1.

Consider the following vectors:

$$\mathbf{x} = \begin{pmatrix} 3 \\ 0 \\ -6 \end{pmatrix} \quad : \quad \mathbf{y} = \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix}$$ [text : x=(3,0,-6) y=(-2,0,4)]

and the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$ [text : A = [0 1 -1 ; 1 0 0 ; 2 1 0]]

A) What is the angle between these two vectors $\mathbf{x}$, $\mathbf{y}$? Answer as best you can without a calculator.

B) Which vector is longest: $\mathbf{x}$, $\mathbf{y}$, *or* $\mathbf{x} \times \mathbf{y}$? [text: x, y, or x-cross-y]

C) Compute $\mathbf{y}^T A\mathbf{x}$. [text : y^T A x]

2.

Consider the function $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\mathbf{u} = f(\mathbf{v}) = \begin{pmatrix} v_1 v_2^2 - v_3^3 \\ 2v_3 - 3v_2 \\ v_3^2 \end{pmatrix}$$ [text : **u** = f(**v**) = (v_1*v_2^2 - v_3^3 ; 2v_3 - 3v_2 ; v_3^2)]

A) What is $f \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$? [text : f(1,2,3)]

B) Compute the derivative $[Df]$.

C) Can one solve for ***v*** as a function of ***u*** for ***v*** close to $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ [text : (1, 2, 3)] ? Explain.

3.

Consider the function *f* given by

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 \ln\left(\frac{y^2}{z+1}\right)$$ [text : f(x,y,z) = x^2*ln(y^2/(z+1))]

A) Compute and simplify the gradient ∇f

B) If, at the point *(1, 2, 3)* each input is increasing at a unit rate, at what rate is the output of *f* changing?

C) Compute the equation of a plane tangent to the level set $f^{-1}(0)$ [text : f^(-1)(0)] at the point *(1,2,3)*.

4.

A real-valued function *f(x,y)* has Taylor expansion about the origin equal to:

$$f = 5 - (C-2)x + x^2 - 3xy + 3y^2 - \frac{2C}{3}x^3 + x^2y + \frac{5(C-1)}{4}xy^2 + \frac{1}{6}y^3 + \cdots$$

[text : f = 5 - (C-2)x + x^2 - 3xy + 3y^2 - (2C/3)x^3 + x^2y + (5(C-1)/4)xy^2 + y^3/6 + ...]

All terms up to and including order 3 are displayed, and *C* is an unknown constant.

A) If *f* has a critical point at the origin, what is the value of *C*? Explain.

B) Compute from this Taylor series the second derivative (or Hessian) matrix of *f* at the origin.

C) Assuming that the origin is a critical point, as in part (A), classify the critical point.

5.

Consider the following probability density function

$$\rho = \frac{2y}{1+x^2}$$ [text : rho = 2y/(1+x^2)]

on the domain given by $0 \leq x \leq L$ and $0 \leq y \leq 1$ [text : x from 0 to L and y from 0 to 1]

A) What is the value of *L* that makes this a probability density function? Explain.

B) Compute the probability that a randomly chosen point in the domain with this density has *x* less than 1/3 ***and*** y greater than 1/2.

6.

Compute the circulation of the planar vector field

$$\vec{F} = \left(x^5 - x^2y - y^3\right)\hat{\imath} + \left(2x^3 + 4xy^2 + 2y^3\right)\hat{\jmath}$$ [text : F = (x^5-x^2y-y^3)i + (2x^3+4xy^2+2y^3)j]

along the circle of radius 2 centered at the origin (assume counterclockwise orientation).

Hint: a direct computation may be unpleasant.

7.

Consider the following parametrized path in 3-d:

$$\gamma(t) = \begin{pmatrix} t \\ t^2 - 2 \\ t^3 \end{pmatrix} \quad : \quad 1 \leq t \leq 2$$ [text : gamma(t) = (t ; t^2-2 ; t^3) for t from 1 to 2]

A) Compute the velocity and acceleration vectors of γ as a function of *t*.

B) Compute the work done along γ by the field

$$\vec{F} = \left(y^2 + \frac{z}{x^2}\right)\hat{\imath} + (2xy)\hat{\jmath} - \left(\frac{1}{x}\right)\hat{k}$$ [text : F = (y^2+z/x^2)i + (2xy)j - (1/x)k]

Hint: think first, then compute.

8.

Consider the vector field

$\vec{F} = (xz^2)\hat{\imath} + (z^3 - 2y)\hat{\jmath} + ((x^2 + y^2)z)\hat{k}$ [text : F = (xz^2)i + (z^3-2y)j + ((x^2+y^2)z)k]

with corresponding flux 2-form

$\varphi_{\vec{F}} = (xz^2)dy \wedge dz + (z^3 - 2y)dz \wedge dx + ((x^2 + y^2)z)dx \wedge dy$

[text : phi_F = (x^2z)dy^dz + (z^3-2y)dz^dx + ((x^2+y^2)z)dx^dy]

A) Compute the derivative of this flux 2-form $\varphi_{\vec{F}}$.

B) Compute the net flux of the vector field $\vec{F}$ across the sphere of radius 3 at the origin (assume outward-pointing normals).