1. a) Find all solutions in integers to the equation $129x + 291y = 1$.
   b) Do the same for the equation $129x + 291y = 3$.
   Justify your assertions.

2. Show $f(x) = x^2$ is not uniformly continuous as a function on the whole real line (i.e. show for some $\epsilon > 0$ there is no $\delta > 0$ so that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$).

3. For each of the following, either give an example or explain why none exists.
   a) A non-abelian group of order 20.
   b) Two non-isomorphic abelian groups of order 30.
   c) A finite field whose non-zero elements form a cyclic group of order 17 under multiplication.
   d) A non-trivial automorphism of a finite field.

4. Let $f$ be a real-valued continuous function defined for all $0 \leq x \leq 1$, such that $f(0) = 1$, $f(1/2) = 2$ and $f(1) = 3$. Show that
   $$\lim_{n \to \infty} \int_0^1 f(x^n)dx$$
   exists and compute this limit. Justify your assertions.

5. Let $V$ be the real vector space consisting of polynomials $f(x) \in \mathbb{R}[x]$ having degree at most 5 (including the 0 polynomial).
   a) Find a basis for $V$, and determine the dimension of $V$.
   b) Define $T: V \to \mathbb{R}^6$ by $T(f) = (f(0), f(1), f(2), f(3), f(4), f(5))$.
      Show $T$ is a linear transformation and find its kernel.
   c) Deduce that for every choice of $a_0, \ldots, a_5 \in \mathbb{R}$ there is a unique polynomial $f(x) \in \mathbb{R}[x]$ of degree at most 5 such that $f(j) = a_j$ for $j = 0, 1, \ldots, 5$.  

6. a) Is there a metric space structure on the set \( \mathbb{Z} \) such that the open sets are precisely the subsets \( S \subset \mathbb{Z} \) such that \( \mathbb{Z} - S \) is finite, and also the empty set?

b) Is there a metric space structure on the set \( \mathbb{Z} \) such that every subset is open?

Justify your assertions.

7. Let \( \vec{F} \) be a vector field defined in \( \mathbb{R}^3 \) minus the origin defined by

\[
\vec{F}(\vec{r}) = \frac{\vec{r}}{||\vec{r}||^3} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{(x^2 + y^2 + z^2)^{3/2}}
\]

for \( \vec{r} \neq 0 \).

a) Compute \( \text{div} \ \vec{F} \).

b) Let \( S \) be the sphere of radius 1 centered at \( (x, y, z) = (2, 0, 0) \). Compute

\[
\int_S \vec{F} \cdot \vec{n} \ dS.
\]

8. Let \( \{a_n\} \) be a bounded sequence of real numbers. Consider the infinite series

\[
f(x) = \sum_{n=1}^{\infty} \frac{a_n}{x^n}
\]

where \( x \) is a real number. Prove that for any \( c > 1 \) this series converges uniformly on \( \{x \in \mathbb{R} \mid x \geq c\} \).

9. Let \( A \) be the ring of continuous functions \( f: \mathbb{R} \to \mathbb{R} \), under (pointwise) addition and multiplication.

a) Determine whether \( A \) is a integral domain.

b) Let \( I \subset A \) be the subset consisting of functions \( f \) such that \( f(0) = 0 \). Is \( I \) an ideal? Is it a maximal ideal? What is \( A/I \)?
10. Suppose \( \{a_n: n = 1, 2, \ldots\} \) is a sequence of real numbers so that
\[
\sum_{n=1}^{\infty} |a_n| = 1.
\]
Let \( f(x) \) be given by the cos series
\[
f(x) = \sum_{n=1}^{\infty} a_n \cos(nx).
\]
Prove that the series for \( f \) converges and that \( f \) is continuous.

11. Let
\[
M = \begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

a) Find the minimal and characteristic polynomials of \( M \).
b) Is \( M \) similar to a diagonal matrix \( D \) over \( \mathbb{R} \)? If so, find such a \( D \).
c) Repeat part (b) with \( \mathbb{R} \) replaced by \( \mathbb{C} \) and also by the field \( \mathbb{Z}/5\mathbb{Z} \).

12. Let \( V \) be the vector space of \( C^\infty \) real-valued functions on \( \mathbb{R} \).
Consider the following maps \( T_i: V \to V \).
\[
T_1(f) = f'' - 6f' + 9f \\
T_2(f) = f' - xf \\
T_3(f) = ff'
\]
a) Which of the maps \( T_i \) are linear transformations?
b) For each one that is, find a basis for the kernel.