(0, 2) Quantum Cohomology

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Outline

1 Background
   - Quantum Cohomology
   - The half-twisted model
   - The Gauged Linear Sigma Model and Toric Geometry

2 Correlation Functions and Quantum Cohomology
   - Correlation Functions
   - Quantum Cohomology
Abstract

In this talk, a mathematical definition is given of the topological correlation functions of a (0, 2) gauged linear sigma model in the geometric phase of a neighborhood of the (2, 2) locus in moduli. The geometric data determining the model is a smooth toric variety $X$ and a deformation $E$ of its tangent bundle $TX$. This definition is consistent with the known results of physics and leads to a proof of the existence of a quantum cohomology ring in complete generality, extending known results of physics.
Consider the topological A-model on a compact Kähler manifold \((X, g)\) (no coupling to gravity, a TQFT)

- Observables \(H^*(X)\)
- Correlation functions
  \[ \langle \omega_1, \ldots, \omega_n \rangle_\beta, \omega_i \in H^{2k_i}(X), \beta \in H_2(X, \mathbb{Z}) \]
- Virtual dimension
  \[ D = c_1(X) \cdot \beta + \text{dim } X, \quad \sum k_i = D \]
- \[ \langle \omega_1, \ldots, \omega_n \rangle = \sum_\beta \langle \omega_1, \ldots, \omega_n \rangle_\beta q^{\beta}, \quad q^{\beta} = \exp(\int_\beta (B + ig)), \]
  where \(B\) is the B-field
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A Model Lagrangian

\[ \mathcal{L} = \frac{1}{\alpha'} \int_{\Sigma} d^2 z \left( (g_{\mu\nu} + iB_{\mu\nu}) \partial \phi^\mu \overline{\partial} \phi^\nu + \frac{i}{2} g_{\mu\nu} \psi^\mu_+ D_z \psi^\nu_+ \right. \\
\left. + \frac{i}{2} g_{\mu\nu} \psi^\mu_- D_z \psi^\nu_- + R_{ijkl} \psi^i_+ \psi^j_+ \psi^k_- \psi^l_- \right) \]

The fields have been twisted from the usual sigma model so that

\[ \psi^i_+ \in C^\infty (\phi^* TX), \quad \psi^i_- \in C^\infty (\overline{K_\Sigma} \otimes (\phi^* \overline{TX})^*), \quad \psi_i^+ \in C^\infty (K_\Sigma \otimes (\phi^* \overline{TX})^*), \quad \psi_i^- \in C^\infty (\phi^* \overline{TX}). \]

where \( \phi : \Sigma \rightarrow X \) is the map from the worldsheet to \( X \).
The observables are in 1-1 correspondence with elements of $H^*(X)$, generating the *quantum cohomology ring* by operator product.

In QFT, an operator is zero by definition if after inserting it into a correlation function, all correlation functions vanish after arbitrary additional insertions.

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The quantum cohomology ring is the algebra generated by a basis for the cohomology, modulo those products which are zero as operators in the above sense.
In particular, a quantum cohomology relation of the form

$$\prod_{i=1}^{r} \alpha_i = q^\gamma \prod_{j=1}^{s} \eta_j, \quad \alpha_i, \eta_j \in H^*(X)$$

is equivalent to the identities

$$\langle \alpha_1, \ldots, \alpha_r, \omega_1, \ldots, \omega_n \rangle = q^\gamma \langle \eta_1, \ldots, \eta_s, \omega_1, \ldots, \omega_n \rangle$$

for any $\omega_1, \ldots, \omega_n$, which is in turn equivalent to

$$\langle \alpha_1, \ldots, \alpha_r, \omega_1, \ldots, \omega_n \rangle_{\beta+\gamma} = \langle \eta_1, \ldots, \eta_s, \omega_1, \ldots, \omega_n \rangle_\beta$$

for any $\omega_l$ and $\beta$. 
The A/2 model

The model is a “half-twist" of the (0, 2) nonlinear sigma model described by the Lagrangian

\[
\mathcal{L} = \frac{1}{\alpha'} \int_{\Sigma} d^2 z \left( (g_{\mu\nu} + iB_{\mu\nu}) \partial \phi^\mu \bar{\partial} \phi^\nu + \frac{i}{2} g_{\mu\nu} \psi^\mu D_\Sigma \psi^\nu + \frac{i}{2} h_{\alpha\beta} \lambda^\alpha D_\Sigma \lambda^\beta + F_{i\bar{j}ab} \psi_i^+ \psi_{\bar{j}}^+ \lambda_a^\alpha \lambda_{\bar{b}}^\beta \right)
\]

with field content

\[
\psi_i^+ \in \mathcal{C}^\infty (\phi^* TX), \quad \lambda_a^\alpha \in \mathcal{C}^\infty \left( \overline{K}_\Sigma \otimes \phi^* \overline{E}^* \right),
\]
\[
\psi_{\bar{i}}^+ \in \mathcal{C}^\infty (K_\Sigma \otimes (\phi^* TX)^*), \quad \lambda_{\bar{a}}^\alpha \in \mathcal{C}^\infty (\phi^* E),
\]

where $E$ is a holomorphic vector bundle on $X$. 
Anomaly Cancellation

- Anomaly cancellation requires
  \[ \Lambda^\text{top} E^* \cong K_X, \quad \text{ch}_2(E) = \text{ch}_2(TX) \]
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The half-twisted model is not topological, but has a sector in which the OPE closes [Adams-Distler-Ernebjerg].

In this sector, the observables are in 1-1 correspondence with the cohomology ring $H^*(X, \Lambda^*E^*)$, which we call the \textit{polymology} of $(X, E)$.

Since $\Lambda^{\text{top}}E^* \simeq K_X$, we have $H^{\text{top}}(X, \Lambda^{\text{top}}E^*) \simeq \mathbb{C}$, providing a mathematical definition of classical correlation functions.

Quantum corrections deform the classical polymology ring. In many situations, we know from physics that this deformation is an associative ring, the $(0, 2)$ quantum cohomology ring.
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A rigorous mathematical definition of correlation functions in the A/2 theory could produce a powerful method for computing Yukawa couplings in heterotic string theory. Alas, this is beyond today’s technology. But we can rigorously define and easily compute correlation functions exactly in the analogous gauged linear sigma model, when $E$ is a deformation of $TX$.

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The (0, 2) gauged linear sigma model (GLSM) is a 2D QFT with (0, 2) SUSY. It is obtained from the (2, 2) GLSM by decomposing the (2, 2) multiplets into (0, 2) multiplets, then varying the (0, 2) multiplets independently.

In a certain regime of FI parameters, the moduli space of vacua corresponds to a toric variety $X$ and holomorphic vector bundle $E$ on $X$.

To be expeditious, I start with a smooth projective toric variety $X$ and a deformation $E$ of the tangent bundle $TX$, then engineer a (0, 2) GLSM from that data.
The Gauged Linear Sigma Model

The \((0, 2)\) gauged linear sigma model (GLSM) is a 2D QFT with \((0, 2)\) SUSY. It is obtained from the \((2, 2)\) GLSM by decomposing the \((2, 2)\) multiplets into \((0, 2)\) multiplets, then varying the \((0, 2)\) multiplets independently.

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Toric Geometry: Notation

- $T$: a torus $T \simeq (\mathbb{C}^*)^r$
- $N = \text{Hom}(\mathbb{C}^*, T) \simeq \mathbb{Z}^r$ the lattice of 1-parameter subgroups
- $M = \text{Hom}(T, \mathbb{C}^*) \simeq \mathbb{Z}^r$ the lattice of characters.
- $\langle \cdot , \cdot \rangle : M \times N \rightarrow \mathbb{Z}$: $(m \circ n)(t) = t^{\langle m, n \rangle}$
- $\Sigma$ complete simplicial fan in $N_\mathbb{R} = N \otimes \mathbb{R}$.
- $X = X_\Sigma$ the associated complete toric variety, assumed smooth.
- $\Sigma(1)$: the set of 1-dimensional cones in $\Sigma$.
- $S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$ homogeneous coordinate ring
- $T$-invariant divisor $D_\rho$ defined by $x_\rho = 0$; $x_\rho \in H^0(O(D_\rho))$
- $W = H^2(X) = \mathbb{C}^{\Sigma(1)}/(M \otimes \mathbb{C})$; divisors mod linear equivalence
Example: $\mathbb{P}^1 \times \mathbb{P}^1$

$$\begin{align*}
D_2 & \quad (0, 1) \\
D_3 & \quad (-1, 0) \\
D_4 & \quad (0, -1)
\end{align*}$$

$$S = \mathbb{C}[x_1, x_2, x_3, x_4], \quad ((x_1 : x_3), (x_2, x_4)) \in \mathbb{P}^1 \times \mathbb{P}^1$$

$$D_1 \sim D_3, \quad D_2 \sim D_4, \quad H_1 = [D_1] = [D_3], \quad H_2 = [D_2] = [D_4]$$

$$W = H^2(X, \mathbb{C}) = \text{span}(H_1, H_2)$$
Relation to GLSM

- There is an open set $U_\Sigma \subset \mathbb{C}^{\Sigma(1)}$ invariant under the complexification $G_\mathbb{C}$ of the gauge group $G$, so that $X_\Sigma = U_\Sigma / G_\mathbb{C}$
- The theory contains gauge fields for $G$
- Have a charged $(0,2)$ chiral field $\Phi_\rho$ for each $\rho \in \Sigma(1)$
- In the situation of a deformation of $TX$, also have fermi fields $\Lambda_\rho$ with the same charges as those of $\Phi_\rho$
- The topological observables are generated by $W = H^2(X, \mathbb{C})$
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Example: $\mathbb{P}^1 \times \mathbb{P}^1$

$G = U(1) \times U(1)$

\[
\Phi_1 \quad \Phi_2 \quad \Phi_3 \quad \Phi_4
\]

\[
\begin{array}{cccc}
U(1)_1 & 1 & 0 & 1 & 0 \\
U(1)_2 & 0 & 1 & 0 & 1
\end{array}
\]
Tangent Bundle and its Deformations

In general,

\[ 0 \to T^*X \to \bigoplus_{\rho \in \Sigma(1)} \mathcal{O}(-D_\rho) \to W \otimes \mathcal{O} \to 0, \]

The rightmost nontrivial map is induced by the canonical sections \( x_\rho \otimes [D_\rho] \) of \( \mathcal{O}(D_\rho \otimes W) \).

Deformation of \( TX \)

\[ 0 \to E^* \to \bigoplus_{\rho \in \Sigma(1)} \mathcal{O}(-D_\rho) \to W \otimes \mathcal{O} \to 0, \]

by deforming \( x_\rho \otimes [D_\rho] \) to sections \( s_\rho \in \mathcal{O}(D_\rho \otimes W) \) sufficiently generic so that the kernel \( E^* \) is still a vector bundle.
Gauge Sectors

- Fix genus 0 worldsheet $\Sigma = \mathbb{P}^1$
- The theory forms sectors according to the topological type of the gauge bundle
- For $\mathbb{P}^1 \times \mathbb{P}^1$, assign chern classes $(d, e)$ to $U(1) \times U(1)$ gauge bundle
- $\Phi_1, \Phi_3, \Lambda_1, \Lambda_3$ become global sections of $\mathcal{O}_{\mathbb{P}^1}(d)$, charge $(1, 0)$
- $\Phi_2, \Phi_4, \Lambda_2, \Lambda_4$ become global sections of $\mathcal{O}_{\mathbb{P}^1}(e)$, charge $(0, 1)$
- The zero modes of the $\Phi_i$ fill out the GLSM moduli space $X_{(d,e)} = \mathbb{P}^{2d+1} \times \mathbb{P}^{2e+1}$, exactly as in the $(2, 2)$ situation
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In general, the topological types of the gauge bundle correspond to homology classes $\beta \in H_2(X, \mathbb{Z})$

- Have a GLSM moduli space $X_\beta$ parametrized by the zero modes of the $\Phi_\beta$
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The General Case

- In general, the topological types of the gauge bundle correspond to homology classes $\beta \in H_2(X, \mathbb{Z})$.
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Correlation Functions

Classical Correlation Functions

\[ 0 \longrightarrow E^* \longrightarrow \bigoplus_{\rho \in \Sigma(1)} \mathcal{O}(-D_{\rho}) \longrightarrow W \otimes \mathcal{O} \longrightarrow 0, \]

\[ H^0(\mathcal{O}(-D_{\rho})) = H^1(\mathcal{O}(-D_{\rho})) = 0 \]
gives

\[ \psi : W = H^0(W \otimes \mathcal{O}) \cong H^1(E^*), \]

hence cup product

\[ \psi : \text{Sym}^k W \rightarrow H^k(\Lambda^k E^*). \]

Fixing a normalization \( \int_X : H^{\text{top}}(X, \Lambda^{\text{top}} E^*) \cong \mathbb{C}, \) for
\( P \in \text{Sym}^{\dim(X)} W, \) define the classical correlation function as

\[ \langle P \rangle_0 = \int_X \psi(P) \]
As we will see, the cup product can be computed explicitly by algebraic geometry. The method builds on the earlier computational methods of [K-Sharpe] and [Guffin-K] developed to verify a conjecture of [Adams-Basu-Sethi], while providing new viewpoints that elucidate more of the structure.
Classical Correlation Functions: $\mathbb{P}^1 \times \mathbb{P}^1$

- Put $Z = \bigoplus \mathcal{O}(-D_i) \simeq \mathcal{O}(-1, 0)^2 \oplus \mathcal{O}(0, -1)^2$. Then the cup product is identified with the extension class of the generalized Koszul complex on the $s_{\rho}$

\[
0 \to \Lambda^2 E^* \to \Lambda^2 Z \to Z \otimes W \to \text{Sym}^2 W \otimes \mathcal{O} \to 0.
\]

- This can be broken up into short exact sequences

\[
0 \to \Lambda^2 E^* \to \Lambda^2 Z \to S_1 \to 0,
\]
\[
0 \to S_1 \to Z \otimes W \to \text{Sym}^2 W \otimes \mathcal{O} \to 0.
\]

- The cup product factors as $\text{Sym}^2 W \to H^1(S_1) \to H^2(\Lambda^2 E^*)$

- We also see from $H^i(Z \otimes W) = 0$ that $\text{Sym}^2 W \to H^1(S_1)$ is an isomorphism, while $H^1(S_1) \to H^2(\Lambda^2 E^*)$ is surjective and has kernel generated by $H^1(\Lambda^2 Z)$. 
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\( P^1 \times P^1 \), Continued

- \( Z = \mathcal{O}(-1, 0)^2 \oplus \mathcal{O}(0, -1)^2 \) gives
  \[ \Lambda^2 Z = \mathcal{O}(-2, 0) \oplus \mathcal{O}(-1, -1)^4 \oplus \mathcal{O}(0, -2) \]
- Only nonzero contributions \( H^1(\mathcal{O}(-2, 0)) \) and \( H^1(\mathcal{O}(0, -2)) \) to \( H^1(\Lambda^2 Z) \) arise from the respective pairs of divisors \( \{D_1, D_3\}; \{D_2, D_4\} \), which do not intersect in \( X \). More generally, these are the primitive collections of toric geometry.
- Chasing through the diagrams gives an explicit polynomial \( Q \in \text{Sym}^2 W \) associated with the generator of \( H^1(\mathcal{O}(-2, 0)) \) and another \( \tilde{Q} \in \text{Sym}^2 W \) associated with the generator of \( H^1(\mathcal{O}(0, -2)) \), which lie in the kernel of the cup product.
- In summary, we have computed the polymology of \((X, E)\) as
  \[ H^*(\Lambda^* E^*) = \text{Sym} W / (Q, \tilde{Q}). \]
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Explicit Calculation

- Since \( H^0(\mathcal{O}(D_1)) = H^0(\mathcal{O}(D_3)) \) is spanned by \( x_1 \) and \( x_3 \), can write

\[
S_1 = w_{11} x_1 + w_{13} x_3, \quad S_3 = w_{31} x_1 + w_{33} x_3
\]

for certain \( w_{ij} \in W \).

- Putting

\[
A = \begin{pmatrix} w_{11} & w_{13} \\ w_{31} & w_{33} \end{pmatrix}
\]

we compute \( Q = \det(A) \).

- The computation of \( \tilde{Q} \) is analogous.

- Since the polymology is finite dimensional, \( Q, \tilde{Q} \) have no common factors and it follows that the top part \( \dim \text{Sym}^2 W / (Q, \tilde{Q}) \) of the polymology is one dimensional.
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The computation of the classical correlation functions $\langle P \rangle_0$ with $P \in \text{Sym}^2 W$ becomes trivial: take the image of $P$ in the quotient $\text{Sym}^2 W / (Q, \tilde{Q})$ and this is the unnormalized correlation function. Choose your favorite isomorphism of this 1 dimensional vector space with $\mathbb{C}$ if you want to normalize.
In general, the same construction gives the classical polymology $H^*(X, \wedge^* E^*)$ as a quotient $\text{Sym} W / I$.

To each primitive collection $\{D_{i_1}, \ldots, D_{i_k}\}$ which we index by $K = \{i_1, \ldots, i_k\}$, get a nonvanishing cohomology $H^{k-1}(X, \mathcal{O}(- \sum D_{i_j})) \simeq \mathbb{C}$ and a generator $Q_K$ of $I$.

Each $Q_K$ is explicitly computable as a product of determinants of the linear coefficients of the $s_\rho$ used to define $E$.

$I$ is simply the ideal generated by all of the $Q_K$, so is independent of nonlinear deformations!

If $E = TX$, then $Q_K$ is just the product $[D_{i_1}] \cdots [D_{i_k}]$ occurring in the definition of the Stanley-Reisner ideal. Thus our computation of the polymology reduces to the familiar toric description of $H^*(X)$, as it should.
Classical Polymology, General Case

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Thus our computation of the polymology reduces to the familiar toric description of $H^*(X)$, as it should.
More Details of the Computation

- Primitive collections are compatible with linear equivalence: if $K$ is a primitive collection and $\rho \in K$, then if $D_\rho \sim D_\rho'$ it can be shown that $\rho' \in K$. Each $K$ can therefore be partitioned into a set $T^+_K$ of linear equivalence classes.

- Among the sections of $H^0(\mathcal{O}(D_\rho))$ are the sections expressed as a linear combination of the $x_{\rho'}$ for $D_{\rho'}$ in the linear equivalence class of $D_\rho$. The deformations of $TX$ with these terms and no others have been called linear deformations in the physics literature. Fixing a linear equivalence class $c$, the linear terms associated with the $s_\rho$ for all $\rho \in c$ form a square matrix. Let $Q_c$ be its determinant.

$$Q_K = \prod_{c \in T^+_K} Q_c$$
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Nonclassical Correlation Functions

Since $X_\beta$ is a toric variety and $E_\beta$ is a deformation of $TX_\beta$, we only have to describe its primitive collections $K_\beta$ and the $Q_{K\beta}$ in terms of the classical data.

Recall from [Morrison-Plesser] that the fan for $X_\beta$ is obtained from that of $X$ by replacing each edge $\rho \in \Sigma(1)$ with $h^0(D_{\rho} \cdot \beta) := h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(D_{\rho} \cdot \beta))$ edges $\rho_1, \ldots, \rho_{h^0(D_{\rho} \cdot \beta)}$, keeping the same gauge group and charges, and charactering the fan by requiring that the primitive collections $K_\beta$ of $X_\beta$ are in 1-1 correspondence with the primitive collections $K$ for $X$, associating to $K$ the collection of divisors

$$K_\beta = \{ \rho_i \mid \rho \in K \}.$$ 

The result is $Q_{K\beta} = \prod_{c \in T^+_K} Q_c^{h^0(D_c \cdot \beta)}$. 
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- The result is $Q_{K_\beta} = \prod_{c \in T_K^+} Q_c^{h^0(D_c \cdot \beta)}$.
Putting everything together, we learn that the polymology of $X_\beta$ is $\text{Sym} W / (I_\beta)$,

$$I_\beta = \left( \prod_{c \in T^+_K} Q_c^{h^0(D_c \cdot \beta)} \mid K \text{ a primitive collection for } X \right)$$
Example: $\mathbb{P}^1 \times \mathbb{P}^1$

- For $\beta = (d, e)$, have $h^0(H_1 \cdot \beta) = d + 1$, $h^0(H_2 \cdot \beta) = e + 1$, so the polynomial in sector $\beta$ is

$$\text{Sym} \mathcal{W} / \left( Q^{d+1}, \tilde{Q}^{e+1} \right).$$

- This is all that is need to compute correlation functions as elements of the 1 dimensional vector space

$$\text{Sym}^{2d+2e+2} \mathcal{W} / (Q^{d+1}, \tilde{Q}^{e+1}).$$
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$$\text{Sym}^{2d+2e+2} \mathcal{W} / (Q^{d+1}, \tilde{Q}^{e+1}).$$
This is still not enough to compute correlation functions: 
\[ \dim X_\beta \neq c_1(X) \cdot \beta + \dim X \] in general. We need \textit{four-fermi terms}, an analogue of the virtual fundamental class of Gromov-Witten theory.
Put $h^1(s) = h^1(\mathcal{O}_{\mathbb{P}^1}(s))$. Then

$$F_\beta = \prod_c Q_c^{h^1(D_c \cdot \beta)},$$

where the product is taken over all linear equivalence classes $c$ (i.e. without regard to any primitive collections).

- We have $c_1(X) \cdot \beta + \dim(X) + \deg(F_\beta) = \dim X_\beta$.
- If $P \in \text{Sym}^k X$ with $k = c_1(X) \cdot \beta + \dim(X)$, then we can define the correlation function as

$$\langle P \rangle_\beta = [P F_\beta],$$

where the brackets denote the equivalence class of $P F_\beta$ in the 1 dimensional vector space $\text{Sym}^{\dim X_\beta} W / I_\beta$. 
Four-fermi Terms and Correlation Functions

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  \[
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  \]
  where the brackets denote the equivalence class of $P F_\beta$ in the 1 dimensional vector space $\text{Sym}^{\dim X_\beta}W / I_\beta$. Would you like me to continue with the rest of the content or ask a specific question based on this information?
Four-fermi Terms and Correlation Functions

- Put $h^1(s) = h^1(O_{\mathbb{P}^1}(s))$. Then

$$F_\beta = \prod_c Q_c^{h^1(D_c \cdot \beta)},$$

where the product is taken over all linear equivalence classes $c$ (i.e. without regard to any primitive collections).

- We have $c_1(X) \cdot \beta + \dim(X) + \deg(F_\beta) = \dim X_\beta$.

- If $P \in \text{Sym}^k X$ with $k = c_1(X) \cdot \beta + \dim(X)$, then we can define the correlation function as

$$\langle P \rangle_\beta = [PF_\beta],$$

where the brackets denote the equivalence class of $PF_\beta$ in the 1 dimensional vector space $\text{Sym}^{\dim X_\beta} W / I_\beta$. 
Comparing sectors: $\mathbb{P}^1 \times \mathbb{P}^1$

- We already know a ring that surjects onto the quantum cohomology ring: $\text{Sym} \mathcal{W}$. We only have to identify the operator identities that they satisfy to identify the quantum cohomology ring.

- Remaining problem: correlation functions from different sectors live in different vector spaces.

- If $\beta = (d, e)$ and $\beta' = (d', e')$ with $d' \geq d$ and $e' \geq e$, there is a natural map of polymomologies

$$\text{Sym} \mathcal{W}/(Q^{d+1}, \tilde{Q}^{e+1}) \xrightarrow{Q^{d'-d} \tilde{Q}^{e'-e}} \text{Sym} \mathcal{W}/(Q^{d'+1}, \tilde{Q}^{e'+1})$$

which restricts to an isomorphism $f_{\beta' \beta}$ from $\text{Sym}^{2d+2e+2} \mathcal{W}/(Q^{d+1}, \tilde{Q}^{e+1})$ to $\text{Sym}^{2d'+2e'+2} \mathcal{W}/(Q^{d'+1}, \tilde{Q}^{e'+1})$. 
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The maps $f_{\beta'}\beta$ are compatible and form a direct system. The direct limit is a one-dimensional vector space $V$ containing all the correlation functions. In particular, correlation functions in different sectors can be compared.
We can now state the quantum cohomology relations for $\mathbb{P}^1 \times \mathbb{P}^1$:

$$Q = q_1, \quad \tilde{Q} = q_2.$$ 

Here, the $q_i$ are the GLSM version of the Kähler terms of the NLSM, depending on the FI parameters.
Verification of QC Relations for $\mathbb{P}^1 \times \mathbb{P}^1$

We have to show that for any $P \in \text{Sym}^{2d+2e+2} W$ we have

$$\langle QP \rangle_{d+1,e} = \langle P \rangle_{d,e}$$

But since $f_{\beta'\beta}$ is multiplication by $Q$ in this case, this amounts to the tautology $QP = Q(P)$. 
General Case

- The general case is similar. We say that \( \beta' \) dominates \( \beta \) if the fan for \( X_{\beta'} \) can be obtained from the fan for \( X_\beta \) by adding more edges to each linear equivalence class.

- Equivalent to \( h^0(D_c \cdot \beta') \geq h^0(D_c \cdot \beta) \) for all \( c \).

- These polymologies can be identified by multiplication by

\[
\prod_c Q_c^{h^0(D_c \cdot \beta') - h^0(D_c \cdot \beta)},
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leading to a direct system.

- Now quantum cohomology relations have a precise mathematical meaning.
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Now quantum cohomology relations have a precise mathematical meaning.
Batyrev’s Relations

- We need a notion due to Batyrev. Let $v_\rho \in N$ be the primitive integral generator of $\rho \in \Sigma(1)$. Let $K$ be a primitive collection.

- Consider $v_K = \sum_{\rho \in K} v_\rho$. Then there is a unique cone $\sigma \in \Sigma$ such that $v_K$ is in the relative interior of $\sigma$.

- Batyrev shows that this gives a unique relation $\sum a_\rho v_\rho = 0$ with the properties $a_\rho = 1$ if $\rho \in K$, $a_\rho < 0$ if $\rho \in \sigma(1)$, and $a_\rho = 0$ otherwise.

- Furthermore, Batyrev shows that there is a unique $\beta_K \in H_2(X, \mathbb{Z})$ such that $D_\rho \cdot \beta_K = a_\rho$ for all $\rho \in \Sigma(1)$. 
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Quantum Cohomology Relations: General Case

We can show that the edges of the cone $\sigma$ in Batyrev’s relation respects linear equivalence in the same way that $K$ does. So we can partition $\sigma(1)$ into linear equivalence classes denoted $T_K^-$. Then the quantum cohomology relations are:

$$\prod_{c \in T_K^+} Q_c = q^{\beta_K} \prod_{c \in T_K^-} Q_c^{-D_c: \beta_K}.$$ 

This agrees with and extends the results of [Melnikov, McOrist].

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Summary

- $(0, 2)$ correlation functions for deformations of tangent bundles of smooth toric varieties are now a mathematically precise notion.
- The deduced quantum cohomology relations agree with those found by techniques in physics.
- The results are independent of nonlinear deformations.
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