Courant Algebroids and Generalizations of Geometry

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- $TM \oplus T^*M \oplus \mathcal{G}$ (type I + YM, heterotic)
Generalizations of geometry

- $TM \oplus T^*M$ (type I or type II in absence of RR fluxes)
- $TM \oplus T^*M \oplus S^\pm$ (type IIA/IIB with RR fluxes)
- $TM \oplus T^*M \oplus G$ (type I + YM, heterotic)
- $TM \oplus \wedge^2 T^*M \oplus \ldots$ (M-theory)
Outline of the talk

- Review of generalized geometry
- Beyond generalized geometry
- $B^2$-geometry
- $M$-geometry
- $T$-duality and generalized geometry
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  - $B_{2n}$-geometry
  - M-geometry
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- Beyond generalized geometry
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  - M-geometry
- T-duality and generalized geometry
Replace structures on $TM$ (such as $[~,~], \iota_x, \mathcal{L}_x, d, ...$) by similar structures on $E = TM \oplus T^*M$
Generalized geometry [Hitchin, Gualtieri, Cavalcanti]

Replace structures on $TM$ (such as $[,]$, $\iota_x$, $\mathcal{L}_x$, $d$, ...) by similar structures on $E = TM \oplus T^*M$

- Bilinear form on sections $x + \xi \in \Gamma(TM \oplus T^*M)$

$$\langle x + \xi, y + \eta \rangle = \frac{1}{2}(\iota_x \eta + \iota_y \xi)$$
Generalized geometry [Hitchin, Gualtieri, Cavalcanti]

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- (Dorfman) Bracket

$$(x + \xi) \circ (y + \eta) = [x, y] + \mathcal{L}_x \eta - \iota_y d\xi$$
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Replace structures on $TM$ (such as $\lbrack \; , \rbrack$, $\iota_x$, $\mathcal{L}_x$, $d$, ...) by similar structures on $E = TM \oplus T^*M$

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  $$(x + \xi) \circ (y + \eta) = [x, y] + \mathcal{L}_x \eta - \iota_y d\xi$$

- Clifford algebra

  $$\{ \gamma x + \xi, \gamma y + \eta \} = 2\langle x + \xi, y + \eta \rangle$$
Clifford module $\Omega^\bullet(M)$

$$\gamma^x + \xi \cdot \omega = \iota_x \omega + \xi \wedge \omega$$
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De-Rham differential on $\Omega^\bullet(M)$

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$
Symmetries of $\langle \ , \ \rangle$ given by sections of the adjoint bundle

\[ \wedge^2 E \cong \wedge^2 TM \oplus \text{End}(TM) \oplus \wedge^2 T^* M \]

[They form the group $O(n, n)$]
Symmetries of \( \langle , \rangle \) given by sections of the adjoint bundle

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\wedge^2 E \cong \wedge^2 TM \oplus \text{End}(TM) \oplus \wedge^2 T^* M
\]

[They form the group \( O(n, n) \)]

In particular, we have the so called B-transform, for \( b \in \Omega^2(M) \)

\[
e^b \cdot (x + \xi) = x + (\xi + \iota_x b)
\]

\[
b \cdot (x + \xi) = \iota_x b \quad \text{(infinitesimally)}
\]
Symmetries of $\langle \ , \ \rangle$ given by sections of the adjoint bundle

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We have

$$e^b \cdot ((x + \xi) \circ (y + \eta)) = e^b \cdot (x + \xi) \circ e^b \cdot (y + \eta) + \iota_x \iota_y db$$
Symmetries of $\langle \ , \ \rangle$ given by sections of the adjoint bundle

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We have

$$e^b \cdot ((x + \xi) \circ (y + \eta)) = e^b \cdot (x + \xi) \circ e^b \cdot (y + \eta) + \iota_x \iota_y db$$

Symmetries of the Dorfman bracket are $\text{Diff}(M) \ltimes \Omega^2_{\text{cl}}(M)$
This suggest the introduction of a twisted Dorfman bracket, with $H \in \Omega^3(M)$, $dH = 0$

$$(x + \xi) \circ_H (y + \eta) = [x, y] + \mathcal{L}_x \eta - \iota_y d\xi + \iota_x \iota_y H$$
Twisting

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$$(x + \xi) \circ_H (y + \eta) = [x, y] + \mathcal{L}_x \eta - \iota_y d\xi + \iota_x \iota_y H$$

such that

$$e^b \cdot ((x + \xi) \circ_H (y + \eta)) = \left(e^b \cdot (x + \xi)\right) \circ_{H+db} \left(e^b \cdot (y + \eta)\right)$$
Twisting

This suggest the introduction of a twisted Dorfman bracket, with $H \in \Omega^3(M)$, $dH = 0$

$$(x + \xi) \circ_H (y + \eta) = [x, y] + \mathcal{L}_x \eta - \imath_y d\xi + \imath_x \imath_y H$$

such that

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and a twisted differential

$$dH \omega = d\omega + H \wedge \omega$$
Properties (for $A, B, C \in \Gamma E, f \in C^\infty(M)$)

(i) $A \circ (B \circ C) = (A \circ B) \circ C + B \circ (A \circ C)$

(ii) $A \circ (fB) = f(A \circ B) + (\rho(A)f)B$
Properties (for $A, B, C \in \Gamma E$, $f \in C^\infty(M)$)

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The Courant bracket is defined as the anti-symmetrization

$$[[A, B]] = \frac{1}{2}(A \circ B - B \circ A)$$

or, conversely,

$$A \circ B = [[A, B]] + d\langle A, B \rangle$$
Recall the usual Cartan relations

\[
\{\iota_x, \iota_y\} = 0 \\
\{d, \iota_x\} = \mathcal{L}_x \\
[\mathcal{L}_x, \iota_y] = \iota_{[x,y]} \\
[\mathcal{L}_x, \mathcal{L}_y] = \mathcal{L}_{[x,y]} \\
[d, \mathcal{L}_x] = 0
\]
Dorfman bracket as a derived bracket

Recall the usual Cartan relations

\{i_x, i_y\} = 0
\{d, i_x\} = \mathcal{L}_x
[\mathcal{L}_x, i_y] = i_{[x,y]}
[\mathcal{L}_x, \mathcal{L}_y] = \mathcal{L}_{[x,y]}
[d, \mathcal{L}_x] = 0

Here we have the analogue

\{\gamma_A, \gamma_B\} = 2\langle A, B \rangle
\{d_H, \gamma_A\} = \mathcal{L}_A
[\mathcal{L}_A, \gamma_B] = \gamma_{A \circ B}
[\mathcal{L}_A, \mathcal{L}_B] = \mathcal{L}_{A \circ B} = \mathcal{L}_{[A,B]}
[d_H, \mathcal{L}_A] = 0

where \(\mathcal{L}_x + \xi \omega = \mathcal{L}_x \omega + (d \xi + i_x \mathcal{H}) \wedge \omega\)
A Leibniz algebroid \((E, \circ, \rho)\) is a vector bundle \(E \rightarrow M\), with a composition (Leibniz/Loday bracket) \(\circ\) on \(\Gamma E\), and a morphism of vector bundles \(\rho : E \rightarrow TM\) (anchor) such that

(L1) \(A \circ (B \circ C) = (A \circ B) \circ C + B \circ (A \circ C)\)

(L2) \(\rho(A \circ B) = [\rho(A), \rho(B)]\)

(L3) \(A \circ (fB) = f(A \circ B) + (\rho(A)f)B\)
A Courant algebroid \((E, \circ, \langle \ , \rangle, \rho)\) is a vector bundle \(E \to M\), with a composition \(\circ\) on \(\Gamma E\), a morphism of vector bundles \(\rho : E \to TM\), and a field of nondegenerate bilinear forms \(\langle \ , \rangle\) on \(\Gamma E\) such that

\[(C1) \ A \circ (B \circ C) = (A \circ B) \circ C + B \circ (A \circ C)\]

\[(C2) \ \rho(A)\langle B, C \rangle = \langle A, B \circ C + C \circ B \rangle\]

\[(C3) \ \rho(A)\langle B, C \rangle = \langle A \circ B, C \rangle + \langle B, A \circ C \rangle\]
A Courant algebroid $(E, \circ, \langle , \rangle, \rho)$ is a vector bundle $E \to M$, with a composition $\circ$ on $\Gamma E$, a morphism of vector bundles $\rho : E \to TM$, and a field of nondegenerate bilinear forms $\langle, \rangle$ on $\Gamma E$ such that

\begin{align*}
(C1) \quad A \circ (B \circ C) &= (A \circ B) \circ C + B \circ (A \circ C) \\
(C2) \quad \rho(A)\langle B, C \rangle &= \langle A, B \circ C + C \circ B \rangle \\
(C3) \quad \rho(A)\langle B, C \rangle &= \langle A \circ B, C \rangle + \langle B, A \circ C \rangle
\end{align*}

It follows that $(E, \circ, \rho)$ is a Leibniz algebroid. Moreover

$$A \circ B + B \circ A = 2D\langle A, B \rangle$$

where $D = \frac{1}{2} \rho^* d : C^\infty(M) \to \Gamma E$, i.e. $\langle Df, A \rangle = \frac{1}{2} \rho(A)f$
An exact Courant algebroid $E$ is a Courant algebroid that fits in the exact sequence

$$0 \rightarrow T^*M \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \rightarrow 0$$
Exact Courant algebroids

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Such a Courant algebroid admits an isotropic splitting $s : TM \rightarrow E$, which allows us to identify $E \cong TM \oplus T^*M$. The composition on $x + \xi \in \Gamma(TM \oplus T^*M)$ is uniquely determined by

$$H(x, y, z) = \langle x \circ y, z \rangle$$

It turns out $H \in \Omega^3_{\text{cl}}(M)$.
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**Theorem (Ševera)**

*Equivalence classes of exact Courant algebroids are in 1–1 correspondence with $H^3(M, \mathbb{R})$.***
We now consider the vector bundle

\[ E = TM \oplus 1 \oplus T^*M \]

with nondegenerate bilinear form

\[ \langle x + f + \xi, y + g + \eta \rangle = \frac{1}{2}(\iota_x \eta + \iota_y \xi) + fg \]
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Dorfman bracket

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and anchor map

\[ \rho(x + f + \xi) = x \]
Adjoint bundle

\[ \wedge^2 E = \wedge^2 TM \oplus TM \oplus \text{End}(TM) \oplus T^* M \oplus \wedge^2 T^* M \]
Adjoint bundle

$$\wedge^2 E = \wedge^2 TM \oplus TM \oplus \text{End}(TM) \oplus T^* M \oplus \wedge^2 T^* M$$

In particular

$$a \cdot (x + f + \xi) = \iota_x a - fa, \quad a \in \Omega^1$$

$$b \cdot (x + f + \xi) = \iota_x b, \quad b \in \Omega^2$$
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Note

$$[a_1, a_2] = a_1 \wedge a_2$$
$B_{2n}$-geometry (cont’d)

Adjoint bundle

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Note

$$[a_1, a_2] = a_1 \wedge a_2$$

Symmetries of the Dorfman bracket iff \(da = 0 = db\).
Consider $F \in \Omega^2(M)$, $H \in \Omega^3(M)$, and a twisted Dorfman bracket

$$(x + f + \xi) \circ (y + g + \eta) = [x, y] + (x(g) - y(f)) + \iota_x \iota_y F + \mathcal{L}_x \eta - \iota_y d\xi + 2gdf + \iota_x \iota_y H + \iota_x Fg - \iota_y Ff$$

This defines a Courant algebroid provided $dF = 0$ and $dH + F \wedge F = 0$.
Consider $F \in \Omega^2(M)$, $H \in \Omega^3(M)$, and a twisted Dorfman bracket

$$(x + f + \xi) \circ (y + g + \eta) = [x, y] + (x(g) - y(f)) + \iota_x \iota_y F$$

$$+ \mathcal{L}_x \eta - \iota_y d\xi + 2gdf + \iota_x \iota_y H + \iota_x Fg - \iota_y Ff$$

This defines a Courant algebroid provided

$$dF = 0$$

$$dH + F \wedge F = 0$$
The bracket can be obtained as a derived bracket using

\[ d_{F,H} \omega = d\omega + (-1)^{|\omega|} F \wedge \omega + H \wedge \omega \]

and

\[ \gamma_{x+f+\xi} \cdot \omega = \iota_x \omega + (-1)^{|\omega|} f \omega + \xi \wedge \omega \]
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and

\[ \gamma_{x+f+\xi} \cdot \omega = \iota_x \omega + (-1)^{|\omega|} f \omega + \xi \wedge \omega \]

Note

\[ d_{F,H}^2 = 0 \]

\[ \{ \gamma_{x+f+\xi}, \gamma_{y+g+\eta} \} = 2\langle x + f + \xi, y + g + \eta \rangle \]

iff \( F \) and \( H \) satisfy the Bianchi identities as before.
The bracket can be obtained as a derived bracket using

\[ d_{F,H} \omega = d\omega + (-1)^{|\omega|} F \wedge \omega + H \wedge \omega \]

and

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Note

\[ d_{F,H}^2 = 0 \]

\[ \left\{ \gamma_{x+f+\xi}, \gamma_{y+g+\eta} \right\} = 2\langle x + f + \xi, y + g + \eta \rangle \]

iff \( F \) and \( H \) satisfy the Bianchi identities as before.

This is an example of non-exact transitive Courant algebroid

\[
T^* M \longrightarrow TM \oplus 1 \oplus T^* M \longrightarrow TM \longrightarrow 0
\]
Transitive Courant algebroids are of the form

\[ E = TM \oplus \mathfrak{g} \oplus T^* M \]

where \( \mathfrak{g} = \ker \rho / (\ker \rho)^\perp \) is a bundle of Lie algebras with bracket

\[ [r, s]_{\mathfrak{g}} = \text{pr}_{\mathfrak{g}}(r \circ s) \]

and

\[ \rho(x + r + \xi) = x \]
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\[ [r, s]_{\mathfrak{g}} = \text{pr}_{\mathfrak{g}}(r \circ s) \]

and

\[ \rho(x + r + \xi) = x \]

Suppose

\[ \langle x + r + \xi, y + s + \eta \rangle = \frac{1}{2} (\iota_x \eta + \iota_y \xi) + \langle r, s \rangle_{\mathfrak{g}} \]
Then the Dorfman bracket on $E$ is completely determined by

$$H(x, y, z) = \langle \text{pr}_{T^*M}(x \circ y), z \rangle$$

$$R(x, y) = \text{pr}_{\mathcal{G}}(x \circ y)$$

$$\nabla_x r = \text{pr}_{\mathcal{G}}(x \circ r).$$

It turns out $H \in \Omega^3(M)$, $R \in \Omega^2(M, g)$, and $\nabla_x$ a $TM$-connection on $\mathcal{G}$.
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$$H(x, y, z) = \langle \text{pr}_{T^*M}(x \circ y), z \rangle$$
$$R(x, y) = \text{pr}_G(x \circ y)$$
$$\nabla_x r = \text{pr}_G(x \circ r).$$

It turns out $H \in \Omega^3(M)$, $R \in \Omega^2(M, g)$, and $\nabla_x$ a $TM$-connection.

Namely

$$(x + r + \xi) \circ (y + s + \eta) = [x, y]$$
$$- \iota_x \iota_y R + [r, s]_G + \nabla_x s - \nabla_y r$$
$$- \iota_x \iota_y H + \langle s, d\nabla r \rangle_G + \mathcal{L}_x \eta - \iota_y d\xi + \langle \iota_x R, s \rangle_G - \langle \iota_y R, r \rangle_G$$
where we have defined a ‘twisted differential’

\[ d_{\nabla} : \Omega^k(M, \Gamma G) \to \Omega^{k+1}(M, \Gamma G) \]

by

\[
(d_{\nabla} \omega)(x_0, \ldots, x_k) = \sum_{i=0}^{k} (-1)^i \nabla_{x_i} \omega(x_0, \ldots, \hat{x_i}, \ldots, x_k) \\
+ \sum_{0 \leq i < j \leq k} (-1)^{i+j} \omega([x_i, x_j], x_0, \ldots, \hat{x_i}, \ldots, \hat{x_j}, \ldots, x_k)
\]

[Note that \( d_{\nabla}^2 = 0 \) iff the curvature corresponding to \( \nabla_x \) vanishes]
where

\[ L_x \langle r, s \rangle = \langle \nabla_x r, s \rangle + \langle r, \nabla_x s \rangle \]

\[ \nabla_x [r, s] = [\nabla_x r, s] + [r, \nabla_x s] \]

\[ d_\nabla R = 0 \]

\[ d^2_\nabla r = (\nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x, y]}) r = [R, r], \]

\[ dH = \langle R \wedge R \rangle. \]
In M-theory we have a 3-form $C_3$ with

$$F_4 = dC_3$$

satisfying

$$dF_4 = 0 \quad \text{(Bianchi)}$$

and

$$d(*F_4) + \frac{1}{2} F_4 \wedge F_4 = 0 \quad \text{(e.o.m.)}$$
In M-theory we have a 3-form $C_3$ with

$$F_4 = dC_3$$

satisfying

$$dF_4 = 0 \quad \text{(Bianchi)}$$

and

$$d(\ast F_4) + \frac{1}{2} F_4 \wedge F_4 = 0 \quad \text{(e.o.m.)}$$

After putting $F_7 = \ast F_4$ we have

$$d(F_7 + \frac{1}{2} C_3 \wedge F_4) = 0, \quad F_7 + \frac{1}{2} C_3 \wedge F_4 = dC_6$$
Summarizing

\[ F_4 = dC_3 \]

\[ F_7 = dC_6 - \frac{1}{2} C_3 \wedge F_4 \]

with

\[ dF_4 = 0 \]

\[ dF_7 + \frac{1}{2} F_4 \wedge F_4 = 0 \]
Symmetries by $z_3 \in \Omega^3_{\text{cl}}, z_6 \in \Omega^6_{\text{cl}}$

\[
C'_3 = C_3 + z_3 \\
C'_6 = C_6 + z_6 + \frac{1}{2} C_3 \wedge z_3
\]

Group law

\[
(z_3, z_6) \cdot (z'_3, z'_6) = (z_3 + z'_3, z_6 + z'_6 - \frac{1}{2} z_3 \wedge z'_3)
\]
The relevant bundle in this case is

\[ E = TM \oplus \wedge^2 T^* M \oplus \wedge^5 T^* M \]

with Dorfman bracket

\[(x + a_2 + a_5) \circ (y + b_2 + b_5) = [x, y] + \mathcal{L}_x b_2 - \iota_y da_2 + \mathcal{L}_x b_5 - \iota_y da_5 + da_2 \wedge b_2\]
The relevant bundle in this case is

\[ E = TM \oplus \wedge^2 T^* M \oplus \wedge^5 T^* M \]

with Dorfman bracket

\[(x + a_2 + a_5) \circ (y + b_2 + b_5) = [x, y] + \mathcal{L}_x b_2 - \iota_y da_2 + \mathcal{L}_x b_5 - \iota_y da_5 + da_2 \wedge b_2\]

The bracket is invariant under infinitesimal symmetries generated by \( z_3 \in \Omega^3_{\text{cl}}, z_6 \in \Omega^6_{\text{cl}} \)

\[ z_3 \cdot (x + a_2 + a_5) = \iota_x z_3 - z_3 \wedge a_2 \]
\[ z_6 \cdot (x + a_2 + a_5) = -\iota_x z_6 \]
It can be twisted by $F_4 \in \Omega^4$, $F_7 \in \Omega^7$

$$(x + a_2 + a_5) \circ (y + b_2 + b_5) = [x, y]$$

$$+ \mathcal{L}_x b_2 - \iota_y da_2 + \iota_x \iota_y F_4$$

$$+ \mathcal{L}_x b_5 - \iota_y da_5 + da_2 \wedge b_2 + \iota_x F_4 \wedge b_2 + \iota_x \iota_y F_7$$
M-geometry (cont’d)

It can be twisted by $F_4 \in \Omega^4$, $F_7 \in \Omega^7$

$$(x + a_2 + a_5) \circ (y + b_2 + b_5) = [x, y]$$

$$+ \mathcal{L}_x b_2 - \iota_y da_2 + \iota_x \iota_y F_4$$

$$+ \mathcal{L}_x b_5 - \iota_y da_5 + da_2 \wedge b_2 + \iota_x F_4 \wedge b_2 + \iota_x \iota_y F_7$$

and is a Leibniz algebroid iff

$$dF_4 = 0$$

$$dF_7 + \frac{1}{2} F_4 \wedge F_4 = 0$$
M-geometry (cont’d)

It can be twisted by $F_4 \in \Omega^4$, $F_7 \in \Omega^7$

$$(x + a_2 + a_5) \circ (y + b_2 + b_5) = [x, y]$$

$$+ \mathcal{L}_x b_2 - \iota_y da_2 + \iota_x \iota_y F_4$$

$$+ \mathcal{L}_x b_5 - \iota_y da_5 + da_2 \wedge b_2 + \iota_x F_4 \wedge b_2 + \iota_x \iota_y F_7$$

and is a Leibniz algebroid iff

$$dF_4 = 0$$

$$dF_7 + \frac{1}{2} F_4 \wedge F_4 = 0$$

Note that we have $\langle \ , \ \rangle : E \otimes E \to T^* M \oplus \wedge^4 T^* M$

$$\langle x + a_2 + a_5, y + b_2 + b_5 \rangle = (\iota_x b_2 + \iota_y a_2) + (\iota_x b_5 + \iota_y a_5 + a_2 \wedge b_2)$$

(cf. notion of E-Courant algebroid [Chen-Liu-Sheng])
T-duality for principal $S^1$-bundles

Suppose we have a pair $(E, H)$, consisting of a principal circle bundle

$$
\begin{array}{c}
S^1 \\
\downarrow \pi \\
\downarrow \\
M
\end{array} 
\xrightarrow{\quad} 
\begin{array}{c}
E
\end{array}
$$

and a so-called H-flux $H$, a Čech 3-cocycle.

Topologically, $E$ is classified by an element in $F \in H^2(M, \mathbb{Z})$ while $H$ gives a class in $H^3(E, \mathbb{Z})$.
The T-dual of \((E, H)\) is given by the pair \((\hat{E}, \hat{H})\), where the principal \(S^1\)-bundle

\[
\begin{array}{ccc}
\hat{S}^1 & \longrightarrow & \hat{E} \\
\hat{\pi} & \downarrow & \\
& M & \\
\end{array}
\]

and the dual H-flux \(\hat{H} \in H^3(\hat{E}, \mathbb{Z})\), satisfy

\[
\hat{F} = \pi_* H, \quad F = \hat{\pi}_* \hat{H}
\]

where \(\pi_* : H^3(E, \mathbb{Z}) \to H^2(M, \mathbb{Z})\), and \(\hat{\pi}_* : H^3(\hat{E}, \mathbb{Z}) \to H^2(M, \mathbb{Z})\) are the pushforward maps (‘integration over the \(S^1\)-fiber’).
The ambiguity in the choice of $\hat{H}$ is removed by requiring that

$$p^*H - \hat{p}^*\hat{H} \equiv 0$$

in $H^3(E \times_M \hat{E}, \mathbb{Z})$, where $E \times_M \hat{E}$ is the correspondence space

$$E \times_M \hat{E} = \{(x, \hat{x}) \in E \times \hat{E} \mid \pi(x) = \hat{\pi}(\hat{x})\}$$
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**Theorem (B-Evslin-Mathai)**

*This T-duality gives rise to an isomorphism between the twisted cohomologies and twisted K-theories of $(E, H)$ and $(\hat{E}, \hat{H})$ (with a shift in degree by 1)*
Given a principal circle bundle $E$ with H-flux $H \in \Omega^3_{\text{cl}}(E)^{S^1}$

$$S^1 \overset{\pi}{\longrightarrow} E \overset{H = H(3) + A \wedge H(2), \ F = dA}{\longrightarrow} M$$
Given a principal circle bundle $E$ with H-flux $H \in \Omega^3_{\text{cl}}(E)^{S^1}$

\[ H = H_{(3)} + A \wedge H_{(2)}, \quad F = dA \]

there exists a T-dual principal circle bundle

\[ \hat{H} = H_{(3)} + \hat{A} \wedge F, \quad \hat{F} = H_{(2)} = d\hat{A} \]
Theorem [B-Evslin-Mathai, Cavalcanti-Gualtieri]

(a) We have an isomorphism of differential complexes

\[ \tau : (\Omega^\bullet(E)^{S^1}, d_H) \rightarrow (\Omega^\bullet(\hat{E})^{S^1}, d_{\hat{H}}) \]
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\[ \tau : (\Omega^\bullet(E)^{S^1}, d_H) \rightarrow (\Omega^\bullet(\hat{E})^{S^1}, d_{\hat{H}}) \]

\[ \tau(\Omega_k + A \wedge \Omega_{k-1}) = -\Omega_{k-1} + \hat{A} \wedge \Omega_k \]

\[ \tau \circ d_H = -d_{\hat{H}} \circ \tau \]
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Hence, \( \tau \) induces an isomorphism on twisted cohomology

(b) We can identify \((X + \Xi \in \Gamma(TE \oplus T^*E)^{S^1}\) with a quadruple \((x, f; \xi, g)\)

\[ X = x + f \partial A, \quad \Xi = \xi + gA \]

and define a map \( \phi : \Gamma(TE \oplus T^*E)^{S^1} \to \Gamma(T\hat{E} \oplus T^*\hat{E})^{S^1} \)

\[ \phi(x + f \partial A + \xi + gA) = x + g \partial \hat{A} + \xi + f \hat{A} \]
(b) The map $\phi$ is orthogonal wrt pairing on $TE \oplus T^*E$, hence $\tau$ induces an isomorphism of Clifford algebras.
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(c) For $X + \Xi \in \Gamma((TE \oplus T^*E)^{S^1})$ we have

$$\tau(\gamma_{X + \Xi} \cdot \Omega) = \gamma_{\phi(X + \Xi)} \cdot \tau(\Omega)$$
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Hence $\tau$ induces an isomorphism of Clifford modules.

(d) For $X_i + \Xi_i \in \Gamma((TE \oplus T^*E)^{S^1})$ we have

$$\phi(\llbracket X_1 + \Xi_1, X_2 + \Xi_2 \rrbracket_H) = \llbracket \phi(X_1 + \Xi_1), \phi(X_2 + \Xi_2) \rrbracket_{\tilde{H}}$$
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Hence $\phi$ gives a homomorphism of twisted Courant brackets.
(b) The map $\phi$ is orthogonal wrt pairing on $TE \oplus T^*E$, hence $\tau$ induces an isomorphism of Clifford algebras.

(c) For $X + \Xi \in \Gamma((TE \oplus T^*E)^{S^1})$ we have

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Hence $\phi$ gives a homomorphism of twisted Courant brackets.

It follows that T-duality acts naturally on generalized complex structures, generalized Kähler structures, generalized Calabi-Yau structures, ...
The dimensionally reduced Dorfman bracket

\[(x_1, f_1; \xi_1, g_1) \circ (x_2, f_2; \xi_2, g_2) =
\]
\[([x_1, x_2], x_1(f_2) - x_2(f_1) + \iota_{x_1} \iota_{x_2} F;
\]
\[\{ L_{x_1} \xi_2 - \iota_{x_2} d\xi_1 \} + \iota_{x_1} \iota_{x_2} H(3) + (df_1 g_2 + f_2 dg_1)
\]
\[+ \left( g_2 \iota_{x_1} F - g_1 \iota_{x_2} F \right) + \left( f_2 \iota_{x_1} H(2) - f_1 \iota_{x_2} H(2) \right),
\]
\[x_1(g_2) - x_2(g_1) + \iota_{x_1} \iota_{x_2} H(2) \]

is that of the transitive Courant algebroid

\[E = TM \oplus (t \oplus t^*) \oplus T^*M \text{ with } R = -(F, H(2)), \ H = -H(3) \text{ and } \langle \ , \rangle_{\psi} \text{ the canonical pairing between } t \text{ and } t^*.\]
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\[([x_1, x_2], x_1(f_2) - x_2(f_1) + \iota_{x_1} \iota_{x_2} F;
\]
\[(\mathcal{L}_{x_1} \xi_2 - \iota_{x_2} d\xi_1) + \iota_{x_1} \iota_{x_2} H(3) + (df_1 g_2 + f_2 dg_1)
\]
\[+ (g_2 \iota_{x_1} F - g_1 \iota_{x_2} F) + (f_2 \iota_{x_1} H(2) - f_1 \iota_{x_2} H(2)),
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[Doubling of the Atiyah algebroid corresponding to the principal \( S^1 \)-bundle]
Generalization to principal torus bundles

We have

\[ H = H_3 + A_i \wedge H_i^{(2)} + \frac{1}{2} A_i \wedge A_j \wedge H_{ij}^{(1)} + \frac{1}{6} A_i \wedge A_j \wedge A_k \wedge H_{ijk}^{(0)} \]
Generalization to principal torus bundles

We have

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such that

\[ dH = \bar{d} + H_{(3)} + F_{(2)i} \partial A_i \]

\[ + A_i \wedge H^i_{(2)} + \frac{1}{2} A_i \wedge A_j \wedge H^{ij}_{(1)} + \frac{1}{6} A_i \wedge A_j \wedge A_k \wedge H^{ijk}_{(0)} \]
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such that

\[ dH = \bar{d} + H(3) + F(2)i \partial A_i + \frac{1}{2} F(1)^{ij} \partial A_i \wedge \partial A_j + \frac{1}{6} F(0)^{ijk} \partial A_i \wedge \partial A_j \wedge \partial A_k \\
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The \( F(1)ij \) and \( F(0)ijk \) are known as nongeometric fluxes.
Generalization to principal torus bundles

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Theorem (B-Garretson-Kao)

\( T \)-duality provides an isomorphism of (certain) Courant algebroids
THANKS