114 2022A FINAL EXAM **NAME:** **PENNID:**

INSTRUCTIONS:

No book, calculator, or formula sheet.

Use a writing utensil and logic.

If you need to write on the back of a page, go ahead, but indicate clearly that you are doing so please.

Show your work!
Explain yourself clearly to receive partial credit. I recommend you attempt all the problems for partial credit.

These questions are written carefully: **please do not ask for clarification during the exam.** If anything is unclear, use your best judgement and explain yourself. We will grade accordingly. If we have made a mistake, we'll fix it in the grading.

Cheating, or the appearance of cheating, will be dealt with severely. By placing your name on this page, you agree to abide by the rules.

Stay calm. All of these problems are doable. You can make it.

Best wishes!

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PROBLEM 1: Consider the following matrices:

$$A = \begin{bmatrix} 2 & -1 \\ 3 & -1 \\ 1 & 1 \end{bmatrix} \quad : \quad B = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 6 \end{bmatrix}$$

A) Compute AB and BA. *Be sure to indicate which is which.*

B) Compute the inverse of AB and the inverse of BA, if they exist. If they do not exist, explain why not.

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PROBLEM 2: A general quadratic function of two variables has equation

$$f(x,y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

where the six coefficients A, B, C, D, E, F are all constants.

A) Compute the derivative $[Df]$. Under what conditions on the constants is the origin a critical point?

B) Compute the 2nd derivative (or Hessian) $[D^2 f]$. Assuming the origin is a critical point, under what conditions on the constants is it a (local) maximum?

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PROBLEM 3: Consider the unit cube in 4-D with coordinates x_1, x_2, x_3, x_4 ranging from zero to one.

A) Assuming this cube has a probability density function on it of the form

$$\rho = C x_1 x_2^2 x_3^3 x_4^4$$

what is the value of C? *Show all work and explain what you are doing.*

B) If a point is chosen at random with this probability density, what is the probability that $x_1 \leq x_3$ and $x_2 \leq x_4$? Set up, ***but do not solve*** the integral for computing this probability.

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PROBLEM 4: Evaluate the integral

$$\int_{\gamma} ye^z dx + (xe^z + e^y) dy + e^z(xy+1) dz$$

where γ is the straight-line path from the origin to the point $\left(\frac{1}{\ln 3}, \ln 3, \ln 2\right)$.

Show all work and explain.

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PROBLEM 5: Consider the region that is described in spherical coordinates as

$$0 \leq \rho \leq 2 \quad : \quad \frac{\pi}{2} \leq \phi \leq \pi \quad : \quad 0 \leq \theta \leq \frac{\pi}{2}$$

A) Describe carefully and/or draw a careful picture of this domain.

B) Compute the average of the function $f = \frac{1}{\rho}$ on this domain. *Show all work.*

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PROBLEM 6: Consider the surface S given by

$$x^2 + y^2 + z^2 = 25 \quad : \quad z \leq 0 \quad : \quad x^2 + y^2 \leq 4$$

This can be described as a circular region about the "South Pole" of a sphere of radius 5 where $z \leq -3$. Compute the flux of the curl of the vector field

$$\vec{F} = -yz^2\,\hat{\imath} + xz^2\,\hat{\jmath} + e^{-xyz}\,\hat{k}$$

across this surface S, using an upward-pointing normal (along the $+z$ axis).

Explain. Hint: there is more than one way to solve this problem.

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<u>PROBLEM 7:</u> Let

$$f = xz^2 - y^2 \quad : \quad \alpha = 3\,dx - x^2dy \quad : \quad \beta = x^2y\,dy \wedge dz + xy^2\,dz \wedge dx + dx \wedge dy$$

A) Fill in the blanks: *(no justification needed)*

1) f is a ____-form
2) df is a ____-form
3) $d\beta$ is a ____-form
4) $df \wedge \alpha$ is a ____-form

B) Calculate and simply $d\beta$ as much as possible, showing work.

C) Calculate and simplify $df \wedge \alpha$ as much as possible, showing all steps below.

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PROBLEM 8: Consider the function given by

$$f(x, y, z) = \frac{1}{4}x^2 + y^2 + \frac{1}{9}z^2$$

A) Show that the level set $f^{-1}(1)$ (that is, the set of inputs where $f = 1$) contains the point $\left(\sqrt{2}, 0, \frac{3}{2}\sqrt{2}\right)$.

B) Find a vector that is *orthogonal* (or *normal*) to the level set from (A) at the point $\left(\sqrt{2}, 0, \frac{3}{2}\sqrt{2}\right)$. *Explain your answer/reasoning.*

C) Write down and simplify an equation of the tangent plane to the level set from (A) at the point $\left(\sqrt{2}, 0, \frac{3}{2}\sqrt{2}\right)$. *Show work please.*

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PROBLEM 9: Let γ be the closed (parallelogram) path in the plane which follows along straight-line segments from the points:

$$(-1,1) \longrightarrow (-3,1) \longrightarrow (-1,3) \longrightarrow (1,1) \longrightarrow (-1,1)$$

Compute the circulation of the vector field

$$\vec{F} = (\cos x^2 + x \sin 2y + 3x^2)\hat{\imath} + \left(\frac{1}{1+y^2} + x^2 \cos 2y + 2x\right)\hat{\jmath}$$

along this path γ. *Show all work/explain.*

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PROBLEM 10: Consider the following vectors in $\mathbb{R}^6$:

$$\boldsymbol{s} = \begin{pmatrix} 0 \\ 5 \\ -1 \\ -2 \\ 6 \\ 0 \end{pmatrix} : \boldsymbol{t} = \begin{pmatrix} 1 \\ 0 \\ 3 \\ -2 \\ 0 \\ -6 \end{pmatrix} : \boldsymbol{u} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 2 \\ -1 \end{pmatrix} : \boldsymbol{v} = \begin{pmatrix} 0 \\ 7 \\ 4 \\ 3 \\ 4 \\ 1 \end{pmatrix} : \boldsymbol{w} = \begin{pmatrix} 0 \\ 2 \\ 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

A) Identify which pair of vectors above is *orthogonal* (i.e., *perpendicular*). *Explain.*

B) Compute (as best you can without a calculator) the angle between vectors $\boldsymbol{u}$ and $\boldsymbol{w}$. *Explain and show work.*

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PROBLEM 11: Consider the functions

$$f\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} u^2 - 3v \\ uv - w^2 \end{pmatrix} \quad : \quad g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy^2 \\ x^2 - y \\ 3y \end{pmatrix}$$

A) Compute the derivatives of f and g.

B) Compute the derivative of the composition $f \circ g$ at the point where all its inputs equal $+1$. *Show work/explain.*

C) If all inputs of $f \circ g$ equal $+1$ and are *decreasing* at a unit rate, at what rate is the last output changing?

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PROBLEM 12: Evaluate the double integral

$$\iint_D 5\, x^3 y^4 \, dx\, dy$$

where D is the domain given by inequalities $1 \le xy^2 \le 2$ and $3 \le x^2y \le 5$, using the change of variables $u = xy^2 : v = x^2y$. *Show all work / explain.*

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PROBLEM 13: Is the function

$$F\begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix} = \begin{pmatrix} 1 - 2u + 3v \\ e^u - v \\ 2y - \sin x \\ e^x + \ln(1 - 2y) \end{pmatrix}$$

locally invertible near the origin? *Explain and show all work.*

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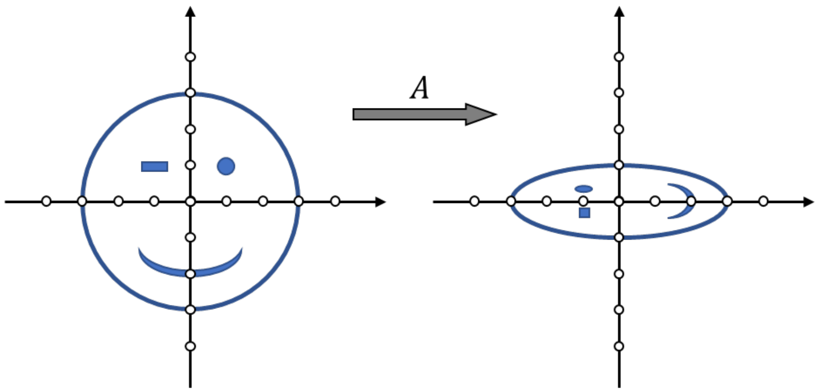
PROBLEM 14: Consider the linear transformation A from $\mathbb{R}^2$ to $\mathbb{R}^2$ which acts as per the figure below. *Assume the dots are at integer values & that the figure is drawn accurately.*



A) What is the matrix that represents A as per the figure above? *Explain carefully.*

B) Which point in the plane is sent by A to the point $(0,2)$? *Explain.*

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