MATH 350: HOMEWORK #1

DUE IN LECTURE MONDAY, SEPT. 8, 2014.

1. Infinite descent

A proof of a fact by infinite descent involves showing that if the fact is not true, then there would be an infinite sequence of descending positive integers. Such a sequence does not exist because every non-empty set of positive integers has a smallest element.

1. Rewrite the proof that $\sqrt{2}$ is irrational given in Theorem 1.1 of Rosen's book so that is a proof by infinite descent. Here is one way to do this. Suppose to the contrary that $\sqrt{2}$ is rational, so that $\sqrt{2} = a/b$ for some positive integers a and b. There is then a positive integer $b_0 = b$ such that $\sqrt{2}b_0 = a$ is an integer. Show by induction on $i \geq 0$ that
$$b_{i+1} = (2 - \sqrt{2})b_i = 2b_i - \sqrt{2}b_i$$
is a positive integer less than b_i. Conclude that $\{b_i : i \geq 0\}$ is an infinite descending sequence of positive integers, and such a sequence cannot exist.

2. How would you modify the proof in (1) to show that $\sqrt{n}$ is not rational if n is a positive integer which is not a square of another positive integer? (Hint: The hypothesis is that n is not in the set $W = \{m^2 : 0 < m \in \mathbb{Z}\}$ of squares of positive integers. Show that the set
$$\{n - m^2 : 0 < m \in \mathbb{Z} \quad \text{and} \quad 0 < n - m^2 \in \mathbb{Z}\}$$
has a smallest element, which we can write as $n - m_0^2$. Prove that
$$1 \leq m_0^2 < n < (m_0 + 1)^2.$$
You can assume the result from calculus that the function $x \to \sqrt{x}$ is increasing for $x > 0$, which then implies
$$1 \leq m_0 < \sqrt{n} < (m_0 + 1).$$
If $\sqrt{n}b_i$ is an integer for some positive integer b_i, consider $b_{i+1} = ((m_0 + 1) - \sqrt{n})b_i$.)

3. Why does an argument of the kind in (1) and (2) fail to show that $\sqrt{4}$ is not rational?

2. Mathematical induction.

4. Do problem 4 of section 1.3 of Rosen's book.

5. Do problem 32 of section 1.3 of Rosen's book.