Towards A Global Mirror Symmetry

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Plan of talk:

(I) What is "global"?

(II) A conjectural physical package $\text{BCOV}$

(III) An approach to hypersurface $\text{LG-model}$

(IV) Unexpected bonus! $\uparrow$ orbifold $\text{GW}$-Theory
(1) WHAT IS "GLOBAL"?

"Local" Mirror symmetry

Mirror symmetry of local Calabi-Yau

$X - CY 3$-fold $\leftrightarrow X' - another CY 3$-fold

A-model

U

kahler str

GW- Theory $(g=0)$

B-model

U

complex str

periods

Famous example:

$X = \{ \sum_{i=1}^{5} x_i^5 = 0 \} \leftrightarrow X' = \{ \sum_{i=1}^{5} x_i^5 - s y^{\frac{1}{5}} \sum_{i=1}^{5} x_i = 0 \}$

kahler str

t $\in$ kahler cone

NO complex str-

match $\gamma \in$

$\Pi_1 = 1$
A temporary solution at the time (20 years ago) ↓

"local mirror ↔ symmetry

\[ t \leftarrow \gamma \in \text{a neigh of } \gamma = \infty \uparrow \text{large complex str limit} \]

**Now:** Restore "Global" structure of Mirror symmetry

**Benefit:**
1. Compute higher genus Gromov-Witten Theory
2. Study modularity of " \( \cdot \)"
3. Prove Landau-Ginzburg/Calabi-Yau correspondence
4. More, ...

"GLOBAL" = allow \( \gamma \) to move around in the entire moduli space of complex str
(II) A conjectural physical package

Background: B-model \((g=0)\)
   \[ 5 \text{ periods} \]

A toy model

\[ E(\xi) = \left\{ \sum_{i=1}^{3} x_i^3 - 3 \xi^{\frac{1}{3}} \sum_{i=1}^{3} x_i = 0 \right\} / \mathbb{Z}_3 \]

\[ \forall \xi \in \{ 0, 1, \infty \} \]

CY-form:

\[ H^{1,0}(E(\xi)) = \mathbb{C} \]

\[ \omega_\xi \to \text{holomorphic} (1,0) \text{ form} \]

\[ \text{vary holomorphically as we vary } \xi \]

Another choice:

\[ \omega(\xi) \rightarrow f(\xi) \omega(\xi) \]

\[ \text{holomorphic funct} \]
\( H_1(\mathbb{E}(\mathcal{V}), \mathbb{Z}) \) has a symplectic basis

\[
A, B \text{ with } A^2 = B^2 = 0 \quad A \cdot B = 1 \quad B \cdot A = -1
\]

change basis

\[
\begin{pmatrix} A \\ B \end{pmatrix} \rightarrow \begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}
\]

\( \quad \text{SL}_2 \mathbb{Z} \)

\[\text{PERIOD}: \quad q = \sum_A \omega(\nu) \quad p = \sum_B \omega(\nu) \]

vary \( \nu \) \( \rightarrow \) \( q(\nu), p(\nu) \) = multi-valued

\( \text{funct of } \mathbb{P}^1 \{0, 1, \infty\} \)

\[ \tau = \frac{p(\nu)}{q(\nu)} \in \mathbb{H}_+ \quad \text{upper half-plane} \]

\( \nu \)

\( \downarrow \quad \text{universal cover} \quad \mathbb{P}^1 \{0, 1, \infty\} \)

\( \tau^{\prime} \quad \text{monodromy} \rightarrow T(3) \text{SL}_2 \mathbb{Z} \)

\text{GW-theory} \iff \text{periods of }

\text{parameter } t

\text{near } \{ \infty \} \text{ of basis } A, B

\( \tau = i \infty \quad \text{"large complex str" limit} \)
Going Back to quintic 3-fold (B-model)

\[ \mathbb{P} \left( 30, 1, 00 \right) \]

\[ \mathbb{H}_4 \cong \mathbb{C} \text{- modular coord} \]

\[ \mathbb{H}_3 (X, z) = 4 \]

GW-theory of quintic

"local" periods near \( \tau = i \infty \)

Mirror

Remark:

1. Above "local" mirror symmetry for \( g=0 \) has been verified by Givental, Liu-Liang-Yau in the middle of 90's.

Our interest:

2. Higher genus \( (g>0) \) GW-theory

(1) **GLOBAL PROPERTY:**

1. exist a global \( F_g^B(z, \bar{z}) \) - genus \( g \) generating function of \( B \)-model GW Theory

   (i) defined for all \( z \)

   (ii) non- holomorphic

(2) **Modular Invariance**

\[
F_g^B(hz, h\bar{z}) = F_g^B(z, \bar{z}) \cdot j(h, z)^k
\]

\( h \in P \)- monodromy group

(3) \( \partial \bar{z} F_g^B = \not= 0 \) - holomorphic anomaly equation

\( \uparrow \) BCOV, Klemm, ..... 

\[
F_g^B(z, \bar{z}) = \sum_{j} F_{g, j}(z) (i\text{m}\bar{z})^{-j}
\]

(2) + (3) \( \Rightarrow F_g^B(z) \) - quasi-modular form
(1) **SPECIAL LIMIT (Mirror Conjectures)**

- **B-model** \[\text{CY-to-CY}\]  \[\text{A-model}\]
- (1) near \(\tau = i\infty\) \[\longleftarrow\]\ Gromov-Witten Theory of \(X\)
  
  - Original "local" mirror symmetry
  - Small/Large duality
  
  \(\text{landau-Ginzburg/CY correspondence}\)

- (2) near \(\tau = 0\) \[\longleftarrow\]  "conjectural" Landau-Ginzburg model

- (3) near \(\tau = 1\) \[\longleftarrow\]  "conjectural" matrix model

(4) Beyond quintic, other limits.

**Klemm's group**

Assume the existence of above package + general properties of known for mathematician special limits only for \(g = 0, 1\).

A STRIKING computation of GW-theory of \(g \leq 51\).
Goal of Remaining talk

- Describe an approach for a mathematical construction of above package for hypersurface. Most of steps are conjectural at this moment.
- Present some theorems in dimension one.
First advance: Gepner limit
Conjectural LG-model = Theory of Fan-Jarvis-Ruan-Witten (2007)

LG-model:

(i) \( W: \mathbb{C}^n \to \mathbb{C} \) "non-degenerate" quasi-homogeneous poly
(ii) \( G \subset \text{Aut}(W) \) - finite abelian symmetry

Theory of Fan-Jarvis-Ruan-Witten:

- A complete A-model theory of LG-model based on solving Witten equations:
  \[ \overline{\partial} \Omega_i + \overline{\partial} \Omega^j W = 0 \]

- A GW-type curve counting theory
  - based on \( \overline{\mathcal{M}}_{g,n} \)
  - 2D TFT
  - satisfies axioms of GW-theory

- Much easier to calculate (ADE, elliptic singularity, \( g=0 \) quintic, expanding rapidly)
What happens for (1): build a rigorous theory of $F_B^g(z, \bar{z})$ with expected properties

- A hard problem
- Progress on $X = T^{2m}$ (Costello - Li)
- Our approach for hypersurfaces such as quintic 3-fold (Milanov - Krausze Shen)

**Key Observation**

CY-deformation: $x_1^3 + x_2^3 + x_3^3 - 3y^{\frac{1}{3}}x_1x_2x_3$

\[\nabla \text{subset} \]

$\sum_{i=1}^{3} x_i^3 + x_2^3 + x_3^3 - 3y^{\frac{1}{3}}x_1x_2x_3 + t_0 + t_1x_1$

$+ t_2x_2 + t_3x_3 + t_4x_1x_2 + t_5x_1x_3 + t_6x_2x_3$

minimal deformation of singularity $W = x_1^3 + x_2^3 + x_3^3$
A "baby" B-model of singularity/\text{LG-model}

\[ W = x_1^3 + x_2^3 + x_3^3 \]

Milnor ring \[ Q_W = \frac{\mathbb{C}[x_1, x_2, x_3]}{\partial_{x_1} W, \partial_{x_2} W, \partial_{x_3} W} \] 8-dim

\[ = \{ 1, x_1, x_2, x_3, x_1 x_2, x_1 x_3, x_2 x_3, x_1 x_2 x_3 \} \]

Universal deformation \[ W(t; \sigma) = x_1^3 + x_2^3 + x_3^3 + \sigma x_1 x_2 x_3 + t_0 + t_1 x_1 + t_2 x_2 + t_3 x_3 + t_4 x_1 x_2 + t_5 x_1 x_3 + t_6 x_2 x_3 \]

B-model parameter space \[ = \{ (t; \sigma), \ |t| < 3, \ |\sigma|^3 < 27 \} \]

For each \((t; \sigma)\),

\[ Q_{W(t; \sigma)} = \frac{\mathbb{C}[x_1, x_2, x_3]}{\partial_{x_3} W(t; \sigma)} \]

a family of Frobenius algebras

Pairing: \( \langle \phi_1, \phi_2 \rangle = \text{Res} \frac{\phi_1 \phi_2 dx_1 dx_2 dx_3}{dW(t; \sigma)} \)
Main properties

1. For generic $\tau \neq 0$, $W(t, \tau)$ is holomorphic Morse function.
   Frobenius algebra of such $(t, \tau)$ is Semi-Simple.

2. Pairing is NOT flat.

Saito - Givental Theory:

1. Saito - Theory:
   - Primitive form
   - Replace $dx_1 dx_2 dx_3 \rightarrow \frac{1}{\rho(\sigma)} \cdot dx_1 dx_2 dx_3$
   - Flat pairing $\Rightarrow$ Frobenius manifold structure $\mathcal{F}_0^{B}(t, \sigma, \rho)$

2. Givental Theory:
   - On semi-simple Frobenius model, exist $\mathcal{F}_g^{B}(t, \sigma, \rho)$
what we know about primitive form \( \frac{1}{p} \) ?

- \( P \) always exist locally \( \implies \) Saito-Givental Theory
- explicit formula for ADE, elliptic singularities
- difficult to get explicit formula in general
- along CY-direction (marginal deformation), related to periods
  \( \Downarrow \) leads to
  Global Saito-Givental Theory
  (under developed by Milanov, ...)
Global Saito-Givental Theory in dim = 1
(Milanov, 

Starting point: \( P(\sigma) \) - period, i.e. \( P(\sigma) = \sum_{A} \omega(\sigma) \)

Recall \( \tau = \frac{\sum_{B} \omega(\sigma)}{\sum_{A} \omega(\sigma)} \in H^+ \)

\( A, B \) - symplectic basis

flat coord along \( \sigma \)-direction

\( \mathcal{B} \) - model

Parameter space = \( \{ (t_i, \tau), \; |t_i| \leq \epsilon, \; \tau \in H^+ \} \)

\[ \downarrow \text{monodromy group} \]
\[ \{ (t_i, \sigma), \; |t_i| \leq \epsilon, \; \sigma^{\pm 27} \} \]

\( t_i \rightarrow \) CY - deformation

\( (t_i, \tau) \rightarrow \) Frobenius mod str \( \rightarrow \mathcal{F}_B^\sigma(t_i, \tau) \)

\( (t_i \neq 0, \tau) \rightarrow \mathcal{F}_B^\sigma(t_i, \tau) \rightarrow \) semi-simple

holomorphic

with respect to \( t_i, \tau \)
Let
\[ D^B_{sg}(t_i, \tau) = \exp \left( \sum_{h \in H} \tau^{2g-2} \hat{X}_h D_{sg}^B \right) \]
where \( \hat{X}_h \) - differential operator defined out of \( h \in H(\mathbb{Z}) \).

Corollary: \( \hat{F}_g^B(t_i, \tau) \) is not modular.

A magic trick: Anti-holomorphic completion

We found an explicit way to complete
\[ \hat{F}_g^B(t_i, \tau) \rightarrow \hat{F}_g^B(t_i, \tau, \overline{\tau}) \]

\[ \hat{F}_g^B(t_i, \tau, \overline{\tau}) = \sum_{j=1}^{K} \hat{F}_g^B(t_i, \tau, \overline{\tau}) \left( (\text{in } \tau) \right)^{-j} \]
defined via Feynmann diagram expansion

Quasi-modular form

Easy Fact: \( \hat{F}_g^B(t_i, \tau, \overline{\tau}) \) satisfies holomorphic anomaly equation.
Modular Invariance: $F^B_g(t, \tau, \bar{\tau})$ is modular invariant

Assume: $F^B_g(t_i=0, \tau, \bar{\tau})$ extends to $t_i=0$

$somehow\ a\ difficult\ problem!$

Corollary (a):

$F^B_g(t_i, \tau, \bar{\tau}) = \sum a_1(\tau, \bar{\tau}) t_i$

then $a_1(\tau, \bar{\tau})$ are classical modular form

Corollary (b):

$F^B_g(\tau, \bar{\tau})$ satisfies holomorphic anomaly eqn
desired $B$-model theory
(IV) An unexpected bonus!

\[ \mathcal{F}_g^\mathcal{B}(t_i, \tau) \text{ has a mirror of its own.} \]

A-model

Mirror:

\[ \xi x_1^3 + x_2^3 + x_3^3 = 0 \]

\[ \mathbb{P}^1 \]

orbifold \( \mathbb{P}^1 \)

\[ \mathbb{P}^1_{x_0, x_1, x_2, x_3} \]

LG-dual: \( (W = x_1^3 + x_2^3 + x_3^3, z_3^3) \)

Theorem: (Krawitz-Shen)

1. Near \( \tau = 0 \), \( \mathcal{F}_g^\mathcal{B}(t_i, \tau) = \mathcal{F}_g^{\mathcal{F}_3\mathcal{R}W}(t'_i, \tau') \) extends to \( t_i = 0 \)

2. Near \( \tau = i\pi \), \( \mathcal{F}_g^\mathcal{B}(t_i, \tau) = \mathcal{F}_g^{6W}(t'_i, \theta = \frac{2\pi i}{3}) \)

3. Same holds for \( x_9 = x_1^2 + x_2^4 + x_3^4 \rightarrow \mathbb{P}^1_{\mathcal{R}^1}, 4, 4 \)
   \( J_6 = x_1^2 + x_2^3 + x_3^6 \rightarrow \mathbb{P}^1_{2, 3, 6} \)
Two Bonuses:

(I) GW-theories of $\mathbb{P}^1_{3,3,3}$, $\mathbb{P}^1_{2,4,4}$, $\mathbb{P}^1_{2,3,6}$ are quasi-modular

\[ \uparrow \]

wanted very much by mathematician

(II) LG/CY - correspondence holds for all genera for these examples.

\[ \uparrow \]

First example of all genera

A less important Result:

Restrict to $t_i = 0$ \[ \Rightarrow \] recover elliptic curve

\[ \uparrow \]

Known already by Okounkov Pandharipande