

THE GEOMETRY OF CAPILLARY AND CONSTANT MEAN
CURVATURE SURFACES

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ABSTRACT

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Constant mean curvature (CMC) surfaces are critical points of the area functional for variations that preserve the volume of the region enclosed by the surface. Capillary surfaces are defined in a similar way, but instead of the area functional, one considers a functional that is the sum of the surface area with a boundary term. Both of these types of surfaces arise in nature as the interface between a liquid and air. The index of a CMC or a capillary surface is an integer that measures how far the surface is from minimizing the functional. In this thesis, we explore the relationship between the index and the geometry of capillary and CMC surfaces.

We begin by showing that the index together with the area bound the genus of compact CMC surfaces embedded in a compact 3-manifold. We also show that in the case where the surface is not minimal and the 3-manifold has finite fundamental group, the index and the mean curvature are sufficient to bound the genus. Then we move on to study capillary surfaces immersed in 3-manifolds. Amongst other results, we describe the conformal structure of noncompact capillary surfaces with finite index, one consequence of this description is that the only noncompact capillary surface immersed in a half-space with acute contact angle and zero index is the half-plane.

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Chapter 1

Preliminary

Constant mean curvature (CMC) surfaces often appear in nature as the interface between fluids in equilibrium, think of, for example, the surface of a small water drop in suspension. Capillary surfaces can be seen in similar contexts, except that now we assume that the fluid touches some container. Examples of this type of surface can be seen in small drops of liquid in a smooth tube or the interface between fluids inside a thin tube. In fact, these surfaces are so natural to consider that they have been studied by Gauss [29] and Laplace [33] in the early 19th century, before the modern formulation of differential geometry.

Capillary and CMC surfaces are equilibrium solutions, however this equilibrium might be unstable. The index of a surface is a way of quantifying instability by, roughly, counting the number of directions where deforming the surface leads to a decrease in energy. It is expected that more complex surfaces have higher index than simpler surfaces, as the more complex surface have regions where a deformation could

decrease energy. As we will see in many points along the text, there are many results that relate the index to the geometry and topology of minimal surfaces. Less is known for the more general case of CMC surfaces, and even less is known for capillary surfaces. Here we aim to decrease this knowledge gap.

We begin this work by briefly introducing these two types of surfaces in general Riemannian 3-manifolds. We will also define the weak and the strong Morse index of these surfaces, study the relationship between these indices, and some properties exhibited by surfaces of finite index. Most of the results and concepts presented in this chapter can be easily generalized to higher dimensions, however since our main results in later chapters regard surfaces in 3-manifolds, we will state our results and definitions for this particular case.

Our results in CMC surfaces have appeared in [46] and can be found in Chapter 2, while Chapter 3 contains the results on capillary surfaces, which have been obtained in collaboration with Han Hong and have appeared in [32].

1.1 Basic Definitions

1.1.1 Constant mean curvature surfaces

Let M be a Riemannian 3-manifold and Σ be a 2-sided surface immersed in M by $X : \Sigma \rightarrow M$. If Σ has a boundary, we assume the boundary is not contained in Σ .

Fix an $\epsilon > 0$, we say that a smooth map

$$\bar{X} : \Sigma \times (-\epsilon, \epsilon) \longrightarrow M$$

$$\bar{X}(p, t) = X_t(p)$$

is a compact variation of X if

1. For all $t \in (-\epsilon, \epsilon)$, the map $X_t : \Sigma \rightarrow M$ is an immersion of Σ in the interior of M .
2. The immersion X_0 coincides with X .
3. \bar{X} is supported on a compact set $K \subset \Sigma$, that is, $t \mapsto X_t(p)$ is constant for all $p \in \Sigma \setminus K$.

Note that since Σ does not contain its boundary, any K as above must be bounded away from the boundary of Σ .

Fix a normal vector field ν along Σ . We define the signed volume functional associated to the variation \bar{X} as

$$V_t = \int_{[0, t] \times K} \bar{X}^*(dV) \tag{1.1}$$

where dV is the volume form of M . Note that in the case where X is an embedded bounding a compact region of M and t is small, V_t is the difference between the volume bounded by $X(\Sigma)$ and $X_t(\Sigma)$. It is easy to see that the derivative of (1.1) is

$$\left. \frac{d}{dt} \right|_{t=0} V_t = \int_{\Sigma} \langle Y, \nu \rangle \tag{1.2}$$

where the measure used in the integral to the right is induced by X , and $Y = \partial_t|_{t=0}X_t$ is the variational vector field associated to X_t . In case $\frac{d}{dt}|_{t=0}V_t = 0$ we say that the variation \bar{X} is *volume-preserving*.

Fix a compact set K containing the support of \bar{X} and let A_t be the area of K by the measure induce by X_t . It is well-known that (see e.g [21, p. 7])

$$\frac{d}{dt}\Big|_{t=0} A_t = - \int_{\Sigma} \langle Y, \vec{H}_{\Sigma} \rangle, \quad (1.3)$$

where \vec{H}_{Σ} is the mean curvature of Σ associated to X , which we define as

$$\vec{H}_{\Sigma} = D_{E_1}E_1 + D_{E_2}E_2$$

where D is the pull-back of the Levi-Civita connection on M by X and $\{E_1, E_2\}$ are vector fields in Σ that form an orthonormal basis in a neighborhood of Σ .

We say that Σ is a *constant mean curvature* surface if for any volume-preserving compact variation of X have that $\frac{d}{dt}|_{t=0}A_t = 0$. Note that this condition is equivalent to $H_{\Sigma} = |\langle \vec{H}_{\Sigma}, \nu \rangle|$ being constant along Σ . Alternatively, when the value of the mean curvature is important, we say that Σ is a H_{Σ} -surface. A 0-surface is called a *minimal surface*. It is possible for a minimal surface to be one-sided, however we will not consider this case.

Let Σ be a CMC surface immersed in M by $X : \Sigma \rightarrow M$ and take a volume-preserving compact variation \bar{X} . Since a CMC surface is a critical point for area it is natural to take the second variation of this functional. Standard calculations yield

(see e.g. [21, p. 39])

$$\left. \frac{d^2}{dt^2} \right|_{t=0} A_t = \int_{\Sigma} |\nabla u|^2 - (|A_{\Sigma}|^2 + \text{Ric}_M(\nu, \nu))u^2$$

where $u = \langle Y, \nu \rangle$ and A_{Σ} is the second fundamental form of Σ . For this reason, the stability operator of Σ is the symmetric bilinear form on $C_0^{\infty}(\Sigma)$ defined by

$$Q(u, u) = \int_{\Sigma} |\nabla u|^2 - (|A_{\Sigma}|^2 + \text{Ric}_M(\nu, \nu))u^2 = - \int_{\Sigma} uJu \quad (1.4)$$

for $u \in C_0^{\infty}(\Sigma)$, where J is the Jacobi operator

$$J = \Delta + |A_{\Sigma}|^2 + \text{Ric}_M(\nu, \nu). \quad (1.5)$$

Now assume additionally that Σ is bounded. Let $B(\Sigma) = \{u \in C_0^{\infty}(\Sigma) : \int_{\Sigma} u = 0\}$ be the set of balanced (mean zero) functions on Σ with compact support, then the *weak (Morse) index* of Σ is

$$\text{Index}_w(\Sigma) = \max\{V \subset B(\Sigma) : V \text{ is a subspace and } Q|_{V \times V} \text{ is negative-definite}\}. \quad (1.6)$$

Informally, the weak index counts the number of directions where one can decrease the area of the surface while keeping the volume functional constant to first degree. Similarly, the *strong (Morse) index* of Σ is defined as

$$\text{Index}_s(\Sigma) = \max\{V \subset C_0^{\infty}(\Sigma) : V \text{ is a subspace and } Q|_{V \times V} \text{ is negative-definite}\}. \quad (1.7)$$

It is easy to see that the weak index is no larger than the strong index. However, as

we will see in Section 1.2, they cannot differ by more than one. When the weak (or strong) index of Σ is zero we say that Σ is zero we say that Σ is *weakly (resp. strongly) stable*. This particular case plays a central role in the study of these surfaces.

Barbosa and Berard [11, Proposition 2.2] show that the weak index of a bounded CMC surface can also be understood as the maximum k such that $\tilde{\lambda}_k < 0$ where $\tilde{\lambda}_1 < \tilde{\lambda}_2 \leq \tilde{\lambda}_3 \leq \dots$ are the $L^2(\Sigma)$ eigenvalues associated to the problem

$$\begin{cases} Ju + \tilde{\lambda}u = \frac{1}{|\Sigma|} \int_{\Sigma} Ju & \text{in } \Sigma, \\ u = 0 & \text{along } \partial\Sigma, \\ \int_{\Sigma} u = 0. \end{cases} \quad (1.8)$$

Similarly, the strong index of a bounded CMC surface is the index of the largest negative eigenvalue associated to the problem

$$\begin{cases} Ju + \lambda u = 0 & \text{in } \Sigma, \\ u = 0 & \text{along } \partial\Sigma, \end{cases} \quad (1.9)$$

So it follows from standard elliptic theory that the weak and the strong index of a bounded CMC surface must be finite.

1.1.2 Capillary surfaces

Now let M be a Riemannian 3-manifold with boundary and let Σ be a 2-sided surface with boundary immersed in M by $X : \Sigma \rightarrow M$. Assume additionally that the interior of Σ does not touch the boundary of M , that is $\partial M \cap X(\Sigma) \subseteq X(\partial\Sigma)$ and

that Σ meets the boundary of M transversally. We assume that $\Gamma = X(\partial\Sigma) \cap \partial M$ is smooth, but allow $\partial\Sigma \setminus \Gamma$ to be only continuous. Following Carlotto and Franz [12] we will refer to $\partial\Sigma \setminus \Gamma$ as the *edge* of Σ . Topologically, we assume that Σ contains Γ but not its edge. Similarly to as we had in Subsection 1.1.1, we say that a smooth map

$$\begin{aligned}\bar{X} : \Sigma \times (-\epsilon, \epsilon) &\longrightarrow M \\ \bar{X}(p, t) &= X_t(p)\end{aligned}$$

is an admissible variation of X if

1. For all $t \in (-\epsilon, \epsilon)$, the map $X_t : \Sigma \rightarrow M$ is an immersion of Σ in M such that $\partial M \cap X_t(\Sigma) \subseteq X_t(\partial\Sigma)$.
2. The immersion X_0 coincides with X .
3. \bar{X} is supported on a compact set K . That is, $t \mapsto X_t(p)$ is constant for all $p \in \Sigma \setminus K$.

Note that since Σ does not contain its edge but contains Γ , an admissible variation must fix the edge of Σ , but can move Γ .

Fix a normal vector field ν along Σ . We define the volume and area functionals A_t and V_t exactly as in the CMC case, additionally we define the wetting area functional by

$$W_t = \int_{[0, t] \times (K \cap \partial\Sigma)} \bar{X}^*(dA)$$

where dA is the area element of ∂M . In the case where $X(\Gamma)$ is embedded and bounds a compact region of ∂M , the value of W_t for t small is the difference between the area bounded by $X(\Gamma)$ and $X_t(\Gamma)$.

It is easy to see that

$$\left. \frac{d}{dt} \right|_{t=0} W_t = \int_{\partial \Sigma} \langle Y, T \rangle$$

where Y is the variational vector field associated to \bar{X} and T is the unit normal vector field of Γ in ∂M chosen according to the orientation given by ν .

Again we have that

$$\left. \frac{d}{dt} \right|_{t=0} V_t = \int_{\Sigma} \langle Y, \nu \rangle$$

but the variation of the area functional does include a boundary term, more specifically,

$$\left. \frac{d}{dt} \right|_{t=0} A_t = - \int_{\Sigma} \langle Y, \vec{H}_{\Sigma} \rangle + \int_{\partial \Sigma} \langle Y, \eta \rangle$$

where η is the conormal of Σ .

Fix a $\theta \in (0, \pi)$. We define the energy functional

$$E_t = A_t - \cos \theta W_t,$$

then

$$\left. \frac{d}{dt} \right|_{t=0} E_t = - \int_{\Sigma} \langle Y, \vec{H}_{\Sigma} \rangle + \int_{\partial \Sigma} \langle Y, \eta - \cos \theta T \rangle.$$

We say that Σ is an *edged capillary surface* at a constant angle θ if for every volume-preserving admissible variation we have that $\left. \frac{d}{dt} \right|_{t=0} E_t = 0$. It is clear that this condition holds if, and only if, Σ is CMC and the angle between η and T is constant

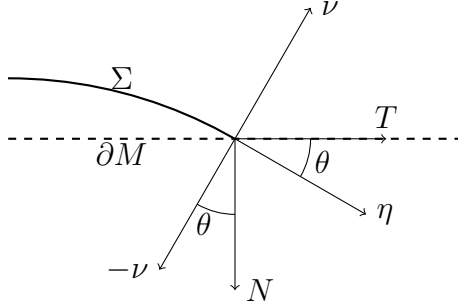


Figure 1.1: Vectors and angles for a spherical cap in a half-space of \mathbb{R}^3 .

equal θ . In our convention, we will choose ν such that $\langle \vec{H}_\Sigma, \nu \rangle$ is nonpositive, for the second fundamental form of Σ we adopt the convention $A_\Sigma(\cdot, \cdot) = \langle D\nu, \cdot \rangle$. We will use N to refer to the outwards normal of the ambient space M . Figure 1.1 exemplifies our conventions.

Note that, by our definition, an edged capillary surface is always two-sided. A *capillary surface* is an edged capillary surface with empty edge, this implies that a capillary surface is complete in the sense of metric spaces. A CMC surface can be seen as an edged capillary surface where the capillary boundary (that is, $\Gamma = X(\Sigma) \cap \partial M$) is empty. A *free boundary CMC surface* is a capillary surface at a constant angle $\pi/2$, note that in this case the energy functional equals the area functional.

Consider an edged capillary surface Σ immersed in M by $X : \Sigma \rightarrow M$ at a constant angle θ and take an admissible variation \bar{X} . Then second variation of area is

$$\left. \frac{d^2}{dt^2} \right|_{t=0} A_t = \int_{\Sigma} |\nabla u|^2 - (|A_\Sigma|^2 + \text{Ric}_M(\nu, \nu))u^2 - \int_{\partial\Sigma} qu^2$$

where $u = \langle Y, \nu \rangle$, Y is the variational vector field associated to \bar{X} , and q is defined

by

$$q = \frac{1}{\sin \theta} h_{\partial M}(T, T) + \cot \theta A_{\Sigma}(\eta, \eta). \quad (1.10)$$

In the above $h_{\partial M}(T, T) = \langle D_T N, T \rangle$ (for a proof, see [42, p 357-360]). Note that

$$\eta = \frac{1}{\sin \theta} N + \cot \theta \nu, \quad (1.11)$$

so if we let $H_{\partial M} = \text{tr} h_{\partial M}$, then it follows from (1.11) that q can be re-written as

$$q = \frac{1}{\sin \theta} H_{\partial M} + \cot \theta H_{\Sigma} - \kappa_{\partial \Sigma}, \quad (1.12)$$

where $\kappa_{\partial \Sigma}$ is the geodesic curvature of $\partial \Sigma$ in Σ , that is, $\kappa_{\partial \Sigma} = \langle D_{\tau} \eta, \tau \rangle$ with τ unit tangent vector field in $\partial \Sigma$. It is also interesting to note that any function $u \in C_0^{\infty}(\Sigma)^1$ can be realized as the product the normal to Σ with the variational vector field of some admissible variation [4, Proposition 2.1].

The stability operator of Σ is defined as the quadratic form on $C_0^{\infty}(\Sigma)$ such that for all $u \in C_0^{\infty}(\Sigma)$ we have

$$\begin{aligned} Q(u, u) &= \int_{\Sigma} |\nabla u|^2 - (\text{Ric}_M(\nu, \nu) + |A_{\Sigma}|^2)u^2 - \int_{\partial \Sigma} q u^2 \\ &= - \int_{\Sigma} u J u + \int_{\partial \Sigma} u \left(\frac{\partial u}{\partial \eta} - q u \right), \end{aligned} \quad (1.13)$$

where J the Jacobi operator.

The weak and the strong indices of a bounded edged capillary surface are defined by (1.6) and (1.7) respectively. Similarly to the CMC case, the weak and the strong

¹Recall that by our definition an edged capillary surface contains the capillary part of its boundary, hence a compactly supported function on Σ can take nonzero values in this part of the boundary.

indices can be characterized by an eigenvalue problem, but in this case the associated eigenvalue problem has mixed Dirichlet and Robin boundary conditions. Namely, the eigenvalue problem associated to the weak index becomes

$$\begin{cases} Ju + \tilde{\lambda}u = \frac{1}{|\Sigma|} \int_{\Sigma} Ju & \text{in } \Sigma, \\ \frac{\partial u}{\partial \eta} = qu & \text{along } \Gamma, \\ u = 0 & \text{along } \partial\Sigma \setminus \Gamma, \\ \int_{\Sigma} u = 0. \end{cases} \quad (1.14)$$

And the eigenvalue problem associated to the strong index becomes

$$\begin{cases} Ju + \lambda u = 0 & \text{in } \Sigma, \\ \frac{\partial u}{\partial \eta} = qu & \text{along } \Gamma, \\ u = 0 & \text{along } \partial\Sigma \setminus \Gamma. \end{cases} \quad (1.15)$$

So it follows from standard elliptic theory that when Σ is bounded the weak index and the strong index must be finite.

1.2 Capillary surfaces of finite index

In this section we will define the (weak and strong) index for unbounded (not pre-compact) edged capillary surfaces, and examine some fundamental properties of surfaces with finite index. We will state our results for edged capillary surfaces, as they are the more general case. To apply these results for CMC surfaces the boundary

term must be ignored.

Let Σ be an unbounded edged capillary surface immersed in M . Note that every bounded open subset $\Omega \subset \Sigma$ is an edged capillary surface, so we can define the weak and the strong index of Ω as in (1.6) and (1.7), respectively. Furthermore, it is clear from the definition that if $\tilde{\Omega} \supset \Omega$ is open and bounded, then $\text{Index}_w(\tilde{\Omega}) \geq \text{Index}_w(\Omega)$ and $\text{Index}_s(\tilde{\Omega}) \geq \text{Index}_s(\Omega)$. Take an exhaustion of Σ by bounded open sets $\Omega_1 \subset \Omega_2 \subset \dots \subset \Sigma$ we define the weak index of Σ as

$$\text{Index}_w(\Sigma) = \lim_{n \rightarrow \infty} \text{Index}_w(\Omega_n), \quad (1.16)$$

and the strong index of Σ as

$$\text{Index}_s(\Sigma) = \lim_{n \rightarrow \infty} \text{Index}_s(\Omega_n). \quad (1.17)$$

Note that these limits can tend to positive infinity. Now we show that the index is well-defined.

Proposition 1.1. *The limits $\lim_{n \rightarrow \infty} \text{Index}_w(\Omega_n)$ and $\lim_{n \rightarrow \infty} \text{Index}_s(\Omega_n)$ are independent of the choice of exhaustion by bounded open sets $\Omega_1 \subset \Omega_2 \subset \dots \subset \Sigma$.*

Proof. Let $\tilde{\Omega}_1 \subset \tilde{\Omega}_2 \subset \dots \subset \Sigma$ be another exhaustion as the one above, note that since each on the sets Ω_n are bounded, for all $n \in \mathbb{N}$ there is a number $k(n) \in \mathbb{N}$ such that $\Omega \subset \tilde{\Omega}_{k(n)}$, because $\lim_{n \rightarrow \infty} k(n) = \infty$ we have

$$\lim_{n \rightarrow \infty} \text{Index}_w(\Omega_n) \leq \lim_{n \rightarrow \infty} \text{Index}_w(\tilde{\Omega}_{k(n)}) = \lim_{n \rightarrow \infty} \text{Index}_w(\tilde{\Omega}_n).$$

The same argument shows that $\lim_{n \rightarrow \infty} \text{Index}_w(\Omega_n) \geq \lim_{n \rightarrow \infty} \text{Index}_w(\tilde{\Omega}_n)$,

concluding the proof for the weak index. The same arguments also work for the strong index. \square

The following result will be useful to decompose a capillary surface into components of smaller index.

Proposition 1.2. *Let $\Omega_1, \Omega_2 \subset \Sigma$ be open and disjoint, then $\text{Index}_s(\Omega_1 \cup \Omega_2) = \text{Index}_s(\Omega_1) + \text{Index}_s(\Omega_2)$.*

Proof. We will only consider the case where Ω_1 and Ω_2 are bounded, since the other cases follow by considering an exhaustion and taking a limit. Note that we can extend any function in $C_0^\infty(\Omega_k)$, $k = 1, 2$ to a function in $C_0^\infty(\Omega_1 \cup \Omega_2)$ by setting the extension to be zero outside the original domain. Since Ω_1 and Ω_2 are open and disjoint, restricting any function in $C_0^\infty(\Omega_1 \cup \Omega_2)$ to Ω_k , where $k = 1, 2$, gives a function in $C_0^\infty(\Omega_k)$. So we can decompose $C_0^\infty(\Omega_1 \cup \Omega_2) \equiv C_0^\infty(\Omega_1) \oplus C_0^\infty(\Omega_2)$. Furthermore, Q respects this decomposition in the sense that for any pair $\varphi \in C_0^\infty(\Omega_1), \psi \in C_0^\infty(\Omega_2)$ we have that $Q(\varphi, \psi) = 0$ because the supports of φ and ψ are disjoint. So any subspace of $C_0^\infty(\Omega_1 \cup \Omega_2)$ where Q is negative-definite is equivalent to a sum of two subspaces, one in $C_0^\infty(\Omega_1)$, other in $C_0^\infty(\Omega_2)$, where Q is negative-definite. \square

It is also interesting to note that the strong index and the weak index are closely related.

Proposition 1.3. *For any edged capillary surface Σ exactly one of the following must hold:*

(a) $\text{Index}_w(\Sigma) = \text{Index}_s(\Sigma) = \infty$; or

(b) $\text{Index}_w(\Sigma), \text{Index}_s(\Sigma) < \infty$ and $\text{Index}_w(\Sigma) \leq \text{Index}_s(\Sigma) \leq \text{Index}_w(\Sigma) + 1$.

Proof. Note that it is enough to show that for every bounded open set $\Omega \subset \Sigma$ we have

$$\text{Index}_w(\Omega) \leq \text{Index}_s(\Omega) \leq \text{Index}_w(\Omega) + 1.$$

The inequality $\text{Index}_w(\Omega) \leq \text{Index}_s(\Omega)$ follows from the fact that $B(\Omega) \subset C_0^\infty(\Omega)$.

To show the other inequality assume by contradiction that $\text{Index}_s(\Omega) \geq \text{Index}_w(\Omega) + 2$ and let $V \subset C_0^\infty(\Omega)$ be such that $\dim V = \text{Index}_w(\Omega) + 2$ and $Q|_{V \times V}$ is negative-definite. Note that there must be a function $\varphi \in V$ such that $\int_\Sigma \varphi = 1$, else we would have that $V \subset B(\Omega)$. Now, note that the linear map $L : V \rightarrow V \cap B(\Omega)$ defined by

$$L(u) = u - \varphi \int_\Omega u$$

has kernel generated by φ , hence $\dim V \cap B(\Omega) \geq \text{Index}_w(\Omega) + 1$, a contradiction. \square

In the cases where the weak and the strong indices of Σ are both finite we will say that Σ has finite index. Note that if Σ has finite index, then for a large enough open bounded set $\Omega \subset \Sigma$ we must have $\text{Index}_s(\Omega) = \text{Index}_s(\Sigma)$, and hence by Proposition 1.2 we can conclude that $\Sigma \setminus \bar{\Omega}$ must be strongly stable.

We say that a function $u \in C^2(\Sigma)$ is a Jacobi function if

$$\begin{cases} Ju = 0 & \text{in } \Sigma, \\ \frac{\partial u}{\partial \eta} = qu & \text{along } \Gamma = \partial\Sigma \cap \partial M. \end{cases} \quad (1.18)$$

Jacobi functions play an important role in the study of (edged) capillary surfaces with finite index. Fisher-Colbrie and Schoen have shown that the existence of a positive Jacobi function characterizes strong stability in complete, noncompact minimal surfaces [27, Theorem 1]. Here we will show an analogous result for edged capillary surfaces. In the following result we denote the first eigenvalue of the stability operator Q in $\Omega \subset \Sigma$ by $\lambda_1(\Omega)$.

Proposition 1.4. *Let Σ be an unbounded edged capillary surface in a 3-manifold M and let $C \subset \Sigma$ be a compact subset. The followings are equivalent:*

1. $\lambda_1(\Omega) \geq 0$ for all bounded open sets $\Omega \subset \Sigma \setminus C$.
2. $\lambda_1(\Omega) > 0$ for all Ω as above.
3. There exists a positive Jacobi function u in $\Sigma \setminus C$.

Proof. This proof is very similar to the one used by Fisher-Colbrie and Schoen for complete, noncompact minimal surfaces [21, 27]. We will show the proof here for the sake of completeness.

To show that (1) \Rightarrow (2) assume by contradiction that there is a bounded open set $\Omega \subset \Sigma \setminus C$ such that $\lambda_1(\Omega) = 0$. Let $\tilde{\Omega} \subset \Sigma \setminus C$ be a bounded open set strictly containing Ω , since $0 \leq \lambda_1(\tilde{\Omega}) \leq \lambda_1(\Omega) = 0$ we can conclude that $\lambda_1(\tilde{\Omega}) = 0$. Let ϕ be eigenfunction associated to $\lambda_1(\Omega)$ and let $H_0^1(\Omega)$ be the closure of $C_0^\infty(\Omega)$ under the Sobolev H^1 -norm. Since $|\phi| \in H_0^1(\Omega)$ is also a minimizer for the Rayleigh quotient of Q , we can assume that $\phi \geq 0$. Notice that the function $\tilde{\phi} : \tilde{\Omega} \rightarrow \mathbb{R}$ that extends

ϕ by zero outside Ω is in the space $H_0^1(\tilde{\Omega})$ and is also a minimizer for this Rayleigh quotient. So it is also a weak solution to the Jacobi equation (1.18). However, since $\tilde{\phi}$ is zero at some point it follows from the Harnack inequality (see the version in [16]) that $\tilde{\phi}$ vanishes everywhere, a contradiction.

Now, to show that (3) \Rightarrow (1) let us write $p = |A_\Sigma|^2 + \text{Ric}_M(\nu, \nu)$ and let $w = \log u$ where u is a positive Jacobi function. Then

$$\Delta w = -p - |\nabla w|^2,$$

and along the capillary boundary Γ :

$$\frac{\partial w}{\partial \eta} = q.$$

Let $f \in C_0^\infty(\Omega)$ where $\Omega \subset \Sigma \setminus C$ is open and bounded. Then

$$\begin{aligned} \int_\Sigma f^2 p + f^2 |\nabla w|^2 &= - \int_\Sigma f^2 \Delta w \\ &= 2 \int_\Sigma f \langle \nabla f, \nabla w \rangle - \int_\Gamma f^2 \frac{\partial w}{\partial \eta} \\ &\leq 2 \int_\Sigma |f| |\nabla f| |\nabla w| - \int_\Gamma f^2 q \\ &\leq \int_\Sigma f^2 |\nabla w| + |\nabla f|^2 - \int_\Gamma f^2 q. \end{aligned}$$

So $Q(f, f) \geq 0$.

Finally we show that (2) \Rightarrow (3). Note that a weak solution in $H_0^1(\Omega)$ to the

problem

$$\begin{cases} Jv = f_1 & \text{in } \Omega \\ \frac{\partial v}{\partial \eta} - qv = f_2 & \text{on } \Gamma \cap \Omega \\ v = 0 & \text{on } \partial\Omega \setminus \Gamma. \end{cases}$$

can be characterized as a function $v \in H_0^1(\Omega)$ such that for all $\varphi \in C_0^\infty(\Omega)$ we have:

$$Q(v, \varphi) = \int_{\Omega} f_1 \varphi + \int_{\Gamma \cap \Omega} f_2 \varphi.$$

The existence of weak solutions for any $f_1 \in L^2(\Omega)$, $f_2 \in L^2(\Gamma \cap \Omega)$ can be established through the Fredholm alternative using (2) and standard arguments (for example, by following the arguments in [25, Section 6.2.3]). In particular, there is a weak solution v to

$$\begin{cases} Jv = -\text{Ric}_M(\nu, \nu) - |A_\Sigma|^2 & \text{in } \Omega \\ \frac{\partial v}{\partial \eta} - qv = q & \text{on } \Gamma \cap \Omega \\ v = 0 & \text{on } \partial\Omega \setminus \Gamma. \end{cases}$$

So $u = v + 1$ is a Jacobi function in Ω . Furthermore, u must be positive, since if u is not positive, then $u|_{u < 0}$ is an eigenfunction of Q in the set $\{u < 0\}$ associated to the eigenvalue zero, contradiction the hypotheses.

Consider an exhaustion $\Omega_1 \subset \Omega_2 \subset \dots \subset \Sigma \setminus C$ of $\Sigma \setminus C$ by bounded open sets. Fix a point $x \in \Omega_1$, for each $n \in \mathbb{N}$ let u_n be a positive Jacobi function, by rescaling u_n we can assume that $u_n(x) = 1$. So it follows from the Harnack inequality [16] that for all compact sets $K \subset \Sigma \setminus C$ containing x there is a constant C_K such that, for all

n

$$\sup_{y \in K, n \in \mathbb{N}} u_n(x) \leq C_K.$$

Using the Schauder estimates of Agmon, Douglis and Nirenberg [2, Theorem 7.3] we can conclude that for all $\alpha \in (0, 1)$ there is a constant C'_K such that all $u_n|_K$ have $C^{2,\alpha}$ -Hölder norm bounded by C'_K . So, passing to a subsequence, we can assume that the u_n converge in $C^{2,\alpha/2}$ -Hölder norm to a positive Jacobi function in K . Using a standard diagonal argument we can construct a positive Jacobi function on $\Sigma \setminus C$. \square

1.3 Harmonic coordinates

Harmonic coordinates will be a very important tool in showing the curvature estimates (theorems 2.10 and 3.6) because they allow for control of the metric tensor in terms of curvature and injectivity radius. We will say that a Riemannian 3-manifold M , possibly with boundary, has *bounded geometry* if there are positive constants ι and Λ such that:

- (i) M has absolute sectional curvature bound $|K_M| \leq \Lambda$;
- (ii) the boundary of M has second fundamental form bound $|h_{\partial M}| \leq \Lambda$;
- (iii) every geodesic of M and every geodesic of ∂M with length at most ι is minimizing;
- (iv) there is a collar neighborhood U of ∂M in M and a function $f \in C^2(U)$ such that $f|_{\partial M} = 0$, $|\nabla f| = 1$ and $f(U) \supset [0, \iota)$.

More specifically, when (i)-(iv) hold we will say that M has *curvature bounded above by* Λ and *injectivity radius bounded below by* ι . In a local sense, these quantities control how far M is from looking like a part of \mathbb{R}^3 . This is formalized by the theorem below, which is essentially the same as a result by Anderson et al. [9, Theorem 3.2.1] (see also [43]). Here d_M is the intrinsic distance in M and $B_r^M(p)$ is the intrinsic ball in M of radius $r > 0$ centered at $p \in M$.

Theorem 1.5. *Suppose M is a Riemannian 3-manifold with curvature bounded above by Λ and injectivity radius bounded below by ι . Fix an $\alpha \in (0, 1)$, then there are constants $r_0 > 0$ and $Q_0 > 1$ depending only on Λ, ι and α such that for all $p \in M$ we have:*

(i) *If $d_M(p, \partial M) > r_0$, then there is a neighborhood U of $\vec{0}$ in \mathbb{R}^3 and a coordinate chart $\varphi : U \rightarrow B_{r_0}^M(p)$ such that $\varphi(\vec{0}) = p$, and, in these coordinates, the metric of M is bounded by*

$$Q_0^{-1}\delta_{ij} \leq g_{ij} \leq Q_0\delta_{ij} \tag{1.19}$$

as a quadratic form, where δ_{ij} represents the Euclidean norm. Additionally, the metric has Hölder bounds

$$\|g_{ij}\|_{C^{1,\alpha}} \leq Q_0. \tag{1.20}$$

(ii) *If $d_M(p, \partial M) \leq r_0$, there is a $b \leq 0$, a neighborhood U of $\vec{0}$ in $\{x \in \mathbb{R}^3 : x_1 \geq b\}$ and a coordinate chart $\varphi : U \rightarrow B_{2r_0}^M(p)$ such that $U \cap \{x_1 = b\}$ is mapped to ∂M , $\varphi(\vec{0}) = p$ and (1.19)–(1.20) still hold in these coordinates.*

We will often refer to the charts given by the theorem above as *harmonic coordinates*. Note that since the result above is local in nature, it also holds in the interior of any region where curvature and injectivity radius are bounded. In the case where M is complete, the conditions involving the boundary are ignored.

Chapter 2

On the Genus and Area of Constant Mean Curvature Surfaces

In this chapter we will present results published by the author in [46]. We will begin in Section 2.1 by contextualizing and introducing our main results. Then we will discuss the local parametrization of CMC surfaces as graphs in Section 2.2, introduce a blow-up technique in Section 2.4, before finally showing our main results in Section 2.7.

2.1 Introduction

As we established in Chapter 1, the (weak and strong) index is a natural variational quantity associated to CMC surfaces, hence it is expected to be controlled in CMC surfaces produced through variational methods. For example, it has been shown that

the index of minimal surfaces associated to the volume spectrum are controlled by the order of the associated spectrum element [34, 55], and analogous results are believed to hold for CMC surfaces produced through the Zhou-Zhu min-max procedure [34, 56].

On the other hand, the geometry and topology of surfaces produced through variational methods tend to be hard to control directly. With the objective of bridging this gap, we are interested in the relation between the index of CMC surfaces and classical geometric and topological quantities. This relation has been well explored for minimal surfaces and hypersurfaces (see, for example [6, 17, 18, 40, 47, 52]), although the following conjecture from Marques and Neves' 2014 ICM lectures is still open:

Conjecture 2.1 ([37]). *If the ambient manifold has positive Ricci curvature, then an index I embedded orientable compact minimal hypersurface has first Betti number bounded by a fixed multiple of I .*

The relation between the index and the geometry of CMC surfaces and hypersurfaces is well known in some special cases (e.g [3, 14, 44]). Here we will study the area and genus of CMC surfaces of bounded index by describing the degeneration of sequences of such surfaces in a similar way as Chodosh, Ketover and Maximo have done for minimal surfaces [17]. We will show that in any closed 3-manifold, the genus of a CMC surface is controlled by its index and area independently of the value of its mean curvature:

Theorem 2.2. *Let $I, A_0 \geq 0$ and suppose M is a closed Riemannian 3-manifold. There is a constant C depending only on M, I and A_0 such that any closed, connected*

CMC surface embedded in M with area at most A_0 and strong index at most I has genus at most C .

If M is spherical, that is, M is closed with finite fundamental group, we will show that every sequence of closed, connected, CMC surfaces embedded in M with an uniform lower mean curvature bound and an uniform upper index bound has a subsequence that does not accumulate away from a finite set or points. Using this fact we are able to show:

Theorem 2.3. *Let $I \geq 0$, $\eta > 0$, and suppose M is a spherical Riemannian 3-manifold. There are constants B and C depending only on M, I and η such that any closed, connected CMC surface embedded in M with strong index at most I and mean curvature at least η has area at most B and genus at most C .*

The compactness assumption in Theorem 2.3 is necessary [22], as is the lower bound on the mean curvature [19]. However it is not clear if the assumption on the fundamental group of M is essential.

Here we always assume the surfaces are connected, unless stated otherwise. Recall that we also assume that that CMC surfaces are one-sided, however it is also possible to have one-sided minimal surfaces. In this case, the proof of Theorem 2.2 is given in [17]. Now we will now present the main ideas in the proof of theorems 2.2 and 2.3.

2.1.1 Outline of the proofs of the main theorems

Theorems 2.2 and 2.3 are shown by studying the degeneration of sequences of closed CMC surfaces, possibly with varying mean curvature, with strong index at most I embedded in a closed 3-manifold M . Let $\{\Sigma_n\}_{n \in \mathbb{N}}$ be such a sequence, assume by contradiction that the genus of the Σ_n form a divergent sequence, and assume for simplicity that, passing to a subsequence, the Σ_n have uniformly bounded mean curvature (if this is not the case, an extra rescaling is needed).

By Theorem 2.10, we can pass the sequence $\{\Sigma_n\}$ to a subsequence with uniformly bounded second fundamental form away from a set of at most I points in M , called blow-up points. Near the blow-up points we follow the arguments of Chodosh, Ketover and Maximo [17] to show that, after rescaling the surfaces Σ_n , a subsequence must converge to a minimal surface in \mathbb{R}^3 with index at most I . This fact allows us to bound the genus and area of the surfaces Σ_n near the blow-up points.

Away from the blow-up points, the surfaces Σ_n must subconverge to a weak CMC lamination of M ². If this limit lamination is formed by properly embedded leafs, we can use these limit leafs to bound the genus of the surfaces Σ_n away from the blow-up points. In order to show that the limit lamination is formed by properly embedded leafs, we have to bound the number of sheets of any surface Σ_n that pass through a small region away from the blow-up points. At this part, the proofs of theorems 2.2 and 2.3 diverge.

²Even though we will not use this fact, it is interesting to note that by Theorem 2.10 and the Local Removable Singularity Theorem for CMC laminations [36, Theorem 1.2] we can choose a subsequence so this limit lamination extends to the blow-up points.

In the case of Theorem 2.2 we have uniform area bound for the surfaces Σ_n , so the bound of the number of sheets follows from a standard argument. On the other hand, to proof Theorem 2.3 we will to show that if M is spherical, the direction of the mean curvature vectors of the sheets associated to any surface Σ_n must alternate in a certain sense. This, together with an uniform lower bound on the mean curvature of these surfaces, will give uniform bounds on the maximum number of sheets of any of the surfaces Σ_n that can pass tough a small region away from the blow-up points.

2.2 Graph parametrization of CMC surfaces

In order to show the main results of this chapter, we need to understand CMC surfaces as graphs and their convergence. The results on this section are valid for CMC hypersurfaces of arbitrary dimension, however, to be consistent with the rest of the text, we will consider only consider the 2-dimensional case.

Let $U \subset \mathbb{R}^3$ be a open set and let suppose g is a metric on U such that (1.19) and (1.20) hold for some $\alpha \in (0, 1)$ and $Q_0 > 1$. Suppose Σ is a CMC surface in U which is parametrized by the graph of a function $u : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ near some point p . Let $\nu_u : D \rightarrow \mathbb{R}^3$ be a parametrization for the normal of the graph of u , we assume that ν_u points upward in the sense that its product with the third coordinate vector field

is positive. The mean curvature of Σ in the direction of ν_u is given by

$$\begin{aligned} \langle \nu_u, \vec{H}_\Sigma(\cdot, \cdot, u(\cdot, \cdot)) \rangle &= a^{ij} \left(\frac{\partial u}{\partial x_i \partial x_j} + \Gamma_{ij}^3 + \frac{\partial u}{\partial x_i} \Gamma_{3j}^3 + \frac{\partial u}{\partial x_j} \Gamma_{i3}^3 + \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \Gamma_{33}^3 \right) \\ &\quad - \sum_{m=1}^2 \frac{\partial u}{\partial x_m} h^{ij} \left(\Gamma_{ij}^m + \frac{\partial u}{\partial x_i} \Gamma_{3j}^m + \frac{\partial u}{\partial x_j} \Gamma_{i3}^m + \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \Gamma_{33}^m \right) \end{aligned} \quad (2.1)$$

where the Γ_{ij}^k are the Christoffel symbols associated to g and

$$a^{ij} = g^{ij} - \left(1 + g^{nm} \frac{\partial u}{\partial x_n} \frac{\partial u}{\partial x_m} \right)^{-1} g^{ik} g^{j\ell} \frac{\partial u}{\partial x_k} \frac{\partial u}{\partial x_\ell}$$

(for details see [20, Ch. 5])

Note that the quantity of the left hand side of (2.1) is constant and equal to $\pm H_\Sigma$, depending on the direction of the mean curvature vector \vec{H}_Σ . Then, assuming that $|\nabla u| < C$ for some constant C , we have that the ellipticity of a^{ij} is controlled by C and Q_0 . Using this fact and the $C^{2,\alpha}$ bounds on the metric g we can apply the Schauder estimates [30, Thm 6.2] to find $C^{2,\alpha}$ bounds for u in the interior of D in terms of C, Q_0 and the maximum of $|u|$. This fact is central to the proof of lemma below.

Lemma 2.4 (Uniform Graph Lemma). *Let $U \subset \mathbb{R}^3$ be an open set and consider a metric g in U such that (1.19) and (1.20) hold for some $\alpha \in (0, 1)$ and $Q_0 > 1$. Let $\Sigma \subset U$ be a properly embedded CMC surface and suppose the second fundamental form of Σ is bounded above by a constant C . Then there are $\epsilon > 0$, $\rho > \epsilon$ and $C' > 0$ depending only on Q_0, C and α , such that for every x at euclidean distance at least ρ from ∂U there is a rotation $R \in O(3)$ such that:*

1. Every connected component of $R(\Sigma \cap B_\epsilon^{\mathbb{R}^3}(x))$ is part of the graph of a function u over the disk of euclidean radius ρ .
2. For all such functions u we have

$$\|u\|_{C^{2,\alpha}} \leq C'.$$

Proof. We will assume without loss of generality that $x = \vec{0}$. From (1.19) and (1.20) we can conclude that there is a $K > 0$ depending only on Q_0 and C such that the norm of the euclidean second fundamental form of Σ is at most K . Fix an $\epsilon > 0$ and let $\Sigma' = \Sigma \cap B_\epsilon^{\mathbb{R}^3}(\vec{0})$. In the following, $G : \Sigma \rightarrow \mathbb{R}\mathbb{P}^2$ will be the euclidean unoriented Gauss map of Σ .

We first claim that there is an $\epsilon > 0$ depending only on C and Q_0 such that

$$\sup_{p,q \in \Sigma'} d_{\mathbb{R}\mathbb{P}^2}(G(p), G(q)) < \frac{\pi}{4}.$$

For p and q in the same connected component of Σ' , the existence of ϵ follows immediately from the bounds on the second fundamental form. For p, q in distinct connected components, we can use the bound on the second form to show that if ϵ is sufficiently small and the distance between $G(p)$ and $G(q)$ is more than $\pi/4$, then the connect components of Σ containing p and q must cross.

Choose a rotation R so that the tangent planes to $R(\Sigma')$ are bounded away from any vertical plane by a distance of at least $\pi/6$. It follows that there is a constant $\rho > \epsilon$ depending only on K and ϵ such that the connected components of

$R(\Sigma) \cap B_\rho^{\mathbb{R}^2}(\vec{0}) \times \mathbb{R}$ intersecting $B_\epsilon^{\mathbb{R}^3}(\vec{0})$ are graphs of a functions over $B_\rho^{\mathbb{R}^2}(\vec{0})$. The bounds on the euclidean second form and on the distance from the tangent planes to vertical planes give bounds on the C^2 norms for these functions. Making ρ smaller and applying interior Schauder estimates (see e.g. [30, Theorem 6.2]) to the mean curvature equation (2.1) we can obtain $C^{2,\alpha}$ bounds for these functions. \square

Remark 2.5. *Lemma 2.4 can be easily extended for general manifolds with bounded geometry by using harmonic coordinates. In this case ϵ, ρ and C' depend only on C , the sectional curvature bound and the injectivity radius bound of the ambient manifold.*

As we have previously noted, the quantity in (2.1) can have different signs depending on the direction of the mean curvature vector. We will show that if two graphs parametrizing distinct parts of a CMC surface have mean curvature vectors pointing in distinct directions, then these two graphs cannot be close together in the C^0 metric.

Proposition 2.6. *Consider $\Sigma, U, g, \alpha, Q_0$ and C as in the statement of Lemma 2.4. Let $\rho > 0$ and write $D(\rho)$ for the disk around the origin in \mathbb{R}^2 with radius ρ . Suppose the graphs of $u, v : D(\rho) \rightarrow \mathbb{R}$ parametrize parts of Σ and let $h_u, h_v : D(\rho) \rightarrow \mathbb{R}$ be the mean curvature of the graphs of u and v respectively in the direction of the upwards pointing normal vector field. Assume that $H_\Sigma > \eta$ for some $\eta > 0$ and*

1. h_u and h_v have distinct signs;
2. $u \leq v$ point-wise;

3. $\|u\|_{C^{2,\alpha}(D(\rho))}, \|v\|_{C^{2,\alpha}(D(\rho))} \leq C'$ for some $C' > 0$.

Then there is a $c > 0$ depending only on Q_0, α, ρ, η and C such that

$$\|u - v\|_{C^0(D(\rho))} > c.$$

Proof. Since u and v have $C^{2,\alpha}$ norm bounded above by C' , we can conclude that for all $\epsilon > 0$ there is a $\delta > 0$ depending only on C', α, ϵ and ρ such that

$$\|u - v\|_{C^2(D(\rho/2))} < \epsilon$$

whenever $\|u - v\|_{C^0(D(\rho))} < \delta$. So if u and v can be made arbitrarily close in $D(\rho)$, their mean curvatures can be made to be arbitrarily close in $D(\rho/2)$.

On the other hand, we can conclude from the mean curvature equation (2.1) that there is a constant C'' depending only on Q_0, α, ρ and C' such that

$$2\eta = \|h_u - h_v\|_{C^0(D(\rho/2))} \leq C'' \|u - v\|_{C^2(D(\rho/2))}$$

whenever u and v are close enough in the C^2 metric. It then follows that u and v cannot be made arbitrarily close in the C^0 metric. \square

Finally, we use the result above to show that, if the direction of the mean curvarvature vector alternates between graphs, then it is possible to control the number of graphs passing tough a small region of space.

Proposition 2.7. *Consider $\Sigma, U, g, \alpha, Q_0$ and C as in the statement of Lemma 2.4.*

Let $\rho > 0$ and suppose $u_1, \dots, u_n : D(\rho) \rightarrow \mathbb{R}$ parametrize parts of Σ . For $k =$

$1, \dots, n$ let $h_{u_k} : D(\rho) \rightarrow \mathbb{R}$ be the mean curvature of the graph of u_k in the direction of the upwards points normal vector field. Assume that there is $\eta > 0$ such that $H_\Sigma > \eta$ and

1. h_{u_k} and $h_{u_{k+1}}$ have distinct signs for all $k = 1, \dots, n-1$;
2. $u_1 < u_2 < \dots < u_n$ point-wise;
3. $\|u_k\|_{C^{2,\alpha}(D(\rho))} \leq C'$ for some $C' > 0$ and all $k = 1, \dots, n$;
4. $\inf_{D(\rho/2)} |u_k| < \epsilon$ for some $\epsilon > 0$ and all $k = 1, \dots, n$.

Then there is a number N depending only on Q_0, α, C', ρ and ϵ such that $n < N$.

Proof. Let $k \in \{1, \dots, n-2\}$, the difference $w_{k,k+2} = u_{k+2} - u_k$ follows an elliptic equation with ellipticity and coefficients controlled by the C^1 norms of the metric, u_{k+2} and u_k (see [20, Chapter 5; 21, Chapter 7]). It follows that the functions $w_{k,k+2}$ obey a Harnack inequality in $D(\rho/2)$ with a constant C'' that only depends on Q_0, C' and ρ [30, Theorem 8.20]. Using the $C^{2,\alpha}$ bounds for the functions u_k , we conclude that there must be some $k \in \{1, \dots, n-2\}$ such that

$$w_{k,k+2}(\vec{0}) \leq \frac{2C'}{n}.$$

So it follows from the Harnack inequality that

$$\|w_{k,k+2}\|_{C^0(D(\rho/2))} < \frac{2C'C''}{n}.$$

So we must have $n < 2C'C''c^{-1}$ where c is as in Proposition 2.6. □

2.3 CMC laminations

A *lamination* of a 3-manifold M is the union of a collection \mathcal{L} of smooth, pairwise disjoint embedded surfaces called Leaves such that $\bigcup_{L \in \mathcal{L}} L$ is closed. Furthermore, we assume that \mathcal{L} has a local product structure in the sense that for every $p \in M$ there is a neighborhood U of p in M and a continuous coordinate chart $\Psi : U \rightarrow \mathbb{R}^3$ such that each leaf $L \in \mathcal{L}$ passing through U is the union of sheets of the form

$$\Psi(U) \cap \{t\}$$

where $t \in \mathbb{R}$. A *CMC lamination* is a lamination where each leaf is a CMC surface, and an *H-lamination* is a CMC lamination where each leaf has mean curvature H . A *minimal lamination* is a laminations where the leafs are minimal.

We are also interested in the case where leafs can touch at isolated points, since these cases appear naturally in the limit of CMC laminations. In this case, the lamination is called a *weak CMC lamination*, the definition of a weak CMC lamination is similar to that of a CMC lamination, except that instead of requiring the leafs to be pairwise disjoint we require that near any point of contact between two leafs or between a leaf and itself, the surfaces lie at one side of the other. And instead of assuming that there is a coordinate chart that rectifies the lamination, we assume that second fundamental form of the leafs is uniformly bounded on compact sets of M (see e.g. [36]). A weak *H-lamination* is a weak CMC lamination where each leaf has mean curvature H . Note that it follows from the maximum principle for CMC

surfaces that if a point belongs to two leaves of a weak H -lamination, then the mean curvature vector of the two leaves at p must point in opposite directions.

We say that a sequence of CMC surfaces $\{\Sigma_n\}_{n \in \mathbb{N}}$ properly embedded in M converges to a weak CMC lamination \mathcal{L}_∞ if the union of the leaf of \mathcal{L}_∞ is the limit set of the sequence of surfaces, that is, if every convergent sequence $\{p_n\}_{n \in \mathbb{N}}$ where $p_n \in \Sigma_n$ converges to a point in a leaf of \mathcal{L}_∞ and if every point in a leaf of \mathcal{L}_∞ can be obtained by taking such a limit. We say that the convergence occurs with multiplicity at most m if, for all $L \in \mathcal{L}_\infty$ and all $p \in L$ there is a neighborhood U of p in M such that, in U , L is parametrized as a graph and for all n large enough Σ_n is parametrized as a union of at most m graphs. If there is such a m , we say that convergence occurs with finite multiplicity.

Theorem 2.8. *Let M be a complete Riemannian 3-manifold with bounded geometry. Consider a sequence of CMC surfaces $\{\Sigma_n\}_{n \in \mathbb{N}}$ properly embedded in M . Suppose there is a constant $C > 0$ and an open set $\Omega \subset M$ such that for all $n \in \mathbb{N}$ and $x \in \Sigma_n \cap \Omega$ we have that $|A_{\Sigma_n}(x)| < C$. Then there is a constant $H \geq 0$ and a weak H -lamination \mathcal{L}_∞ of Ω such that a subsequence $\{\Sigma_{n_k}\}_{k \in \mathbb{N}}$ converges to \mathcal{L}_∞ in Ω .*

Proof. We can assume there are points $p_n \in \Sigma_n \cap \Omega$ such that, passing to a subsequence, they converge to some point $p \in \Omega$ (else the set of limit points is empty in Ω , so the intersection of Ω with the leaves of \mathcal{L}_∞ must be empty). Passing to a subsequence we can assume that the tangent planes of the surfaces Σ_n at the points p_n must converge to some plane at p . Fix a parametrization φ of M near p

with harmonic coordinates as described in Theorem 1.5.

Using Lemma 2.4 we have that that, after possibly making U smaller, for every $\alpha \in (0, 1)$ there is $\rho > 0$ depending only on ambient curvature bound Λ , the ambient injectivity radius bound ι , C and α such that, for n large enough, $\varphi^{-1}(\Sigma_n)$ is parametrized by graphs of functions $u_{n,1}, \dots, u_{n,K_n}$ over some disk D_ρ with $C^{2,\alpha}$ bounds that do not depend on n .

First we assume that for all $n \in \mathbb{N}$ the mean curvature vectors of all $u_{n,1}, \dots, u_{n,K_n}$ point upwards, later we will deal with the more general case. Set $M_{n,k} = u_{n,k}(\vec{0})$ where $k = 1, \dots, K_n$, then we define a coordinate chart $\Psi_n : U \rightarrow \mathbb{R}^3$ by letting $\Psi_n^{-1}(x_1, x_2, x_3) = (x_1, x_2, f_n(x_1, x_2))$ where

$$f_n(x_1, x_2, x_3) = u_{n,k}(x_1, x_2) + \frac{x_3 - M_{n,k}}{M_{n,k+1} - M_{n,k}}(u_{n,k+1}(x_1, x_2) - u_{n,k}(x_1, x_2))$$

for $(x_1, x_2, x_3) \in U \cap \mathbb{R}^2 \times [M_{n,k}, M_{n,k+1}]$. Using similar arguments to the one used in Proposition 2.7, we can apply the Harnack inequality to bound the oscillation of $u_{n,k+1}(x_1, x_2) - u_{n,k}(x_1, x_2)$ and the Schauder inequality to bound the gradient of the $u_{n,k}$. Using that

$$\begin{aligned} \nabla f_n(x_1, x_2, x_3) &= \nabla u_{n,k}(x_1, x_2) + \frac{x_3 - M_{n,k}}{M_{n,k+1} - M_{n,k}} \nabla(u_{n,k+1}(x_1, x_2) - u_{n,k}(x_1, x_2)) \\ &\quad + \frac{u_{n,k+1}(x_1, x_2) - u_{n,k}(x_1, x_2)}{M_{n,k+1} - M_{n,k}} \frac{\partial}{\partial x_3} \end{aligned}$$

and the mentioned above, it is possible to obtain a bi-Lipschitz bound for Ψ_n in a region of U that does not depend on n (see e.g. [20, Prop. B1]). Then, making U smaller and passing to a subsequence we can conclude that the Ψ_n converge in C^α to

some Lipschitz map $\Psi : U \rightarrow \mathbb{R}^3$. The leaves of \mathcal{L}_∞ in U are the accumulation set of the sheets $\Psi_n(\varphi^{-1}(\Sigma_n))$. Note that, using the Schauder estimates for the function $u_{n,k}$ we can conclude that L_∞ is a H -lamination of U where $H = \lim_{n \rightarrow \infty} H_{\Sigma_n}$.

In case there are graphs in U with mean curvature vectors pointing upwards and downwards we can follow a similar procedure as above, first only considering the graphs pointing in one direction and then only considering the graphs pointing the other other direction. This means that we get two maps rectifying the \mathcal{L}_∞ , one for each direction. In this case, leaves with mean curvature vectors pointing in distinct directions can touch, for this reason we can only guarantee a weak lamination. Finally, we can extend the lamination to all of Ω by a standard continuity procedure followed by taking a diagonal sequence. □

Remark 2.9. *In the case where we can bound the number of graphs parametrizing the Σ_n in small neighborhoods, the limit weak lamination must be a finite collection of properly embedded CMC surfaces.*

2.4 Curvature bounds for CMC surfaces with finite index

In 1983, Schoen [49] proved that the second fundamental form of a strongly stable two-sided minimal surface immersed in a 3-manifold is bounded above by a constant multiplying the reciprocal of the distance to the boundary, and this constant only

depends on the curvature of the ambient manifold and its covariant derivative. In particular, he proved that

$$|A_\Sigma(p)| \leq \frac{C}{d_\Sigma(p, \partial\Sigma)} \quad \forall p \in \Sigma, \quad (2.2)$$

where Σ is a strongly stable two-sided minimal surface immersed in \mathbb{R}^3 and $C > 0$ is a constant independent of Σ . This curvature estimate gives a different proof to the fact that a two-sided stable complete minimal surface in \mathbb{R}^3 must be a plane.

Schoen's curvature estimates have many generalizations, those more relevant to our work are the generalization to strongly stable free boundary minimal surfaces by Guang, Li and Zhou [31], generalization to strongly stable immersed CMC surfaces by Rosenberg, Souam and Toubiana [43] and the generalization of embedded minimal surfaces of finite index by Chodosh, Ketover and Maximo [17].

An estimate like (2.2) cannot hold for strongly stable CMC surfaces in general 3-manifolds. For instance, horospheres are strongly stable complete CMC surfaces in hyperbolic 3-space but are not totally geodesic. However, Rosenberg-Souam-Toubiana [43] showed that far away from its boundary, the second fundamental form of a strongly stable immersed CMC surface is bounded above by an universal constant C that does not depend on the ambient space or the surface divided by the square root of an absolute sectional curvature bound of the ambient space. More precisely, they obtained the following curvature estimate: There is a constant C such that for any complete 3-manifold M with sectional curvature bounds $|K_M| < \Lambda$, and for any

strongly stable two-sided CMC surface Σ immersed in M

$$|A_\Sigma(p)| \leq \frac{C}{\min\{d_\Sigma(p, \partial\Sigma), (\sqrt{\Lambda})^{-1}\}} \quad \forall p \in \Sigma. \quad (2.3)$$

In order to show this inequality they use a blow-up procedure with harmonic coordinates. Using harmonic coordinates usually requires a lower injectivity radius bound, in order to avoid any assumptions on injectivity radius they first pull a neighborhood of the ambient manifold back to its tangent space through the exponential map, where there are lower injectivity radius bounds that depend only on Λ .

We will show a version of this inequality for CMC surfaces of finite index drawing from ideas of [43] and [17]. In this chapter we will only use this result in the case where the surfaces are embedded, however we will prove the result for the case of immersed surfaces since we will need to use this case for Theorem 3.6.

Theorem 2.10. *For all $I \geq 0$ there is a constant C depending only on I such that the following holds: Let M be a complete Riemannian 3-manifold with curvature bounded above by Λ and injectivity radius bounded below by $\iota > 0$. Suppose Σ is a CMC surface immersed in M with strong index at most I . Then there is a set $\mathcal{B} \subset \Sigma$ with at most I points such that*

$$|A_\Sigma(p)| \min \left\{ d_\Sigma(p, \partial\Sigma \cup \mathcal{B}), \iota, (\sqrt{\Lambda})^{-1} \right\} \leq C \quad (2.4)$$

for all $p \in \Sigma$.

This Theorem will be shown later on in this section. It is possible to improve the

result above so it does not depend on the injectivity radius of the ambient manifold (see [46]). But since we will not need this fact, we opt to show the inequality with the injectivity radius factor since it simplifies the proof and makes it more similar to Theorem 3.6.

One consequence of the fact that the constant C in Theorem 2.3 does not depend on the mean curvature of Σ is that we can use this result to bound the diameter of CMC surfaces with large mean curvature.

Corollary 2.11. *Let C be the constant given by Theorem 2.10 in the case $I = 0$. Let Σ be a CMC surface with finite index immersed in M and assume $H_\Sigma \geq \sqrt{2}C \max\{\iota^{-1}, \sqrt{\Lambda}\}$. Then,*

- *in case Σ has nonempty boundary,*

$$d_\Sigma(p, \partial\Sigma) \leq (\text{Index}_s(\Sigma) + 1) \frac{2\sqrt{2}C}{H_\Sigma} \quad (2.5)$$

for all $p \in \Sigma$; and

- *in case Σ is closed*

$$\text{diam}_\Sigma(\Sigma) \leq \text{Index}_s(\Sigma) \frac{2\sqrt{2}C}{H_\Sigma}, \quad (2.6)$$

where $\text{diam}_\Sigma(\Sigma)$ is the intrinsic diameter of Σ .

Proof. We will only show (2.5) since the proof of (2.6) is very similar. Let $I = \text{Index}_s(\Sigma)$, first we consider the case where $I = 0$. Since $|A_\Sigma| \geq H_\Sigma/\sqrt{2}$, it follows from Theorem 2.10 that

$$H_\Sigma d_\Sigma(p, \partial\Sigma) \leq \sqrt{2}C$$

for all $p \in \Sigma$.

Now consider the case when $I > 0$. Let $p \in \Sigma$ and take $\gamma : [0, L] \rightarrow \Sigma$ to be an unit speed geodesic minimizing the distance from p to $\partial\Sigma$. Assume by contradiction that γ has length $(I + 1)(2\sqrt{2}CH_\Sigma^{-1} + \epsilon)$ for some $\epsilon > 0$. Fix numbers $0 = t_0 < t_1 < \dots < t_{I+1} = L$ such that $t_{i+1} - t_i \geq 2\sqrt{2}C/H_\Sigma + \epsilon$ for $i = 0, \dots, I$. Let \mathcal{D}_i be the geodesic disks around $\gamma(t_i)$ with radius $\sqrt{2}C/H_\Sigma + \epsilon/2$. Note that, by construction, these disks are pair-wise disjoint and they are disjoint from the edge of Σ for $i = 0, \dots, I$. The case $I = 0$ implies that these disks cannot be strongly stable. So we conclude from Proposition 1.2 that Σ must have strong index at least $I + 1$, a contradiction. \square

Before showing Theorem 2.10 we will define what we mean by a *sequence of rescaled surfaces*. Suppose $\{M_n\}_{n \in \mathbb{N}}$ is a sequence of 3-manifolds with uniform sectional curvature bounds $|K_n| \leq \Lambda$ and injectivity radius bounds $\iota_n \geq \iota > 0$. For each $n \in \mathbb{N}$, let Σ_n be an H_n -surface (possibly with boundary) immersed in M_n . Take a sequence of points $p_n \in \Sigma_n$. By abuse of notation, denote image of p_n under the immersion of Σ_n in M_n by p_n .

Fix an $\alpha \in (0, 1)$ and let r_0, Q_0 be as in Theorem 1.5. For every $n \in \mathbb{N}$, take an open set $U_n \subset \mathbb{R}^3$ containing $\vec{0}$ and harmonic coordinates $\varphi_n : U_n \rightarrow B_{r_0}^{M_n}(p_n)$. We define $\Sigma_n - p_n$ to be the immersed surface in U_n obtained by pulling back the connected component of Σ_n in $B_{r_0}^{M_n}(p_n)$ containing p_n .

Take a sequence of positive numbers $\{\sigma_n\}_{n \in \mathbb{N}}$ with $\sigma_n \rightarrow \infty$. Let $\mu_\sigma : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the dilation $x \mapsto \sigma^{-1}x$. Define $\tilde{g}_n = (\varphi_n \circ \mu_{\sigma_n})^*g_n$ where g_n is the metric in M_n .

Then we can also dilate $\Sigma_n - p_n$ to get a surface immersed in $\sigma_n U_n$, which we will denote by $\sigma_n(\Sigma_n - p_n)$. We refer to any sequence of surfaces constructed through this procedure as a sequence of rescaled surfaces.

Note that in the \tilde{g}_n metric, $\sigma_n(\Sigma_n - p_n)$ is a $\frac{H_n}{\sigma_n}$ -surface. Furthermore, it follows from (1.19) and (1.20) that $(\sigma_n U_n, \tilde{g}_n) \rightarrow (\mathbb{R}^3, \delta_{ij})$ locally in $C^{1,\alpha}$ as $n \rightarrow \infty$ (for a proof, see [43]).

Proof of Theorem 2.10

We will use induction on I . The case $I = 0$ has been done by Rosenberg, Soam and Toubiana [43], we will present the arguments here for the sake of completeness. Assume by contradiction that there is a sequence of complete 3-manifolds with bounded geometry M_n and immersed, strongly stable H_n -surfaces Σ_n immersed in M_n with points $q_n \in \Sigma_n$ such that

$$|A_n(q_n)| \min\{d_n(q_n, \partial\Sigma_n), \iota_n, (\sqrt{\Lambda_n})^{-1}\} > n$$

where

- d_n and A_n are the intrinsic distance and the second fundamental form on Σ_n , respectively;
- Λ_n is an upper bound on the absolute value of the sectional curvature of M_n ;
- ι_n is a lower bound on the injectivity radius of M_n .

Since the quantity on the left hand side of (2.4) is scale-invariant, we can rescale the metrics of the M_n we can assume that $\Lambda_n = \iota_n = 1$. Fix an $\alpha \in (0, 1)$ and

consider the values r_0, Q_0 given by Theorem Theorem 1.5 with $\iota = \Lambda = 1$, we can assume without loss of generality that $r_0 < 1$. Note that

$$|A_n(q_n)| \min\{d_n(q_n, \partial\Sigma_n), r_0\} \geq r_0 |A_n(q_n)| \min\{d_n(q_n, \partial\Sigma_n), 1\} > r_0 n.$$

Let \mathcal{D}_n be the intrinsic disk in Σ_n around q_n with radius $\min\{d_n(q_n, \partial\Sigma_n), r_0\}$. Take p_n to be the point in \mathcal{D}_n maximizing

$$|A_n(p)| d_{\mathcal{D}_n}(p, \partial\mathcal{D}_n).$$

Let $\lambda_n = |A_n(p_n)|$ and $R_n = d_{\mathcal{D}_n}(p_n, \partial\mathcal{D}_n)$. By construction we have that $\lambda_n R_n > r_0 n$ and $\lambda_n > n$ because $R_n \leq r_0$.

Let $\tilde{\mathcal{D}}_n$ be the intrinsic disk in \mathcal{D}_n centered at p_n with radius $R_n/2$. Note that by construction

$$d_{\mathcal{D}_n}(p, \partial\mathcal{D}_n) \geq \frac{R_n}{2}$$

for all $p \in \tilde{\mathcal{D}}_n$. It follows that for all such p we have

$$|A_n(p)| \leq \frac{\lambda_n R_n}{d_{\mathcal{D}_n}(p, \partial\mathcal{D}_n)} \leq 2\lambda_n.$$

Consider the sequence of rescaled surfaces

$$\lambda_n(\tilde{\mathcal{D}}_n - p_n).$$

immersed in $\lambda_n U_n$ where $U_n \subset \mathbb{R}^3$ is as described above. Since $\lambda_n U_n$ equipped with the pull-back metric converges locally in $C^{1,\alpha}$ to the euclidean 3-space, it follows that for any $m \in \mathbb{N}$ there is a $n \in \mathbb{N}$ such that the euclidean ball of radius m centered at

the origin, which we write as $B_m(\vec{0})$, is contained in $\lambda_n U_n$. Following [43], we define $\Delta_{n,m}$ to be the component of $\lambda_n(\tilde{\mathcal{D}}_n - p_n) \cap B_m(\vec{0})$ containing the point associated with $p_n \in \Sigma_n$, which we will denote by $0_n \in \lambda_n(\tilde{\mathcal{D}}_n - p_n)$. Since $\lambda_n R_n > r_0 n$, the intrinsic distance between 0_n and the boundary of $\lambda_n(\tilde{\mathcal{D}}_n - p_n)$ is at least $r_0 n/2$, and since we are scaling the immersion by λ_n , the norm of the second fundamental form of $\lambda_n(\tilde{\mathcal{D}}_n - p_n)$ in the pull-back metric is bounded above by 2 and equals 1 at 0_n . Hence, by making $n = n_m$ large enough in relation to m , we can assume that the boundary of $\lambda_n(\tilde{\mathcal{D}}_n - p_n)$ is disjoint from $B_m(\vec{0})$ and that $\Delta_{n_m,m}$ has euclidean second fundamental form with norm bounded above by 5 everywhere and bounded below by $1/2$ at 0_n .

Let $\Delta_n = \Delta_{n_m,m}$, up to passing to a subsequence we can assume that the tangent planes $T_{0_n} \Delta_n$ converge. Rotate the ambient space so this limit plane is $\{x_3 = 0\}$. It follows from standard arguments (see Lemma 2.4, [43, Proposition 2.3], [35, Lemma 4.35]) that there are positive constants C and δ such that a part of Δ_n is the graph of a function u_n over the disk $D_\delta = \{x \in B_\delta(\vec{0}) : x_3 = 0\}$ and

$$|\nabla u_n| < 1, \|u_n\|_{C^2(D_\delta)} < C.$$

Applying the Schauder estimates to the mean curvature equation (2.1) (see Lemma 2.4, [43, Lemma 2.4]) it is easy to see that there is a $\delta' \in (0, \delta)$ such that, after possibly making C larger

$$\|u_n\|_{C^{2,\alpha}(D_{\delta'})} < C.$$

Passing to a subsequence we can assume there is a function $u : D_\delta \rightarrow \mathbb{R}$ such that $u_n \rightarrow u$ in $C^2(D_\delta)$ and $C^{2,\alpha/2}(D_{\delta'})$. Fix a $\delta'' \in (\delta', \delta)$ and let $x_0 \in D_{\delta'}$ be at euclidean distance of $\delta'/2$ from the origin. By repeating the arguments above we have that, for n large, the surfaces Δ_n are parametrized by graphs over the disk of radius δ'' in the tangent plane of the graph of u at $(x_0, u(x_0))$. Passing to a subsequence, graphs must converge in the $C^{2,\alpha/2}$ topology, extending the graph of u to a larger surface. Continuing this construction and using a diagonal argument we encounter a complete noncompact CMC surface S immersed in euclidean 3-space with second fundamental form bounded above by 5 everywhere and passing through the origin with nonzero second fundamental form. Now we will show that S is strongly stable by showing it accepts a positive Jacobi function. It follows that S must be a plane [23], which contradicts the fact that S passes through the origin with nonzero second fundamental form.

Let \tilde{S} be the universal cover of S . We will show that \tilde{S} accepts a positive Jacobi function. It follows from the construction used in Proposition 1.4 that we only have to show that every bounded open set $\tilde{U} \subset \tilde{S}$ accepts a positive Jacobi function. Let $\Pi : \tilde{S} \times (-\epsilon, \epsilon)$ be the immersion $\Pi(p, t) = \tilde{f}(p) + t\tilde{\nu}(p)$ where \tilde{f} is the immersion of \tilde{S} in \mathbb{R}^3 and $\tilde{\nu}$ is a choice of normal in \tilde{S} . Let U be the image of \tilde{U} in S . By construction, for n large, a piece of Δ_n will be close enough to U everywhere so that this piece is parametrized by a function $\Pi(\cdot, \tilde{u}_n(\cdot))$ where $\tilde{u}_n : \tilde{U} \rightarrow (-\epsilon, \epsilon)$. Since any part of a strongly stable CMC surface is also strongly stable, it follows that Δ_n is

strongly stable, and hence it accepts a positive Jacobi function v_n . Let $\tilde{v}_n : \tilde{S} \rightarrow \mathbb{R}^+$ be such that $\tilde{v}_n(\cdot) = v_n(\Pi(\cdot, \tilde{u}_n(\cdot)))$, through an analogous construction, the Jacobi operator on Δ_n defines an elliptic operator on \tilde{U} . Since the metrics are converging in $C^{1,\alpha}$ to the Euclidean metric and $\tilde{u}_n \rightarrow 0$ in $C^{2,\alpha/2}$, we can conclude that these operators converge to the Jacobi operator on \tilde{U} and that the functions \tilde{v}_n converge to a positive Jacobi function in \tilde{U} . This concludes the base case.

Now assume that the statement holds for all $I \leq k - 1$ where $k \in \mathbb{N}$. Let Σ_n be sequence of CMC surfaces with strong index k immersed in manifolds with bounded geometry. We will use the same construction and the same notation used in the base case. Consider the surfaces Δ_n , it follows that there must be a $\rho > 0$ such that $\Delta_n \cap B_\rho(\vec{0})$ is unstable for all n large enough, since else we can follow the same steps from the base case to derive a contradiction. This implies that there is a $\rho' > 0$ such that, for all n large, the disk $B_{(\rho'/\lambda_n)}^{\Sigma_n}(p_n)$ is unstable. So $\hat{\Sigma}_n = \Sigma_n \setminus B_{(\rho'/\lambda_n)}^{\Sigma_n}(p_n)$ has strong index at most $k - 1$, then there must be a set $\hat{\mathcal{B}}_n \subset \hat{\Sigma}_n$ making (2.4) hold. We will show that we can take $\mathcal{B}_n = \hat{\mathcal{B}}_n \cup \{p_n\}$.

Assume by contradiction that there are points $z_n \in \Sigma_n$ such that

$$|A_n(z_n)| \min\{d_n(z_n, \partial\Sigma_n \cup \mathcal{B}_n), 1\} \rightarrow \infty$$

We will consider two cases. First assume

$$\liminf_{n \rightarrow \infty} \lambda_n d_n(z_n, p_n) < \infty.$$

Passing to a subsequence we have

$$\frac{d_n(z_n, p_n)}{R_n} = \frac{\lambda_n d_n(z_n, p_n)}{\lambda_n R_n} \rightarrow 0,$$

so from n large enough we can deduce that $z_n \in \tilde{\mathcal{D}}_n$, and hence $|A_n(z_n)| \leq 2\lambda_n$ so we conclude

$$\lim_{n \rightarrow \infty} |A_n(z_n)| d_n(z_n, p_n) \leq \lim_{n \rightarrow \infty} 2\lambda_n d_n(z_n, p_n) < \infty,$$

which contradicts the choice of z_n .

Now assume

$$\liminf_{n \rightarrow \infty} \lambda_n d_n(z_n, p_n) = \infty.$$

It follows that for n large enough $d_n(z_n, p_n) > \rho'/\lambda_n$, hence $z_n \in \hat{\Sigma}_n$. Since $d_n(z_n, \partial B_{\rho'/\lambda_n}^{\Sigma_n}(p_n)) = d_n(z_n, p_n) - \rho'/\lambda_n$ are positive and bounded below for n large enough, we must have

$$|A_n(z_n)| \min\{d_n(z_n, \partial \hat{\Sigma}_n \cup \hat{\mathcal{B}}_n), 1\} \rightarrow \infty,$$

contradicting the inductive hypothesis.

Remark 2.12. *As we have observed, Theorem 2.10 also holds without any injectivity radius assumptions. The proof in this case is very similar, except that in this case, before constructing the sequence of rescaled surfaces one must first pull back the surface to the tangent space of the ambient manifold. For more details see [43, 46].*

2.5 Picture near blow-up points

In this section we will consider sequences of CMC surfaces $\{\Sigma_n\}_{n \in \mathbb{N}}$ embedded in a closed 3-manifold M . Following Chodosh, Ketover and Maximo [17] we say that a sequence of finite sets of points $\mathcal{B}_n \subset \Sigma_n$ is a *sequence of blow-up sets* if:

1. The curvature blows up at the points in \mathcal{B}_n , that is, taking $\lambda_n(p) = |A_n(p)|$ we have

$$\liminf_{n \rightarrow \infty} \min_{p \in \mathcal{B}_n} \lambda_n(p) \rightarrow \infty.$$

2. Taking any sequence of points $\{p_n \in \mathcal{B}_n\}_{n \in \mathbb{N}}$ we can pass to a subsequence so

$$\lambda_n(p_n)(\Sigma_n - p_n) \rightarrow \mathcal{L}_\infty,$$

where \mathcal{L}_∞ is a weak CMC lamination of euclidean 3-space with $\|A_{\mathcal{L}_\infty}\|_{C^0} \leq 5$ and $|A_{\mathcal{L}_\infty}|(\vec{0}) = 1$.

3. The blow-up points do not appear in the blow-up limit of the other points, that is

$$\liminf_{n \rightarrow \infty} \min_{p, q \in \mathcal{B}_n, p \neq q} \lambda_n(p) d_M(p, q) = \infty$$

Proposition 2.13. *Suppose that the strong index of the surfaces Σ_n is uniformly bounded above by a constant I . Then there is a sequence of blow-up sets $\mathcal{B}_n \subset \Sigma_n$ with cardinality at most I , a constant C depending only on I and a constant c_M depending only on M such that*

$$|A_n|(x) \min\{d_M(x, \partial\Sigma_n \cup \mathcal{B}_n), c_M\} \leq C \tag{2.7}$$

for all n and all $x \in \Sigma_n$.

Proof. We will show that one can take the sets \mathcal{B}_n as in the proof of Theorem 2.10 by changing the distance function from the intrinsic distance to the extrinsic distance. It is clear that these sets have cardinality at most I and that the $\lambda_n(p_n)(\Sigma_n - p_n)$ are embedded with bounded second fundamental form and the distance from the origin to boundary of $\lambda_n(p_n)(\Sigma_n - p_n)$ diverges. So it follows from Theorem 2.8 that these surfaces must subconverge to a weak CMC lamination of euclidean 3-space.

To show the third item in the definition holds, let $\hat{\mathcal{B}}_n$ be as constructed in the proof of Theorem 2.10 and assume by contradiction that

$$\liminf_{n \rightarrow \infty} \min_{q \in \hat{\mathcal{B}}_n} \lambda_n(p_n) d_M(p_n, q) < \infty.$$

Take q_n to be a point in $\hat{\mathcal{B}}_n$ closest to p_n and let $\lambda_n(q_n) = |A_n(q_n)|$, repeating the arguments from the proof of Theorem 2.10 we have that, passing to a subsequence, for n large enough

$$\lambda_n(q_n) \leq 2\lambda_n(p_n),$$

so $\liminf_{n \rightarrow \infty} \lambda_n(p_n) d_M(q_n, p_n) < \infty$. This clearly implies

$$\lim_{n \rightarrow \infty} |A_n(q_n)| d_M(q_n, \partial \hat{\Sigma}_n) < \infty$$

contradicting the choice of q_n . □

In the case where the mean curvature of the Σ_n are uniformly bounded, repeating the arguments in [17, Section 4] we can conclude:

Lemma 2.14. *Suppose $\{\Sigma_n\}$ is a sequence of CMC surfaces embedded in a close Riemannian 3-manifold M with mean curvature uniformly bounded above by a constant H and strong index at most I . Suppose the blow-up sets \mathcal{B}_n accumulate on a set \mathcal{B}_∞ . Then there is a $\delta > 0$ such that the set*

$$B_\delta^M(\mathcal{B}_\infty) = \bigcup_{p \in \mathcal{B}_\infty} B_\delta^M(p)$$

is a union of disjoint balls making the following hold for all n large enough. Let Σ'_n be the (possibly disconnected) surface formed by the connected components of $B_\delta^M(\mathcal{B}_\infty) \cap \Sigma_n$ passing through \mathcal{B}_n and let Σ''_n be the complement of Σ'_n in $B_\delta^M(\mathcal{B}_\infty) \cap \Sigma_n$, then:

- 1. The components of Σ''_n are CMC disks with second fundamental form uniformly bounded above.*
- 2. Σ'_n intersects $B_\delta^M(\mathcal{B}_\infty)$ transversely in at most m circles.*
- 3. The genus of Σ'_n is at most r ;*
- 4. The surfaces Σ'_n have uniformly bounded area;*

where m and r depend only on H and I .

2.6 Picture away from blow-up points

In this section we will show that a sequence of CMC surfaces embedded in a space with finite fundamental group and with second fundamental form bounded above and mean curvature bounded away from zero cannot accumulate in a dense region. The

results presented here will be used to proof Theorem 2.3 and will also be needed to remove any upper mean curvature bound assumption on Theorem 2.2. The results here are valid for CMC hypersurfaces of arbitrary dimensions, however we will only state the results in dimension three in order to be consistent with the rest of the text.

Let M be a Riemannian 3-manifold (possibly with boundary) with finite fundamental group. Let $\Sigma \subset M$ be a properly embedded two-sided surface and let ν be a normal vector field along Σ . Suppose $\gamma : [0, 1] \rightarrow M$ is a differentiable curve between points in $M \setminus \Sigma$ which is transversal to Σ . Write $\gamma^{-1}(\Sigma) = \{t_1, \dots, t_n\}$ where $t_1 < \dots < t_n$. We say that Σ is ℓ -alternating if for any such curve γ one of the following holds:

- Either $\ell > n - 1$;
- or there is a $j \leq \ell + 1$ such that

$$\langle \gamma'(t_1), \nu(\gamma(t_1)) \rangle \text{ and } \langle \gamma'(t_j), \nu(\gamma(t_j)) \rangle$$

have opposite signs.

Proposition 2.15. *Suppose $|\pi_1(M)| = \ell$. Then any properly embedded two-sided surface $\Sigma \subset M$ is ℓ -alternating.*

Proof. Let \tilde{M} be the universal cover of M and let $\tilde{\Sigma}$ be the lift of Σ in the sense that, for any evenly covered open set $U \subset M$ and any sheet $\tilde{U} \subset \tilde{M}$ over U , $\tilde{\Sigma} \cap \tilde{U}$ is mapped isometrically to $\Sigma \cap U$ by the covering map. Note that we can lift a normal vector field ν of Σ to a normal vector field $\tilde{\nu}$ of $\tilde{\Sigma}$.

Let $\gamma : [0, 1] \rightarrow M$ be a differentiable curve with ends away from Σ which is transversal to Σ and intersects Σ in at least $\ell + 1$ points. Choose any lift $\tilde{\gamma}$ of γ . Note that $\gamma^{-1}(\Sigma) = \tilde{\gamma}^{-1}(\tilde{\Sigma})$ and for each t_j in this set

$$\langle \gamma'(t_j), \nu(\gamma(t_j)) \rangle = \langle \tilde{\gamma}'(t_j), \tilde{\nu}(\tilde{\gamma}(t_j)) \rangle.$$

So we only need to show that there are $j, j' \leq \ell + 1$ such that

$$\langle \tilde{\gamma}'(t_j), \tilde{\nu}(\tilde{\gamma}(t_j)) \rangle \text{ and } \langle \tilde{\gamma}'(t_{j'}), \tilde{\nu}(\tilde{\gamma}(t_{j'})) \rangle$$

have opposite signs.

Since the order of the cover $\tilde{M} \rightarrow M$ is ℓ , $\tilde{\Sigma}$ has at most ℓ connected components. So there must be a connected component $\tilde{\Sigma}_0$ of $\tilde{\Sigma}$ intersecting $\tilde{\gamma}$ at least twice. Let $\tilde{\gamma}(s_1) = \tilde{x}, \tilde{\gamma}(s_2) = \tilde{y} \in \tilde{\Sigma}_0$ be the first and second points of intersection of $\tilde{\gamma}$ with $\tilde{\Sigma}_0$, respectively. Assume by contradiction that the product of the velocity of $\tilde{\gamma}$ with the normal of $\tilde{\Sigma}_0$ has the same sign at both these points. By changing the direction of the normal of $\tilde{\Sigma}_0$ we can assume that these are both negative. Choose a path $\eta : [0, 1] \rightarrow \tilde{\Sigma}_0$ between \tilde{x} and \tilde{y} and for $r \geq 0$ set

$$\eta_r(t) = \exp_{\tilde{M}}(r\tilde{\nu}(\eta(t))).$$

Fix $\rho > 0$ small enough so that η_ρ does not intersect $\tilde{\Sigma}_0$. Taking $\epsilon > 0$ small enough, we can join $\tilde{\gamma}|_{[s_1-\epsilon, s_2-\epsilon]}$ to η_ρ by two smooth curves that do not intersect $\tilde{\Sigma}_0$. Let β be the concatenation of these 4 curves. Then β is piecewise smooth and intersects $\tilde{\Sigma}_0$ exactly at \tilde{x} . On the other hand, since \tilde{M} is simply-connected and $\tilde{\Sigma}_0$ is properly

embedded, the intersection number of β and Σ_0 has to be even. \square

Using the result above and Proposition 2.7 we conclude that, under the hypothesis of Theorem 2.8, if M has finite fundamental group, the lamination \mathcal{L}_∞ must be a finite collection of properly embedded CMC surfaces.

2.7 Proof of the main theorems

Now we are ready to proof theorems 2.2 and 2.3. Since the proofs are very similar we present the proof of Theorem 2.2 in more detail and reference it in the proof of Theorem 2.3.

Proof of Theorem 2.2

Let M be a closed Riemannian 3-manifold and let $\{\Sigma_n\}_{n \in \mathbb{N}}$ be sequence of closed CMC surfaces embedded in M with index at most I and area at most A_0 . Suppose by contradiction that the genus of the Σ_n form a divergent sequence. Passing to a subsequence, we can assume that the sequence of blow-up sets $\{\mathcal{B}_n \subset \Sigma_n\}_{n \in \mathbb{N}}$ accumulate. Let H_n be the mean curvature of Σ_n . We will consider two cases depending on the behavior of the series $\{H_n\}_{n \in \mathbb{N}}$.

Case 1: $\liminf_{n \rightarrow \infty} H_n < \infty$.

Pass to a subsequence so the H_n are uniformly bounded and divide the Σ_n into three parts: Σ'_n, Σ''_n as in Lemma 2.14 and

$$\Sigma_n^b = \Sigma_n \setminus B_{\delta/2}^M(\mathcal{B}_\infty).$$

By Theorem 2.10, the norm of the second fundamental form of the surface Σ_n is uniformly bounded away from $B_{\delta/3}^M(\mathcal{B}_\infty)$. Since the area of the Σ_n is uniformly bounded above, the number of graphs in the local parametrization of Lemma 2.4 is uniformly bounded. So we can pass to a subsequence such that the Σ_n^b converge with finite multiplicity to a (possibly disconnected) CMC surface Σ_∞^b (see Theorem 2.8). Hence for n large enough Σ_n^b is a cover of Σ_∞^b with uniformly bounded degree. Using this fact it is easy to bound the Betti numbers of the surfaces Σ_n^b .

Lemma 2.14 together with the area bound limits the Betti numbers of Σ'_n and Σ''_n . By [17, Lemma 3.1] the surfaces Σ_n^b intersects Σ'_n and Σ''_n in annuli, so we can use a Mayer-Vietoris sequence to bound on the Betti numbers of Σ_n ³. This contradicts the assumption that the sequence of the genus of the Σ_n diverges.

Case 2: $\lim_{n \rightarrow \infty} H_n = \infty$.

It follows from Corollary 2.11 that the surfaces Σ_n converge to a point $p_\infty \in M$ in the Hausdorff sense. So for n large enough

$$\Sigma_n \subset B_r^M(p_\infty).$$

where r is smaller than the injectivity radius of M . Using the exponential map of M at p_∞ we can pull Σ_n back to a properly embedded H_n -surface

$$\tilde{\Sigma}_n \subset (B_r^{\mathbb{R}^3}(p_\infty), \exp_{p_\infty}^* g).$$

³We can also use a surgery argument [17, Proposition 5.1]

Scaling the $\tilde{\Sigma}_n$ by H_n we obtain a sequence of 1-surfaces

$$\bar{\Sigma}_n = H_n \tilde{\Sigma}_n$$

with uniformly bounded diameter in an increasing sequence of balls with metrics converging smoothly to the euclidean metric in compact sets.

Consider a decomposition of the $\bar{\Sigma}_n$ into $\bar{\Sigma}'_n, \bar{\Sigma}''_n$ and $\bar{\Sigma}^b_n$ analogous to the decomposition of the Σ_n in Case 1. By Lemma 2.7 and Proposition 2.15, there are uniform bounds on the number of sheets of the surfaces $\bar{\Sigma}^b_n$ on any small open set. Hence we can conclude that, passing to a subsequence, the surfaces $\bar{\Sigma}^b_n$ converge to a 1-surface Σ^b_∞ . These same arguments can be used to bound the number of connected components of the $\bar{\Sigma}''_n$. Finally we can conclude using the same arguments from Case 1. □

Proof of Theorem 2.3

First note that by Theorem 2.2 we only have to show that there is an uniform area bound. Let M be a spherical Riemannian 3-manifold and let $\{\Sigma_n\}_{n \in \mathbb{N}}$ be a sequence of closed CMC surfaces embedded in M with index at most I and mean curvature at least $\eta > 0$. Assume by contradiction that the area of the Σ_n form a divergent sequence and pass to a subsequence so the sequence of blow-up sets $\{\mathcal{B}_n \subset \Sigma_n\}_{n \in \mathbb{N}}$ accumulates.

Let H_n be the mean curvature of Σ_n . In the case where $\limsup_{n \rightarrow \infty} H_n = \infty$, we can pass to a subsequence so $\lim_{n \rightarrow \infty} H_n = \infty$. Following the arguments used in Case

2 of the proof of Theorem 2.2 we can conclude that the area of the mean-curvature rescaled surfaces $\bar{\Sigma}_n$ is uniformly bounded, hence the area of the surfaces Σ_n must converge to 0, a contradiction.

So we may assume $\limsup_{n \rightarrow \infty} H_n < \infty$. In this case we can apply the same decomposition used in Case 1 on the proof of Theorem 2.2 and use Lemma 2.15 to bound the number of sheets of any Σ_n^b that can pass through small regions. A similar argument can be used to bound the number of connected components of Σ_n'' , and hence we conclude that there are uniform area bounds for these two parts. Finally, Lemma 2.14 allows us to conclude that there are uniform area bounds for the surfaces Σ_n' . □

Chapter 3

Capillary Surfaces: Index, Stability and Curvature Estimates

Here we will present results that were obtained in collaboration with Han Hong and appeared in [32]. In this chapter we show several results that relate the index to the geometry and topology of capillary surfaces. We begin in Section 3.1 presenting and contextualizing our main results. Sections 3.2 and 3.3 deal with compact and noncompact capillary surfaces respectively. Finally we obtain curvature bounds for strongly stable capillary surfaces in Section 3.4.

3.1 Introduction

3.1.1 Index estimates for compact capillary surfaces

There are many results that, in the spirit of Conjecture 2.1, give an index bound for free boundary minimal surfaces that depend on an affine function of the genus and the number of boundary components (see e.g. the work of Sargent Sargent [45] and Ambrozio-Carlotto-Sharp [8]). In these papers, harmonic one forms are used to construct test functions for the second variation formula of area functional. This idea was first discovered by Ros [41], and further developed by Savo [48] and Ambrozio-Carlotto-Sharp [7] in the case when the surfaces in question are closed minimal surfaces. Aiex and Hong applied a similar idea to the settings where the surfaces are closed CMC surfaces and free boundary CMC surfaces in a general three-dimensional Riemannian manifold [3] and obtained lower index bounds in terms of the topology. In particular, as a byproduct, they showed that the index of free boundary CMC surfaces in a mean convex domain of \mathbb{R}^3 is bounded below by $(2g - r - 4)/6$ where g is the genus of the surface and r is the number of boundary components of the surface. This particular result was also obtained by Cavalcante-de Oliveira in [13]. Here we generalize these results to compact capillary surfaces. More generally, we show the following:

Theorem 3.1. *Let M be a 3-dimensional oriented Riemannian manifold with boundary isometrically embedded in \mathbb{R}^d and let Σ be a compact capillary surface*

immersed in M at a constant angle θ with genus g and r boundary components.

Suppose that every non-zero $\xi \in \mathcal{H}_T^1(\Sigma, \partial\Sigma)$ satisfies

$$\begin{aligned} & \int_{\Sigma} \sum_{i=1}^2 |II_M(e_i, \xi)|^2 + |II_M(e_i, \star\xi)|^2 dA - \int_{\Sigma} R_M |\xi|^2 dA \\ & - \int_{\partial\Sigma} \frac{2}{\sin\theta} H_{\partial M} |\xi|^2 d\ell < \int_{\Sigma} H_{\Sigma}^2 |\xi|^2 dA + \int_{\partial\Sigma} 2 \cot\theta H_{\Sigma} |\xi|^2 d\ell. \end{aligned}$$

Then

$$\text{Index}_w(\Sigma) \geq \frac{2g + r - 1 - d}{2d}.$$

In particular, in \mathbb{R}^3 we have

Corollary 3.2. *Let M be a domain in \mathbb{R}^3 with smooth boundary and let Σ be a compact capillary surface immersed in M at a constant angle θ with genus g and r boundary components. Suppose that $H_{\partial M} + H_{\Sigma} \cos\theta \geq 0$ along $\partial\Sigma$ and that one of the following holds:*

$$H_{\Sigma} > 0,$$

or

$$H_{\partial M} > 0$$

at some point in $\partial\Sigma$. Then

$$\text{Index}_w(\Sigma) \geq \frac{2g + r - 4}{6}.$$

Remark 3.3. *When M is a closed half-space in \mathbb{R}^3 , $H_{\partial M} + H_{\Sigma} \cos\theta$ reduces to $H_{\Sigma} \cos\theta$. As compact capillary surfaces in a closed half-space cannot be minimal due to maximum principle, the assumption of Corollary 3.2 simply becomes $\theta \in (0, \pi/2]$.*

The ideas of the proof of Theorem 3.1 are essentially the same as those in [3], though the calculations are more involved because of the angle θ is not necessarily $\pi/2$.

3.1.2 Noncompact capillary surfaces

Fisher-Colbrie [26, Theorem 1] has shown that a two-sided complete noncompact minimal surface with finite index in an oriented ambient space of nonnegative scalar curvature is conformally equivalent to a compact Riemann surface with finitely many points removed. We will show an analogous result for noncompact capillary surfaces, which will be essential to all of our other results for this type of surface.

Theorem 3.4. *Let M be an oriented Riemannian 3-manifold with smooth boundary and let Σ be a noncompact capillary surface with finite index immersed in M at a constant angle θ . Assume that $R_M + H_\Sigma^2 \geq 0$ and that one of the following holds:*

$$\partial\Sigma \text{ is compact,}$$

or

$$H_{\partial M} + H_\Sigma \cos \theta \geq 0 \text{ along } \partial\Sigma.$$

Then Σ is conformally equivalent to a compact Riemann surface with boundary and finitely many points removed, each associated to an end of the surface. Moreover,

$$\int_\Sigma R_M + H_\Sigma^2 + |A_\Sigma|^2 + \int_{\partial\Sigma} H_{\partial M} + H_\Sigma \cos \theta < \infty. \quad (3.1)$$

Let $\bar{\Sigma}$ be the compact Riemann surface that describes the topology of the noncompact capillary surface Σ as in Theorem 3.4. Points removed from the interior of $\bar{\Sigma}$ are associated to *interior ends* of Σ , and points removed from the boundary of $\bar{\Sigma}$ are associated to *boundary ends* of Σ . These are described in more details in Remark 3.10 and Remark 3.11. Two corollaries of Theorem 3.4 are proven. Namely, Corollary 3.12 gives conditions under which noncompact capillary surfaces in domains of \mathbb{R}^3 are minimal, and Corollary 3.13 describes the conformal type of strongly stable capillary surfaces under suitable curvature assumptions.

Capillary surfaces in a half-space of \mathbb{R}^3 are particularly important because they serve as local models for capillary surfaces near the boundary. This fact will be essential to the proof of our curvature bounds for capillary surfaces (Theorem 3.6). In particular, we will need to understand stable noncompact capillary surfaces in a half-space of \mathbb{R}^3 .

The only weakly stable complete CMC surfaces in \mathbb{R}^3 are planes, this was first proved for two-sided strongly stable minimal surfaces independently by do Carmo-Peng [24], Fischer-Colbrie-Schoen [27] and Pogorelov [38], later for weakly stable CMC surfaces by da Silveira [23] and for one-sided strongly stable minimal surfaces by Ros [41]. It is then natural to ask whether half-planes intersecting the boundary of a half-space at a constant angle θ are the only weakly stable capillary surfaces immersed in a half-space. When the angle is $\pi/2$, namely, the surface has free boundary, we are able to reflect the surface to get a smooth stable complete noncompact CMC

surface, thus the result in the case when the surface has no boundary is helpful to study the free boundary case. In particular, a reflection argument has been used by Ambrozio-Buzano-Carlotto-Sharp [5, Corollary 2.2] to show that a strongly stable free boundary minimal surface in a half-space must be a half-plane. When the angle is not $\pi/2$, the reflection analysis apparently fails, moreover, the boundary term in the second variation formula does not vanish. Nevertheless, we are able to show:

Theorem 3.5. *Let Σ be a noncompact capillary surface immersed in a half-space of \mathbb{R}^3 at constant angle θ . Assume that $H_\Sigma \cos \theta \geq 0$. Then Σ is weakly stable if and only if it is a half-plane.*

It is easy to see that the assumption $H_\Sigma \cos \theta \geq 0$ translates to Σ has nonzero mean curvature and $\theta \in (0, \pi/2]$ or Σ is minimal. We will also observe in Remark 3.16 that there is no weakly stable noncompact capillary surface with compact boundary immersed in a half-space of \mathbb{R}^3 , even with $H_\Sigma \cos \theta < 0$.

In order to prove Theorem 3.5 we first construct a test function involving angle θ . Thanks to Theorem 3.4, we can construct a cutoff function such that its product with $u = \frac{1}{\sin \theta} + \cot \theta \langle \nu, -E_3 \rangle$ becomes an admissible test function. This function u is designed so that the boundary integral in the quadratic form vanishes, this is essential because we have no control over the sign or magnitude of the boundary integral when the boundary is noncompact. If the surface is not totally geodesic, we can show that this test function leads to negative second variation, thus contradicting weak stability. When $\theta = \pi/2$, the function u is constant. In particular, in the case where the surface

is assumed to have free boundary, these arguments give a new and direct proof to some results in [5].

Notice that there are noncompact capillary surfaces in a half-space besides half-planes. For any angle $\theta \in (0, \pi)$ and any constant $H > 0$, one can construct a noncompact capillary surface with mean curvature H and contact angle θ by cutting an unduloid horizontally (by a plane orthogonal to its axis). Noncompact free boundary surfaces with any constant mean curvature can also be obtained by cutting an unduloid vertically (by a plane that contains its axis). These examples however have infinite index. Examples of minimal capillary surfaces with finite index and any contact angle $\theta \in (0, \pi)$ can be obtained by cutting a catenoid horizontally. Cutting a catenoid vertically gives a noncompact free boundary minimal surface with index one. Observe that another significant difference between cutting a catenoid horizontally or vertically is that the former has compact boundary and one interior ends while the latter has no interior ends and two boundary ends.

3.1.3 Curvature estimates for strongly stable capillary surfaces

As we have already seen in Chapter 2, curvature estimates for strongly stable CMC surface play an important role in the study of these surfaces. Here we will show a curvature bound similar to Theorem 2.10 for strongly stable capillary surfaces. Note

that, using the same arguments used to show Theorem 2.10, it is also possible to extend this result to capillary surfaces of finite index.

Theorem 3.6. *Let $\theta \in (0, \pi)$. Then there is a constant $C = C(\theta)$ such that the following holds: Let M be a 3-manifold with smooth boundary. Assume that M has curvature bounded above by Λ and injectivity radius bounded below by ι . Let Σ be a strongly stable edged capillary surface immersed in M at a constant angle θ . Then,*

- if $H_\Sigma \cos \theta \geq 0$, we have

$$|A_\Sigma(p)| \min\{d_\Sigma(p, \partial\Sigma \setminus \partial M), \iota, (\sqrt{\Lambda})^{-1}\} \leq C$$

for all $p \in \Sigma$; and

- if $H_\Sigma \cos \theta < 0$, we have

$$|A_\Sigma(p)| \min\{d_\Sigma(p, \partial\Sigma \setminus \partial M), \iota, (\sqrt{\Lambda})^{-1}, -(H_\Sigma \cos \theta)^{-1}\} \leq C$$

for all $p \in \Sigma$.

Since these bounds do not depend of the area of the surface, to the best of the author's knowledge they are also new for the special case where Σ is an immersed strongly stable free boundary minimal surface, and hence it gives a proof to a conjecture of Guang, Li and Zhou [31, Conjecture 1.4].

When $\theta \in (0, \pi/2]$ and the mean curvature H_Σ of Σ is large in comparisons with ι^{-1} and Λ , it is possible to show that there is a bound on the diameter of Σ that depends linearly on its strong index, in fact this is the content of Corollary 3.22. In

the special case where the ambient manifold M is a half-space of \mathbb{R}^3 , ι^{-1} and Λ can be taken to be arbitrarily small, so the bounds in Corollary 3.22 are valid for all $H_\Sigma > 0$ as long as $\theta \in (0, \pi/2]$.

3.2 Index estimates for compact capillary surfaces

In this section we estimate the weak index of compact capillary surfaces in 3-manifolds with boundary, generalizing results in [3]. Since we will use harmonic vector fields in the proof, we begin by defining these objects.

3.2.1 Harmonic one-forms and vector fields

On an orientable surface Σ , let

$$\Delta^{[1]}w = (d\delta + \delta d)w$$

be the Hodge Laplacian acting on one-forms. Here d is the exterior differential and δ is the codifferential. A one-form w is harmonic if $\Delta^{[1]}w = 0$. In particular, if a one-form w is closed and coclosed, i.e., $dw = \delta w = 0$, then it is harmonic. However, the converse is not true for surfaces with boundary.

The metric on Σ induces a correspondence between one-forms and vector fields, let $\xi = w^\#$ be the vector field corresponding to the one-form w . Define the Hodge Laplacian on vector fields as $\Delta_{[1]}\xi = \Delta^{[1]}w$. We say ξ is a harmonic vector field if $\Delta_{[1]}\xi = 0$, i.e., the corresponding one-form w is harmonic. Let $\star\xi = (\star w)^\#$ where \star is

the Hodge operator with respect to the metric on Σ . If $\operatorname{div}_\Sigma(\xi) = 0$ and $\operatorname{div}_\Sigma(\star\xi) = 0$, then ξ is harmonic. This is because $\star dw = \operatorname{div}_\Sigma(\star\xi)$ and $\delta w = -\operatorname{div}_\Sigma(\xi)$. The classical Weitzenböck's formula relates the Hodge Laplacian and rough Laplacian of a vector field, that is

$$\Delta_{[1]}\xi = \nabla^*\nabla\xi + \operatorname{Ric}_\Sigma(\xi), \quad (3.2)$$

where $\nabla^*\nabla\xi = -\sum_{i=1}^2 \nabla_{e_i}\nabla_{e_i}\xi$ under a local geodesic frame $\{e_1, e_2\}$ and $\operatorname{Ric}_\Sigma(\xi)$ is defined by $\langle \operatorname{Ric}_\Sigma(\xi), X \rangle = \operatorname{Ric}_\Sigma(\xi, X)$ for any vector field $X \in T\Sigma$.

We will consider the following space:

$$\mathcal{H}_T^1(\Sigma, \partial\Sigma) = \{\xi \in T\Sigma : \operatorname{div}_\Sigma(\xi) = \operatorname{div}_\Sigma(\star\xi) = 0 \text{ on } \Sigma \text{ and } \xi \text{ is tangential along } \partial\Sigma\}.$$

It is known that, when Σ is compact, $\dim \mathcal{H}_T^1(\Sigma, \partial\Sigma) = 2g + r - 1$ where g is the genus and r is the number of boundary components of Σ (see [3, 45]).

The following Lemma will be used in the proof of Corollary 3.2.

Lemma 3.7 ([50, Theorem 3.4.4]). *Let Σ be a complete connected, orientable Riemannian surface with non-empty boundary $\partial\Sigma$. If a harmonic vector field vanishes identically on $U \cap \partial\Sigma \neq \emptyset$ for some open subset $U \subset \Sigma$, then it vanishes identically on Σ .*

3.2.2 Index of a capillary surfaces and harmonic vector fields

Here we will show Theorem 3.1, we will use $\langle \cdot, \cdot \rangle$ and D for the Euclidean product and connection respectively, aside from that we will use the notation established in

Section 1.1.

Proposition 3.8. *Let M be a 3-dimensional Riemannian manifold isometrically embedded in some Euclidean space \mathbb{R}^d . Let Σ be a compact capillary surface immersed in M at a constant angle θ . Given a vector field $\xi \in \mathcal{H}_T^1(\Sigma, \partial\Sigma)$, denote*

$$u_j = \langle \xi, E_j \rangle,$$

where $\{E_j\}_{j=1}^d$ is the canonical basis of \mathbb{R}^d . Then

$$\begin{aligned} \sum_{j=1}^d Q(u_j, u_j) &= \int_{\Sigma} \sum_{i=1}^2 |II_M(e_i, \xi)|^2 + |A_{\Sigma}(e_i, \xi)|^2 - \int_{\Sigma} \frac{|A_{\Sigma}|^2 + R_M + H_{\Sigma}^2}{2} |\xi|^2 \\ &\quad - \frac{1}{\sin \theta} \int_{\partial\Sigma} H_{\partial M} |\xi|^2 - \cot \theta \int_{\partial\Sigma} H_{\Sigma} |\xi|^2, \end{aligned} \quad (3.3)$$

where $\{e_i\}_{i=1}^2$ is an orthonormal frame on Σ , R_M is the scalar curvature of M , II_M is the second fundamental form of M in \mathbb{R}^d and $H_{\partial M}$ denotes the mean curvature of ∂M with respect to the inner normal vector $-N$.

Proof. Using a local orthonormal basis $\{e_i\}_{i=1}^2$ on Σ we have

$$\nabla u_j = \sum_{i=1}^2 e_i \langle \xi, E_j \rangle e_i = \sum_{i=1}^2 \langle D_{e_i} \xi, E_j \rangle e_i.$$

Since

$$D_{e_i} \xi = \nabla_{e_i} \xi - A_{\Sigma}(e_i, \xi) \nu + II_M(e_i, \xi),$$

it follows that

$$\begin{aligned}
\sum_{j=1}^d |\nabla u_j|^2 &= \sum_{j=1}^d \sum_{i=1}^2 |\langle D_{e_i} \xi, E_j \rangle|^2 \\
&= \sum_{i=1}^2 \sum_{j=1}^d \langle \nabla_{e_i} \xi, E_j \rangle^2 + \langle A_\Sigma(e_i, \xi), E_j \rangle^2 + \langle II_M(e_i, \xi), E_j \rangle^2 \\
&= |\nabla \xi|^2 + \sum_{i=1}^2 |A_\Sigma(e_i, \xi)|^2 + |II_M(e_i, \xi)|^2.
\end{aligned} \tag{3.4}$$

Gauss' equation for Σ in M gives us

$$2K_\Sigma = R_M - 2 \operatorname{Ric}_M(\nu, \nu) - |A_\Sigma|^2 + H_\Sigma^2.$$

Hence,

$$\int_\Sigma \operatorname{Ric}_M(\nu, \nu) u_j^2 = \int_\Sigma \left(\frac{R_M}{2} - \frac{|A_\Sigma|^2}{2} + \frac{H_\Sigma^2}{2} - K_\Sigma \right) u_j^2. \tag{3.5}$$

Using Weitzenbock's formula (3.2) and the fact that ξ is harmonic we conclude that

$$\nabla^* \nabla \xi = -K_\Sigma \xi.$$

By computing the exterior derivative along $\partial\Sigma$ we have

$$d\xi^\flat(\eta, \xi) = \langle \nabla_\eta \xi, \xi \rangle - \langle \nabla_\xi \xi, \eta \rangle.$$

Since $d\xi^\flat = 0$,

$$\langle \nabla_\eta \xi, \xi \rangle = \langle \nabla_\xi \xi, \eta \rangle = -k_{\partial\Sigma} |\xi|^2,$$

then

$$\frac{\partial |\xi|^2}{\partial \eta} = -2k_{\partial\Sigma} |\xi|^2.$$

It follows from the divergence theorem and (1.12) that

$$\begin{aligned}
\int_{\Sigma} \Delta |\xi|^2 &= \int_{\partial\Sigma} \frac{\partial |\xi|^2}{\partial \eta} \\
&= -2 \int_{\partial\Sigma} k_{\partial\Sigma} |\xi|^2 \\
&= 2 \int_{\partial\Sigma} q |\xi|^2 - \left(\frac{1}{\sin \theta} H_{\partial M} + \cot \theta H_{\Sigma} \right) |\xi|^2
\end{aligned} \tag{3.6}$$

which together with

$$\Delta |\xi|^2 = -2 \langle \nabla^* \nabla \xi, \xi \rangle + 2 |\nabla \xi|^2$$

implies that

$$\begin{aligned}
\int_{\Sigma} |\nabla \xi|^2 &= \int_{\Sigma} \frac{\Delta |\xi|^2}{2} + \int_{\Sigma} \langle \nabla^* \nabla \xi, \xi \rangle \\
&= \int_{\partial\Sigma} q |\xi|^2 - \left(\frac{1}{\sin \theta} H_{\partial M} + \cot \theta H_{\Sigma} \right) |\xi|^2 - \int_{\Sigma} K_{\Sigma} |\xi|^2.
\end{aligned} \tag{3.7}$$

Combining (1.13), (3.5), (3.7) and (3.4) gives the desired result. \square

Theorem 3.9. *Let M be a 3-dimensional oriented Riemannian manifold with boundary isometrically embedded in \mathbb{R}^d and let Σ be a compact capillary surface immersed in M at a constant angle θ . Assume there exist a real number β and a q -dimensional subspace $\mathbb{W}^q \subseteq \mathcal{H}_T^1(\Sigma, \partial\Sigma)$ such that any non-zero $\xi \in \mathbb{W}^q$ satisfies*

$$\begin{aligned}
&\int_{\Sigma} \sum_{i=1}^2 |II_M(e_i, \xi)|^2 + |II_M(e_i, \star \xi)|^2 - \int_{\Sigma} (R_M + H_{\Sigma}^2) |\xi|^2 \\
&\quad - \int_{\partial\Sigma} \left(\frac{2}{\sin \theta} H_{\partial M} + 2 \cot \theta H_{\Sigma} \right) |\xi|^2 < 2\beta \int_{\Sigma} |\xi|^2.
\end{aligned}$$

Then

$$\#\{\text{eigenvalues of } \tilde{J} \text{ that are smaller than } \beta\} \geq \frac{q-d}{2d}.$$

Proof. Denote $u_j^* = \langle \star \xi, E_j \rangle$. Since $|\star \xi| = |\xi|$, it follows from (3.6) that

$$\int_{\Sigma} \Delta |\star \xi|^2 = 2 \int_{\partial \Sigma} q |\xi|^2 - \left(\frac{1}{\sin \theta} H_{\partial M} + \cot \theta H_{\Sigma} \right) |\xi|^2.$$

Thus following the proof of Proposition 3.8 gives

$$\begin{aligned} \sum_{j=1}^d Q(u_j^*, u_j^*) &= \int_{\Sigma} \sum_{i=1}^2 |II_M(e_i, \star \xi)|^2 + |A_{\Sigma}(e_i, \star \xi)|^2 - \int_{\Sigma} \frac{|A_{\Sigma}|^2 + R_M + H_{\Sigma}^2}{2} |\xi|^2 \\ &\quad - \frac{1}{\sin \theta} \int_{\partial \Sigma} H_{\partial M} |\xi|^2 - \cot \theta \int_{\partial \Sigma} H_{\Sigma} |\xi|^2. \end{aligned} \quad (3.8)$$

Note that whenever $\xi \neq 0$ we may pick $e_1 = \frac{\xi}{|\xi|}$, $e_2 = \frac{\star \xi}{|\star \xi|}$ as an orthonormal basis.

Using this fact it is easy to see that

$$\sum_{i=1}^2 |A_{\Sigma}(e_i, \xi)|^2 + |A_{\Sigma}(e_i, \star \xi)|^2 = \sum_{i,j=1,2} |A_{\Sigma}(e_i, e_j)|^2 |\xi|^2 = |A_{\Sigma}|^2 |\xi|^2. \quad (3.9)$$

Thus summing (3.8) and (3.3) gives

$$\begin{aligned} \sum_{j=1}^d Q(u_j, u_j) + Q(u_j^*, u_j^*) &= \int_{\Sigma} \sum_{i=1}^2 |II_M(e_i, \xi)|^2 + |II_M(e_i, \star \xi)|^2 \\ &\quad - \int_{\Sigma} (R_M + H_{\Sigma}^2) |\xi|^2 - \int_{\partial \Sigma} \left(\frac{2}{\sin \theta} H_{\partial M} + 2 \cot \theta H_{\Sigma} \right) |\xi|^2. \end{aligned}$$

Let $k = \#\{\text{eigenvalues of } \tilde{J} \text{ that are smaller than } \beta\}$ and let $\tilde{\phi}_1, \dots, \tilde{\phi}_k$ the eigenfunctions of \tilde{J} corresponding to eigenvalues $\tilde{\lambda}_1 \leq \dots \leq \tilde{\lambda}_k < \beta$. Consider the linear map defined by

$$\begin{aligned} F : \mathbb{W}^q &\longrightarrow \mathbb{R}^{2dk+d} \\ \xi &\longmapsto \left[\int_{\Sigma} u_j \phi_{\alpha}, \int_{\Sigma} u_j^* \phi_{\alpha}, \int_{\Sigma} u_j^* \right], \end{aligned}$$

where $\alpha = 1, \dots, k$ and $j = 1, \dots, d$. By the Rank-Nullity Theorem, if $2dk + d < q$,

then there exists a nonzero harmonic tangential vector field $\xi \in \mathcal{H}_T^1(\Sigma, \partial\Sigma)$ such that u_j, u_j^* are orthogonal to the first k eigenfunctions of \tilde{J} . Moreover, Lemma 3.1 in [3] shows that $\int_\Sigma u_j = 0$ for any $j = 1, \dots, d$. Thus, it follows from the Min-max principle for \tilde{J} that

$$\sum_{j=1}^d Q(u_j, u_j) + Q(u_j^*, u_j^*) \geq 2\lambda_{k+1} \int_\Sigma |\xi|^2 \geq 2\beta \int_\Sigma |\xi|^2$$

which contradicts the assumption in the proposition. We conclude that $2dk + d \geq q$, that is, $k \geq \frac{q-d}{2d}$ as claimed. \square

Theorem 3.1 follow by setting $\beta = 0$ in Theorem 3.9 and using the fact that $\dim \mathcal{H}_T^1(\Sigma, \partial\Sigma) = 2g + r - 1$.

We now give a proof to Corollary 3.2 in the Introduction. Let M be a domain of \mathbb{R}^3 with smooth boundary and assume that $H_{\partial M} + H_\Sigma \cos \theta \geq 0$ along $\partial\Sigma$. If $H_\Sigma = 0$ and $H_{\partial M} > 0$ at some point of $\partial\Sigma$, then the assumption in Theorem 3.1 is satisfied because $II_M = R_M = 0$, and, by Lemma 3.7, a nonzero harmonic vector field ξ cannot vanish in a segment of $\partial\Sigma$. Otherwise, if $H_\Sigma > 0$, the assumption in Theorem 3.1 automatically holds.

3.3 The structure of noncompact capillary surfaces with finite index

In this section we show, amongst other results, theorems 3.4 and 3.5. We begin it by studying noncompact capillary surfaces in general 3-manifolds with smooth

boundary.

3.3.1 Noncompact capillary surfaces in general 3-manifolds

We begin by showing Theorem 3.4, that is, we will show that under certain curvature assumptions on Σ and M , noncompact capillary surfaces with finite index share a key property of minimal surfaces with finite index in ambient spaces with nonnegative scalar curvature (see [26, Theorem 1]): They are conformal to compact Riemann surfaces with a finite number of punctures.

Proof of Theorem 3.4

Since Σ has finite index, there exists a compact set $C \subset \Sigma$ such that $\Sigma \setminus C$ is strongly stable. For the rest of this proof we will assume that $H_{\partial M} + H_{\Sigma} \cos \theta \geq 0$ along $\partial \Sigma$. It is easy to see that the proof in case where $\partial \Sigma$ is compact follows from the same arguments used below by taking C large enough so that $\partial \Sigma \subset C$.

From equation (1.12) we obtain that for all $\varphi \in C_0^\infty(\Sigma \setminus C)$

$$\int_{\Sigma} |\nabla \varphi|^2 - (\text{Ric}_M(\nu, \nu) + |A_{\Sigma}|^2) \varphi^2 + \int_{\partial \Sigma} \varphi^2 \kappa_{\partial \Sigma} \geq Q(\varphi, \varphi) \geq 0.$$

Applying the same techniques used to prove Proposition 1.4 we conclude that there is a positive function v defined on $\Sigma \setminus C$ such that

$$\begin{cases} Jv = 0 & \text{in } \Sigma \setminus C \\ \frac{\partial v}{\partial \eta} + v \kappa_{\partial \Sigma} = 0 & \text{on } \partial \Sigma \setminus C. \end{cases}$$

Extend v to all of Σ so that it is positive and $\frac{\partial v}{\partial \eta} + v\kappa_{\partial\Sigma} = 0$ along all of $\partial\Sigma$. Let $d\tilde{s}^2 = v^2 ds^2$ where ds^2 is the metric of Σ . Notice that the metric $d\tilde{s}$ has Gaussian curvature

$$\tilde{K}_\Sigma = v^{-2}(K_\Sigma - \Delta \log v), \quad (3.10)$$

where K_Σ is the Gaussian curvature of ds^2 . From the Gauss' equation

$$\Delta v + (|A_\Sigma|^2 + \text{Ric}_M(\nu, \nu))v = \Delta v - K_\Sigma v + \frac{1}{2}(R_M + H_\Sigma^2 + |A_\Sigma|^2)v = 0,$$

then

$$K_\Sigma - \Delta \log v = K_\Sigma - \frac{v\Delta v - |\nabla v|^2}{v^2} = \frac{1}{2}(R_M + H_\Sigma^2 + |A_\Sigma|^2) + \frac{|\nabla v|^2}{v^2} \geq 0. \quad (3.11)$$

Hence $\tilde{K}_\Sigma \geq 0$ on $\Sigma \setminus C$. The geodesic curvature of $\partial\Sigma$ in the metric $d\tilde{s}^2$ is

$$\tilde{\kappa}_{\partial\Sigma} = v^{-2} \frac{\partial v}{\partial \eta} + v^{-1} \kappa_{\partial\Sigma}, \quad (3.12)$$

so $\tilde{\kappa}_{\partial\Sigma} = 0$.

Now we will show that $(\Sigma, d\tilde{s}^2)$ is complete in the sense of metric spaces, that is, every geodesic of $(\Sigma, d\tilde{s}^2)$ either exits for infinite time or hits the boundary $\partial\Sigma$. Following [26] we can construct a divergent geodesic γ of $(\Sigma, d\tilde{s}^2)$ starting at a point $p \in \partial\Sigma$ that minimizes distance to the boundary of any disk of $(\Sigma, d\tilde{s}^2)$ around p . To be more precise, γ is constructed as the limit of geodesics of a family of metrics that interpolates between ds^2 and $d\tilde{s}^2$, each of which equals $d\tilde{s}^2$ in a neighborhood of p and equals ds^2 far away from p . It is enough to show that such γ has infinite length, which can be done by the same arguments used by Fischer-Colbrie [26]. We conclude

that $(\Sigma, d\tilde{s}^2)$ is complete.

Since Σ has smooth boundary we can double it to obtain a smooth surface without boundary $\check{\Sigma}$. Because $(\Sigma, d\tilde{s}^2)$ has totally geodesic boundary, the double accepts a metric $d\check{s}^2$ that restricts to $d\tilde{s}^2$ in both copies of Σ contained in $\check{\Sigma}$. Note that $d\check{s}^2$ might not be smooth, but it is C^2 (see e.g. [54, Lemma 2.4]). We will argue that $(\check{\Sigma}, d\check{s}^2)$ is geodesically complete. Let $\Phi : \check{\Sigma} \rightarrow \check{\Sigma}$ be the involution coming from the doubling process, and let γ be a minimizing geodesic in $(\check{\Sigma}, d\check{s}^2)$ that is not contained in $\partial\Sigma$. Note that the interior of γ can intersect $\partial\Sigma$ at most once, since otherwise we can construct a minimizing geodesic that is not C^1 by using Φ to reflect the part of γ between the first two intersections of γ with $\partial\Sigma$ and joining this reflected curve to the rest of γ . Now we can use the fact that $(\Sigma, d\tilde{s}^2)$ is complete to show that if γ is divergent, it must have infinite length.

Since $(\check{\Sigma}, d\check{s}^2)$ is complete with nonnegative Gaussian curvature away from a compact set, it follows from the Huber Theorem that $\check{\Sigma}$, and hence also Σ , is finitely connected. The same arguments used by Fischer-Colbrie [26, p. 128] show that the ends of $(\check{\Sigma}, d\check{s}^2)$ are conformal to puncture disks.

Let $d\hat{s}^2 = u^2 ds^2$ where $u > 0$ is such that $u|_{\Sigma \setminus C}$ is a Jacobi function. The same arguments used to show that $(\Sigma, d\tilde{s}^2)$ is complete also show that $(\Sigma, d\hat{s}^2)$ is complete. Furthermore, (3.10)–(3.12) apply to this conformal change and imply that in $\Sigma \setminus C$

$$\hat{K}_\Sigma \geq \frac{1}{2u^2}(R_M + H_\Sigma^2 + |A_\Sigma|^2),$$

where \hat{K}_Σ is the Gaussian curvature associated to $d\hat{s}^2$, and in $\partial\Sigma \setminus C$

$$\hat{\kappa}_{\partial\Sigma} = \frac{1}{u \sin \theta} (H_{\partial M} + H_\Sigma \cos \theta),$$

where $\hat{\kappa}_{\partial\Sigma}$ is the geodesic curvature of $\partial\Sigma$ associated to $d\hat{s}^2$. We then conclude that

$$\begin{aligned} \int_{\Sigma \setminus C} R_M + H_\Sigma^2 + |A_\Sigma|^2 dA + \int_{\partial\Sigma \setminus C} H_{\partial M} + H_\Sigma \cos \theta dl \\ \leq 2 \int_{\Sigma \setminus C} \hat{K}_\Sigma d\hat{A} + \sin \theta \int_{\partial\Sigma \setminus C} \hat{\kappa}_{\partial\Sigma} d\hat{\ell}, \end{aligned} \quad (3.13)$$

where dA , $d\hat{A}$ are the area elements associated to ds^2 and $d\hat{s}^2$, respectively, and dl , $d\hat{\ell}$ are the length elements associated to ds^2 and $d\hat{s}^2$, respectively.

Since \hat{K}_Σ and $\hat{\kappa}_{\partial\Sigma}$ are nonnegative outside C , we have that $(\Sigma, d\hat{s}^2)$ accepts a total curvature and a total geodesic curvature (as defined in [51]). Thus we obtain from the Cohn-Vossen Theorem with boundary ([51, Theorem 2.2.1]) that

$$\int_{\Sigma \setminus C} \hat{K}_\Sigma d\hat{A} + \int_{\partial\Sigma \setminus C} \hat{\kappa}_{\partial\Sigma} d\hat{\ell} < \infty.$$

Combining (3.13) with the inequality above completes the proof of the theorem. \square

Remark 3.10. *One can think of a capillary surface Σ as in the statement of Theorem 3.4 as a compact Riemann surface $\bar{\Sigma}$ with boundary and with points p_1, \dots, p_k removed. We can order these points so that p_1, \dots, p_ℓ are in the interior of $\bar{\Sigma}$ (there might be no interior punctures, in which case we take $\ell = 0$) and $p_{\ell+1}, \dots, p_k$ are in the boundary of $\bar{\Sigma}$. Then each p_i for $i \in \{1, \dots, \ell\}$ corresponds to an end of Σ that is away from its boundary, we will refer to these ends as **interior***

ends. For $i \in \{\ell + 1, \dots, k\}$, each p_i corresponds to an end of Σ that contains a noncompact part of $\partial\Sigma$, we refer to these ends as **boundary ends**.

Remark 3.11. It is clear that each interior end of Σ is conformal to a punctured disk $D_* = \{z \in \mathbb{C} : 0 < |z| < 1\}$. The boundary ends of Σ are conformal to a semi-open punctured half-disk $D_*^+ = \{z \in D_* : \operatorname{Re}(z) \geq 0\}$. To see this fact, note that a neighborhood of a point p_i for $i \in \{\ell + 1, \dots, k\}$ in the double $\tilde{\Sigma}$ is conformal to a punctured disk. We can assume that this neighborhood is symmetric with respect to the isometry Φ , and hence Φ induces a conformal, orientation-reversing diffeomorphism on the punctured disk. The composition of this transformation with complex conjugation must be an injective holomorphic map from the punctured disk to itself, and hence must be a rotation. So we can assume that the part of this punctured disk corresponding to Σ is D_*^+ .

In what follows we show two consequences of Theorem 3.4. The first such consequence is that, under some conditions on the contact angle, noncompact capillary surfaces in weakly convex domains of \mathbb{R}^3 must be minimal.

Corollary 3.12. Let M be a weakly mean-convex domain of \mathbb{R}^3 and let Σ be a noncompact capillary surface with finite index immersed in M at a constant angle $\theta \in (0, \pi/2)$. Then Σ must be a minimal surface. If M is a half-space, the assumption on the angle can be weakened to $\theta \in (0, \pi/2]$.

Proof. It follows from Theorem 3.4 that Σ is conformally equivalent to a compact

Riemann surface $\bar{\Sigma}$ punctured at finitely many points, and

$$\int_{\Sigma} H_{\Sigma}^2 + \int_{\partial\Sigma} H_{\Sigma} \cos \theta < \infty. \quad (3.14)$$

Suppose Σ has an interior end. We choose a geodesic ray $\gamma : [0, \infty) \rightarrow \Sigma$ contained in an interior end. Fix small $\epsilon_0 > 0$ and consider a sequence of points $p_j = \gamma(2j\epsilon_0)$ such that the geodesic disks $D_{\epsilon_0}(p_j)$ of radius ϵ_0 around p_j satisfy $D_{\epsilon_0}(p_j) \cap D_{\epsilon_0}(p_k) = \emptyset$ whenever $j \neq k$. From the uniform lower bound on the area of geodesic disks in [28, Theorem 3] it follows that

$$A(\Sigma) \geq \sum_{j=0}^n A(D_{\epsilon_0}(p_j)) \geq (n+1)C(\epsilon_0),$$

where A is the area and $C(\epsilon_0)$ is a positive constant depending only on ϵ_0 . Since above inequality holds for any $n \in \mathbb{N}$, we obtain that Σ has infinite area, thus it follows from (3.14) that $H_{\Sigma} = 0$.

If Σ has no interior end, it has at least one boundary end since it is noncompact. Then it is easy to see that $\partial\Sigma$ has infinite length. Thus, if $\theta \in (0, \pi/2)$, we must have $H_{\Sigma} = 0$. This completes the first part of the theorem.

If $\theta \in (0, \pi/2]$ and M is a half-space, we only need to deal with the special case where $\theta = \pi/2$ and Σ has no interior ends. Denote a noncompact boundary component of Σ by E , choose a sequence of disjoint points $p_j \in E$ and a small $\epsilon_0 > 0$ such that geodesic disks $D_{\epsilon_0}(p_j)$ are disjoint. By using a similar idea to the use used by Frensel [28], we show that $A(D_{\epsilon_0}(p_j)) \geq C(\epsilon_0)$ where $C(\epsilon_0)$ is a positive constant depending only on ϵ_0 . Indeed, without loss of generality, we assume that p_j is the

origin and let D_μ be the geodesic disk centered at origin with radius μ . For small μ , denote $\Gamma_\mu = \partial D_\mu \cap P$ and $I_\mu = \partial D_\mu \setminus \Gamma_\mu$ where P is the plane, i.e., boundary of the half-space. It is well known that $\Delta_\Sigma |x|^2 = 4 + 2H_\Sigma \langle \nu, x \rangle$. We integrate both sides of this equation on the disk D_μ to obtain

$$(4 - 2H_\Sigma \mu)A(D_\mu) \leq 4|D_\mu| - 2H_\Sigma \int_{D_\mu} |x| \leq \int_{D_\mu} \Delta_\Sigma |x|^2.$$

By the divergence theorem,

$$\int_{D_\mu} \Delta_\Sigma |x|^2 = \int_{\partial D_\mu} 2\langle \eta, x \rangle \leq 2\mu l(I_\mu),$$

where l denotes the length. In addition, from the co-area formula it follows that

$$\frac{\partial A(D_\mu)}{\partial \mu} \geq l(I_\mu).$$

Combining above equations results in

$$\left(\frac{A(D_\mu)}{\mu^2} \right)' \geq -H_\Sigma \frac{A(D_\mu)}{\mu^2},$$

equivalently,

$$\left(\log \frac{A(D_\mu)}{\mu^2} \right)' \geq -H_\Sigma.$$

We then show the claim by integrating both sides over (a, ϵ_0) and letting a tend to zero.

Hence the surface has infinite area and thus (3.14) implies again that $H_\Sigma = 0$. \square

The second corollary classifies the conformal types of strongly stable noncompact capillary surfaces.

Corollary 3.13. *Let M be an oriented 3-manifold with smooth boundary and let Σ*

be a strongly stable noncompact capillary surface immersed in M at a constant angle θ . Assume that $R_M + H_\Sigma^2 \geq 0$ in Σ and $H_{\partial M} + H_\Sigma \cos \theta \geq 0$ along $\partial\Sigma$. Then the compact Riemann surface $\bar{\Sigma}$ is a disk and the ends of Σ can only have one of the following configurations:

1. There are two boundary ends and no interior ends.
2. There are no boundary ends and a single interior end.
3. There is a single boundary end and no interior ends.

Moreover, if (1) or (2) holds, then Σ is totally geodesic, $R_M = 0$ in Σ and $H_{\partial M} = 0$ along $\partial\Sigma$.

Proof. Since Σ is strongly stable, it accepts a globally defined positive Jacobi function u . Considering a metric $d\hat{s}^2$ as in the proof of Theorem 3.4 we can conclude that $\hat{K}_\Sigma \geq 0$ on Σ and $\hat{\kappa}_{\partial\Sigma} \geq 0$ in all of $\partial\Sigma$, furthermore

$$\int_{\Sigma} \hat{K}_\Sigma d\hat{A} \geq 0$$

with equality if and only if Σ is totally geodesic and $R_M = 0$ on Σ , additionally we have that

$$\int_{\partial\Sigma} \hat{\kappa}_{\partial\Sigma} d\hat{\ell} \geq 0$$

with equality if and only if $H_{\partial M} + H_\Sigma \cos \theta = 0$ along $\partial\Sigma$.

It follows from the Cohn-Vossen Theorem [51, Theorem 2.2.1] that

$$\int_{\Sigma} \hat{K}_\Sigma d\hat{A} + \int_{\partial\Sigma} \hat{\kappa}_{\partial\Sigma} d\hat{\ell} \leq \pi(4 - 4g - 2r - k - \ell),$$

where g and r are the genus and number of boundary components of $\bar{\Sigma}$, respectively, and k, ℓ are the total number of ends and the number of interior ends of Σ , respectively. Since Σ has a boundary and is noncompact, we conclude that $k \geq 1$ and $r \geq 1$. It then follows that $g = 0, r = 1$, and $k + \ell = 1$ or 2 . Note that

$$k + \ell = \#\{\text{boundary ends}\} + 2 \times \#\{\text{interior ends}\},$$

so the only possible configurations of ends are the ones listed in (1)–(3). In the case where $k + \ell = 2$, we conclude that \hat{K}_Σ and $\hat{\kappa}_{\partial\Sigma}$ vanish, hence Σ is totally geodesic, $R_M = 0$ along Σ and $H_{\partial M} = 0$ along $\partial\Sigma$, proving the rigidity statement. \square

Remark 3.14. *Observe that each of the three possibilities of Corollary 3.13 does occur. For an example of:*

1. *consider $M = \mathbb{R}^2 \times [0, 1]$ and let Σ be an infinite flat strip in M meeting the boundary at a constant angle $\theta \in (0, \pi)$;*
2. *let $M = \mathbb{S}^1 \times \mathbb{R}^+ \times \mathbb{R}$ and take $\Sigma = \mathbb{S}^1 \times \mathbb{R}^+ \times \{0\}$;*
3. *consider a half-plane in a half-space of \mathbb{R}^3 .*

Remark 3.15. *Combining the inequality in Theorem 3.4 and Corollary 3.13, we can also show that there are no strongly stable noncompact capillary minimal surfaces Σ in a 3-manifold M with nonnegative scalar curvature and uniformly mean-convex boundary, i.e., $R_M \geq 0$ and $H_{\partial M} \geq c > 0$. Indeed, with such curvature assumptions the inequality in Theorem 3.4 implies that $\partial\Sigma$ must be compact.*

However, the arguments in the proof of Corollary 3.13 show that Σ has a single boundary end and no interior ends, which is a contradiction.

3.3.2 Weakly stable capillary surfaces in a half-space of \mathbb{R}^3

Capillary surfaces in a half-space of \mathbb{R}^3 are analogous to complete CMC surfaces in \mathbb{R}^3 in the sense that these are local models for capillary surfaces in any 3-manifold. Hence the characterization of these capillary surfaces plays an important role in the study of capillary surfaces in general 3-manifolds. The characterization of the stable surfaces is particularly important, and that is the content of Theorem 3.5, namely, we show that the only noncompact capillary surface immersed in a half-space of \mathbb{R}^3 at a constant angle θ such that $H_\Sigma \cos \theta \geq 0$ is a half-plane.

Proof of Theorem 3.5

It is clear from (1.13) and (1.12) that half-planes are weakly stable in a half-space of \mathbb{R}^3 . To show the other implication, let us assume without loss of generality that the half-space is $\{x_3 \geq 0\}$. It suffices to prove the following claim:

Claim 1. *If $|A_\Sigma|$ is not identically zero, then there exists a compactly supported function v on Σ such that*

$$Q(v, v) < 0, \quad \text{and} \quad \int_{\Sigma} v = 0.$$

Since Σ is conformally equivalent to a compact Riemann surface $\bar{\Sigma}$ with boundary and finite punctures p_1, \dots, p_k . Without loss of generality, we assume that p_1, \dots, p_ℓ

are punctures in the interior while $p_{\ell+1}, \dots, p_k$ are punctures on the boundary. Thus there exists a compact subset $\Sigma_0 \subset \Sigma$ such that each component Σ_i of $\Sigma \setminus \Sigma_0$ is conformally equivalent to either a semi-cylinder $\mathbb{S}^1 \times (0, \infty)$ or a half-semi-cylinder $\mathbb{S}_+^1 \times (0, \infty)$ with the standard product metric.

Since $|A_\Sigma|$ is not identically zero, there exists $p \in \Sigma$ such that $|A_\Sigma(p)| \neq 0$. We can choose Σ_0 such that $p \in \Sigma_0$ and

$$\int_{\Sigma_0} |A_\Sigma|^2 > \frac{12(k + \ell)\pi}{r(1 - \cos \theta)} \quad (3.15)$$

for some fixed $r > 0$ large enough that we will choose later.

We parametrize each end Σ_i by conformal coordinates $(\omega, y_i) \in \mathbb{S}^1 \times (0, \infty)$. For each $a > 0$, we can define functions $\varphi_i : \Sigma_i \rightarrow \mathbb{R}_+$ by

$$\varphi_i(y_i) = \begin{cases} 1 - \frac{y_i}{r}, & 0 \leq y_i \leq 2r \\ -1, & 2r \leq y_i \leq 2r + a \\ -1 + \frac{y_i - (2r + a)}{r}, & 2r + a \leq y_i \leq 3r + a \\ 0, & 3r + a \leq y_i. \end{cases} \quad (3.16)$$

We then define a function $\varphi_a : \Sigma \rightarrow \mathbb{R}_+$ such that $\varphi_a \equiv 1$ on Σ_0 and $\varphi_a = \varphi_i$ on Σ_i for $i = 1, \dots, k$. Let $u : \Sigma \rightarrow \mathbb{R}_+$ be given by

$$u = \frac{1}{\sin \theta} + \cot \theta \langle \nu, -E_3 \rangle.$$

where ν is the unit normal vector field of Σ and E_3 is the 3rd standard coordinate vector field. Note that $0 < \frac{1 - \cos \theta}{\sin \theta} \leq u \leq \frac{2}{\sin \theta}$. We claim that there exists an $a = a_0 > 0$

such that

$$\int_{\Sigma} u\varphi_{a_0} < 0.$$

In fact,

$$\begin{aligned} \int_{\Sigma} u\varphi_a &\leq \int_{\Sigma_0 \cup (\cup_{i=1}^n \{0 \leq y_i \leq 2r\})} u\varphi_a + \int_{\cup_{i=1}^n \{2r \leq y_i \leq 3r+a\}} -u \\ &\leq \int_{\Sigma_0 \cup (\cup_{i=1}^n \{0 \leq y_i \leq 2r\})} u\varphi_a - \frac{1 - \cos \theta}{\sin \theta} \sum_{i=1}^n |\{2r \leq y_i \leq 3r+a\}| \end{aligned}$$

where the first term on the right-hand side is independent of a , thus a fixed number.

Since the area of Σ is infinite, we can choose $a_0 > 0$ such that the desired inequality holds.

We now define new functions $\psi_i : \Sigma_i \rightarrow \mathbb{R}_+$ by

$$\psi_i(y_i) = \begin{cases} 1 - \frac{y_i}{r}, & 0 \leq y_i \leq r + b \\ -b/r, & r + b \leq y_i \leq 3r + a_0 - b \\ -1 + \frac{y_i - (2r + a_0)}{r}, & 3r + a_0 - b \leq y_i \leq 3r + a_0 \\ 0, & 3r + a_0 \leq y_i. \end{cases} \quad (3.17)$$

We then define a function $\psi_{a_0, b} : \Sigma \rightarrow \mathbb{R}_+$ such that $\psi_{a_0, b} = 1$ on Σ_0 and $\psi_{a_0, b} = \psi_i$ on Σ_i . Note that $\psi_{a_0, r} = \varphi_{a_0}$ and $\psi_{a_0, 0} \geq 0$. Since the integral $\int_{\Sigma} u\psi_{a_0, b}$ is continuous in b and $\int_{\Sigma} u\psi_{a_0, 0} > 0$, there exists a $0 < b_0 \leq r$ such that

$$\int_{\Sigma} u\psi_{a_0, b_0} = 0.$$

Let $v = u\psi_{a_0, b_0}$, notice it is a piece-wise smooth function with compact support.

We now calculate $Q(v, v)$. First note that

$$\begin{aligned}
u|_{\partial\Sigma} &= \frac{1}{\sin\theta} + \cot\theta\langle \sin\theta T + \cos\theta E_3, -E_3 \rangle \\
&= \frac{1}{\sin\theta} - \frac{\cos^2\theta}{\sin\theta} \\
&= \sin\theta \\
&= \langle \eta, -E_3 \rangle.
\end{aligned}$$

It follows from simple computations that η is a principal direction of Σ (see e.g. [4, Lemma 2.2], [53, Prop 2.1]). Using this fact we obtain

$$\begin{aligned}
\frac{\partial u}{\partial \eta} &= \cot\theta\langle D_\eta \nu, -E_3 \rangle \\
&= \cot\theta A_\Sigma(\eta, \eta)\langle \eta, -E_3 \rangle \\
&= \cot\theta A_\Sigma(\eta, \eta)u|_{\partial\Sigma}. \\
&= qu.
\end{aligned} \tag{3.18}$$

Since $\Delta\nu = -|A_\Sigma|^2\nu$ in the coordinate-wise sense (see e.g [10, Proposition 2.24]), we have

$$\Delta u + |A_\Sigma|^2 u = \frac{|A_\Sigma|^2}{\sin\theta}. \tag{3.19}$$

Then we have that

$$\begin{aligned}
Q(v, v) &= \int_{\Sigma} u^2 |\nabla \psi_{a_0, b_0}|^2 + \psi_{a_0, b_0}^2 |\nabla u|^2 + 2u \psi_{a_0, b_0} \langle \nabla u, \nabla \psi_{a_0, b_0} \rangle - \int_{\partial \Sigma} q u^2 \psi_{a_0, b_0}^2 \\
&= \int_{\Sigma} u^2 |\nabla \psi_{a_0, b_0}|^2 + \frac{1}{2} \psi_{a_0, b_0}^2 \Delta(u^2) - \psi_{a_0, b_0}^2 u \Delta u + \frac{1}{2} \langle \nabla(u^2), \nabla(\psi_{a_0, b_0}^2) \rangle \\
&\quad - \int_{\partial \Sigma} q u^2 \psi_{a_0, b_0}^2 \\
&= \int_{\Sigma} u^2 |\nabla \psi_{a_0, b_0}|^2 - \psi_{a_0, b_0}^2 u \Delta u + \int_{\partial \Sigma} u \psi_{a_0, b_0}^2 \left(\frac{\partial u}{\partial \eta} - q u \right) \\
&= \int_{\Sigma} u^2 |\nabla \psi_{a_0, b_0}|^2 - \psi_{a_0, b_0}^2 u \frac{|A_{\Sigma}|^2}{\sin \theta} \\
&\leq \frac{4}{\sin^2 \theta} \int_{\Sigma} |\nabla \psi_{a_0, b_0}|^2 - \frac{1 - \cos \theta}{\sin^2 \theta} \int_{\Sigma} \psi_{a_0, b_0}^2 |A_{\Sigma}|^2 \\
&\leq \frac{4}{\sin^2 \theta} \sum_{i=1}^n \int_{\Sigma_i} |\nabla \psi_{a_0, b_0}|^2 - \frac{1 - \cos \theta}{\sin^2 \theta} \int_{\Sigma_0} |A_{\Sigma}|^2 \tag{3.20}
\end{aligned}$$

Since the Dirichlet energy is conformally invariant in dimension two, we have that for

$i = 1, \dots, \ell$

$$\int_{\Sigma_i} |\nabla \psi_{a_0, b_0}|^2 = \int_{\Sigma_i} |D\phi_i|^2 d\theta dy_i = \frac{2\pi(r + 2b_0)}{r^2} \leq \frac{6\pi}{r}$$

and for $i = \ell + 1, \dots, k$

$$\int_{\Sigma_i} |\nabla \psi_{a_0, b_0}|^2 \leq \frac{3\pi}{r}.$$

Then it follows from (3.20) and (3.15) that

$$Q(v, v) < 0.$$

This completes the proof of the claim. Since the surface Σ is weakly stable, by the claim we must have that $|A_{\Sigma}| \equiv 0$ and thus Σ is a half-plane. \square

Remark 3.16. *Note that same arguments show that a weakly stable noncompact*

capillary surface (without angle condition) immersed in a half-space of \mathbb{R}^3 cannot have compact boundary. Also, using the same test function we constructed above, one can show directly that there is no strongly stable compact capillary surface immersed in a half-space of \mathbb{R}^3 .

3.3.3 L^2 characterization of the strong index

Fisher-Colbrie [26, Proposition 2] showed that the index of a complete noncompact minimal surface can be realized as the cardinality of a set of L^2 eigenfunctions defined globally on the surface. This fact is important to study the relation between the index and the topology of these surfaces (see e.g. the work of Ros [41, Theorem 17]). In what follows we will show that something similar also holds for capillary surfaces. In order to show this result we need a Poincaré-type inequality (Lemma 3.18)

Let Σ be a surface, if Σ has a boundary, we suppose the boundary is contained in Σ , so it is complete in the sense of metric spaces. Let w be a nonnegative Lipschitz function on Σ with compact support. Note that, by our assumptions, w can be positive on the boundary of Σ . Let Ω be the support of w and $\Gamma = \Omega \cap \partial\Sigma$. For a function $\varphi \in C(\Omega)$ define the norms

$$\|\varphi\|_{\Omega}^2 = \int_{\Omega} \varphi^2, \quad \|\varphi\|_{\Omega, w}^2 = \int_{\Omega} w\varphi^2.$$

In order to show our Poincaré-type inequality we need the following results:

Proposition 3.17. *There is a constant $C > 0$ depending only on Σ and w such that*

$$\int_{\Gamma} w\varphi^2 \leq C\|\varphi\|_{\Omega}(\|\varphi\|_{\Omega} + \|\|\nabla\varphi\|\|_{\Omega,w}) \quad (3.21)$$

for all $\varphi \in C^{\infty}(\Sigma)$.

Proof. Let $\tilde{\eta}$ be a compactly supported smooth vector field in Σ that extends the conormal η . Then for $\varphi \in C^{\infty}(\Sigma)$ we have that

$$\begin{aligned} \int_{\Gamma} w\varphi^2 &= \int_{\Omega} \operatorname{div}(w\varphi^2\tilde{\eta}) \\ &= \int_{\Omega} \varphi^2\langle\nabla w, \tilde{\eta}\rangle + 2w\varphi\langle\nabla\varphi, \tilde{\eta}\rangle + w\varphi^2\operatorname{div}(\tilde{\eta}) \\ &\leq 2\left(\int_{\Omega} w\varphi^2\right)^{\frac{1}{2}}\left(\int_{\Omega} w\langle\nabla\varphi, \tilde{\eta}\rangle^2\right)^{\frac{1}{2}} + \int_{\Omega} \varphi^2(|\nabla w|\|\tilde{\eta}\| + w|\operatorname{div}(\tilde{\eta})|) \\ &\leq C\|\varphi\|_{\Omega}(\|\varphi\|_{\Omega} + \|\|\nabla\varphi\|\|_{\Omega,w}), \end{aligned}$$

where C depends only on w and $\tilde{\eta}$. □

Lemma 3.18. *For every $\epsilon > 0$ there is a constant $C > 0$ depending only on Σ, w and ϵ such that*

$$\int_{\Gamma} w\varphi^2 \leq \epsilon \int_{\Omega} w|\nabla\varphi|^2 + C \int_{\Omega} \varphi^2$$

for all $\varphi \in C^{\infty}(\Sigma)$.

Proof. Suppose by contradiction that the result does not hold. Then, for some $\epsilon_0 > 0$ there is a sequence $\varphi_n \in C^{\infty}(\Sigma)$ such that for all $n \in \mathbb{N}$

$$\int_{\Gamma} w\varphi_n^2 > \epsilon_0 \int_{\Omega} w|\nabla\varphi_n|^2 + n \int_{\Omega} \varphi_n^2. \quad (3.22)$$

By rescaling we can assume that $\|\|\nabla\varphi_n\|\|_{\Omega,w} = 1$.

Firstly, if $\|\varphi_n\|_\Omega \rightarrow \infty$, then by rescaling this sequence we can obtain another sequence of functions $\psi_n \in C^\infty(\Sigma)$ such that $\|\nabla\psi_n\|_{\Omega,w} \rightarrow 0$, $\|\psi_n\|_\Omega = 1$ and $\int_\Gamma w\psi_n^2 > n$. This contradicts (3.21).

Thus we can assume that $\|\varphi_n\|_\Omega$ are uniformly bounded above. It follows from (3.21) that $\int_\Gamma w\varphi_n^2$ are uniformly bounded above, and hence by (3.22) we must have that $\|\varphi_n\|_\Omega \rightarrow 0$. This is impossible since, by (3.21) it would imply that $\int_\Gamma w\varphi_n^2 \rightarrow 0$ but $\int_\Gamma w\varphi_n^2 > \epsilon_0$. \square

Now we are ready to characterize the strong index by eigenfunctions of the Jacobi operator. Note that the functions f_i below are eigenfunctions in the weak sense.

Proposition 3.19. *Let Σ be a noncompact capillary surface immersed in a 3-manifold with smooth boundary and let $I = \text{Index}_s(\Sigma)$. Assume $I < \infty$, then there exists a subspace W of $L^2(\Sigma)$ having an orthonormal basis f_1, \dots, f_I consisting of eigenfunctions for Q with eigenvalues $\lambda_1, \dots, \lambda_I$, respectively, such that each $\lambda_i < 0$. Moreover, for all $\varphi \in C_0^\infty(\Sigma) \cap W^\perp$ we have*

$$Q(\varphi, \varphi) \geq 0.$$

Proof. For all $R > 0$, let $B_R \subset \Sigma$ be a ball of radius R centred at some point that will be fixed through this proof. Take $R_0 \geq 4$ such that $\Sigma \setminus \overline{B_{R_0}}$ is strongly stable.

For $R \geq R_0$, let $\zeta \in C^\infty(\Sigma)$ be such that

$$\zeta = 0 \text{ on } B_R,$$

$$\zeta = 1 \text{ on } \Sigma \setminus B_{2R},$$

and

$$|\nabla\zeta| \leq \frac{6}{R}, \quad |\nabla\zeta|^2 < \frac{4(1-\zeta^2)}{R^2}.$$

For the construction of ζ see [26]. For simplicity we will write $\text{Ric}_M(\nu, \nu) + |A_\Sigma|^2$ as p . Since $\Sigma \setminus \overline{B_R}$ is strongly stable, we have that for all $\varphi \in C_0^\infty(\Sigma)$

$$\int_{\partial\Sigma} q(\zeta\varphi)^2 + \int_\Sigma p(\zeta\varphi)^2 \leq \int_\Sigma |\nabla\zeta\varphi|^2 = \int_\Sigma \zeta^2 |\nabla\varphi|^2 + 2\zeta\varphi \langle \nabla\zeta, \nabla\varphi \rangle + \varphi^2 |\nabla\zeta|^2. \quad (3.23)$$

It follows from the Cauchy-Schwarz inequality and the arithmetic-geometric mean inequality that

$$\int_\Sigma 2\zeta\varphi \langle \nabla\zeta, \nabla\varphi \rangle \leq \int_{B_{2R}} \zeta^2 \varphi^2 + |\nabla\zeta|^2 |\nabla\varphi|^2.$$

Adding $Q(\varphi, \varphi)$ to both sides of (3.23) and rearranging gives

$$\begin{aligned} - \int_{\partial\Sigma} (1-\zeta^2)q\varphi^2 - \int_\Sigma (1-\zeta^2)p\varphi^2 + \int_\Sigma (1-\zeta^2)|\nabla\varphi|^2 \\ \leq Q(\varphi, \varphi) + \int_{B_{2R}} \varphi^2 + \frac{4}{R^2} \int_\Sigma (1-\zeta^2)|\nabla\varphi|^2. \end{aligned} \quad (3.24)$$

Using Lemma 3.18 we conclude

$$\int_{\partial\Sigma} (1-\zeta^2)q\varphi^2 \leq \sup_{B_{2R}} |q| \int_{\partial\Sigma} (1-\zeta^2)\varphi^2 \leq \frac{1}{4} \int_\Sigma (1-\zeta^2)|\nabla\varphi|^2 + C_R \int_{B_{2R}} \varphi^2 \quad (3.25)$$

for some constant C_R depending on R . Since $R \geq 4$ it follows from (3.24) and (3.25)

that

$$\int_{B_R} |\nabla \varphi|^2 \leq 2Q(\varphi, \varphi) + C'_R \int_{B_{2R}} \varphi^2 \quad (3.26)$$

for some new constant C'_R depending only on R .

Let $\rho > R_0$, then there are functions $f_{1,\rho}, \dots, f_{I,\rho} \in L^2(B_\rho)$ forming an orthonormal set such that each $f_{i,\rho}$ is an eigenfunction of Q in the sense that

$$\begin{cases} \Delta f_{i,\rho} + p f_{i,\rho} + \lambda_{i,\rho} f_{i,\rho} = 0 & \text{in } B_\rho \\ \frac{\partial f_{i,\rho}}{\partial \eta} - q f_{i,\rho} = 0 & \text{on } \partial \Sigma \cap B_\rho \\ f_{i,\rho} = 0 & \text{on } \partial B_\rho \setminus \partial \Sigma. \end{cases}$$

for some eigenvalue $\lambda_{i,\rho} < 0$. Extend $f_{i,\rho}$ to be 0 outside B_ρ . Since $\max_i \lambda_{i,\rho}$ is decreasing in ρ we can assume that $\lambda_{i,\rho} \leq -\epsilon_0$ for some constant $\epsilon_0 > 0$. In addition, from (3.26) it follows that $Q(f_{i,\rho}, f_{i,\rho}) \geq -C_R \int_\Sigma f_{i,\rho}^2$, thus there is a constant $C > 0$ such that $\lambda_{i,\rho} \geq -C$ for all i and ρ as above. Notice that

$$\begin{aligned} \int_\Sigma 2\zeta f_{i,\rho} \langle \nabla f_{i,\rho}, \nabla \zeta \rangle &= \frac{1}{2} \int_\Sigma \langle \nabla f_{i,\rho}^2, \nabla \zeta^2 \rangle \\ &= \int_{\partial \Sigma} \zeta^2 f_{i,\rho} \frac{\partial f_{i,\rho}}{\partial \eta} - \int_\Sigma \zeta^2 (f_{i,\rho} \Delta f_{i,\rho} + |\nabla f_{i,\rho}|^2) \\ &= \int_{\partial \Sigma} q (\zeta f_{i,\rho})^2 - \int_\Sigma \zeta^2 (-(p + \lambda_{i,\rho}) f_{i,\rho}^2 + |\nabla f_{i,\rho}|^2). \end{aligned}$$

Substituting it in (3.23) we can conclude that

$$-\lambda_{i,\rho} \int_\Sigma \zeta^2 f_{i,\rho}^2 \leq \int_\Sigma |\nabla \zeta|^2 f_{i,\rho}^2 \leq \frac{36}{R^2}.$$

Since $\lambda_{i,\rho} \leq -\epsilon_0$ we can conclude that

$$\int_{\Sigma \setminus B_{2R}} f_{i,\rho}^2 \leq \frac{C'}{R^2} \quad (3.27)$$

for some constant C' . Inequality (3.26) and the fact that $Q(f_{i,\rho}, f_{i,\rho}) < 0$ imply

$$\int_{B_R} |\nabla f_{i,\rho}|^2 + f_{i,\rho}^2 \leq C'_R + 1. \quad (3.28)$$

By the Rellich-Kondrachov compactness theorem the embedding $H^1(B_R) \rightarrow L^2(B_R)$ and the trace map $H^1(B_R) \rightarrow L^2(B_R \cap \partial\Sigma)$ are both compact (see [1, Theorem 6.3]). So using (3.28) and a standard diagonal argument we can construct a sequence $R_j \rightarrow \infty$ and functions f_1, \dots, f_I in Σ such that for all $i = 1, \dots, I$ and $R > 0$ we have as $j \rightarrow \infty$

$$f_{i,R_j} \rightarrow f_i \text{ strongly in } L^2(B_R),$$

$$f_{i,R_j}|_{\partial\Sigma} \rightarrow f_i|_{\partial\Sigma} \text{ strongly in } L^2(B_R \cap \partial\Sigma),$$

$$f_{i,R_j} \rightarrow f_i \text{ weakly in } H^1(B_R).$$

It follows that the functions f_i are eigenfunctions of Q in the weak sense, and the corresponding eigenvalues are $\lambda_i = \lim_{j \rightarrow \infty} \lambda_{i,R_j}$. Furthermore, it follows from (3.27) that f_1, \dots, f_I form an orthonormal set. Let W be the linear space spanned by these functions and take $\varphi \in C_0^\infty(\Sigma) \cap W^\perp$, then for all $\rho > R$ we can write

$$\varphi = \sum_{i=1}^I a_{i,\rho} f_{i,\rho} + \varphi_\rho$$

where $a_{i,\rho} = \int_\Sigma \varphi f_{i,\rho}$ and $\varphi_\rho \perp f_{i,\rho}$. It is not hard to see that $a_{i,R_j} \rightarrow 0$ as $j \rightarrow \infty$.

Since λ_{i,R_j} are uniformly bounded below, we can conclude that

$$Q(\varphi, \varphi) = Q(\varphi_{R_j}, \varphi_{R_j}) + \epsilon_{R_j} \geq \epsilon_{R_j} \rightarrow 0.$$

Hence $Q(\varphi, \varphi) \geq 0$. This completes the proof. \square

In the following corollary we give an application of Proposition 3.19. It further characterizes the topology of an *index one* free boundary CMC surface in a 3-manifold with boundary.

Corollary 3.20. *Let M be an oriented flat 3-manifold with smooth weakly convex boundary and let Σ be a noncompact free boundary CMC surface with strong index one immersed in M . Then Σ is conformally equivalent to a compact Riemann surface $\bar{\Sigma}$ of genus g and r boundary components punctured at k points, and*

$$k + \ell \leq 4 \left\lfloor \frac{1-g}{2} \right\rfloor - 2r + 8,$$

where ℓ is the number of boundary punctures. In particular, $g \leq 3$, $r \leq 3$, $g + r \leq 4$ and $k + \ell \leq 6$.

Proof. By Theorem 3.4, Σ is conformally equivalent to a compact Riemann surface $\bar{\Sigma}$ of genus g with r boundary components punctured at k points p_1, \dots, p_k . Let ds^2 be the metric on Σ . Let ρ be a positive smooth function such that $(\Sigma, \rho ds^2)$ is complete as a metric space and has finite area. Indeed, since each end is conformal either to a punctured disk or half-disk, we can take conformal coordinates centered at zero near each end and we can let $\rho = 1/|z|$ in each of these neighborhoods, so that the metric is

complete and the surface has finite area. Let $d\tilde{s}^2 = \rho ds^2$. Since the Dirichlet integral is conformally invariant in dimension 2, we have that

$$\tilde{Q}(u, u) = \int_{\Sigma} |\tilde{\nabla}u|^2 - |A_{\Sigma}|^2 \rho^{-1} u^2 d\tilde{A} - \int_{\partial\Sigma} \rho^{-1/2} q u^2 d\tilde{\ell}$$

coincides with the stability operator $Q(u, u)$ for any compactly supported function u on Σ . Since (Σ, ds^2) has index one, \tilde{Q} has index one. By Proposition 3.19, there exists an eigenfunction $f_1 \in L^2(\Sigma, d\tilde{s}^2)$ corresponding to a negative eigenvalue for the following eigenvalue problem:

$$\begin{cases} \tilde{\Delta}u + |A_{\Sigma}|^2 \rho^{-1} u + \lambda u = 0 & \text{in } \Sigma \\ \frac{\partial u}{\partial \tilde{\eta}} = \rho^{-1/2} q u & \text{on } \partial\Sigma. \end{cases}$$

The construction in Proposition 3.19 and standard arguments imply that f_1 is strictly positive and smooth. Furthermore, we have that $Q(u, u) \geq 0$ for any $u \in C_c^\infty(\Sigma)$ that is $L^2(\Sigma, d\tilde{s}^2)$ -orthogonal to f_1 .

We claim that $Q(u, u) \geq 0$ for any $u \in C^\infty(\bar{\Sigma})$ that is $L^2(\Sigma, d\tilde{s}^2)$ -orthogonal to f_1 . As $k_{\partial\Sigma} \geq 0$, (3.1) implies that $|A_{\Sigma}|^2$ and q are integrable, thus $Q(u, u)$ is finite. Note that $u f_1$ is in $L^1(\Sigma, d\tilde{s}^2)$ since $u, f_1 \in L^2(\Sigma, d\tilde{s}^2)$. The proof of the claim follows from the standard cut-off arguments (see e.g. [39, Prop. 1.1]).

We may view $\bar{\Sigma}$ as a compact domain of a closed orientable surface $\hat{\Sigma}$ of genus g by sticking disks to the boundary components of $\bar{\Sigma}$. Let $\Phi = (\Phi_1, \Phi_2, \Phi_3) : \hat{\Sigma} \rightarrow \mathbb{S}^2$ be a holomorphic map with degree less than or equal to $\lfloor \frac{g+3}{2} \rfloor$. Since $(\Sigma, d\tilde{s}^2)$ has finite area it follows that f_1 is integrable, so there exists a conformal diffeomorphism

Ψ of \mathbb{S}^2 such that the composed map $\Psi \circ \Phi$ satisfies

$$\int_{\tilde{\Sigma}} f_1(\Psi \circ \Phi) = \vec{0}.$$

This is usually called the Hersch balancing trick, see [15, Lemma 5.1] for a proof. It then follows that for $i = 1, 2, 3$

$$\int_{\Sigma} |\nabla(\Psi \circ \Phi)_i|^2 - |A_{\Sigma}|^2(\Psi \circ \Phi)_i^2 - \int_{\partial\Sigma} q(\Psi \circ \Phi)_i^2 \geq 0.$$

Summing over $i = 1, 2, 3$ yields

$$\begin{aligned} 8\pi \deg(\Psi) &= \int_{\Sigma} |\nabla(\Psi \circ \Phi)|^2 \\ &\geq \int_{\Sigma} |A_{\Sigma}|^2 + \int_{\partial\Sigma} q \\ &= \int_{\Sigma} H_{\Sigma}^2 - 2K_{\Sigma} + \int_{\partial\Sigma} H_{\partial M} - \kappa_{\partial\Sigma} \\ &\geq - \int_{\Sigma} 2K_{\Sigma} - \int_{\partial\Sigma} 2\kappa_{\partial\Sigma}. \end{aligned}$$

Note that K_{Σ} and $k_{\partial\Sigma}$ are both integrable due to (3.1), then combining above inequality with the Cohn-Vossen inequality (see [51, Theorem 2.2.1]) we obtain

$$8\pi \left\lfloor \frac{g+3}{2} \right\rfloor \geq -2\pi(4 - 4g - 2r - k - l).$$

This completes the proof. □

3.4 Curvature bounds for capillary surfaces

In this section, we obtain curvature estimates for strongly stable capillary surfaces in a 3-manifold with bounded geometry. The methods used to prove Theorem 3.6 are very similar to those used to show Theorem 2.10, for this reason we will concentrate on where the proofs diverge.

Proof of Theorem 3.6

We start as in the proof of Theorem 2.10 by assuming for the sake of contradiction that there is a sequence of 3-manifolds with bounded geometry M_n , a sequence of strongly stable edged capillary surfaces Σ_n immersed in M_n at constant angle θ , and a sequence of points $q_n \in \Sigma_n$ such that

$$|A_n(q_n)| \min\{d_n(q_n, \partial\Sigma_n \setminus \partial M_n), \iota_n, (\sqrt{\Lambda_n})^{-1}, (H_n \cos \theta)_-^{-1}\} > n,$$

where

- A_n , d_n are the second fundamental form and the intrinsic distance of Σ_n , respectively;
- ι_n , Λ_n are the injectivity radius and curvature bounds of M_n , respectively;
- H_n is the mean curvature of Σ_n ;
- $(H_n \cos \theta)_-$ is 0 if $H_n \cos \theta \geq 0$ and is $-H_n \cos \theta$ otherwise.

We can rescale the metrics of the ambient spaces M_n so that we can assume $\Lambda_n = \iota_n = 1$ and $H_n \cos \theta \geq -1$. Fix an $\alpha \in (0, 1)$ and consider the values r_0, Q_0

given by Theorem 1.5 with $\Lambda = \iota = 1$. We assume without loss of generality that $r_0 < 1$. So we have that

$$|A_n(q_n)| \min\{d_n(q_n, \partial\Sigma_n \setminus \partial M_n), r_0\} \geq r_0 n.$$

Let \mathcal{D}_n be the intrinsic disk in Σ_n around q_n with radius $\min\{d_n(q_n, \partial\Sigma_n \setminus \partial M_n), r_0\}$ and take p_n to be the point in \mathcal{D}_n maximizing

$$|A_n(p_n)| d_{\mathcal{D}_n}(p, \partial\mathcal{D}_n \setminus \partial M_n).$$

Writing $\lambda_n = |A_n(p_n)|$ and $R_n = d_{\mathcal{D}_n}(p_n, \partial\mathcal{D}_n \setminus \partial M_n)$, we have that by construction, $\lambda_n R_n > r_0 n$ and $\lambda_n > n$.

Continuing to follow the proof of Theorem 2.10, we let $\tilde{\mathcal{D}}$ be the intrinsic disk in \mathcal{D}_n centered at p_n with radius $R_n/2$. So, by construction we have that for all $p \in \tilde{\mathcal{D}}$

$$|A_n(p)| \leq \frac{\lambda_n R_n}{\min\{d_{\mathcal{D}_n}(p, \partial\mathcal{D}_n \setminus \partial M_n), r_0\}} \leq 2\lambda_n. \quad (3.29)$$

Consider a sequence of rescaled surfaces $\lambda_n(\tilde{\mathcal{D}}_n - p_n)$ as in the proof of Theorem 2.10. Repeating the arguments of this proof, we have that, after passing to a subsequence, we can choose a piece Δ_n of $\lambda_n(\tilde{\mathcal{D}}_n - p_n)$ such that Δ_n has Euclidean second fundamental form with norm bounded above by 5 everywhere and bounded below by 1/2 at the point 0_n that is identified with p_n . Furthermore, the extrinsic Euclidean distance between 0_n and the edge of Δ_n diverges to ∞ as $n \rightarrow \infty$.

In case $\limsup_n \lambda_n d_{\tilde{\mathcal{D}}_n}(p_n, \partial\tilde{\mathcal{D}}_n) = \infty$, we can pass to a subsequence so the distance from $0_n \in \Delta_n$ to its boundary diverges. In this case, we can use the same

arguments from the proof of Theorem 2.10 to find a complete stable limit surface S in Euclidean space that passes through the origin with nonzero second fundamental form, a contradiction to the uniqueness of the plane [23]. The only distinction is that, in this case, the limit ambient space can be a half-space of \mathbb{R}^3 , this however does not require any modification to the arguments.

In case $\limsup_n \lambda_n d_{\tilde{\mathcal{D}}_n}(p_n, \partial\tilde{\mathcal{D}}_n) < \infty$, some modifications to the argument are needed to account for the boundary. Pass to a subsequence so the sequence $\{\lambda_n d_{\tilde{\mathcal{D}}_n}(p_n, \partial\tilde{\mathcal{D}}_n)\}_{n \in \mathbb{N}}$ converges. Note that since the edge of $\lambda_n(\tilde{\mathcal{D}}_n - p_n)$ is diverging to infinity, we must have that $\lim_{n \rightarrow \infty} d_n(p_n, \partial M_n) = 0$. This means that, for n large, the boundary of M_n is identified with a plane in the ambient space of $\lambda_n(\tilde{\mathcal{D}}_n - p_n)$ (case (ii) in Theorem 1.5). Since these planes are at a bounded distance from the origin, we can pass to a subsequence so they converge, hence the ambient space converges to some half-space \mathcal{H} . The construction of the limit surface S proceeds as in the previous case, except that some care needs to be taken near the boundary.

To be more precise, assume there is a sequence of points $x_n \in \partial\Delta_n$ that converge to some point $x \in \partial\mathcal{H}$ and such that the normal vectors of the surfaces Δ_n at x_n converge to some vector v . Then we can parametrize each surface Δ_n near x_n as a graph of a function over the plane passing through x that is normal to $\partial\mathcal{H}$ and contains the component of v that is perpendicular to $\partial\mathcal{H}$. Using arguments analogous to those of Lemma 2.4 we can obtain $C^{2,\alpha}$ bounds for these functions. Since the contact angle θ give control over the derivative of these functions at the boundary, we can use

boundary Schauder estimates [2, Theorem 7.3] to get $C^{2,\alpha}$ estimates at the boundary.

Now, we can again use the Jacobi function to show that the limit surface S is strongly stable, hence a plane by Theorem 3.5. However, this contradicts the fact that S passes through the origin with nonzero second fundamental form.

Remark 3.21. *In the case where Σ is capillary, that is, has empty edge, we interpret $d_\Sigma(p, \partial\Sigma \setminus \partial M)$ to be positive infinity for all $p \in \Sigma$.*

The Corollary below is an analog of Corollary 2.11. Since the proof of both results is essentially the same, we will omit it here.

Corollary 3.22. *Fix $\theta \in (0, \pi/2]$ and take C, M, ι and Λ as in the statement of Theorem 3.6. Let Σ be an edged capillary surface with finite index immersed in M at constant angle θ . Assume $H_\Sigma \geq \sqrt{2}C \max\{\iota^{-1}, \sqrt{\Lambda}\}$. Then,*

- *in case Σ has nonempty edge,*

$$d_\Sigma(p, \partial\Sigma \setminus \partial M) \leq (\text{Index}_s(\Sigma) + 1) \frac{2\sqrt{2}C}{H_\Sigma} \quad (3.30)$$

for all $p \in \Sigma$; and

- *in case Σ is capillary*

$$\text{diam}_\Sigma(\Sigma) \leq \text{Index}_s(\Sigma) \frac{2\sqrt{2}C}{H_\Sigma}, \quad (3.31)$$

where $\text{diam}_\Sigma(\Sigma)$ is the intrinsic diameter of Σ .

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