Vanishing Chiral Algebras and Höhn–Stolz Conjecture

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The chiral algebras of $(0, 2)$ sigma models are

- generalizations of quantum cohomology rings
- related to chiral differential operators, $(0, 2)$ mirror symmetry, geometric Langlands, etc.

I will discuss:

- the chiral algebras can vanish
- implications for the geometry of loop spaces
- esp. Höhn–Stolz conjecture in the Kähler case.
Höhn–Stolz conjecture

$M$: closed, string (spin & $p_1(M)/2 = 0$) manifold

If $\text{Ric} > 0$, then the Witten genus $\phi_W(M) = 0$.


Here

$$\phi_W(M) = \prod_{m=1}^{\infty} (1 - q^m)^{\dim M} \sum_{n=0}^{\infty} q^n \text{index}(D \otimes V_n),$$

$D$: Dirac op; $V_0 = 1$, $V_1 = TM$, $V_2 = TM \oplus S^2TM$, . . .

✓ complete intersections, $G/H$, . . .
Add fermions; then it has (0, 1) SUSY:

\[ Q = Q^*, \quad Q^2 = H - P \quad (H = -\partial_t, \quad P = -i\partial_\sigma). \]

For the theory to be consistent, \( M \) must be string.

Partition function

\[ Z = \phi_W(M)/\eta(q)^{\dim M}, \quad q = e^{i\tau}. \]
### Loop-space viewpoint

View as quantum mechanics on $\mathcal{LM} = \{S^1 \to M\}$:

<table>
<thead>
<tr>
<th>QFT</th>
<th>loop space</th>
</tr>
</thead>
<tbody>
<tr>
<td>${\psi^a_\sigma, \psi^b_\sigma'} = \delta^{ab} \delta_{\sigma\sigma'}$</td>
<td>Clifford algebra states</td>
</tr>
<tr>
<td>parity</td>
<td>spinors</td>
</tr>
<tr>
<td>$Q$</td>
<td>chirality</td>
</tr>
<tr>
<td>SUSY states</td>
<td>Dirac operator</td>
</tr>
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<td></td>
<td>harmonic spinors</td>
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</tbody>
</table>

$Z$ is the $S^1$-equivariant index of Dirac operator:

$$Z = \text{str} q^P,$$

where $\text{str}$ is over $\{\text{harmonic spinors on } \mathcal{LM}\}$. 
Loop-space Lichnerowicz

Lichnerowicz: no harmonic spinors if $R > 0$.

Analogy with the finite-dimensional case:

<table>
<thead>
<tr>
<th>$M$</th>
<th>$\mathcal{L}M$</th>
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</thead>
<tbody>
<tr>
<td>spin</td>
<td>string</td>
</tr>
<tr>
<td>$R &gt; 0$</td>
<td>$\text{Ric} &gt; 0$</td>
</tr>
<tr>
<td>$\hat{A}(M) = 0$</td>
<td>$\phi_W(M) = 0$?</td>
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</tbody>
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Stolz’s idea: “loop-space Lichnerowicz”

no harmonic spinors on $\mathcal{L}M$ if $\text{Ric} > 0$

will imply the Höhn–Stolz conjecture.
If $M$ is Kähler, $D = D + D^*$ with $D^2 = 0$, so we can consider the $D$-cohomology (spinor cohomology).

Similarly, $Q = Q + Q^*$ and we have $(0, 2)$ SUSY:

$$Q^2 = 0, \quad \{Q, Q^*\} = H - P.$$

The $Q$-cohomology of states

$$H^\bullet_Q \cong \{\text{harmonic spinors on } \mathcal{L}M\}.$$

Loop-space Lichnerowicz in the Kähler case will be

$$H^\bullet_Q = 0 \text{ if } c_1 > 0.$$
Now consider the $Q$-cohomology of operators. Its elements

- vary holomorphically:
  
  $$\partial \bar{z} \mathcal{O} = [H - P, \mathcal{O}] = [Q, \ldots]$$

- have operator product expansions (OPE):

  
  $$[\mathcal{O}_i(z)] \cdot [\mathcal{O}_j(w)] \sim c_{ij}^k (z - w) [\mathcal{O}_k(w)]$$

so define a **chiral algebra** $\mathcal{A}$, an OPE algebra of holomorphic fields.
Chiral differential operators

Ignoring instantons,

\[ A = H^\bullet(D_M). \]

\( D_M \): a sheaf of CDO. [Witten, hep-th/0504078]

Locally, \( D_M \) is a vertex alg known as the \( \beta \gamma \) system:

\[ \beta_i(z) \gamma^j(w) \sim \frac{\delta^j_i}{z - w}. \]

Some properties:

- \( H^\bullet(D_{G/B}) \) is a \( \hat{g} \)-module of critical level.
- The energy-momentum tensor \( L \not\in H^\bullet \) if \( c_1 \neq 0 \).
Vanishing theorem

Witten’s prediction: for $M = \mathbb{P}^1$, instantons make

$$1 = 0$$

and the chiral algebra vanishes. [Witten, hep-th/0504078]

Verified. [Tan & JY, 0801.4782; Arakawa & Malikov, 0911.0922]

More generally:

If $\exists \mathbb{P}^1 \subset M$ with trivial normal bundle, then $\mathcal{A} = 0$.

Ex. $G/B$. [JY, 1002.0028; Frenkel–Losev–Nekrasov]
$H^\bullet_Q$ is a module over $\mathcal{A}$:

$$[\mathcal{O}] \cdot [\Psi] = [\mathcal{O}\Psi].$$

If the chiral algebra vanishes, then

$$[\Psi] = [1] \cdot [\Psi] = 0.$$

So the $Q$-cohomology of states vanishes as well.

The spinor cohomology of $\mathcal{L}M$ is zero, there are no harmonic spinors on $\mathcal{L}M$, and

$$\phi_W(M) = 0.$$
**Question**

$G/B$ has $c_1 > 0$ and $\mathcal{A} = 0$. Do we have:

the chiral algebra vanishes if $c_1 > 0$?

This will imply the Kähler loop-space Lichnerowicz.

Renormalization group ($\sim$ Ricci) flow argument:

1. Rescale the 2d metric $g \to tg$. $\mathcal{A}$ is invariant.
2. For $t \sim 0$, quantum effects is small if $c_1 > 0$.
3. Show $\mathcal{A}$ admits no energy-momentum tensor.
4. As $t \to \infty$, the theory flows to a SCFT, so $L \in \mathcal{A}$.
5. The SCFT is trivial, with $\mathcal{A} = 0$. 
“Loop-space Lichnerowicz” implies Höhn–Stolz.
For $M$ Kähler, no harmonic spinors on $\mathcal{L}M$ iff $H_Q = 0$.
$\mathcal{A} = H^\bullet(\mathcal{D}_M)$ ignoring instantons.
$\mathcal{A} = 0$ in some cases in the presence of instantons.
$\mathcal{A} = 0$ implies $H_Q = 0$.
$\mathcal{A} = 0$ if $c_1 > 0$?