W-curves and Mirror Symmetry

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The theory of Fan-Jarvis-Ruan-Witten
- provides an orbifolded LG A-Model for each quasi-homogeneous singularity \( W \)
- is known to satisfy a form of LG-LG mirror symmetry
- fits into the LG-CY correspondence
By *singularity* we mean

- \( W \in \mathbb{C}[x_1, \ldots, x_N] \) is quasi-homogeneous with weights \( q_1, \ldots, q_N \in \mathbb{Q} \cap [0, 1/2] \)
- \( W \) is non-degenerate (isolated singularity at origin)
- The weights of \( W \) are unique (i.e., no term of the form \( xy \))
Examples of Singularities

\[ A_n = x^{n+1} \quad \text{weight} \ q_x = \frac{1}{(n+1)} \]
\[ E_6 = x^3 + y^4 \quad \text{weights} \ q_x = 1/3 \quad q_y = 1/4 \]
\[ E_7 = x^3 + xy^3 \quad \text{weights} \ q_x = 1/3 \quad q_y = 2/9 \]
\[ E_8 = x^3 + y^5 \quad \text{weights} \ q_x = 1/3 \quad q_y = 1/5 \]
\[ D_n = x^{n-1} + xy^2 \quad \text{weights} \ q_x = \frac{1}{(n-1)} \quad q_y = \frac{(n-1)}{2n} \]
\[ P_8 = x^3 + y^3 + z^3 \quad \text{weights} \ q_x = q_y = q_z = 1/3 \]
Automorphism groups

There is a canonical automorphism of $W$ defined by

$$J = (\exp(2\pi iq_1, \ldots, 2\pi iq_N) \in (\mathbb{C}^*)^N).$$

The largest group of automorphisms we allow is

$$G_{max} = \{(\alpha_1, \ldots, \alpha_N) \in (\mathbb{C}^*)^N | W(\alpha_1 x_1, \ldots, \alpha_N x_N) = W(x_1, \ldots, x_N)\}$$

A group $G$ is admissible if

$$J \in G \leq G_{max}$$
An orbifold Landau-Ginzburg A-model

Theorem (Fan, Jarvis, Ruan (after Witten))

For each nondegenerate $W$ and admissible $G$ there is a

- Moduli space $\mathcal{M}_{W,G}^{W,G}$
- State space $\mathcal{H}_{W,G} = \bigoplus_{\gamma \in G} \mathcal{H}_\gamma$
- Pairing $\langle , \rangle : \mathcal{H}_\gamma \times \mathcal{H}_{\gamma^{-1}} \rightarrow \mathbb{C}$
- Virtual Cycle $[\mathcal{M}_{W,G}^{W,G}]^{vir}$
- Cohomological Field Theory $\{\Lambda_{g,k}^{W,G}, \mathcal{H}, \langle , \rangle\}$
Consequences of a CohFT

Associated to the CohFT we get

- $g = 0, n = 3$ Frobenius algebra $\mathcal{H}_{W,G}, \star$
- $g = 0, n \geq 3$ Frobenius manifold
- $g \geq 0$ potential $\Phi^{FJRW}_{W,G}$
Associated to each singularity $W$ we have

- Frobenius algebra: Milnor ring $\mathbb{C}[x_1, \ldots, x_N]/(\frac{\partial W}{\partial x_1}, \ldots, \frac{\partial W}{\partial x_N})$
- Saito Frobenius manifold associated to $W$ (semi-simple)
- Givental formal potential $\Phi_{W}^{Givental}$
- In some cases:
  Kac-Wakimoto/Drinfeld-Sokolov integrable hierarchy
## LG-LG mirror symmetry

<table>
<thead>
<tr>
<th>A-model (FJRW)</th>
<th>B-model</th>
<th>g ≥ 0</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Singularity</strong></td>
<td><strong>Group</strong></td>
<td><strong>Potential</strong></td>
</tr>
<tr>
<td>$A_n$</td>
<td>$\langle J \rangle = G_{max}$</td>
<td>$A_n$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$\langle J \rangle = G_{max}$</td>
<td>$E_6$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$\langle J \rangle = G_{max}$</td>
<td>$E_7$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$\langle J \rangle = G_{max}$</td>
<td>$E_8$</td>
</tr>
<tr>
<td>$D_n$</td>
<td>$\langle J \rangle$</td>
<td>$D_n$</td>
</tr>
<tr>
<td>$n$ even</td>
<td>$[G_{max} : \langle J \rangle] = 2$</td>
<td>$D_n^T / \mathbb{Z}_2$</td>
</tr>
<tr>
<td>$D_n$</td>
<td>$G_{max}$</td>
<td>$D_n^T$</td>
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<td>$D_n$</td>
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W-curves and Mirror Symmetry
For each singularity

\[ W = \sum_{i=1}^{N} c_i \prod_{j=1}^{N} x_i^{a_{ij}} \]

with the same number of monomials as variables (invertible), just transpose the exponent matrix:

\[ A := (a_{ij}) \Rightarrow A^T = (a_{ji}) \]

to get

\[ W^T = \sum_{i=1}^{N} c_i \prod_{j=1}^{N} x_i^{a_{ji}} \]

- \( A_n, E_{6,7,8}, P_8 \) are all self-dual
- \( D_n^T = x^{n-1}y + y^2 \cong A_{2n-3} \)
Krawitz defines $G^T$ in such a way that

- $|G^T| = [G_{max} : G]$ 
- $G^T \leq SL_N$ iff $J \in G$

Orbifold mirror:

- For Frobenius algebras:
  FJRW (A-model) for $W/G$ is isomorphic to IV-K-K orbifold
  Milnor ring for $W^T/G^T$

- Expect same for Frobenius manifolds:
  (Orbifolded Frobenius manifold is not yet fully defined)
**LG-LG mirror symmetry**

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<tr>
<td>( W ) ( G )</td>
<td>( g = 0 ) ( n = 3 ) \begin{itemize} \item Milnor Ring \end{itemize}</td>
</tr>
<tr>
<td>( P_8 ) ( G_{\text{max}} )</td>
<td>( P_8 )</td>
</tr>
<tr>
<td>( X_9 ) ( G_{\text{max}} )</td>
<td>( X_9 )</td>
</tr>
<tr>
<td>( J_{10} ) ( G_{\text{max}} )</td>
<td>( J_{10} )</td>
</tr>
<tr>
<td>( W ) ( G_{\text{max}} )</td>
<td>( W^T ) [AKS]</td>
</tr>
<tr>
<td>( W ) ( G &lt; G_{\text{max}} ) for most inv ( W )</td>
<td>( \left[ W^T / G^T \right] ) [FJJ]</td>
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