Quiz Instructions

This is the final exam.

You have 75 minutes to take the exam.

Please read each problem carefully and follow the instructions.

In particular, each problem is an "essay", but please just write on paper. You can ignore all answer entries on this exam. Only your written and scanned work will be graded. Show all your work on paper, neatly, indicating clearly which problem you are showing work for. There is nothing to enter in this exam -- no numbers, multiple choice, etc. Simply show your work on paper.

By submitting this exam, you explicitly pledge that you have NOT

-> communicated with anyone about the quiz;

-> used notes, books, websites, videos, or calculators; or

-> added any written work after the quiz has ended.

REMEMBER: When you have completed this exam, you must upload a .pdf scan of your written work as soon as possible, within ten (10) minutes of completion.

Please check to make sure that everything has uploaded. If there is a problem, don't panic. Solve the problem and communicate with your professor.

Good luck!

1.

Since every exam should begin with a question that you get right...

Name one thing that we learned this semester that you really liked or that surprised you.

No need to elaborate: just tell us something fun you learned for a point.

2.

Consider the 2-form field on $\mathbb{R}^3$ given by:

$$\beta = (x+z)dx \wedge dz + z\,dx \wedge dy$$

A) Compute and simplify the derivative $d\beta$.

B) Integrate β over the parametrized surface given by

$$G\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} s+t \\ st \\ t-s \end{pmatrix} \quad : \quad 0 \leq s \leq 1 \ : \ 0 \leq t \leq 1$$

3.

Consider the 4-by-4 matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 5 \\ 0 & 0 & 1 & -3 \\ -3 & 5 & 1 & -11 \\ 0 & 0 & 2 & -5 \end{bmatrix}$$

A) Use row-reduction and back-substitution to solve the linear system

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -7 \\ 8 \end{pmatrix}$$

for the variables $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, being sure to show/explain your steps.

B) Compute the determinant of this matrix A.

4.

For some value of C, the following is a probability density on the ***unit ball*** in $\mathbb{R}^3$:

$$f(x, y, z) = C(x^2 + y^2 + z^2)^{3/2}$$

A) Compute the value of C.

B) Compute the probability that a point in the unit ball chosen at random with this density lies within 1/10 of the boundary sphere (that is, $x^2 + y^2 + z^2 \geq \frac{9}{10}$). Your answer may be a little messy: don't try to simplify it (since you do not have access to a calculator).

5.

Consider the vector field on $\mathbb{R}^3$ given by

$$\vec{F} = \left(\left(\frac{2}{3}x^3 + 2xy^2\right)\cos z\right)\hat{\imath} + \left(\left(2x^2y + \frac{2}{3}y^3\right)\cos z\right)\hat{\jmath} + 0\,\hat{k}$$

A) Compute the divergence of $\vec{F}$.

B) Compute the flux of $\vec{F}$ across the entire boundary of the closed cylinder (including end caps) given by $x^2 + y^2 \leq 4$ and $-\dfrac{\pi}{2} \leq z \leq \dfrac{\pi}{2}$. Use the customary outward-pointing normal and explain how you get your answer.

C) Explain carefully why the flux of $\vec{F}$ across each of the ***end caps*** of the cylinder (*i.e.*, where $x^2 + y^2 \leq 4$ and $z = \pm\frac{\pi}{2}$) is zero.

6.

Consider the following functions:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\cos 7x + e^{2y} \\ 2x + \sin 4y \end{pmatrix} \quad : \quad g\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 3u + v - uv^2 \\ -u + v - u^2v^2 \end{pmatrix}$$

Note that each takes the origin to the origin.

A) Compute the derivatives of f and of g.

B) Compute the derivative of $f \circ g$ evaluated at the origin.

C) If the outputs of g at the origin are changing at rates $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$, then at what rates are the two inputs u and v changing?

7.

Consider the function

$f(x, y) = 8x + y^2 - 2x^2 - xy^2$

A) Find and classify all ***three*** critical points of f, showing all steps.

B) If f is ***constrained*** to the curve where $2x^2 - xy + 2y^2 = 9$, then what set of equations would you use to solve this ***constrained optimization problem***? Do not solve the equations: simply set them up and explain what these equations are

8.

Consider the parametrized curve in $\mathbb{R}^4$ given by

$$\gamma(t) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2t\cos t \\ 2t\sin t \\ 3t \\ 4t \end{pmatrix} \quad : \quad -1 \leq t \leq 1$$

A) Compute the velocity vector of this curve.

B) Set up but ***do not solve*** an integral to compute the arclength of this curve.

C) This curve intersects the origin when $t = 0$. What angle does the curve make with the positive x_3-axis? Show all work / explain your reasoning. Your final answer may be messy: since you do not have access to a calculator, do not worry about the precise numerical value.