MATH 210, PROBLEM SET 1

DUE IN LECTURE ON THURSDAY, JAN. 25

1. Frankfurt's theory of lying and bullshit.

Read Frankfurt's book "On Bullshit." In particular, see the description of the distinction he makes between lying and bullshit on pages 50 through 62.

1. Discuss whether the actions in each of the situations below are closer to telling the truth, lying or to bullshitting. What matters is how you justify your answers, since in come cases one could argue that more than one classification could apply.
   a. A poker player deliberately plays his cards in a way that he thinks will convince the other players that he holds a particular extremely high hand, when in fact he has a worthless hand.
   b. After the Gulf of Tonkin incident in Vietnam in August of 1964, Lyndon Johnson said in a press conference that "We still seek no wider war." The American government claimed that North Vietnamese warships had attacked U.S. vessels in the Gulf of Tonkin. Later it was shown that the opposite was true. The Vietnam escalation began shortly after Johnson's speech.
   c. H. R. McMaster, a national security adviser to President Trump, defended Trump after it was revealed that he had given highly classified intelligence to Russia. McMaster said: "I think the real issue is, and I think what I would like to see really debated more, is that our national security has been put at risk by those violating confidentiality."
2. Find an example in the media or in your everyday life which fits Frankfurt's definition of bullshit. You may include excerpts of articles, pictures or other material. Justify why the example you propose fits Frankfurt's definition.

2. Games with saddlepoints

Before trying to do these problems, read section 15.1 of the "For all practical purposes" text, which I will refer to as FAPP. I also summarize at the end of this homework the main points regarding saddlepoints.

Suppose that the payoff matrix to the player 1 of a zero sum game between two players is given by

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \cdots & \cdots & \cdots \\ a_{n,1} & \cdots & a_{n,m} \end{pmatrix}$$

where the rows correspond to the n options of player 1 and the columns correspond to the m options of player 2.

The game is defined to have a saddle point if

$$\max_{1\le i\le n}\left(\min_{1\le j\le m} a_{i,j}\right) = \min_{1\le j\le m}\left(\max_{1\le i\le n} a_{i,j}\right).$$

In this case, there exist optimal pure strategies for each player, which may not be unique. A best pure strategy for player 1 is then called a maximin strategy; a best pure strategy for player 2 is then called a minimax strategy.

The game represented by $A = (a_{i,j})_{1\le i\le n, 1\le j\le m}$ has a dominant strategy for one player or the other if there is a choice of option by one player which is at least as good for them as any other option no matter what option the other player picks.

Please briefly explain your reasoning for each of the problems below; the credit I can give depends on your reasoning.

(1) In the following two-person zero-sum game, the payoffs represent gains to row player I and losses to column player II.

$$\begin{pmatrix} 3 & 8 & 2 \\ 8 & 5 & 1 \\ 6 & 5 & 4 \end{pmatrix}$$

what is the maximin strategy for player 1? What is the minimax strategy for player 2?

(2) Consider the following three two-person zero-sum games. The payoffs represent gains to the row player I and losses to column player II.

$$\begin{pmatrix} 3 & 6 \\ 5 & 4 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 5 & 6 & 5 \\ 1 & 4 & 2 & -1 \\ 8 & 5 & 7 & 5 \\ 0 & 2 & 6 & 2 \end{pmatrix}$$

Which of these games have a saddlepoint? In which of the games does neither player have a dominant strategy?

3. Games with mixed strategies.

To understand this topic, I suggest reading pages 21 to 33 of the book "How math can save your life" as well as section 15.2 of FAPP. I think pages 31 to 33 of "How math can save your life" are actually clearer than FAPP when it comes to describing how to choose mixed strategies.

(1) The governor of a state must decide whether to invest \$10 million this year in hurricane preparedness. He's not fond of science, and he's become tired of scientists telling him how hurricanes are becoming more frequent due to global warming. So he decides to ignore them and pick a strategy which maximizes his worse case political scenario. The (purely political) payoff to the governor depending on what he does are as follows:

$$\begin{pmatrix} & \text{hurricane occurs} & \text{no hurricane} \\ \text{governor invests} & 1.0 & -0.5 \\ \text{governor doesn't} & -2.0 & 0.5 \end{pmatrix}$$

Does this game have a saddlepoint? What mixed strategy is the best option for the governor if he sticks to this purely political approach?

(2) Describe a situation in the news, at Penn or in your own experience which can be viewed as a two-person two-option zero sum game. Formulate and explain the payoff matrix for one player in this game and determine what the optimal strategies of the players should be. Comment on the significance of your answer.

Notes on the theory of saddlepoints

Suppose that two players, named I and II, play a zero-sum game in which player I has n options while player II has m options. If player I plays option i and player II plays option j, the payoff to player I is some number $a_{i,j}$, while the payoff to player II is $-a_{i,j}$. (The game is zero sum because the payoff to each player is exactly the negative of the payoff to the other.)

The Payoff matrix for player I is then

$$A = (a_{i,j})_{1\le i\le n, 1\le j\le m}.$$

Rows in this matrix are options for player I, while columns are options for player II.

A. Definition of the maximin.

The maximin of this matrix is

$$\text{maximin}(A) = \max_{1\le i\le n}(\min_{1\le j\le m}\ a_{i,j}) \tag{3.1}$$

This can be computed by first finding

$$A_{row}(i) = \min_{1\le j\le m}\ a_{i,j} = a_{i,j(i)} = \text{a minimal entry in row } i$$

for some integer $j(i)$ depending on i. There may be more than one $j(i)$ which works. One then finds

$$\text{maximin}(A) = \max_{1\le i\le n}\ A_{row}(i) = a_{i_1,j(i_1)}$$

for some i_1.

Player II would like to minimize the payoff to player I of playing the game, since this maximizes the payoff to them. The significance of $A_{row}(i)$ is that if player II knows that player I is going to play option i, they should choose a stategy $j = j(i)$ for themselves which minimizes $a_{i,j}$ as j runs from 1 to m. The payoff to player I of playing option i is then $A_{row}(i)$.

If player I realizes that someone is going to leak their choice of strategy to player II, then they should choose i so that $A_{row}(i)$ is as large as possible. This leads to the payoff maximin(A) to player I (and payoff −maximin(A) to player II). One should think of this number as the best player I can do if once they choose an option, this choice is told to player II, so that player II can best counter the choice made by player I.

B. Definition of the minimax.

The minimax of the matrix A is

$$\text{minimax}(A) = \min_{1\le j\le m}(\max_{1\le i\le n}\ a_{i,j}) \tag{3.2}$$

This can be computed by first finding

$$A_{col}(j) = \max_{1\le i\le n}\ a_{i,j} = a_{i(j),j} = \text{a maximal entry in column } j$$

for some integer $i(j)$. One then finds

$$\text{minimax}(A) = \min_{1\le j\le m} A_{col}(j) = a_{i(j_2),j_2}$$

for some j_2. This number represents the minimal payoff to player I (and thus the maximal payoff to player II) which player II can force if the option that player II picks is communicated to player I so that player I can best counter it.

C. Why the minimax is at least as large as the maximin.

Here is an argument for why

$$\text{maximin}(A) = \max_{1\le i\le n}\left(\min_{1\le j\le m} a_{i,j}\right) \le \min_{1\le j\le m}\left(\max_{1\le i\le n} a_{i,j}\right) = \text{minimax}(A).$$

The left hand side is the highest outcome player 1 can achieve if player 2 plays their best pure strategy against any choice of pure strategy of player 1. The right hand side is the smallest outcome player 2 can hold player 1 to if player 2 plays a pure strategy and player 1 chooses their best pure strategy against each pure strategy by player 2. This implies the inequality.

D. Saddlepoints.

One says A has a saddlepoint if

$$\text{maximin}(A) = \text{minimax}(A)$$

In this case, suppose $\text{maximin}(A) = a_{i_1,j(i_1)}$ and $\text{minimax}(A) = a_{i(j_2),j_2}$ as above. Then

$$\text{maximin}(A) = a_{i_1,j(i_1)} \le a_{i_1,j_2} \le a_{i(j_2),j_2} = \text{maximin}(A)$$

by the definitions of $j(i_1)$ and $i(j_2)$. But we are supposing A has a saddlepoint, so the far left and far right terms are equal, and we get

$$\text{maximin}(A) = a_{i_1,j(i_1)} = a_{i_1,j_2} = a_{i(j_2),j_2} = \text{maximin}(A).$$

We could have chosen $j(i_1)$ to be any j which minimizes $a_{i,j}$ as j ranges over $1 \le j \le m$, so this shows we might as well have chosen $j(i_1) = j_2$. Similarly we could choose $i(j_2) = i_1$.

The upshot is that if A has a saddlepoint, then players I and II should pick options i_1 and j_2 respectively. Player I is then guaranteed a payoff of $\text{maximin}(A) = \text{minimax}(A)$ and player II can ensure that the payoff is not larger than this. Neither player has an incentive to make another choice, since if they did, the other player could benefit at their expense. For instance, if player I picked some other option i_1', then

$$A_{row}(i_1) \ge A_{row}(i_1') = a_{i_1',j(i_1')}$$

and if this is a strict inequality, then player II could make things worse for player II by picking option $j(i_1')$ rather than j_2.

If A does not have a saddlepoint, there need not be one best option for each player. They may raise their expected returns by choosing randomly between their options giving each one a certain probability of being chosen. This is the subject of §15.2 of FAPP.