}(A) = \sum_{k \in I} (f_{ik}, Af_{jk})$$

for $i, j = 1, \cdots, n$ and $A \in \mathfrak{B}(\mathfrak{H})$ and

$$\sum_{k \in I} \|F_k\|^2 = \sum_{k \in I} \sum_{i=1}^{n} \|f_{ik}\|^2 < \infty.$$

Note that $\Omega \in M_n(\mathfrak{B}(\mathfrak{H})_*)$ is positive if and only if the mapping $A \to [\omega_{ij}(A_{ij})]$ from $M_n(\mathfrak{B}(\mathfrak{H}))$ to $M_n(\mathbb{C})$ is completely positive.

Next suppose $\mathfrak{H} = \mathfrak{K} \otimes L^2(0, \infty)$ and Ω is an $(n \times n)$ matrix of weights on $\cup_{t>0} U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$. We say Ω is a positive boundary weight if there is a positive $\Xi = [\xi_{ij}] \in M_n(\mathfrak{B}(\mathfrak{H})_*)$ so that

$$\omega_{ij}(A) = \xi_{ij}((I - \Lambda)^{-\frac{1}{2}} A (I - \Lambda)^{-\frac{1}{2}})$$

for all $A \in \cup_{t>0} U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$ and $i, j = 1, \cdots, n$.

We begin with the definition.

Definition 3.11. Suppose ω and η are positive boundary weights. We say a linear map γ is a corner from ω to η if the matrix

$$\begin{bmatrix} \omega & \gamma \\ \gamma^* & \eta \end{bmatrix}$$

is positive boundary weight. We say γ is maximal whenever ω' is a boundary weight so that

$$\begin{bmatrix} \omega & \gamma \\ \gamma^* & \eta \end{bmatrix} \geq \begin{bmatrix} \omega' & \gamma \\ \gamma^* & \eta \end{bmatrix} \geq 0$$

then $\omega' = \omega$. We say γ is hyper maximal if ω' and η' are boundary weights so that

$$\begin{bmatrix} \omega & \gamma \\ \gamma^* & \eta \end{bmatrix} \geq \begin{bmatrix} \omega' & \gamma \\ \gamma^* & \eta' \end{bmatrix} \geq 0$$

then $\omega' = \omega$ and $\eta' = \eta$.

Again, suppose ω and η are boundary weights. We say γ is a q-corner from ω to η if the matrix

$$\begin{bmatrix} \frac{\omega_t}{1+\omega_t(\Lambda)} & \frac{\gamma_t}{1+\gamma_t(\Lambda)} \\ \frac{\gamma_t^*}{1+\gamma_t^*(\Lambda)} & \frac{\eta_t}{1+\eta_t(\Lambda)} \end{bmatrix}$$

on $\mathfrak{B}(\mathfrak{H}\oplus\mathfrak{H})$ is positive for all $t>0$. We say ω and η are disjoint if the only corner between ω and η is the zero weight. We say ω and η are connected if they are not disjoint. We say a q-corner γ from ω to η is maximal whenever ω' is boundary weight so that

$$\begin{bmatrix} \frac{\omega_t}{1+\omega_t(\Lambda)} & \frac{\gamma_t}{1+\gamma_t(\Lambda)} \\ \frac{\gamma_t^*}{1+\gamma_t^*(\Lambda)} & \frac{\eta_t}{1+\eta_t(\Lambda)} \end{bmatrix} \geq \begin{bmatrix} \frac{\omega_t'}{1+\omega_t'(\Lambda)} & \frac{\gamma_t}{1+\gamma_t(\Lambda)} \\ \frac{\gamma_t^*}{1+\gamma_t^*(\Lambda)} & \frac{\eta_t}{1+\eta_t(\Lambda)} \end{bmatrix} \geq 0$$

for all $t>0$ then $\omega'=\omega$. We say a q-corner γ from ω to η is hyper maximal if ω' and η' are boundary weights so that

$$\begin{bmatrix} \frac{\omega_t}{1+\omega_t(\Lambda)} & \frac{\gamma_t}{1+\gamma_t(\Lambda)} \\ \frac{\gamma_t^*}{1+\gamma_t^*(\Lambda)} & \frac{\eta_t}{1+\eta_t(\Lambda)} \end{bmatrix} \geq \begin{bmatrix} \frac{\omega_t'}{1+\omega_t'(\Lambda)} & \frac{\gamma_t}{1+\gamma_t(\Lambda)} \\ \frac{\gamma_t^*}{1+\gamma_t^*(\Lambda)} & \frac{\eta_t'}{1+\eta_t'(\Lambda)} \end{bmatrix} \geq 0$$

for all $t>0$ then $\omega'=\omega$ and $\eta'=\eta$.

Note that γ is a q-corner from ω to ρ then γ^* is a q-corner from ρ to ω so saying ω and ρ are connected is the same as saying ρ and ω are connected. Note that if γ is a hyper maximal q-corner from ω to ρ if and only if γ is a maximal q-corner and γ^* is a maximal q-corner from ρ to ω. The next lemma gives a description of ordinary corners (not q-corners) between positive functional ω and ρ in $\mathfrak{B}(\mathfrak{H})_*$.

Lemma 3.12. *Suppose ω_1 and ω_2 are positive elements of $\mathfrak{B}(\mathfrak{H})_*$ and Ω_1 and Ω_2 are density matrices giving ω_1 and ω_2 so that*

$$\omega_1(A)=tr(A\Omega_1) \qquad \textit{and} \qquad \omega_2(A)=tr(A\Omega_2)$$

for $A\in\mathfrak{B}(\mathfrak{H})$ where tr is the trace normalized so that the trace of a rank one projection is one. Let $\mathfrak{M}_1$ be the closure of the range of Ω_1 and let $\mathfrak{M}_2$ be the closure of the range of Ω_1. Then γ is a corner between ω_1 and ω_2 if and only if γ is of the form

$$\gamma(A)=tr(A\Omega_2^{\frac{1}{2}}X\Omega_1^{\frac{1}{2}})$$

for $A\in\mathfrak{B}(\mathfrak{H})$ with X an operator from $\mathfrak{M}_1$ to $\mathfrak{M}_2$ and $\|X\|\leq 1$. A corner γ from ω_1 to ω_2 is maximal if and only if X is an isometry of $\mathfrak{M}_1$ into $\mathfrak{M}_2$. A corner is hyper maximal if and only if X is a unitary from $\mathfrak{M}_1$ to $\mathfrak{M}_2$ so both X and X^* are isometries. It follows that there is a maximal corner from ω_1 to ω_2 if and only if the rank of ω_2 is greater than the rank of ω_1 and there is a hyper maximal corner from ω_1 to ω_2 if and only if the ranks of ω_1 and ω_2 are equal.

Proof. Assume the notation of the theorem. One sees γ is a corner from ω_1 to ω_2 if and only if $\gamma(A)=tr(A\Psi)$ and the matrix

$$\Omega=\begin{bmatrix} \Omega_1 & \Psi^* \\ \Psi & \Omega_2 \end{bmatrix}$$

is positive. Suppose $\Psi = \Omega_2^{\frac{1}{2}} X \Omega_1^{\frac{1}{2}}$. Suppose $F = \{f, g\} \in \mathfrak{H} \oplus \mathfrak{H}$. Then

$$\begin{aligned}(F, \Omega F) &= (f, \Omega_1 f) + (f, \Psi^* g) + (g, \Psi f) + (g, \Omega_2 g) \\ &= (\Omega_1^{\frac{1}{2}} f, \Omega_1^{\frac{1}{2}} f) + (\Omega_1^{\frac{1}{2}} f, X^* \Omega_2^{\frac{1}{2}} g) + (\Omega_2^{\frac{1}{2}} g, X \Omega_1^{\frac{1}{2}} f) + (\Omega_2^{\frac{1}{2}} g, \Omega_2^{\frac{1}{2}} g) \\ &\geq \|\Omega_1^{\frac{1}{2}} f\|^2 + \|\Omega_1^{\frac{1}{2}} g\|^2 - 2\|\Omega_1^{\frac{1}{2}} f\| \, \|\Omega_1^{\frac{1}{2}} g\| \\ &= (\|\Omega_1^{\frac{1}{2}} f\| - \|\Omega_1^{\frac{1}{2}} g\|)^2 \geq 0.\end{aligned}$$

Hence, Ω is positive. Conversely suppose Ω given in terms of Ω_1, Ω_2 and Ψ is positive. Let $F = \{f, cg\}$ with c a complex number. Since Ω is positive we have

$$(F, \Omega F) = (f, \Omega_1 f) + 2Re(c(f, \Psi^* g)) + |c|^2 (g, \Omega_2 g) \geq 0$$

Since c is arbitrary we have $|(f, \Psi^* g)|^2 \leq (f, \Omega_1 f)(g, \Omega_2 g)$. Let $\langle \cdot, \cdot \rangle$ be the bilinear form on (Range $\Omega_1^{\frac{1}{2}}$)$\times$ (Range $\Omega_2^{\frac{1}{2}}$) of the form

$$\langle \Omega_1^{\frac{1}{2}} f, \Omega_2^{\frac{1}{2}} g \rangle = (f, \Psi g)$$

From the estimate above we have $\langle \Omega_2^{\frac{1}{2}} f, \Omega_2^{\frac{1}{2}} g \rangle \leq \|\Omega_1^{\frac{1}{2}} f\| \, \|\Omega_2^{\frac{1}{2}} g\|$. Since the bilinear form $\langle \cdot, \cdot \rangle$ is norm continuous it has a unique norm continuous extension to Cartesian product of the closures of the ranges of $\Omega_1^{\frac{1}{2}}$ and $\Omega_2^{\frac{1}{2}}$. And by the Riesz representation theorem there is a unique linear operator X^* with $\|X^*\| \leq 1$ from $\mathfrak{M}_2$ to $\mathfrak{M}_1$ so that $\langle \Omega_1^{\frac{1}{2}} f, \Omega_2^{\frac{1}{2}} g \rangle = (\Omega_1^{\frac{1}{2}} f, X \Omega_2^{\frac{1}{2}} g) = (f, \Psi^* g)$. Then we have

$$(f, \Psi^* g) = \langle \Omega_1^{\frac{1}{2}} f, \Omega_2^{\frac{1}{2}} g \rangle = (\Omega_1^{\frac{1}{2}} f, X^* \Omega_2^{\frac{1}{2}} g) = (f, \Omega_1^{\frac{1}{2}} X^* \Omega_2^{\frac{1}{2}} g)$$

for all $f, g \in \mathfrak{H}$ and, hence, $\Psi^* = \Omega_1^{\frac{1}{2}} X^* \Omega_2^{\frac{1}{2}}$ and then $\Psi = \Omega_2^{\frac{1}{2}} X \Omega_1^{\frac{1}{2}}$.

Suppose γ is a maximal corner from ω_1 to ω_2. Since γ is a corner we have $\gamma(A) = tr(A \Omega_2^{\frac{1}{2}} X \Omega_1^{\frac{1}{2}})$ for $A \in \mathfrak{B}(\mathfrak{H})$ where X is a bounded operator from $\mathfrak{M}_1$ to $\mathfrak{M}_2$ with norm not greater than one. Let $\omega_1'(A) = \mathrm{tr}(\mathrm{A} \Omega_1^{\frac{1}{2}} X^* X \Omega_1^{\frac{1}{2}})$ for $A \in \mathfrak{B}(\mathfrak{H})$. One checks that

$$0 \leq \begin{bmatrix} \omega_1' & \gamma \\ \gamma^* & \omega_2 \end{bmatrix} \leq \begin{bmatrix} \omega_1 & \gamma \\ \gamma^* & \omega_2 \end{bmatrix}$$

so $\omega' = \omega$ since γ is maximal. Since $\omega' = \omega$ we have $\Omega_1^{\frac{1}{2}} X^* X \Omega_1^{\frac{1}{2}} = \Omega_1 = \Omega_1^{\frac{1}{2}} I \Omega_1^{\frac{1}{2}}$ and X is an isometry.

Now suppose X is an isometry from $\mathfrak{M}_1$ to $\mathfrak{M}_2$ and $\gamma(A) = \mathrm{tr}(\mathrm{A} \Omega_2^{\frac{1}{2}} X \Omega_1^{\frac{1}{2}})$ for $A \in \mathfrak{B}(\mathfrak{H})$ and

$$0 \leq \begin{bmatrix} \omega_1' & \gamma \\ \gamma^* & \omega_2' \end{bmatrix} \leq \begin{bmatrix} \omega_1 & \gamma \\ \gamma^* & \omega_2 \end{bmatrix}$$

Then there are operators T_1 and T_2 with $0 \leq T_i \leq P_i$ with P_i the range projection for $\Omega_i^{\frac{1}{2}}$ so that $\omega_i'(A) = tr(A \Omega_i^{\frac{1}{2}} T_i \Omega_i^{\frac{1}{2}})$ for $A \in \mathfrak{B}(\mathfrak{H})$ and $i = 1, 2$. It follows that

$$\begin{bmatrix} \Omega_1^{\frac{1}{2}} T_1 \Omega_1^{\frac{1}{2}} & \Omega_1^{\frac{1}{2}} X^* \Omega_2^{\frac{1}{2}} \\ \Omega_2^{\frac{1}{2}} X \Omega_1^{\frac{1}{2}} & \Omega_2^{\frac{1}{2}} T_2 \Omega_2^{\frac{1}{2}} \end{bmatrix} \geq 0.$$

Hence, we have

$$|(f, \Omega_2^{\frac{1}{2}} X \Omega_1^{\frac{1}{2}} g)|^2 \leq (f, \Omega_2^{\frac{1}{2}} T_2 \Omega_2^{\frac{1}{2}} f)(g, \Omega_1^{\frac{1}{2}} T_1 \Omega_1^{\frac{1}{2}} g)$$

for all $f, g \in \mathfrak{H}$. Since X is an isometry from the $\mathfrak{M}_1$ to $\mathfrak{M}_2$ it follows that for $g \in \mathfrak{H}$

$$\sup\{|(f, \Omega_2^{\frac{1}{2}} X \Omega_1^{\frac{1}{2}} g)|^2; f \in \mathfrak{H},\ (f, \Omega_2 f) \leq 1\} = (g, \Omega_1 g)$$

From the inequality above for $|(f, \Omega_2^{\frac{1}{2}} X \Omega_1^{\frac{1}{2}} g)|^2$ and the fact that $T_2 \leq P_2$ we have for $g \in \mathfrak{H}$

$$\sup\{|(f, \Omega_2^{\frac{1}{2}} X \Omega_1^{\frac{1}{2}} g)|^2; f \in \mathfrak{H},\ (f, \Omega_2 f) \leq 1\} \leq (g, \Omega_1^{\frac{1}{2}} T_1 \Omega_1^{\frac{1}{2}} g).$$

Hence, we have $(g, \Omega_1 g) \leq (g, \Omega_1^{\frac{1}{2}} T_1 \Omega_1^{\frac{1}{2}} g)$ and since $T_1 \leq P_1$ we have $T_1 = P_1$ and, hence, $\omega_1' = \omega_1$. Hence, γ is maximal and we have shown that γ is maximal if and only if X is an isometry.

One sees γ is hyper maximal if and only if γ is a maximal corner from ω_1 to ω_2 and γ^* is a maximal corner joining ω_2 and ω_1. Hence, we see γ is hyper maximal if and only if X and X^* are both isometries which is the same as saying X is unitary.

Note that if rank $\omega_1 \leq$ rank ω_2 then there is an isometry X from $\mathfrak{M}_1$ to $\mathfrak{M}_2$ and the resulting corner is maximal. Conversely, if γ is a maximal corner from ω_1 to ω_2 then the corresponding X is an isometry and, hence, rank $\omega_1 \leq$ rank ω_2. Similarly, there is a hyper maximal corner from ω_1 to ω_2 if and only if rank $\omega_1 =$ rank ω_2. $\square$

We will derive a computable condition for determining whether boundary weights are connected. First we prove a useful lemma.

Lemma 3.13. *Suppose ω and η are positive boundary weights and $\gamma \neq 0$ is a q-corner from ω to η. Then*

$$h(t) = \frac{(1 + \omega_t(\Lambda))^{\frac{1}{2}}(1 + \eta_t(\Lambda))^{\frac{1}{2}}}{|1 + \gamma_t(\Lambda)|}$$

is an non increasing function of t which is bounded above and, therefore, it has a finite limit $\kappa \geq 1$ as $t \to 0+$ and $\kappa\gamma$ is a corner from ω to η. Also, the function

$$h_1(t) = \frac{(1 + \omega_t(\Lambda))^{\frac{1}{2}}(1 + \eta_t(\Lambda))^{\frac{1}{2}}}{1 + \gamma_t(\Lambda)}$$

converges to a limit as $t \to 0+$. (Note $h(t) = |h_1(t)|$ for $t > 0$).

Conversely, if $\kappa \geq 1$ and $\kappa\gamma$ is a corner from ω to η and

$$\frac{(1 + \omega_t(\Lambda))^{\frac{1}{2}}(1 + \eta_t(\Lambda))^{\frac{1}{2}}}{|1 + \gamma_t(\Lambda)|} \leq \kappa$$

for all $t > 0$ then γ is a q-corner from ω to ρ. If ω and η are unbounded then the corner $\kappa\gamma$ from ω to η is trivially maximal in that $\lambda\kappa\gamma$ is not a corner from ω to η for $\lambda > 1$.

Proof. Assume the notation of the lemma and γ is a q-corner from ω to η. Then from definition 3.11 we have the matrix

$$\Xi_t = \begin{bmatrix} \frac{\omega_t}{1+\omega_t(\Lambda)} & \frac{\gamma_t}{1+\gamma_t(\Lambda)} \\ \frac{\gamma_t^*}{1+\gamma_t^*(\Lambda)} & \frac{\eta_t}{1+\eta_t(\Lambda)} \end{bmatrix}$$

is positive for all $t > 0$. Let $y = 1 + \gamma_t(\Lambda)$ and $z = y/|y|$. It then follows the matrix

$$\begin{aligned}\Omega_t &= \begin{bmatrix} z(1+\omega_t(\Lambda))^{\frac{1}{2}} & 0 \\ 0 & (1+\eta_t(\Lambda))^{\frac{1}{2}} \end{bmatrix} \Xi_t \begin{bmatrix} \overline{z}(1+\omega_t(\Lambda))^{\frac{1}{2}} & 0 \\ 0 & (1+\eta_t(\Lambda))^{\frac{1}{2}} \end{bmatrix} \\ &= \begin{bmatrix} \omega_t & h(t)\gamma_t \\ h(t)\gamma_t^* & \eta_t \end{bmatrix}\end{aligned}$$

is positive for all $t > 0$ with

$$h(t) = \frac{(1+\omega_t(\Lambda))^{\frac{1}{2}}(1+\eta_t(\Lambda))^{\frac{1}{2}}}{|1+\gamma_t(\Lambda)|}$$

Since the matrix Ω_t is positive we have for $A, B \in (I-\Lambda)^{\frac{1}{2}}\mathfrak{B}(\mathfrak{H})(I-\Lambda)^{\frac{1}{2}}$ and $t > 0$

$$|h(t)\gamma_t(A^*B)|^2 \leq \omega_t(A^*A)\eta_t(B^*B).$$

Since we can arrange for $\gamma(A^*B) \neq 0$ it follows that $h(t)$ is bounded as $t \to 0+$.

Next we will show $h(t)$ is a non increasing function of t. Assume $0 < t < s$. Note that

$$\omega_t(\Lambda) - \omega_s(\Lambda) = \omega_t(X(s)) \qquad \text{where} \qquad X(s) = E(s)\Lambda E(s)$$

and similar relations hold for η and γ. Since the matrix Ω_t is positive we have

$$\begin{bmatrix} \omega_t(X_s) & h(t)\gamma_t(X_s) \\ h(t)\gamma_t^*(X_s) & \eta_t(X_s) \end{bmatrix} = \begin{bmatrix} \omega_t(\Lambda) - \omega_s(\Lambda) & h(t)(\gamma_t(\Lambda) - \gamma_s(\Lambda)) \\ h(t)(\gamma_t^*(\Lambda) - \gamma_s^*(\Lambda)) & \eta_t(\Lambda) - \eta_s(\Lambda) \end{bmatrix}$$

is positive. Hence, the determinant is non negative so we have

$$(\omega_t(\Lambda) - \omega_s(\Lambda))(\eta_t(\Lambda) - \eta_s(\Lambda)) \geq h(t)^2|\gamma_t(\Lambda) - \gamma_s(\Lambda)|^2$$

and recalling the definition of $h(t)$ we have

$$\begin{aligned}|1+\gamma_t(\Lambda)|^2(\omega_t(\Lambda) - &\omega_s(\Lambda))(\eta_t(\Lambda) - \eta_s(\Lambda)) \\ &\geq |\gamma_t(\Lambda) - \gamma_s(\Lambda)|^2(1+\omega_t(\Lambda))(1+\eta_t(\Lambda)).\end{aligned}$$

To simplify this inequality we let

$$f(x) = 1 + \omega_x(\Lambda), \qquad g(x) = 1 + \eta_x(\Lambda), \qquad k(x) = 1 + \gamma_x(\Lambda)$$

for $x > 0$. Note f and g are non increasing functions of x. In terms of these functions the above inequality becomes

$$|k(t)|^2(1 - f(s)/f(t))(1 - g(s)/g(t)) \geq |k(t) - k(s)|^2.$$

Now for $a, b \in [0,1]$ we have $a - 2\sqrt{ab} + b = (\sqrt{a} - \sqrt{b})^2 \geq 0$ which yields

$$(1 - \sqrt{ab})^2 = 1 - 2\sqrt{ab} + ab \geq 1 - a - b + ab = (1-a)(1-b).$$

Using this inequality with $a = f(s)/f(t)$ and $b = g(s)/g(t)$ in the above inequality we find

$$(1 - (f(s)g(s))^{\frac{1}{2}}(f(t)g(t))^{-\frac{1}{2}})|k(t)| \geq |k(t) - k(s)|. \tag{3.1}$$

Since $|k(t) - k(s)| \geq |k(t)| - |k(s)|$ we have

$$-(f(s)g(s))^{\frac{1}{2}}(f(t)g(t))^{-\frac{1}{2}})|k(t)| \geq -|k(s)|.$$

Since $k(t)$ and $k(s)$ can not be zero this inequality is equivalent to the inequality

$$h(t) = \frac{f(t)^{\frac{1}{2}}g(t)^{\frac{1}{2}}}{|k(t)|} \geq \frac{f(s)^{\frac{1}{2}}g(s)^{\frac{1}{2}}}{|k(s)|} = h(s)$$

for $0 < t \leq s$. Hence, h is non increasing function of t. Since h is non increasing and $h(t)$ is uniformly bounded it follows that $h(t)$ approaches a limit κ as $t \to 0$.

Now we show the function h_1 in the statement of the lemma has a limit as $t \to 0+$. Since the absolute value of h_1 has a limit and that limit is not zero to show h_1 has a limit it is enough to show the reciprocal has a limit as $t \to 0+$. Let $w(t) = (f(t)g(t))^{\frac{1}{2}}$ for $t > 0$. In terms of the functions k and w we have $h_1(t)^{-1} = k(t)/w(t)$ for $t > 0$. Since $h(t)$ is non increasing $|h_1(t)|^{-1}$ is non decreasing in t. In terms of $w(t)$ inequality (3.1) becomes

$$(1 - w(s)/w(t))|k(t)| \geq |k(t) - k(s)|$$

for $0 < t < s$. Squaring both sides, canceling terms and dividing by $w(s)w(t)$ yield the inequality

$$\frac{-2|k(t)|^2}{w(t)^2} + \frac{w(s)|k(t)|^2}{w(t)^3} \geq \frac{-2Re(k(t)k(s))}{w(s)w(t)} + \frac{|k(s)|^2}{w(s)w(t)}$$

for $0 < t < s$. And this inequality is equivalent to the inequality

$$(1 - w(s)/w(t))(\frac{|k(s)|^2}{w(s)^2} - \frac{|k(t)|^2}{w(t)^2}) \geq \left|\frac{k(t)}{w(t)} - \frac{k(s)}{w(s)}\right|$$

for $0 < t < s$. Since $|k(t)|/w(t)$ is non decreasing and approaches a finite limit as $t \to 0+$ this inequality shows $k(t)/w(t)$ has a limit as $t \to 0+$. Hence, $h_1(t) = w(t)/k(t)$ has a limit as $t \to 0+$.

Note for $0 < t < s$ we have $\omega_s(A) = \omega_t((I - E(s))A(I - E(s))$ for all $A \in \cup_{t>0}U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$. Since Ω_t is positive, it follows that

$$\begin{bmatrix} \omega_s & h(t)\gamma_s \\ h(t)\gamma_s^* & \eta_s \end{bmatrix}$$

is positive for all $0 < t < s$. Then taking the limit as $t \to 0+$ we find the matrix

$$\begin{bmatrix} \omega_s & \kappa\gamma_s \\ \kappa\gamma_s^* & \eta_s \end{bmatrix}$$

is positive for all $s > 0$ and hence the matrix of weights

$$\begin{bmatrix} \omega & \kappa\gamma \\ \kappa\gamma^* & \eta \end{bmatrix}$$

yields a positive weight η on $\mathfrak{B}(\mathfrak{H} \oplus \mathfrak{H})$.

Conversely, suppose $\kappa \geq 1$ and $\kappa\gamma$ is a corner from ω to η and

$$\frac{(1 + \omega_t(\Lambda))^{\frac{1}{2}}(1 + \eta_t(\Lambda))^{\frac{1}{2}}}{|1 + \gamma_t(\Lambda)|} \leq \kappa$$

for all $t > 0$. A straight forward computation shows γ is a q-corner from ω to ρ.

Finally, we show $\kappa\gamma$ is a trivially maximal corner from ω to η if ω and η are unbounded. Suppose $\lambda > 1$ and $\lambda\kappa\gamma$ is a corner from ω to η. From the inequality for κ and the fact that $\omega_t(\Lambda)$, $\eta_t(\Lambda)$ and, therefore, $|\gamma_t(\Lambda)|$ tend to infinity as $t \to 0+$ we have

$$\lim_{t \to 0+} \frac{\omega_t(\Lambda)\eta_t(\Lambda)}{|\gamma_t(\Lambda)|^2} \leq \kappa^2.$$

Since $\lambda\kappa\gamma$ is a corner we have $\lambda^2\kappa^2|\gamma_t(\Lambda)|^2 \leq \omega_t(\Lambda)\eta_t(\Lambda)$ for all $t > 0$ and, therefore,

$$\lim_{t \to 0+} \frac{\omega_t(\Lambda)\eta_t(\Lambda)}{|\gamma_t(\Lambda)|^2} \geq \lambda^2\kappa^2 > \kappa^2.$$

This contradicts the previous limit inequality so $\kappa\gamma$ is a trivially maximal corner from ω to η. $\square$

Theorem 3.14. *Suppose ω and η are non zero boundary weights. Then ω and η are connected if and only if ω and η can be expressed in the form*

$$\omega(A) = \sum_{k \in I} (f_k, (I - \Lambda)^{-\frac{1}{2}} A (I - \Lambda)^{-\frac{1}{2}} f_k)$$

and

$$\eta(A) = \sum_{k \in I} (g_k, (I - \Lambda)^{-\frac{1}{2}} A (I - \Lambda)^{-\frac{1}{2}} g_k)$$

for $A \in \cup_{t>0} U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$ with f_k and g_k in $\mathfrak{H}$ and $g_k = zf_k + h_k$ where $z \neq 0$ with h_k in the domain of $(I - \Lambda)^{-\frac{1}{2}}$ for each $k \in I$ with I a countable index set and

$$\sum_{k \in I} \|(I - \Lambda)^{-\frac{1}{2}} h_k\|^2 < \infty.$$

Proof. Before we begin the proof we remark that in the sums for ω and η we sum over the same index set I. Even though we sum over the same index set the sums for ω and η can have different numbers of non zero terms since some of the $f's$ or $g's$ can be zero.

Assume the setup and notation of the theorem. Assume $g_k = zf_k + h_k$ with $z \neq 0$ and h_k in the domain of $(I - \Lambda)^{-\frac{1}{2}}$ for each $k \in I$ and the sum involving the

h_k given above converges. Since the weight η is unchanged if we multiply the g_k by a complex number of modulus one we can assume without loss of generality that $z > 0$. Since the sum involving the h_k above converges and $0 \leq \Lambda \leq I$ we have

$$r = \sum_{k \in I} \|\Lambda^{\frac{1}{2}}(I - \Lambda)^{-\frac{1}{2}} h_k\|^2 < \infty. \tag{3.2}$$

Let

$$\lambda = \frac{2z}{1 + z^2 + r} \qquad \text{and} \qquad \gamma(A) = \sum_{k \in I} (f_k, (I - \Lambda)^{-\frac{1}{2}} A (I - \Lambda)^{-\frac{1}{2}} g_k)$$

for $A \in \cup_{t>0} U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$. We will show that $\lambda\gamma$ is a q-corner from ω to η. From the form of ω, η and γ it is clear that

$$\begin{bmatrix} \omega_t & \gamma_t \\ \gamma_t^* & \eta_t \end{bmatrix} \geq 0$$

for all $t > 0$. Since the Schur product $X \to A \circ X$ is completely positive if and only if the matrix A is positive. It follows that

$$\begin{bmatrix} \frac{\omega_t}{1+\omega_t(\Lambda)} & \frac{\lambda\gamma_t}{1+\lambda\gamma_t(\Lambda)} \\ \frac{\lambda\gamma_t^*}{1+\lambda\gamma_t^*(\Lambda)} & \frac{\eta_t}{1+\eta_t(\Lambda)} \end{bmatrix}$$

is positive for each $t > 0$ if the matrix

$$\begin{bmatrix} \frac{1}{1+\omega_t(\Lambda)} & \frac{\lambda}{1+\lambda\gamma_t(\Lambda)} \\ \frac{\lambda}{1+\lambda\gamma_t^*(\Lambda)} & \frac{1}{1+\eta_t(\Lambda)} \end{bmatrix}$$

is positive for each $t > 0$. Since the diagonal entries are clearly positive all we need to establish the positivity of this matrix is to show the determinant is positive. Computing the determinant, multiplying by denominators and collecting terms one finds that the determinant of the above matrix is non negative if and only if

$$\lambda^2(1 + \omega_t(\Lambda) + \eta_t(\Lambda) + \omega_t(\Lambda)\eta_t(\Lambda) - |\gamma_t(\Lambda)|^2) \leq 1 + 2\lambda Re(\gamma_t(\Lambda)).$$

To give this inequality a name so we can refer to it we will call this inequality the determinant inequality. We will need some notation. Let

$$\nu(A) = \sum_{k \in I} (f_k, (I - \Lambda)^{-\frac{1}{2}} A (I - \Lambda)^{-\frac{1}{2}} h_k)$$

and

$$\rho(A) = \sum_{k \in I} (h_k, (I - \Lambda)^{-\frac{1}{2}} A (I - \Lambda)^{-\frac{1}{2}} h_k)$$

for $A \in \cup_{t>0} U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$. Then we have

$$\eta = z^2\omega + z\nu + z\nu^* + \rho \qquad \text{and} \qquad \gamma = z\omega + \nu.$$

Assume $t > 0$ and let $a = \omega_t(\Lambda)$, $b = Re(\nu_t(\Lambda))$, $y = Im(\nu_t(\Lambda))$ and $c = \rho_t(\Lambda)$. Then the determinant inequality becomes

$$\lambda^2(1 + a + z^2 a + 2zb + c + ac - b^2 - y^2) \leq 1 + 2\lambda(za + b) \tag{3.3}$$

Recalling the definition (3.2) of r we see $r = \lim_{t\to 0+} \rho_t(\Lambda)$ so $y \geq c$. The we have

$$\lambda = \frac{2z}{1 + z^2 + r} \leq \frac{2z}{1 + z^2 + c}$$

Examining inequality (3.3) one sees that the interval of real numbers λ satisfying the inequality is a closed interval containing zero. Hence, if inequality (3.3) is satisfied for $\lambda = 2z(1+z^2+c)^{-1}$ it will be satisfied for a smaller $\lambda = 2z(1+z^2+r)^{-1}$. Hence, to show inequality (3.3) is satisfied it is enough to show that inequality (3.3) holds with $\lambda = 2z(1 + z^2 + c)^{-1}$. With this value of λ one easily checks that

$$\lambda^2(1 + z^2 + c) = 2\lambda z \tag{3.4}$$

Multiplying both sides by a we have

$$\lambda^2(a + z^2 a + ac) = 2\lambda za \tag{3.5}$$

Since $(\lambda(z - b) - 1)^2 \geq 0$ we have

$$0 \leq \lambda^2(z^2 - 2zb + b^2) - 2\lambda z + 2\lambda b + 1 \tag{3.6}$$

Finally, we have

$$0 \leq \lambda^2 y^2 \tag{3.7}$$

Adding equalities and inequalities (3.4) through (3.7) we arrive at inequality (3.3). Hence, the determinant inequality is satisfied and $\lambda\gamma$ is a non zero q-corner from ω to η. Hence, w and η are connected.

Next assume ω and η are connected. Then there is a non zero q-corner γ from ω to η. Then from lemma 3.13 we have the function $h(t)$ given in lemma 3.13 is non increasing and has limit $\kappa \geq 1$ as $t \to 0+$ and $\kappa\gamma$ is a corner from ω to η so the matrix

$$\Omega_o = \begin{bmatrix} \omega & \kappa\gamma \\ \kappa\gamma^* & \eta \end{bmatrix}$$

is positive. Since Ω_o is a positive boundary weight on $\mathfrak{B}(\mathfrak{H} \oplus \mathfrak{H})$ it can expressed in the form

$$\Omega_o(A) = \sum_{k\in I} (F_k, ((I - \Lambda) \oplus (I - \Lambda))^{-\frac{1}{2}} A((I - \Lambda) \oplus (I - \Lambda))^{-\frac{1}{2}} F_k)$$

for $A \in \mathfrak{B}(\mathfrak{H} \oplus \mathfrak{H})$ where

$$\sum_{k\in I} \|F_k\|^2 < \infty.$$

Now each vector F_k can be expressed in the form $F_k = \{f_k, g_k\}$ for each $k \in I$ so we have

$$\omega(A) = \sum_{k\in I}(f_k, (I-\Lambda)^{-\frac{1}{2}}A(I-\Lambda)^{-\frac{1}{2}}f_k)$$

and

$$\eta(A) = \sum_{k\in I}(g_k, (I-\Lambda)^{-\frac{1}{2}}A(I-\Lambda)^{-\frac{1}{2}}g_k)$$

and

$$\kappa\gamma(A) = \sum_{k\in I}(f_k, (I-\Lambda)^{-\frac{1}{2}}A(I-\Lambda)^{-\frac{1}{2}}g_k)$$

for $A \in \cup_{t>0}U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$ with f_k and g_k in $\mathfrak{H}$ for $k \in I$.

Now consider the family of matrices

$$M_t = \begin{bmatrix} 1+\omega_t(\Lambda) & \kappa+\kappa\gamma_t(\Lambda) \\ \kappa+\kappa\gamma_t^*(\Lambda) & 1+\eta_t(\Lambda) \end{bmatrix}$$

Note that if $0 < t < s$ then $M_t - M_s =$

$$\begin{bmatrix} \omega_t(\Lambda)-\omega_s(\Lambda) & \kappa\gamma_t(\Lambda)-\kappa\gamma_s(\Lambda) \\ \kappa\gamma_t^*(\Lambda)-\kappa\gamma_s^*(\Lambda) & \eta_t(\Lambda)-\eta_s(\Lambda) \end{bmatrix} = \begin{bmatrix} \omega_t(X) & \kappa\gamma_t(X) \\ \kappa\gamma_t^*(X) & \eta_t(X) \end{bmatrix}$$

where $X = E(s)\Lambda E(s)$. Since $\kappa\gamma$ is a corner and X is positive it follows that $M_t \geq M_s$ for $0 < t < s$ so M_t is non increasing in t. Since $h(t) \leq \kappa$ for all $t > 0$ we have

$$(1+\omega_t(\Lambda))(1+\eta_t(\Lambda)) \leq |\kappa+\kappa\gamma_t(\Lambda)|^2$$

In terms of the M_t this inequality tells us the determinant of M_t is not positive. We will show there is a unit vector v so that $(v, M_t v) \leq 0$ for all $t > 0$. Let $\lambda(t)$ be the minimum eigenvalue of M_t and v_t be a unit vector in $\mathbb{C}^2$ so that $(v_t, M_t v_t) = \lambda(t)$ for $t > 0$. Since M_t is not positive we have $\lambda(t) \leq 0$ for all $t > 0$. Since the set of unit vectors in $\mathbb{C}^2$ is compact in the norm topology there is at least one accumulation point of v_t as $t \to 0+$. Let v be such an accumulation point so for each $\epsilon > 0$ there is a $t \in (0,\epsilon)$ with $\|v - v_t\| < \epsilon$. We show $(v, M_t v) \leq 0$ for all $t > 0$. Suppose $t > 0$ and $\epsilon > 0$. Let $\epsilon_1 = \min(\epsilon, \frac{1}{2}\epsilon/\|M_t\|)$. Then there is an $s \in (0, \epsilon_1)$ with $\|v - v_s\| < \epsilon_1$. We have

$$\begin{aligned}(v, M_t v) - (v_s, M_t v_s) &= ((v - v_s), M_t v) + (v_s, M_t(v - v_s)) \\ &\leq 2\|v - v_s\|\,\|M_t\| < 2\epsilon_1\|M_t\| \leq \epsilon.\end{aligned}$$

Since M_t is non increasing and $0 < s < t$ we have

$$(v_s, M_t v_s) - (v_s, M_s v_s) \leq 0.$$

And we have

$$(v_s, M_s v_s) = \lambda(s) \leq 0.$$

Combining the previous four relations we find $(v, M_t v) < \epsilon$ and since ϵ is arbitrary we have $(v, M_t v) \leq 0$ for all t.

We will show there is a constant q and complex number $z \neq 0$ so that

$$|z|^2\omega_t(\Lambda) + \eta_t(\Lambda) - 2\kappa Re(\overline{z}\gamma_t(\Lambda)) \leq q \tag{3.8}$$

We will need to consider four cases depending on whether $\omega_t(\Lambda)$ and $\eta_t(\Lambda)$ are bounded as $t \to 0+$. Let us consider the first case when both $\omega_t(\Lambda)$ and $\eta_t(\Lambda)$ are bounded as $t \to 0+$. In this case M_t has a limit M_o as $t \to 0+$. We can let $z = 1$ and we can let q simply be the limit of the left hand side as $t \to 0+$.

Next we consider the case when $\eta_t(\Lambda)$ is bounded and $\omega_t(\Lambda)$ is unbounded as $t \to 0+$. Now since $h(t)$ is non increasing and approaches a limit κ as $t \to 0+$ we have

$$h(t)^2 = \frac{(1+\omega_t(\Lambda))(1+\eta_t(\Lambda))}{|1+\gamma_t(\Lambda)|^2} \leq \kappa^2$$

for all $t > 0$. Since $\kappa\gamma_t$ is a corner from ω_t to η_t we have $|\kappa\gamma_t(\Lambda)|^2 \leq \omega_t(\Lambda)\eta_t(\Lambda)$. Then we have

$$(1+\omega_t(\Lambda))(1+\eta_t(\Lambda)) \leq \kappa^2|1+\gamma_t(\Lambda)|^2 \leq \kappa^2(1+\kappa^{-1}\sqrt{\omega_t(\Lambda)\eta_t(\Lambda)})^2$$

for $t > 0$. Then it follows that

$$1 + \omega_t(\Lambda) + \eta_t(\Lambda) \leq \kappa^2 + 2\kappa\sqrt{\omega_t(\Lambda)\eta_t(\Lambda)}$$

for $t > 0$. Solving for κ we find

$$\begin{aligned}
\kappa &\geq \sqrt{1+\omega_t(\Lambda)+\eta_t(\Lambda)+\omega_t(\Lambda)\eta_t(\Lambda)} - \sqrt{\omega_t(\Lambda)\eta_t(\Lambda)} \\
&\geq \sqrt{\omega_t(\Lambda)+\omega_t(\Lambda)\eta_t(\Lambda)} - \sqrt{\omega_t(\Lambda)\eta_t(\Lambda)} \\
&= \sqrt{\omega_t(\Lambda)}(\sqrt{1+\eta_t(\Lambda)} - \sqrt{\eta_t(\Lambda)}) \\
&\geq \sqrt{\omega_t(\Lambda)}(\sqrt{1+\eta_o(\Lambda)} - \sqrt{\eta_o(\Lambda)})
\end{aligned}$$

for all $t > 0$ where $\eta_o(\Lambda)$ is the limit of $\eta_t(\Lambda)$ as $t \to 0+$. Since $\omega_t(\Lambda) \to \infty$ as $t \to 0+$ it follows that κ can not be bounded which is a contradiction. Hence, if $\omega_t(\Lambda)$ is unbounded and $\eta_t(\Lambda)$ is bounded as $t \to 0+$ it follows that ω and η are disjoint. Interchanging the roles of ω and η we see that if $\omega_t(\Lambda)$ is bounded and $\eta_t(\Lambda)$ is unbounded as $t \to 0+$ then ω and η are disjoint. So if ω and η are connected then $\omega_t(\Lambda)$ and $\eta_t(\Lambda)$ must be both bounded or both unbounded as $t \to 0+$.

Finally, we consider the case when both $\omega_t(\Lambda)$ and $\eta_t(\Lambda)$ are unbounded as $t \to 0+$. In this case the vector v so that $(v, M_tv) \leq 0$ for all $t > 0$ can not be a multiple of $(1,0)$ or $(0,1)$ because in the first case we have $(v, M_tv) = 1 + \omega_t(\Lambda)$ in the second case $(v, M_tv) = 1 + \eta_t(\Lambda)$ and neither of these are less than zero. So the vector v must be a multiple of a vector of the form $w = (z, -1)$ with $z \neq 0$. Then we have $(w, M_tw) \leq 0$ which yields the inequality

$$|z|^2\omega_t(\Lambda) + \eta_t(\Lambda) - 2\kappa Re(\overline{z}\gamma_t(\Lambda)) \leq 2\kappa Re(z) - 1 - |z|^2 = q \tag{3.8'}$$

for all $t > 0$. Since Ω_o is positive we have

$$\begin{aligned}
|z|^2\omega_t(I-\Lambda) - 2\kappa Re(\overline{z}\gamma_t(I-\Lambda)) + \eta_t(I-\Lambda) \\
&\leq (1+|z|^2)(\omega_t(I-\Lambda) + \eta_t(I-\Lambda)) \\
&\leq (1+|z|^2)(\omega(I-\Lambda) + \eta(I-\Lambda))
\end{aligned}$$

for $t > 0$ and since ω and η are boundary weights $\omega_t(I - \Lambda)$ and $\eta_t(I - \Lambda)$ are bounded as $t \to 0+$ so the right hand side of the above inequality is bounded for all $t > 0$. Combining the two inequalities above we have

$$\begin{aligned}|z|^2\omega_t(I) - 2\kappa Re(\overline{z}\gamma_t(I)) + \eta_t(I)\\ \leq (1 + |z|^2)(\omega(1 - \Lambda) + \eta(I - \Lambda)) + q = K_o\end{aligned}$$

where the last equality is just the definition of K_o. Now we have

$$\omega_t(I) = \sum_{k \in I}(f_k, (I - \Lambda)^{-\frac{1}{2}}(I - E(t))(I - \Lambda)^{-\frac{1}{2}} f_k)$$

and

$$\eta_t(I) = \sum_{k \in I}(g_k, (I - \Lambda)^{-\frac{1}{2}}(I - E(t))(I - \Lambda)^{-\frac{1}{2}} g_k)$$

and

$$\kappa\gamma_t(I) = \sum_{k \in I}(f_k, (I - \Lambda)^{-\frac{1}{2}}(I - E(t))(I - \Lambda)^{-\frac{1}{2}} g_k)$$

Then we find

$$\begin{aligned}&|z|^2\omega_t(I - \Lambda) - 2\kappa Re(\overline{z}\gamma_t(I - \Lambda)) + \eta_t(I - \Lambda)\\ &\quad = \sum_{k \in I}((zf_k - g_k), (I - \Lambda)^{-\frac{1}{2}}(I - E(t))(I - \Lambda)^{-\frac{1}{2}}(f_k - zg_k)) \leq K_o\end{aligned}$$

for $t > 0$. Let $h_k = g_k - zf_k$ for $k \in I$. Then $g_k = zf_k + h_k$ for $k \in I$ and

$$\sum_{k \in I}(h_k, (I - \Lambda)^{-\frac{1}{2}}(I - E(t))(I - \Lambda)^{-\frac{1}{2}} h_k) \leq K_o$$

for all $t > 0$. Hence, we have

$$\sum_{k \in I} \|(I - \Lambda)^{-\frac{1}{2}}(I - E(t))h_k\|^2 \leq K_o$$

Then for each $k \in I$ we have $\|(I - \Lambda)^{-\frac{1}{2}}(I - E(t))h_k\|$ is bounded as $t \to 0$ from which it follows that h_k is in the domain of $(I - \Lambda)^{-\frac{1}{2}}$ and $\|(I - \Lambda)^{-\frac{1}{2}}(I - E(t))h_k\| \to \|(I - \Lambda)^{-\frac{1}{2}} h_k\|$ as $t \to 0+$. Then from the above inequality we have

$$\sum_{k \in I} \|(I - \Lambda)^{-\frac{1}{2}} h_k\|^2 \leq K_o$$

This completes the proof of the theorem. $\square$

We have the following corollary.

Corollary 3.15. *Suppose ω and η are boundary weights. If ω and η are connected then either $\omega_t(\Lambda)$ and $\eta_t(\Lambda)$ are both bounded or both unbounded as $t \to 0+$. Furthermore, if $\omega_t(\Lambda)$ and $\eta_t(\Lambda)$ are both bounded as $t \to 0+$ then they are connected.*

Proof. The first statement of the corollary was proved in the course of proving theorem 3.14. Now suppose $\omega_t(\Lambda)$ and $\eta_t(\Lambda)$ are bounded as $t \to 0+$. Then if ω and η are defined in terms of vectors f_k and g_k as in the statement of theorem 3.14 one sees that the f_k and g_k are in the domain of $(I-\Lambda)^{-\frac{1}{2}}$ and

$$\sum_{k \in I} \|(I-\Lambda)^{-\frac{1}{2}} f_k\|^2 < \infty \qquad \text{and} \qquad \sum_{k \in I} \|(I-\Lambda)^{-\frac{1}{2}} g_k\|^2 < \infty$$

so for any $z \neq 0$ if we let $h_k = g_k - z f_k$ we have

$$\sum_{k \in I} \|(I-\Lambda)^{-\frac{1}{2}} h_k\|^2 < \infty$$

and by theorem 3.14 it follows that ω and η are connected. $\square$

Lemma 3.16. *Suppose ω and η are boundary weights. Then ω and η are connected if and only if there is a $\lambda > 0$ and a positive element ρ of $\mathfrak{B}(\mathfrak{H})_*$ (so $\rho(I) < \infty$) so that $\eta = \lambda\omega + \gamma + \gamma^* + \rho$ and the matrix*

$$\Omega = \begin{bmatrix} \lambda\omega & \gamma \\ \gamma^* & \rho \end{bmatrix}$$

is a positive boundary weight on $\mathfrak{B}(\mathfrak{H} \oplus \mathfrak{H})$.

Proof. Suppose ω and ρ are connected boundary weights. Then from theorem 3.14 it follows that ω and η can be expressed in the form

$$\omega(A) = \sum_{k \in I} (f_k, (I-\Lambda)^{-\frac{1}{2}} A (I-\Lambda)^{-\frac{1}{2}} f_k)$$

and

$$\eta(A) = \sum_{k \in I} (g_k, (I-\Lambda)^{-\frac{1}{2}} A (I-\Lambda)^{-\frac{1}{2}} g_k)$$

for $A \in \cup_{t>0} U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$ with f_k and g_k in $\mathfrak{H}$ and $g_k = z f_k + h_k$ where $z \neq 0$ with h_k in the domain of $(I-\Lambda)^{-\frac{1}{2}}$ for each $k \in I$ and

$$\sum_{k \in I} \|(I-\Lambda)^{-\frac{1}{2}} h_k\|^2 < \infty.$$

Let $\lambda = |z|^2$ and

$$\gamma(A) = \overline{z} \sum_{k \in I} (g_k, (I-\Lambda)^{-\frac{1}{2}} A (I-\Lambda)^{-\frac{1}{2}} h_k)$$

and

$$\rho(A) = \sum_{k \in I} (h_k, (I - \Lambda)^{-\frac{1}{2}} A (I - \Lambda)^{-\frac{1}{2}} h_k)$$

for $A \in \cup_{t>0} U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$. Then $\eta = \lambda\omega + \gamma + \gamma^* + \rho$ and since $\eta(I) < \infty$ we have η is a positive element of $\mathfrak{B}(\mathfrak{H})_*$ and clearly the matrix Ω is positive.

Conversely, suppose ω and η are boundary weights and $\lambda > 0$ and $\eta = \lambda\omega + \gamma + \gamma^* + \rho$ where ρ is a positive element of $\mathfrak{B}(\mathfrak{H})_*$ and the matrix Ω given above is positive. Since Ω is a boundary weight on $\mathfrak{B}(\mathfrak{H} \oplus \mathfrak{H})$ we have it can be expressed in the form

$$\Omega(A) = \sum_{k \in I} (F_k, (I - \Lambda)^{-\frac{1}{2}} A (I - \Lambda)^{-\frac{1}{2}} F_k)$$

where

$$\sum_{k \in I} \|F_k\|^2 < \infty$$

for $A \in \cup_{t>0} U(t) \oplus U(t)\mathfrak{B}(\mathfrak{H} \oplus \mathfrak{H})U(t)^* \oplus U(t)^*$. Now each vector F_k can be expressed in the form $F_k = \{f'_k, h_k\}$ for each $k \in I$ so we have

$$\lambda\omega(A) = \sum_{k \in I} (f'_k, (I - \Lambda)^{-\frac{1}{2}} A (I - \Lambda)^{-\frac{1}{2}} f'_k)$$

and

$$\rho(A) = \sum_{k \in I} (h_k, (I - \Lambda)^{-\frac{1}{2}} A (I - \Lambda)^{-\frac{1}{2}} h_k)$$

and

$$\gamma(A) = \sum_{k \in I} (f'_k, (I - \Lambda)^{-\frac{1}{2}} A (I - \Lambda)^{-\frac{1}{2}} h_k)$$

for $A \in \cup_{t>0} U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$ with g_k and h_k in $\mathfrak{H}$ for $k \in I$. Since $\rho(I) < \infty$ it follows that h_k is in the domain of $(I - \Lambda)^{-\frac{1}{2}}$ and

$$\sum_{k \in I} \|(I - \Lambda)^{-\frac{1}{2}} h_k\|^2 < \infty.$$

Now let $f_k = \lambda^{-\frac{1}{2}} f'_k$ and $z = \lambda^{\frac{1}{2}}$ and $g_k = z f_k + h_k$ for $k \in I$. Then ω and η can be written in terms of the $f's$ and $g's$ as stated in the lemma. □

Theorem 3.17. *Suppose ω, ρ and η are boundary weights and ω and ρ are connected and ρ and η are connected. Then ω and η are connected.*

Proof. Assume the hypothesis of the theorem. Since ω and ρ are connected it follows from the proof of theorem 3.13 that there is a functional γ so the boundary weight Ω_1 and the matrix M_t for $t > 0$

$$\Omega_1 = \begin{bmatrix} \omega & \gamma \\ \gamma^* & \rho \end{bmatrix} \qquad M_t = \begin{bmatrix} \omega_t(\Lambda) & \gamma_t(\Lambda) \\ \gamma_t^*(\Lambda) & \rho_t(\Lambda) \end{bmatrix}$$

are positive and there is a vector $v = (z_1, -1)$ with $z_1 \neq 0$ so that $(v, M_t v)$ is bounded as $t \to 0+$. (Note γ is $\kappa\gamma$ of the argument of theorem 3.14.) Since ρ and

η are connected there is by the same argument a functional ν so that the boundary weight Ω_2 and the matrix N_t for $t > 0$

$$\Omega_2 = \begin{bmatrix} \rho & \nu \\ \nu^* & \eta \end{bmatrix} \qquad N_t = \begin{bmatrix} \rho_t(\Lambda) & \nu_t(\Lambda) \\ \nu_t^*(\Lambda) & \eta_t(\Lambda) \end{bmatrix}$$

are positive and there is a vector $u = (z_2, -1)$ with $z_2 \neq 0$ so that $(u, N_t u)$ is bounded as $t \to 0+$. Next we show there is functional μ so that the boundary weight Ω_3 and the matrix Q_t

$$\Omega_3 = \begin{bmatrix} \omega & \gamma & \mu \\ \gamma^* & \rho & \nu \\ \mu^* & \nu^* & \eta \end{bmatrix} \qquad Q_t = \begin{bmatrix} \omega_t(\Lambda) & \gamma_t(\Lambda) & \mu_t(\Lambda) \\ \gamma_t^*(\Lambda) & \rho_t(\Lambda) & \nu_t(\Lambda) \\ \mu_t^*(\Lambda) & \nu_t^*(\Lambda) & \eta_t(\Lambda) \end{bmatrix}$$

is positive for $t > 0$. One sees that Ω_3 is positive as a boundary weight if one replaces the functional ξ_{ij} in the i^{th} row and j^{th} column by the bounded functional ξ'_{ij} defined by

$$\xi'_{ij}(A) = \xi_{ij}((I - \Lambda)^{\frac{1}{2}} A (I - \Lambda)^{\frac{1}{2}})$$

for $A \in \mathfrak{B}(\mathfrak{H})$. So the problem of finding a functional μ is the same as finding a functional μ in the case where the functionals ω, ρ and η are bounded. Suppose then that

$$\Omega'_1 = \begin{bmatrix} \omega' & \gamma' \\ \gamma'^* & \rho' \end{bmatrix} \qquad \Omega'_2 = \begin{bmatrix} \rho' & \nu' \\ \nu'^* & \eta' \end{bmatrix}$$

are positive functionals on $\mathfrak{B}(\mathfrak{H} \oplus \mathfrak{H})$. From the Gelfand Segal construction we have there are normal cyclic $*$-representations π_1, π_2 and π_3 of $\mathfrak{B}(\mathfrak{H})$ on $\mathfrak{H}_1$, $\mathfrak{H}_2$ and $\mathfrak{H}_3$ so that

$$\omega'(A) = (f_1, \pi_1(A) f_1), \quad \rho'(A) = (f_2, \pi_2(A) f_2) \quad \text{and} \quad \eta'(A) = (f_3, \pi_3(A) f_3)$$

for all $A \in \mathfrak{B}(\mathfrak{H})$. One checks that there are linear operators C_1 from $\mathfrak{H}_2$ to $\mathfrak{H}_1$ and C_2 from $\mathfrak{H}_3$ to $\mathfrak{H}_2$ so that

$$\gamma'(A^*B) = (\pi_1(A) f_1, C_1 \pi_2(B) f_2) \qquad \text{and} \qquad \nu'(A^*B) = (\pi_2(A) f_2, C_2 \pi_3(B) f_3)$$

for all $A, B \in \mathfrak{B}(\mathfrak{H})$. One checks C_1 intertwines π_1 and π_2 and C_2 intertwines π_2 and π_3. Furthermore, one has $\|C_1\| \leq 1$ and $\|C_2\| \leq 1$. Let $C_3 = C_1 C_2$ and let $\mu'(A) = (f_3, C_1 C_2 \pi_1(A) f_1)$. One sees that C_3 intertwines π_1 and π_3. We show the matrix

$$\Omega'_3 = \begin{bmatrix} \omega' & \gamma' & \mu' \\ \gamma'^* & \rho' & \nu' \\ \mu'^* & \nu'^* & \eta' \end{bmatrix}$$

is positive. As we have seen,Ω'_3 is positive if and only

$$\begin{aligned} S =& \omega'(A^*A) + \gamma'(A^*B) + \mu'(A^*C) \\ &+ \gamma'^*(B^*A) + \rho'(B^*B) + \nu'(B^*C) \\ &+ \mu'^*(C^*A) + \nu'^*(C^*B) + \eta'(C^*C) \geq 0 \end{aligned}$$

for all A, B and $C \in \mathfrak{B}(\mathfrak{H})$. Evaluating S in terms of the $\pi's$ and $C's$ we find $S = (F, GF)$ where

$$F = f \oplus g \oplus h = \pi_1(A)f_1 \oplus \pi_2(B)f_2 \oplus \pi_3(C)f_3$$

and G is the matrix

$$G = \begin{bmatrix} I & C_1 & C_1C_2 \\ C_1^* & I & C_2 \\ C_2^*C_1^* & C_2^* & I \end{bmatrix}.$$

Then

$$\begin{aligned}(F, GF) =&(f, f) + (g, g) + (h, h) \\ &+ 2Re(f, C_1g) + 2Re(f, C_1C_2h) + 2Re(g, C_2h)\end{aligned}$$

Minimizing with respect to h we find the minimum occurs for $h = -C_2^*(C_1^*f + g)$ so

$$\begin{aligned}(F, GF) \geq&\|f\|^2 + \|g\|^2 + 2Re(f, C_1g) - \|C_2^*(C_1^*f + g)\|^2 \\ =&\|f\|^2 - \|C_1^*f\|^2 + \|C_1^*f + g\|^2 - \|C_2^*(C_1^*f + g)\|^2 \geq 0\end{aligned}$$

where the last inequality follows from the facts that $\|C_1\| \leq 1$ and $\|C_2\| \leq 1$. Then the weight Ω_3 whose entries ξ_{ij} are related to the entries of Ω_3' by the relation $\xi_{ij}(A) = \xi_{ij}'((I - \Lambda)^{-\frac{1}{2}}A(I - \Lambda)^{-\frac{1}{2}})$ for $A \in \cup_{t>0}U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$ is a positive boundary weight on $\mathfrak{B}(\mathfrak{H} \oplus \mathfrak{H} \oplus \mathfrak{H})$ and $i, j = 1, 2, 3$ and Q_t defined above is positive for $t > 0$.

Let $x = (z_1, -1, 0)$ and $y = (0, z_2, -1)$. As we have seen above (x, Q_tx) and (y, Q_ty) are bounded as $t \to 0+$ so there are positive constants K_1 and K_2 so that $(x, Q_tx) \leq K_1^2$ and $(y, Q_ty) \leq K_2^2$ for all $t > 0$. Since (w, Q_tw) is a positive quadratic form $\|w\|_t = (w, Q_tw)^{\frac{1}{2}}$ is a norm for each $t > 0$. Let $w = z_2x + y = (z_1z_2, 0, -1)$. Now we have

$$(w, Q_tw) = \|w\|_t^2 = \|z_2x + y\|_t^2 \leq (|z_2|\,\|x\|_t + \|y\|_t)^2 \leq (|z_2|K_1 + K_2)^2$$

for all $t > 0$. Hence, (w, Q_tw) is bounded as $t \to 0+$ and repeating the argument of theorem 3.14 we have that ω and η can be expressed in the form

$$\omega(A) = \sum_{k \in I}(f_k, (I - \Lambda)^{-\frac{1}{2}}A(I - \Lambda)^{-\frac{1}{2}}f_k)$$

and

$$\eta(A) = \sum_{k \in I}(g_k, (I - \Lambda)^{-\frac{1}{2}}A(I - \Lambda)^{-\frac{1}{2}}g_k)$$

for $A \in \cup_{t>0}U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$ with f_k and g_k in $\mathfrak{H}$ and $g_k = z_1z_2f_k + h_k$ where $z_1z_2 \neq 0$ with h_k in the domain of $(I - \Lambda)^{-\frac{1}{2}}$ for each $k \in I$ and

$$\sum_{k \in I}\|(I - \Lambda)^{-\frac{1}{2}}h_k\|^2 < \infty$$

Hence, ω and η are connected. $\square$

Theorem 3.18. *Suppose ω and η are positive boundary weights and either ω or η is unbounded and $\gamma \neq 0$ is a q-corner from ω to η. Let*

$$h(t) = \frac{(1+\omega_t(\Lambda))^{\frac{1}{2}}(1+\eta_t(\Lambda))^{\frac{1}{2}}}{|1+\gamma_t(\Lambda)|}$$

for $t > 0$ and let κ be the limit of $h(t)$ as $t \to 0+$ which we know exists from lemma 3.13. Then there is a number $r \geq 0$ and a complex number $z \neq 0$ so that the weight

$$\Omega_o = \begin{bmatrix} \omega/a & \gamma/b \\ \gamma^*/\overline{b} & \eta/c \end{bmatrix}$$

is q-positive if and only if $a > 0$, $c > 0$ and $b \in \mathbb{C}$ satisfies

$$a|z|^2 + c + r \leq 2\kappa Re(\overline{z}b).$$

Since γ is a q-corner from ω to η the numbers r and z satisfy

$$|z|^2 + 1 + r \leq 2\kappa Re(z).$$

Proof. Assume the hypothesis and notation of the lemma. Since ω and η are connected it follows from corollary 3.15 that both ω and η are unbounded. From the proof of theorem 3.14 it follows that there is a complex number $z \neq 0$ so that if M_t is family of matrices

$$M_t = \begin{bmatrix} 1+\omega_t(\Lambda) & \kappa + \kappa\gamma_t(\Lambda) \\ \kappa + \kappa\gamma_t^*(\Lambda) & 1+\eta_t(\Lambda) \end{bmatrix}$$

and $v = (z, -1)$ then $(v, M_t v) \leq 0$ for all $t > 0$.

Let Ω_o be the weight given in the statement of the lemma. If Ω_o is q-positive then a and c are positive and if a or c is not positive Ω_o can not be q-positive. Then we will assume that $a > 0$ and $c > 0$. From lemma 3.13 we have $\kappa\gamma$ is a trivially maximal corner from ω to η. Then $\kappa a^{-\frac{1}{2}}c^{-\frac{1}{2}}|b|\gamma/b$ is a trivially maximal corner from ω/a to η/c. Then from lemma 3.13 we have Ω_o is q-positive if and only if

$$\frac{(1+\omega_t(\Lambda)/a)^{\frac{1}{2}}(1+\eta_t(\Lambda)/c)^{\frac{1}{2}}}{|1+\gamma_t(\Lambda)/b|} \leq \frac{\kappa|b|}{\sqrt{ac}}$$

for all $t > 0$ and this is equivalent to the inequality

$$\frac{(a+\omega_t(\Lambda))^{\frac{1}{2}}(c+\eta_t(\Lambda))^{\frac{1}{2}}}{|\kappa b + \kappa\gamma_t(\Lambda)|} \leq 1$$

for $t > 0$. This inequality is equivalent to the statement that the determinant of N_t is not positive where

$$N_t = \begin{bmatrix} a+\omega_t(\Lambda) & \kappa b + \kappa\gamma_t(\Lambda) \\ \kappa\overline{b} + \kappa\gamma_t^*(\Lambda) & c+\eta_t(\Lambda) \end{bmatrix}.$$

Note N_t is non increasing in t so by the argument of theorem 3.14 we find that inequality (3.8) is equivalent to the existence of a unit vector w so that $(w, N_t w) \leq 0$ for all $t > 0$. Note if $a = b = c = 1$ then $N_t = M_t$ and there is a vector $v = (|z|^2+1)^{-\frac{1}{2}}(z, -1)$ so that $(v, M_t v) \leq 0$ for all t. We note that if there is a unit vector w so that $(w, N_t w) \leq 0$ for all $t > 0$ then w is a multiple of v and, therefore, $(v, N_t v) \leq 0$ for all $t > 0$. To see this note

$$\lambda I + N_t = \begin{bmatrix} \lambda + a & \kappa b \\ \kappa \overline{b} & \lambda + c \end{bmatrix} + \begin{bmatrix} \omega_t(\Lambda) & \kappa\gamma_t(\Lambda) \\ \kappa\gamma_t^*(\Lambda) & \eta_t(\Lambda) \end{bmatrix}.$$

Let

$$\lambda = \max(0, \frac{1}{2}(\sqrt{(a-c)^2 + 4\kappa^2|b|^2} - a - c).$$

With this choice for λ we insure the matrix on the left in the expression for $\lambda I + N_t$ is positive. Since the matrix on the right in the expression for $\lambda I + N_t$ is positive we have $\lambda I + N_t$ is positive for all $t > 0$. Suppose w is a unit vector and $(w, N_t w) \leq 0$ for all $t > 0$ so $(w, (\lambda I + N_t)w) \leq \lambda$ for all $t > 0$. Note

$$\lambda I + N_t - M_t = Q = \begin{bmatrix} \lambda + a - 1 & \kappa(b-1) \\ \kappa(\overline{b}-1) & \lambda + c - 1 \end{bmatrix}.$$

Then $(v, (\lambda I + N_t)v) = (v, M_t v) + (v, Qv) \leq (v, Qv) = K$ where the last equality is just the definition of K which is given by

$$K = \frac{|z|^2(\lambda + a - 1) - 2\kappa Re(z(b-1) + \lambda + c - 1}{|z|^2 + 1}$$

Now suppose w is not a multiple of v. Then w and v span $\mathbb{C}^2$ and so any unit vector in $u \in \mathbb{C}^2$ can be expressed in the form $u = xv + yw$ with x and y complex numbers satisfying $|x|^2 + |y|^2 \leq (1 - |(v, w)|)^{-1}$. Then we have

$$\begin{aligned} (u, (\lambda I + N_t)u) &= |x|^2(v, (\lambda I + N_t)v) + |y|^2(w, (\lambda I + N_t)w) \\ &\quad + 2Re(\overline{x}y(v, (\lambda I + N_t)w) \\ &\leq K|x|^2 + \lambda|y|^2 + 2|x|\,|y|K^{\frac{1}{2}}\lambda^{\frac{1}{2}} \\ &\leq 2\max(K, \lambda)(|x|^2 + |y|^2) \\ &\leq 2\max(K, \lambda)(1 - |(v, w)|)^{-1} \end{aligned}$$

for all $t > 0$. Hence, we have shown that $\|(\lambda I + N_t)^{\frac{1}{2}}\|$ is uniformly bounded. But this is impossible since $\omega_t(\Lambda)$ and $\eta_t(\Lambda)$ tend to infinity as $t \to 0+$. Hence, w is a multiple of v so there is a unit vector w so that $(w, N_t w) \leq 0$ for all $t > 0$ if and only if $(v, N_t v) \leq 0$ for all t. Since $(v, N_t v) \leq 0$ if and only if $(|z|^2 + 1)(v, N_t v) \leq 0$ we have the weight Ω_o is q-positive if and only if

$$a|z|^2 + c - 2\kappa Re(\overline{z}b) + q(t) \leq 0$$

for all $t > 0$ where

$$q(t) = |z|^2\omega_t(\Lambda) + \eta_t(\Lambda) - 2\kappa Re(\overline{z}\gamma_t(\Lambda))$$

for all $t > 0$. Since we are given that Ω_o is q-positive when a, b and c are all equal to one we have

$$|z|^2 + 1 - 2\kappa Re(z) + q(t) \leq 0$$

for all $t > 0$. Note q is a non increasing function of t which is bounded above so $q(t)$ converges to a limit r as $t \to 0+$ and $r \geq 0$. Hence, we see that Ω_o is q-positive if and only if the inequality stated in the lemma is satisfied. □

Suppose ω and η are positive boundary weights. We say γ is a trivially maximal q-corner between ω and η if $\lambda\gamma$ is not a q-corner for any $\lambda > 1$. Another possible definition of trivially maximal is to say γ is a trivially maximal q-corner from ω to η if the q-inequality

$$0 \leq_q \begin{bmatrix} \lambda\omega & \gamma \\ \gamma^* & \eta \end{bmatrix} \leq_q \begin{bmatrix} \omega & \gamma \\ \gamma^* & \eta \end{bmatrix}$$

implies $\lambda = 1$. The next corollary shows these two notions of being trivially maximal are the same. In fact, the notion of being trivially maximal is the same as being trivially hyper maximal.

Corollary 3.19. *Suppose ω and η are positive unbounded boundary weights and γ is a q-corner from ω to η. Let*

$$\kappa = \lim_{t \to 0+} \frac{(1 + \omega_t(\Lambda))^{\frac{1}{2}}(1 + \eta_t(\Lambda))^{\frac{1}{2}}}{|1 + \gamma_t(\Lambda)|}$$

which exists from lemma 3.13 and let M_t be the family of matrices

$$M_t = \begin{bmatrix} 1 + \omega_t(\Lambda) & \kappa + \kappa\gamma_t(\Lambda) \\ \kappa + \kappa\gamma_t^*(\Lambda) & 1 + \eta_t(\Lambda) \end{bmatrix}$$

for $t > 0$. Then the following statements are equivalent.

(i) *$\lambda\gamma$ is not a q-corner from ω to η for any $\lambda > 1$.*

(ii) *The following q-inequality*

$$0 \leq_q \begin{bmatrix} \lambda\omega & \gamma \\ \gamma^* & \eta \end{bmatrix} \leq_q \begin{bmatrix} \omega & \gamma \\ \gamma^* & \eta \end{bmatrix}$$

implies $\lambda = 1$.

(iii) *The following q-inequality*

$$0 \leq_q \begin{bmatrix} \omega & \gamma \\ \gamma^* & \lambda\eta \end{bmatrix} \leq_q \begin{bmatrix} \omega & \gamma \\ \gamma^* & \eta \end{bmatrix}$$

implies $\lambda = 1$.

(iv) *The following q-inequality*

$$0 \leq_q \begin{bmatrix} \lambda_1\omega & \gamma \\ \gamma^* & \lambda_2\eta \end{bmatrix} \leq_q \begin{bmatrix} \omega & \gamma \\ \gamma^* & \eta \end{bmatrix}$$

implies $\lambda_1 = \lambda_2 = 1$.

(v) *There is a vector $v = (z, -1) \in \mathbb{C}^2$ so that $(v, M_t v) \leq 0$ for $t > 0$ and $(v, M_t v) \to 0$ as $t \to 0$.*

Proof. Assume the hypothesis and notation of the corollary. Since ω, η and γ satisfy the hypothesis of theorem 3.17 let r and z be the numbers associated with these weight by theorem 3.17. Given the proof of theorem 3.18 it is a routine exercise to show that each of the statements (i) through (v) above is equivalent to the statement

$$|z|^2 + 1 + r = 2\kappa Re(z)$$

(i.e. the inequality of theorem 3.17 is an equality). $\square$

Theorem 3.20. *Suppose ω and η are positive boundary weights. Then γ is a maximal q-corner from ω to η if and only if γ is trivially maximal in that $\lambda\gamma$ is not a q-corner from ω to η for $\lambda > 1$ and if*

$$\kappa = \lim_{t\to 0+} \frac{(1+\omega_t(\Lambda))^{\frac{1}{2}}(1+\eta_t(\Lambda))^{\frac{1}{2}}}{|1+\gamma_t(\Lambda)|}$$

(where the limit exist from the proof of lemma 3.13) then $\kappa\gamma$ a corner from ω to η with the property that if $\rho \in \mathfrak{B}(\mathfrak{H})_*$ is positive (so $\rho(I) < \infty$) and

$$\begin{bmatrix} \omega & \kappa\gamma \\ \kappa\gamma^* & \eta \end{bmatrix} \geq \begin{bmatrix} \omega - \rho & \kappa\gamma \\ \kappa\gamma^* & \eta \end{bmatrix} \geq 0$$

then $\rho = 0$.

Proof. Assume ω and η are positive boundary weights and γ is a q-corner from ω to η. It follows from the previous corollary that if γ is a maximal q-corner from ω to η then γ is trivially maximal and if γ is not trivially maximal the γ can not be a maximal q-corner from ω to η. With this said we only have to prove the theorem in the case where γ is trivially maximal, which we now assume. Let κ be the constant given in the statement of the theorem. Suppose ω' is a boundary weight so that

$$0 \leq_q \begin{bmatrix} \omega' & \gamma \\ \gamma^* & \eta \end{bmatrix} \leq_q \begin{bmatrix} \omega & \gamma \\ \gamma^* & \eta \end{bmatrix}$$

It follows that $\omega' \leq_q \omega$ so it follows from theorem 3.9 that there is a bounded boundary weight ρ with $\omega \geq \rho \geq 0$ and a $\lambda \in [0,1]$ so that $\omega' = \lambda(1+\rho(\Lambda))^{-1}(\omega-\rho)$. Since $\lambda\omega \geq_q \omega'$ by theorem 3.9 we have

$$0 \leq_q \begin{bmatrix} \lambda\omega & \gamma \\ \gamma^* & \eta \end{bmatrix} \leq_q \begin{bmatrix} \omega & \gamma \\ \gamma^* & \eta \end{bmatrix}.$$

Since γ is trivially maximal we have $\lambda = 1$ and $\omega' = (1+\rho(\Lambda))^{-1}(\omega - \rho)$.

Now suppose $\kappa\gamma$ does not have the property given at the end of the statement of the theorem so there is a non zero bounded boundary weight ρ with $\omega \geq \rho \geq 0$ so that $\kappa\gamma$ is a corner from $\omega - \rho$ to η. Let $\omega' = (1+\rho(\Lambda))^{-1}(\omega - \rho)$. We claim γ is a q-corner from ω' to η and, therefore, γ is not q-maximal. Now from lemma 3.13 γ is a q-corner from ω' to η if and only if there is a constant κ' so that $\kappa'\gamma$ is a corner from ω' to η and

$$\frac{(1+\omega'_t(\Lambda))^{\frac{1}{2}}(1+\eta_t(\Lambda))^{\frac{1}{2}}}{|1+\gamma_t(\Lambda)|} \leq \kappa' \tag{3.9}$$

for all $t > 0$. We claim $\kappa' = (1+\rho(\Lambda))^{-\frac{1}{2}}\kappa$. First we note that since $\kappa\gamma$ is a corner from $\omega - \rho$ to η then $\kappa'\gamma$ is a corner from ω' to η. Next we have to show inequality (3.9) holds for all $t > 0$. Replacing primed expression in terms of the unprimed expression and squaring both sides inequality (3.9) is equivalent to the inequality

$$(1+\rho(\Lambda) - \rho_t(\Lambda) + \omega_t(\Lambda))(1+\eta_t(\Lambda)) \leq |\kappa + \kappa\gamma_t(\Lambda)|^2$$

for all $t > 0$. Recalling the arguments of theorem 3.18 we see this inequality is equivalent to the existence of a unit vector $w \in \mathbb{C}^2$ so that $(w, N_t w) \leq 0$ for all t where

$$N_t = \begin{bmatrix} 1+\rho(\Lambda) - \rho_t(\Lambda) + \omega_t(\Lambda) & \kappa + \kappa\gamma_t(\Lambda) \\ \kappa + \kappa\gamma_t^*(\Lambda) & 1+\eta_t(\Lambda) \end{bmatrix}$$

Since γ is a q-corner from ω to η there is a unit vector $v \in \mathbb{C}^2$ so there is a vector $v \in \mathbb{C}^2$ so that $(v, M_t v) \leq 0$ for all t where

$$M_t = \begin{bmatrix} 1+\omega_t(\Lambda) & \kappa + \kappa\gamma_t(\Lambda) \\ \kappa + \kappa\gamma_t^*(\Lambda) & 1+\eta_t(\Lambda) \end{bmatrix}.$$

Since N_t and M_t are non increasing and $\|N_t - M_t\| \to 0$ as $t \to 0+$ we have

$$(v, N_t v) \leq \lim_{s\to 0+} (v, N_s v) = \lim_{s\to 0+} (v, M_s v) \leq 0$$

for all $t > 0$. Hence, γ is a q-corner from ω' to η so γ is not q-maximal.

Next suppose γ is a trivially q-maximal corner from ω to η but γ is not q-maximal. Then there is a positive boundary weight ω' so that $\omega' \leq_q \omega$ and γ is a q-corner from ω' to η. Then as we have seen above there is a bounded boundary weight ρ (i.e. $\rho(I) < \infty$) so that $\omega \geq \rho \geq 0$ and $\omega' = (1+\rho(\Lambda))^{-1}(\omega - \rho)$. Since γ is a q-corner from ω' to η it follows from lemma 3.13 that there is a constant κ' so that inequality (3.9) holds and the limit of the expression on the left as $t \to 0+$ is κ'. Using the facts that $\omega_t(\Lambda)$, $\eta_t(\Lambda)$ and $|\gamma_t(\Lambda)|$ tend to infinity and $\rho_t(\Lambda)$ tends to $\rho(\Lambda)$ as $t \to 0+$ we can easily calculate the limit κ' in terms of the limit κ given in the statement of the theorem and we find $\kappa' = (1+\rho(\Lambda))^{\frac{1}{2}}\kappa$. Since $\kappa'\gamma$ is a corner from ω' to η we have $\kappa\gamma$ is a corner from $\omega - \rho$ to η. Hence, $\kappa\gamma$ does not have the property given at the end of the statement of the theorem. □

Corollary 3.21. *Suppose ω and η are positive boundary weights. Then γ is a hyper maximal q-corner from ω to η if and only if γ is trivially maximal in that $\lambda\gamma$ is not a q-corner from ω to η for $\lambda > 1$ and if*

$$\kappa = \lim_{t\to 0+} \frac{(1+\omega_t(\Lambda))^{\frac{1}{2}}(1+\eta_t(\Lambda))^{\frac{1}{2}}}{|1+\gamma_t(\Lambda)|}$$

(where the limit exist from the proof of lemma 3.13) then $\kappa\gamma$ a corner from ω to η with the property that if $\rho, \nu \in \mathfrak{B}(\mathfrak{H})_*$ are positive (so $\rho(I) < \infty$ and $\nu(I) < \infty$) and

$$\begin{bmatrix} \omega & \kappa\gamma \\ \kappa\gamma^* & \eta \end{bmatrix} \geq \begin{bmatrix} \omega - \rho & \kappa\gamma \\ \kappa\gamma^* & \eta - \nu \end{bmatrix} \geq 0$$

then $\rho = \nu = 0$.

Proof. Since γ is hyper maximal if and only if γ and γ^* are maximal the proof follows directly from the previous theorem. □

Theorem 3.22. *Suppose ω is q-pure positive boundary weight and ρ is a positive bounded boundary weight so that $\eta = \omega + \rho$ is normalized so that $\eta(I - \Lambda) = 1$. Suppose η' is a positive boundary weight and $\eta'(I - \Lambda) = 1$. Then η and η' induce cocycle conjugate E_o-semigroups if and only if η' is of the form $\eta' = \omega' + \rho'$ where ω' is a q-pure positive boundary weight and ρ' is a bounded positive boundary weight and ω and ω' are connected and ρ and ρ' are of the same rank.*

Proof. Assume ω and ρ satisfy the statement of the theorem and the E_o-semigroup induced by $\eta = \omega + \rho$ is cocycle conjugate to the E_o-semigroup induced by a positive boundary weight η'. Since the subordinates of η and η' are a cocycle conjugacy invariant of the induced E_o-semigroups it follows that there is an order isomorphism from the subordinates of η to the subordinates of η'. Since $(1 + \rho(\Lambda))^{-1}\omega$ is a q-pure subordinate of η it follows that η' must have a q-pure subordinate μ. We assume μ is a trivially maximal subordinate of η'. Then from theorem 3.9 we have there exists a bounded positive boundary weight ρ' so that $\eta' \geq \rho' \geq 0$ and $\mu = (1 + \rho'(\Lambda))^{-1}(\eta' - \rho')$. Then we have $\eta' = (1 + \rho'(\Lambda))\mu + \rho'$. Let $\omega' = (1 + \rho'(\Lambda))\mu$ and we have $\eta' = \omega' + \rho'$ where ω' is a q-pure positive boundary weight and ρ' is a bounded positive boundary weight. Since η and η' induce cocycle conjugate E_o-semigroups it follows that η and η' are connected. From lemma 3.16 it follows that η and ω are connected as are η' and ω'. Then from theorem 3.17 it follows that ω and ω' are connected. Since there is an order isomorphism from the boundary weights μ with $\eta \geq_q \mu \geq 0$ to the boundary weights μ' with $\eta' \geq_q \mu' \geq 0$ and these sets are determined by the sets of functionals ν and ν' with $\rho \geq \nu \geq 0$ and $\rho' \geq \nu' \geq 0$ it follows that these convex sets must be of the same dimension. Since the dimensions of these convex sets are solely determined by the ranks of ρ and ρ' it follows that the ranks of ρ and ρ' are equal.

Conversely, suppose $\eta = \omega + \rho$ and $\eta' = \omega' + \rho'$ are normalized boundary weights and ω and ω' are q-pure positive boundary weights which are connected and ρ and ρ' are bounded positive boundary weights of the same rank. We show there is a hyper maximal q-corner γ from η to η'. Since ω and ω' are connected there is a q-corner ψ from ω to ω'. Scaling ψ we may assume ψ is trivially maximal. Since ω and ω' are q-pure it follows from corollary 3.19 and theorem 3.20 that ψ is hyper maximal. Since ρ and ρ' are of the same rank it follows from lemma 3.13 there is a hyper maximal corner ν from ρ to ρ'. Let

$$\kappa = \lim_{t \to 0+} \frac{(1 + \omega_t(\Lambda))^{\frac{1}{2}}(1 + \omega_t'(\Lambda))^{\frac{1}{2}}}{|1 + \psi_t(\Lambda)|}$$

which exists by lemma 3.13. Since ψ is hyper maximal it follows from theorem 3.20 that $\kappa\psi$ is a maximal corner from ω to ω'. Let M_t be the family of matrices

$$M_t = \begin{bmatrix} 1 + \omega_t(\Lambda) & \kappa + \kappa\psi_t(\Lambda) \\ \kappa + \kappa\psi_t^*(\Lambda) & 1 + \omega_t'(\Lambda) \end{bmatrix}.$$

Since ψ is maximal it follows from corollary 3.19 there is a vector $v = (z, -1) \in \mathbb{C}^2$ so that $(v, M_t v) \leq 0$ for $t > 0$ and $(v, M_t v) \to 0$ as $t \to 0+$. Let $\gamma = b^{-1}(\psi + \kappa^{-1}\nu)$ where $b > 0$ is a non zero constant we will determine later. Since $\kappa\psi$ is a corner from ω to ω' and ν is a corner from ρ to ρ' it follows that $b\kappa\gamma = \kappa\psi + \nu$ is a corner from $\eta = \omega + \rho$ to $\eta' = \omega' + \rho'$. As we saw in the proof of theorem 3.14 the inequality

$$\frac{(1+\eta_t(\Lambda))^{\frac{1}{2}}(1+\eta_t'(\Lambda))^{\frac{1}{2}}}{|1+\gamma_t(\Lambda)|} \leq \kappa b$$

for $t > 0$ is equivalent to the existence of a non zero vector w so that $(w, N_t w) \leq 0$ for all t where

$$N_t = \begin{bmatrix} 1+\omega_t(\Lambda)+\rho_t(\Lambda) & \kappa b + \kappa\psi_t(\Lambda)+\nu_t(\Lambda) \\ \kappa\overline{b}+\kappa\psi_t^*(\Lambda)+\nu_t^*(\Lambda) & 1+\omega_t'(\Lambda)+\rho_t'(\Lambda) \end{bmatrix}.$$

Recall that $v = (z, -1)$ has the property that $(v, M_t v) \to 0$ as $t \to 0+$. We have for $t > 0$ that

$$(v, N_t v) = (v, M_t v) + a(t) - 2Re(\overline{z}\kappa(b-1))$$

where

$$a(t) = |z|^2\rho_t(\Lambda) + \rho_t'(\Lambda) - 2Re(\overline{z}\nu_t(\Lambda)).$$

Since ν is a corner from ρ to ρ' it follows that $a(t) \geq 0$ for $t > 0$. Since ρ and ρ' are bounded it follows that $a(t)$ has a finite limit a_o as $t \to 0+$. Then we have $(v, N_t v) \to a_o - 2Re(\overline{z}\kappa(b-1))$. Since the real part of z is positive (see theorem 3.18) we can make this limit zero be setting $b = 1 + a_o/(2Re(\kappa z))$. With this choice for b we have from corollary 3.19 that γ is a trivially maximal q-corner from η to η'. The constant κ' associated with the corner γ is

$$\kappa' = \lim_{t\to 0+} \frac{(1+\eta_t(\Lambda))^{\frac{1}{2}}(1+\eta_t'(\Lambda))^{\frac{1}{2}}}{|1+\gamma_t(\Lambda)|} = \kappa b$$

Now suppose μ and μ' are bounded boundary weights so that

$$\begin{bmatrix} \eta & \kappa'\gamma \\ \kappa'\gamma^* & \eta' \end{bmatrix} \geq \begin{bmatrix} \eta-\mu & \kappa'\gamma \\ \kappa'\gamma^* & \eta'-\mu' \end{bmatrix} \geq 0.$$

Then we have

$$\begin{bmatrix} \omega+\rho & \kappa\psi+\nu \\ \kappa\psi^*+\nu^* & \omega'+\rho' \end{bmatrix} \geq \begin{bmatrix} \omega-\rho-\mu & \kappa\psi+\nu \\ \kappa\psi^*+\nu^* & \omega'+\rho'-\mu' \end{bmatrix} \geq 0.$$

Suppose $\mu \neq 0$. Then there is a pure positive functional $\zeta \neq 0$ of the form $\zeta(A) = (h, Ah)$ for $A \in \mathfrak{B}(\mathfrak{H})$ with $h \in \mathfrak{H}$ and $\mu \geq \zeta \geq 0$. Then we have

$$\begin{bmatrix} \omega & \kappa\psi \\ \kappa\psi^* & \omega' \end{bmatrix} + \begin{bmatrix} \rho & \nu \\ \nu^* & \omega' \end{bmatrix} \geq \begin{bmatrix} \zeta & 0 \\ 0 & 0 \end{bmatrix} \geq 0$$

Let Ω, Φ and Ξ be the three positive boundary weights on $\mathfrak{B}(\mathfrak{H} \oplus \mathfrak{H})$ appearing above in the order given. As we saw in the proof of theorem 3.14 there is a countable index set I and vectors $F_i = \{f_{1i}, f_{2i}\} \in \mathfrak{H} \oplus \mathfrak{H}$ for $i \in I$ so that

$$\Omega(A) = \sum_{i\in I}(F_i, (I-\Lambda)^{-\frac{1}{2}}A(I-\Lambda)^{-\frac{1}{2}}F_i)$$

for $A \in \cup_{t>0} U(t) \oplus U(t)\mathfrak{B}(\mathfrak{H}\oplus\mathfrak{H})U(t)^* \oplus U(t)^*$. Similarly there is a countable index set J and vectors $G_j = \{g_{1j}, g_{2j}\} \in \mathfrak{H}\oplus\mathfrak{H}$ so that

$$\Phi(A) = \sum_{j\in J} (G_j, (I-\Lambda)^{-\frac{1}{2}} A (I-\Lambda)^{-\frac{1}{2}} G_j)$$

for $A \in \cup_{t>0} U(t)\mathfrak{B}(\mathfrak{H}\oplus\mathfrak{H})U(t)^*$. Since Φ is bounded the G_j are in the domain of $(I-\Lambda)^{-\frac{1}{2}}$ for each $j \in J$. Recalling the argument of theorem 3.10 we see there are complex numbers z_i and y_j for $i \in I$ and $j \in J$ so the sum of the squares of the absolute values of the $z's$ and $y's$ is bounded and

$$\{(I-\Lambda)^{\frac{1}{2}} h, 0\} = \sum_{i\in I} z_i F_i + \sum_{j\in J} y_i G_i. \tag{3.10}$$

Since ω and ω' are q-pure it follows from theorem 3.10 that the vectors

$$\sum_{i\in I} z_i f_{1i} \qquad \text{and} \qquad \sum_{i\in I} z_i f_{2i}$$

can only be in the domain of $(I-\Lambda)^{-\frac{1}{2}}$ if they are zero. Since in equation (3.10) all the other terms are in the domain of $(I-\Lambda)^{-\frac{1}{2}}$ it follows that all sum over the index set I is zero. Then we have

$$\{(I-\Lambda)^{\frac{1}{2}} h, 0\} = \sum_{j\in J} y_i G_i.$$

But this is equivalent to the statement that

$$\begin{bmatrix} \rho & \nu \\ \nu^* & \omega' \end{bmatrix} \geq \begin{bmatrix} \zeta & 0 \\ 0 & 0 \end{bmatrix} \geq 0$$

But since ν is hyper maximal this implies $\zeta = 0$. Hence $\mu = 0$. A similar argument shows $\mu' = 0$ and, therefore, the q-corner γ satisfies the hypothesis of theorem 2.21 and so γ is a hyper maximal q-corner from η to η'. Hence, η and η' induce E_o-semigroups that are cocycle conjugate. $\square$

The next theorem gives a fairly computable condition that two positive normalized boundary weights induce cocycle conjugate E_o-semigroups.

Theorem 3.23. *Suppose ω and η are positive normalized boundary weights associated with positive trace one operators Ω and H by the formulae*

$$\omega(A) = tr((I-\Lambda)^{-\frac{1}{2}} A (I-\Lambda)^{-\frac{1}{2}} \Omega)$$

and

$$\eta(A) = tr((I-\Lambda)^{-\frac{1}{2}} A (I-\Lambda)^{-\frac{1}{2}} H)$$

for all $A \in \cup_{t>0} U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$. Then the E_o-semigroups induced by ω and η are cocycle conjugate if and only if there is a unitary operator X from the range of Ω to *the range of H and a complex number z with $Re(z) > 0$ so that $(I-\Lambda)^{-\frac{1}{2}}(zX\Omega^{\frac{1}{2}} - H^{\frac{1}{2}})$ is Hilbert Schmidt which means there is a constant K so that*

$$tr((zX\Omega^{\frac{1}{2}} - H^{\frac{1}{2}})^* E(t,\infty)(I-\Lambda)^{-1}(zX\Omega^{\frac{1}{2}} - H^{\frac{1}{2}})) \leq K$$

for all $t > 0$.

Proof. Suppose ω and η are as stated in the theorem and Ω and H are the associated density matrices. In the case when one of the weights ω or η is bounded the condition of the theorem is just the statement that ω and η induce cocycle conjugate E_o-semigroups if and only if they have the same rank. Since in the bounded case the induced E_o-semigroup is completely spatial and the index is just the rank of the inducing functional it follows from Arveson's result that completely spatial E_o-semigroups are cocycle conjugate if and only if they have the same index. Also the proof of theorem 3.22 shows this result in the bounded case. With this said we will assume ω and η are unbounded.

Let us suppose that there is a complex number z and a unitary operator X with the properties given in the statement of the theorem. Let

$$\kappa\gamma(A) = tr((I-\Lambda)^{-\frac{1}{2}}A(I-\Lambda)^{-\frac{1}{2}}H^{\frac{1}{2}}X\Omega^{\frac{1}{2}})$$

for all $A \in \cup_{t>0}U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$ where κ is a positive constant we will determine shortly. Note $\kappa\gamma$ is a corner from ω to η. As we have seen in the proof of theorem 3.14 we have γ is a q-corner from ω to η if there is a non zero vector $v = (v_1, v_2) \in \mathbb{C}^2$ with $v_1v_2 \neq 0$ and $(v, M_tv) \leq 0$ for $t > 0$ where

$$M_t = \begin{bmatrix} 1+\omega_t(\Lambda) & \kappa + \kappa\gamma_t(\Lambda) \\ \kappa + \kappa\gamma_t^*(\Lambda) & 1+\eta_t(\Lambda) \end{bmatrix}.$$

Clearly, we choose $v = (\overline{z}, -1)$ which gives the result,

$$\begin{aligned}(v, M_tv) =& |z|^2 + 1 - 2\kappa Re(z) \\ & + tr(\Lambda(I-\Lambda)^{-1}E(t,\infty)(zX\Omega^{\frac{1}{2}} - H^{\frac{1}{2}})^*(zX\Omega^{\frac{1}{2}} - H^{\frac{1}{2}})).\end{aligned}$$

Since $(I-\Lambda)^{-\frac{1}{2}}(zX\Omega^{\frac{1}{2}} - H^{\frac{1}{2}})$ is Hilbert Schmidt the expression on the right hand side is bounded as $t \to 0+$. Since $Re(z) > 0$ we can by adjust κ so that $(v, M_tv) \leq 0$ for all $t > 0$ and, furthermore, $(v, M_tv) \to 0$ as $t \to 0+$. We assume κ has been so chosen. Then by theorem 3.19 we have that γ is trivially maximal. From the proof of lemma 3.12 with the modifications of working with $(I-\Lambda)^{-\frac{1}{2}}$ and boundary weights we see that $\kappa\gamma$ is a hyper maximal corner from ω to η in that if ω' and η' are positive boundary weights so that

$$\begin{bmatrix} \omega & \kappa\gamma \\ \kappa\gamma^* & \eta \end{bmatrix} \geq \begin{bmatrix} \omega' & \kappa\gamma \\ \kappa\gamma^* & \eta' \end{bmatrix} \geq 0,$$

then $\omega' = \omega$ and $\eta' = \eta$. It then follows that γ satisfies the condition of theorem 2.20 so γ is a hyper maximal q-corner from ω to η. Hence, ω and η induce cocycle conjugate E_o-semigroups.

Conversely, suppose ω and η induce cocycle conjugate E_o-semigroups. Then there is a hyper maximal q-corner γ from ω to η. Let κ be the limit

$$\kappa = \lim_{t\to 0+} \frac{(1+\omega_t(\Lambda))^{\frac{1}{2}}(1+\eta_t(\Lambda))^{\frac{1}{2}}}{|1+\gamma_t(\Lambda)|}$$

Since $\kappa\gamma$ is a corner from ω to η we have from the proof of lemma 3.12 adapted to boundary weights that there is a operator X from the range of $\Omega^{\frac{1}{2}}$ to the range of $H^{\frac{1}{2}}$ and $\|X\| \leq 1$ so that

$$\kappa\gamma(A) = tr((I-\Lambda)^{-\frac{1}{2}}A(I-\Lambda)^{-\frac{1}{2}}H^{\frac{1}{2}}X\Omega^{\frac{1}{2}})$$

for $A \in \cup_{t>0}U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$. Since γ is a q-corner there is a complex number z with Re(z) > 0 so that if $v = (\overline{z}, -1)$ then $(v, M_t v) \leq 0$ for all $t > 0$ and $(v, M_t v) \to 0$ as $t \to 0+$ where M_t is the matrix given above. Then we have

$$\begin{aligned}(v, M_t v) =& |z|^2 + 1 - 2\kappa Re(z)\\ &+ tr(\Lambda(I-\Lambda)^{-1}E(t,\infty)(zX\Omega^{\frac{1}{2}} - H^{\frac{1}{2}})^*(zX\Omega^{\frac{1}{2}} - H^{\frac{1}{2}})).\\ &+ |z|^2 tr(\Lambda(I-\Lambda)^{-1}E(t,\infty)\Omega^{\frac{1}{2}}(I-X^*X)\Omega^{\frac{1}{2}}).\end{aligned}$$

Since $(v, M_t v) \leq 0$ it follows that $\Lambda^{\frac{1}{2}}(I-\Lambda)^{-\frac{1}{2}}(zX\Omega^{\frac{1}{2}} - H^{\frac{1}{2}})$ is Hilbert Schmidt. We have

$$(I-\Lambda)^{-\frac{1}{2}}(zX\Omega^{\frac{1}{2}} - H^{\frac{1}{2}}) = \Lambda^{\frac{1}{2}}(\Lambda^{\frac{1}{2}}(I-\Lambda)^{-\frac{1}{2}}(zX\Omega^{\frac{1}{2}} - H^{\frac{1}{2}})) + (I-\Lambda)^{\frac{1}{2}}(zX\Omega^{\frac{1}{2}} - H^{\frac{1}{2}})$$

so $(I-\Lambda)^{-\frac{1}{2}}(zX\Omega^{\frac{1}{2}} - H^{\frac{1}{2}})$ is the linear combination of Hilbert Schmidt operators and, therefore, it is Hilbert Schmidt.

Next we show X^*X is the range projection for $\Omega^{\frac{1}{2}}$. Let ρ be the boundary weight given by

$$\rho(A) = tr((I-\Lambda)^{-\frac{1}{2}}A(I-\Lambda)^{-\frac{1}{2}}\Omega^{\frac{1}{2}}(I-X^*X)\Omega^{\frac{1}{2}})$$

for $A \in \cup_{t>0}U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$. From the expression for $(v, M_t v) \leq 0$ above it follows that $\rho_t(\Lambda)$ is bounded so ρ is a bounded weight. From the proof of lemma 3.12 we see that

$$\begin{bmatrix} \omega & \kappa\gamma \\ \kappa\gamma^* & \eta \end{bmatrix} \geq \begin{bmatrix} \omega-\rho & \kappa\gamma \\ \kappa\gamma^* & \eta \end{bmatrix} \geq 0.$$

Since γ is hyper maximal we have $\rho = 0$ by theorem 3.21. Hence, it follows that $\Omega^{\frac{1}{2}}(I-X^*X)\Omega^{\frac{1}{2}} = 0$ and X is an isometry. Exchanging the roles of ω and η and γ and γ^* we see by the argument we have just completed that $H^{\frac{1}{2}}(I-XX^*)H^{\frac{1}{2}} = 0$ so X is unitary. $\square$

We define the bounded rank of a positive boundary weight.

Definition 3.24. If ω is a positive boundary weight then the bounded rank of ω is the least upper bound of the rank of ρ where ρ is a positive element of $\mathfrak{B}(\mathfrak{H})_*$ (so $\rho(I) < \infty$) with $\omega \geq \rho$.

We conjecture that two positive normalized boundary weights induced cocycle conjugate E_o-semigroups if and only if they are connected and have the same bounded rank.

In summary we can say that although the case where $\mathfrak{K}$ is one dimensional is not completely understood it appears we already have the tools in hand to settle this case in the near future. To demonstrate the power of the techniques we have for computing whether the induced E_o-semigroups are cocycle conjugate we consider the following problem. Suppose α is an E_o-semigroup. We can form new E_o-semigroups by scaling. Let $\alpha^{(\lambda)}$ be given in terms of α by $\alpha_t^{(\lambda)} = \alpha_{\lambda t}$. The question has been posed whether α and $\alpha^{(\lambda)}$ are cocycle conjugate. We show how we can construct an E_o-semigroup so that α and $\alpha^{(\lambda)}$ are cocycle conjugate if and only if $\lambda = 2^n$ for some integer n. Let $h_o(x) = x - 1$ for $x \in (1, 2]$. Now we define the function h_1 on the whole positive real line by defining $h_1(x) = 0$ for $x > 2$ and for $x \epsilon (2^{-n}, 2^{1-n}]$ we define $h_1(x) = 2^{n/2} h_o(2^n x)$. One sees that

$$\int_0^\infty (1 - e^{-x})|h_1(x)|^2\,dx \leq \sum_{n=0}^\infty (1 - e^{-2^{1-n}}) \int_1^2 |h_o(x)|^2\,dx \leq 4/3$$

It follows that the above integral is finite and by multiplying h_1 by a suitable constant we can arrange it so the above integral equals one. Let h be the function obtained by multiplying by this constant. Let ω be the weight given by $\omega(A) = (h, Ah)$ for $A \in \cup_{t>0} U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$. Let α be the E_o-semigroup induced by this weight. If $\alpha^{(\lambda)}$ is the scaled E_o-semigroup one calculates the weight associated with the scaled E_o-semigroup is given by $\omega_\lambda(A) = s_\lambda^{-1}(h_\lambda, Ah_\lambda)$ for $A \in \cup_{t>0} U(t)\mathfrak{B}(\mathfrak{H})U(t)^*$ where $h_\lambda(x) = h(x/\lambda)$ and

$$s_\lambda = \int_0^\infty (1 - e^{-x})|h_\lambda(x)|^2\,dx$$

From theorem 3.14 we see that ω and ω_λ are connected if and only if $h_\lambda - zh \in L^2(0, \infty)$ for some complex number z and it is easily seen that this is the case if and only if $\lambda = 2^n$ for n an integer. It follows theorem 3.22 that α and $\alpha^{(\lambda)}$ are cocycle conjugate if and only if $\lambda = 2^n$ for some integer n.

IV. Higher Dimensions.

We end with a discussion of the case CP-flows over $\mathfrak{K}$ where the dimension of $\mathfrak{K}$ is greater than one. Going from one dimension to two is already a big step. We have nothing like a complete theory even for the case where $\mathfrak{K}$ is of dimension two. We begin with a simple problem. Suppose ω is a q-pure positive normalized unbounded boundary weight for $\dim(\mathfrak{K}) = 1$ and A is an hermitian matrix with non zero entries. Suppose Ω is a matrix with entries $\{\omega/a_{ij}\}$ for $i, j = 1, \cdots, n$. When is Ω q-positive? The answer is that Ω is q-positive if and only if the diagonal entries $\{a_{ii} : i = 1, \cdots, n\}$ of A are strictly positive and A is conditionally negative. A matrix A is conditionally positive if A is hermitian and $(x, Ax) \geq 0$ for all vectors $x \in \mathbb{C}^n$ so that $x_1 + x_2 + \cdots + x_n = 0$.

Next suppose A with coefficients $\{a_{ij}\}$ and B with coefficients $\{b_{ij}\}$ have positive entries on the diagonal and are conditionally negative. We form the matrix of weights Ω_A with entries ω/a_{ij} and Ω_B with entries ω/b_{ij} for $i,j=1,\cdots,n$ which are q-positive. Then we find $\Omega_A \geq_q \Omega_B$ if and only if $B \geq A$.

Let us consider the simple case where ω is a q-pure normalized weight on $L^2(0,\infty)$. The unital q-positive 2×2 matrices with ω the diagonal entries are of the form

$$\Omega = \begin{bmatrix} \omega & \omega/(1+x^2+iy) \\ \omega/(1+x^2-iy) & \omega \end{bmatrix} \qquad \text{or} \qquad \Omega_o = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}$$

with x,y arbitrary real numbers. What are the equivalence classes? If $x=0$ then Ω is equivalent to $[\omega]$ the weight ω on $L^2(0,\infty)$. To show this we need a hyper maximal q-corner which is the $(1,2)$ and $(1,3)$ entries in the matrix below.

$$\Omega_1 = \begin{bmatrix} \omega & \omega/(1-iy/2) & \omega/(1+iy/2) \\ \omega/(1+iy/2) & \omega & \omega/(1+iy) \\ \omega/(1-iy/2) & \omega/(1-iy) & \omega \end{bmatrix}.$$

To check that the corner above is a hyper maximal q-corner one must show that the matrix below is conditionally negative if and only if $a=0$ and $b_{ij}=0$ for $i,j=1,2$ where $a\geq 0$ and the matrix $B=\{b_{ij}\}$ is non negative.

$$\begin{bmatrix} 1 & (1-iy/2) & (1+iy/2) \\ (1+iy/2) & 1 & (1+iy) \\ (1-iy/2) & (1-iy) & 1 \end{bmatrix} + \begin{bmatrix} a & 0 & 0 \\ 0 & b_{11} & b_{12} \\ 0 & b_{21} & b_{22} \end{bmatrix}$$

We leave this as an exercise. This then shows that the E_o-semigroup induced by $[w]$ and Ω with $x=0$ are cocycle conjugate.

What about when $x\neq 0$. We show that the two weights

$$\Omega = \begin{bmatrix} \omega & \omega/(1+x^2+iy) \\ \omega/(1+x^2-iy) & \omega \end{bmatrix} \qquad \text{and} \qquad \Omega_2 = \begin{bmatrix} \omega & \omega/2 \\ \omega/2 & \omega \end{bmatrix}$$

induce cocycle conjugate E_o-semigroups. We display the hyper maximal q-corner below where we write the matrix of denominators.

$$\begin{bmatrix} 1 & 1+x^2+iy & 1+(1-x)^2+iy & 1+x^2+iy \\ 1+x^2-iy & 1 & 2 & 1 \\ 1+(1-x)^2-iy & 2 & 1 & 2 \\ 1+x^2-iy & 1 & 2 & 1 \end{bmatrix}.$$

One checks that corner above is a hyper maximal q-corner by showing that the matrix above when added to the matrix below is conditionally negative if and only $a_{ij}=0$ and $b_{ij}=0$ for $i,j=1,2$ where $A=\{a_{ij}\}$ and $B=\{b_{ij}\}$ are non negative matrices.

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{bmatrix}$$

Again we leave this as an exercise. This once checked then shows that the weights Ω_1 and Ω_2 yield cocycle conjugate E_o-semigroups. Then we find the cocycle classes of the unital matrix of weights with ω on the diagonal fall into three classes. The first is when $x = 0$ and y is arbitrary which is equivalent to $[\omega]$, the second is when $x \neq 0$ and otherwise arbitrary and y is arbitrary which is equivalent to $x = 1$, $y = 0$ and the third is given by Ω_o where the of diagonal entries are zero or the denominators of the off diagonal entries are infinite.

The main point of this section is that there are ways of constructing and determining the cocycle conjugacy classes in the case where $\mathfrak{K}$ has dimension greater than one. At this point we must admit that the best way to proceed is unclear, but what is clear is that this is a fruitful way to construct spatial E_o-semigroups.

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Department of Mathematics, University of Pennsylvania, Philadelphia, Pa. 19104