(Geometric Langlands correspondence Series)

Conformal field theory
& Geometric Langlands

Part I : The protagonists
– $\mathcal{D}$-modules & opers

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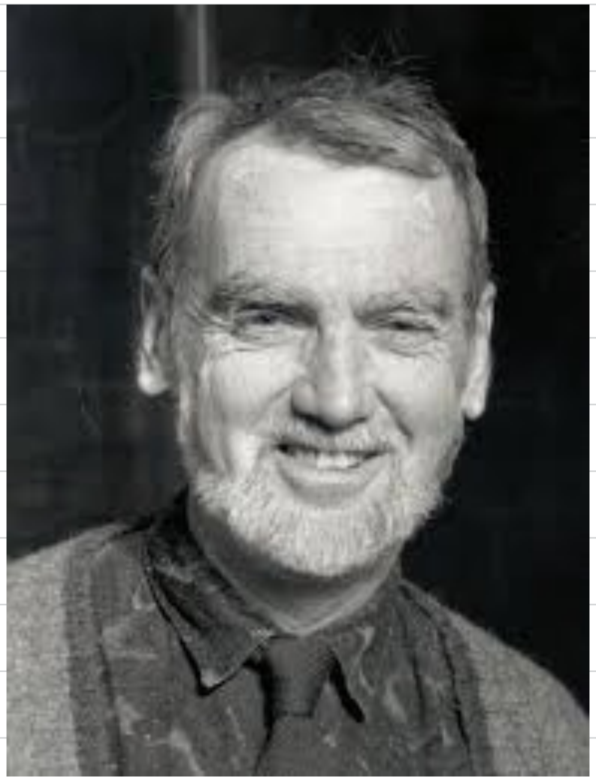
Outline

① Conformal blocks

② $\mathcal{D}$-modules & Opers



Robert Langlands

Alexander Beilinson

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Part I: Conformal Blocks

§1. Conformal blocks over $\mathcal{M}_{g,n}$

§2. Conformal blocks over Bun_G

§1. Conformal blocks over $\mathcal{M}_{g,n}$

Affine Lie algebra $\hat{\mathfrak{g}}$

$\mathfrak{g}$ = simple lie algebra

$$0 \to \mathbb{C}\mathbb{1} \longrightarrow \hat{\mathfrak{g}} \longrightarrow \mathfrak{g} \otimes \mathbb{C}((t)) \longrightarrow 0$$

$\mathbb{1} \in C(\hat{\mathfrak{g}})$ central extension.

$$[A \otimes f(t), B \otimes g(t)] = [A,B]_{\mathfrak{g}} \otimes fg - \kappa_0(A,B) \int f\, dg \cdot \mathbb{1}$$

normalized non-degen inner product

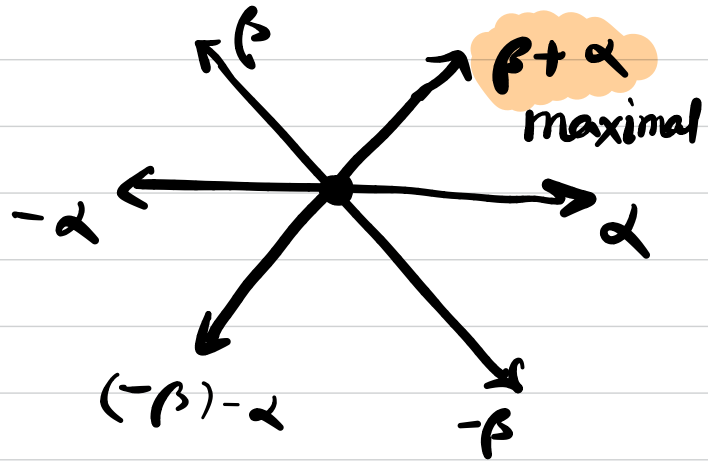
Ex: $\mathfrak{g} = \mathfrak{sl}_2$

$$\kappa_0(A,B) = \operatorname{tr}(AB)$$

Some Root system facts

$$\begin{array}{cccc} & e & f & h \\ & x & y & h \\ sl_2 = \langle & \begin{pmatrix}0&1\\0&0\end{pmatrix}, & \begin{pmatrix}0&0\\1&0\end{pmatrix}, & \begin{pmatrix}1&0\\0&-1\end{pmatrix} \rangle \end{array}$$

$\mathfrak{g} = A_2 = sl_3$



$(x|y) = tr(ad(x)ad(y))$

$ad(x): \mathfrak{g}\to\mathfrak{g}: a\mapsto [x,a]$

$(\theta|\theta) = 2$

weight lattice

$P = \{\lambda\in\mathfrak{h}^* \mid \frac{2(\lambda|\alpha)}{(\lambda|\alpha)}\in\mathbb{Z} \mid \forall\alpha\in\overset{\text{root system}}{\Delta}\}$

$U(sl_2) = \bigoplus_{i\geq 0} U_i$

$U_0 = \mathbb{C}$

$U_1 = \mathbb{C}\oplus\mathbb{C}x\oplus\mathbb{C}y\oplus\mathbb{C}h$

$U_2 = U_1\oplus\mathbb{C}xy\oplus\mathbb{C}xh\oplus\mathbb{C}yh$
$\oplus\mathbb{C}x^2\oplus\mathbb{C}y^2\oplus\mathbb{C}h^2$

$U_i/U_{i-1} = \mathbb{C}[x_1,x_2,x_3]_i$

$\bigoplus U_i/U_{i-1} = \mathbb{C}[x_1,x_2,x_3]$

PBW Basis

λ weight

V_λ = irred. $\mathfrak{g}$-rep'n of highest weight λ

$\mathfrak{b} = \mathfrak{h}\oplus\mathfrak{n}_+ \curvearrowright v^+$ by $\mathfrak{n}_+\cdot v^+ = 0$
$h\cdot v^+ = \lambda(h)v^+$

$Z_\lambda = U(\mathfrak{g})\otimes_{U(\mathfrak{b})}\mathbb{C}v^+$

$\cup$

I_λ maximal proper submod

$V_\lambda = Z_\lambda/I_\lambda$

Its rep'ns

let $\lambda \in P_\ell^+$

$P_\ell^+ = \{ \lambda \in P_+ \mid (\lambda | \theta) \leq \ell \}$

positive weight

highest root

V_λ = irred. $\mathfrak{g}$-rep'n of highest weight λ

$$M_\lambda = \underset{U(\hat{\mathfrak{g}} \otimes t\mathfrak{g}[[t]] \oplus \mathbb{C}\mathbb{1})}{U(\hat{\mathfrak{g}}) \otimes V_\lambda}$$

$\hat{\mathfrak{g}} \otimes t\mathfrak{g}[[t]] \overset{=0}{\curvearrowright} V_\lambda$

$\ell \cdot \mathrm{Id}$

$\mathbb{C}\mathbb{1} \curvearrowright V_\lambda$

$\ell \in \mathbb{C}$

$$L_\lambda = M_\lambda / \underbrace{I_\lambda}_{\text{Maximal proper submodule}}$$

Irreducible

$\boxed{\hat{\mathfrak{g}}\text{-rep'n of level } \ell}$

Conformal blocks & sheaf of coinvariants

X = smooth curve / $\mathbb{C}$

$x_1, \dots, x_n$ = distinct points of X

$t_1, \dots, t_n$ = local coordinates

$L_{\lambda_1}, \dots, L_{\lambda_n}$ = level k irred. $\hat{\mathfrak{g}}$-modules

$L_{\vec{\lambda}} = \bigotimes_{i=1}^{n} L_{\lambda_i}$ is an irreducible rep'n of

$$\hat{\mathfrak{g}}_n = \mathfrak{g} \otimes \left(\bigoplus_{i=1}^{n} \mathbb{C}((t_i)) \right) \oplus \mathbb{C}\mathbb{1}$$

__Ex:__ $(n=2)$ $\hat{\mathfrak{g}}_2 \times L_{\vec{\lambda}} \to L_{\vec{\lambda}}$

$$(g_1, g_2) \cdot (m_1 \otimes m_2) = (g_1 m_1) \otimes m_2 + m_1 \otimes (g_2 \cdot m_2)$$

$$x_i \mapsto t_i$$

$$\mathfrak{g}_{out} := \mathfrak{g} \otimes \mathbb{C}[X - \{x_1, \dots, x_n\}]$$

$\subset$ $\mathfrak{g}$-valued meromorphic fxns on X with poles only an $x_1, \dots, x_n$.

$$\mathfrak{g}_{out} \xrightarrow{\varphi} \bigoplus_{i=1}^{n} \mathfrak{g}((t_i))$$

WANT: Lie alg. homomorphism.

NEED: $\oint_{D_{x_i}} f_i \, dg_i = 0$

$$[A \otimes f(t), B \otimes g(t)] = [A, B]_{\mathfrak{g}} \otimes fg - \underbrace{\boxed{\kappa_0(A,B) \int f\, dg \cdot \mathbb{1}}}_{\text{error term}}$$

SPONSORED BY: RESIDUE THEOREM

(=0) killing it

$$\mathfrak{g}_{out} \curvearrowright \mathcal{L}_{\vec{\lambda}} = \bigotimes_{i=1}^{n} \mathcal{L}_{\lambda_i}$$

$$\mathcal{H}_{\mathfrak{g}}(\lambda_1, \cdots, \lambda_n) := \mathcal{L}_{\vec{\lambda}} / \mathfrak{g}_{out} \cdot \mathcal{L}_{\vec{\lambda}}$$

sheaf of coinvariants

$\mathcal{C}_{\mathfrak{g}}(\lambda_1, \cdots, \lambda_n) :=$ space of linear forms $\mathcal{L}_{\vec{\lambda}} \to \mathbb{C}$ invariant under action of $\mathfrak{g}_{out}$

$$\parallel$$

$$\mathcal{H}_{\mathfrak{g}}^{+}(\lambda_1, \cdots, \lambda_n)$$

space of conformal blocks.

$$\begin{array}{ccc} H_g(\lambda_1, \dots, \lambda_n) & \qquad & \Delta_g(\lambda_1, \dots, \lambda_n) \\ \downarrow & & \downarrow \\ (C, x_1, \dots, x_n, \underbrace{t_1, \dots, t_n}_{\text{NOT NEEDED}}) & & \mathcal{M}_{g,n} \end{array}$$

[TUY '89]

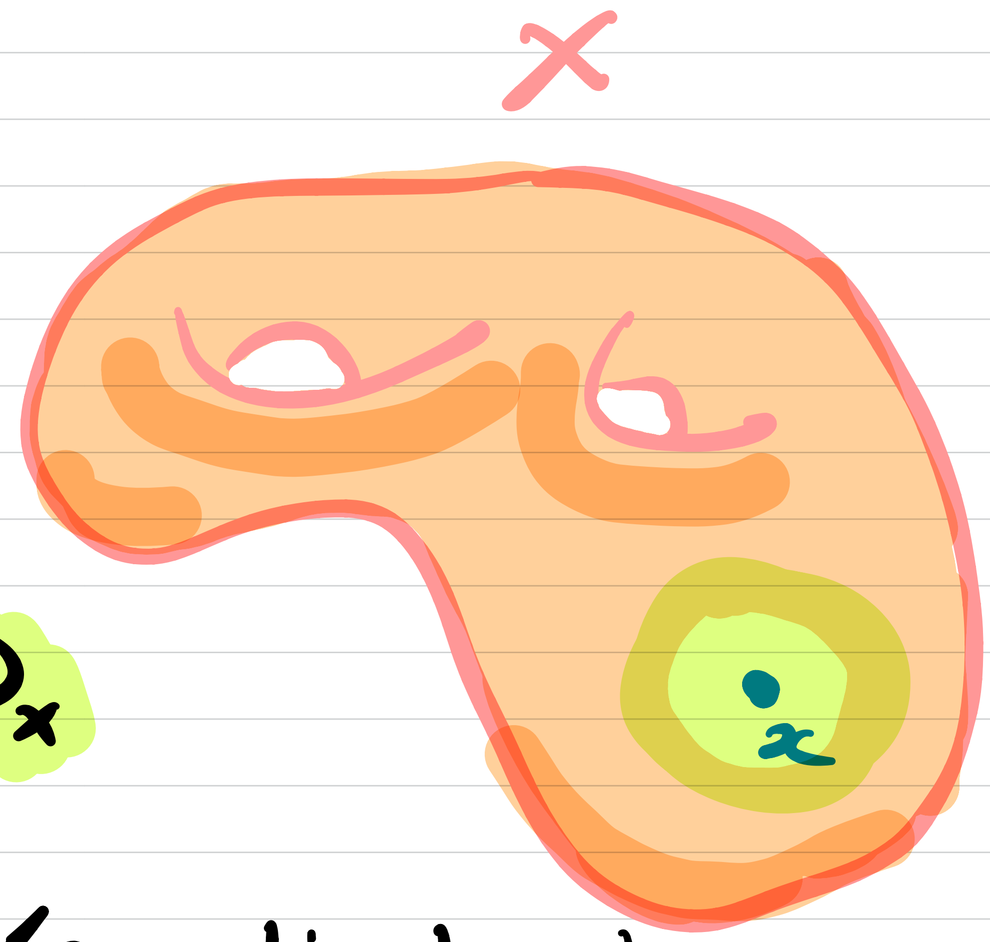
§2. Conformal blocks over Bun_G^x

What is Bun_G^X ?

G = semi-simple reductive lie group

P = G-bundle on X

trivialize P at $X\setminus\{x\}$



trivialize P at a small disc D_x

$$P = \underbrace{\text{coord. change for } X\setminus\{x\}}_{G_{out} = \{X-\{x\}\to G\}} \Big\backslash \underbrace{\text{transition fxn}}_{G((t))} \Big/ \underbrace{\text{coordinate change for } D_x}_{G[[t]]}$$

$$Bun_G = G_{out}\backslash G((t))/G[[t]] = G_{out}\backslash Gr$$

Coinvariants over Bun_G

$P \in \mathrm{Bun}_G^X$ $\qquad\qquad$ $\mathfrak{g}$ = corr. Lie algebra

$\mathfrak{g}_P = P \times_G \mathfrak{g}$ $\leftarrow$ v.b of lie alg over X

$$\mathfrak{g}_P^{out} = \Gamma(X \setminus \{x\}, \mathfrak{g}_P) \curvearrowright \mathcal{L}_{\vec{\lambda}}$$

$$\begin{array}{c} \mathfrak{g}_P^{out} \\ \downarrow \\ \bigoplus_{i=1}^{n} \mathfrak{g}((t)) \end{array}$$

(=0) killing it

$$\mathcal{L}_{\vec{\lambda}} / \mathfrak{g}_P^{out} \cdot \mathcal{L}_{\vec{\lambda}} = H_{\mathfrak{g}}^P(\vec{\lambda})$$

fiber over $P \in \mathrm{Bun}_G^X$ of the v.b. of coinvariant over Bun_G^X.

Part II. $\mathcal{D}$-modules & Opers

§1. Constructing twisted $\mathcal{D}$-modules

§2. $V_k(\hat{\mathfrak{g}})$ and it center

§3. $\mathfrak{g}$-opers on a curve X

§1. Constructing $\mathcal{D}$-modules

What is a $\mathcal{D}$-module?

X = smooth curve / $\mathbb{C}$

$\mathcal{D}_X$ = sheaf of differential operators

(If $U \subseteq X$, $\mathcal{D}_X(U) = \langle z, \partial/\partial z \rangle$)

A $\mathcal{D}$-module if a sheaf of $\mathcal{D}_X$-modules

<u>Ex</u>: $\mathcal{D}_X(U) \curvearrowright \mathcal{O}_X(U)$

$$\left(f_0(z) + f_1(z)\frac{\partial}{\partial^n z}\right) \cdot g(z) = f_0(z)g(z) + f_1(z)\frac{\partial}{\partial^n z}g(z)$$

~~What is a $\mathcal{D}$-module?~~ MOST IMPORTANTLY

WHY am I talking about it ??

$$X = \mathrm{Bun}_G = G_{out} \backslash Gr = G_{out} \backslash G((t)) / G[[t]]$$

$$\mathcal{D}_X = \Omega_{\mathcal{L}}^{\otimes k}$$

$$G_{out} \backslash \tilde{\mathcal{L}} = \mathcal{L} \qquad \tilde{\mathcal{L}} = \hat{G} / G[[t]]$$

$$\begin{array}{ccc} \mathcal{L} & & \tilde{\mathcal{L}} \\ \downarrow & & \downarrow \\ \mathrm{Bun}_G & & Gr \end{array}$$

$$(\mathcal{D}_X \text{ mod}) \ni \Delta(V)$$

Harish-Chandra module

localization operator

(Almost a) General construction

$\mathfrak{g}$ simple lie algebra

$\cup\!\!\shortmid$

$\mathfrak{k}$ lie subalgebra $\leftrightsquigarrow$ K lie group

$(K, \mathfrak{g})$ a Harish-Chandra pair

Z G-variety with a $(K, \mathfrak{g})$-action

$\alpha : \mathfrak{g} \times \Theta_Z \to \Theta_Z$ & $f : K \times Z \to Z$

data

$\alpha : \mathfrak{g} \longrightarrow \Theta_Z$ Lie alg homomorphism

sheaf of holomorphic vector fields

$\Downarrow$

$\mathfrak{g} \otimes \mathcal{O}_Z \xrightarrow{a} \Theta_Z$ Mor. of $\mathcal{O}_Z$-modules

sheaf of holomorphic sections

Anchor map (Assume surjective)

$$\Theta_Z \cong (\mathfrak{g} \otimes \mathcal{O}_Z)/\ker a$$

$Z = H\backslash G$

G = red. lie grup
$\cup\!|$
H lie subgrup.

$(\mathfrak{g}, K) \curvearrowright H\backslash G$
transitive action on right

$V = (\mathfrak{g}, K)$-module

[How do you get such a module?]

[BD 1991]

Answer: Use isomorphism $\mathcal{C}_{\mathfrak{g}} \cong \mathrm{Fun}\,\mathrm{Proj}\,D$

G semi-simple reductive lie group

$A \ni \mathfrak{g} \rightsquigarrow \{e^{tA} \mid t \in \mathbb{R}\}$

↑ one param. subgroup of G

$$\mathfrak{g} \longrightarrow \Theta_{H\backslash G}$$

$$A \longmapsto \frac{d}{dt}\Big|_{t=0} (e^{tA}.p) \quad \text{at each point } p \in H\backslash G$$

$$a : \mathfrak{g} \otimes \mathcal{O}_{H\backslash G} \to \Theta_{H\backslash G}$$

$$\ker(a)_p = \mathrm{Stab}_p(\mathfrak{g})$$

$$= \left\{ A \in \mathfrak{g} \;\Big|\; \frac{d}{dt}\Big|_{t=0} e^{tA}.p = 0 \right\}$$

$V \in (\mathfrak{g}, K)$ module $\left\{ \begin{array}{l} V \text{ is a rep'n of } \mathfrak{g} \\ \mathfrak{k} \curvearrowright V \text{ exponentiated} \\ \quad \text{to } K \curvearrowright V \end{array} \right.$

$V \otimes \mathcal{O}_{H\backslash G} \in \mathrm{Mod}(\mathcal{O}_{H\backslash G})$

$$\frac{\mathfrak{g} \otimes \mathcal{O}_{H\backslash G}}{\ker a} \qquad V \otimes \mathcal{O}_{H\backslash G} \big/ \ker a \cdot (V \otimes \mathcal{O}_{H\backslash G})$$

$$\begin{array}{ccc} \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \end{array}$$

$$\Theta_{H\backslash G} \curvearrowright \tilde{\Delta}(V) \in \mathrm{Mod}(\mathcal{D}_{H\backslash G})$$

space of coinvariants

$$\frac{V\otimes\mathcal{O}_{H\backslash G}}{\ker a\cdot(V\otimes\mathcal{O}_{H\backslash G})}$$

$$\begin{tikzcd}
\tilde{\Delta}(V) \arrow[d] & \dfrac{V}{\mathrm{Stab}_p(\mathfrak{g})\cdot V} \arrow[l, hook'] \arrow[d, mapsto] \\
H\backslash G & p \arrow[l, phantom, "\ni"]
\end{tikzcd}$$

$$\begin{tikzcd} \tilde{\Delta}(V) \arrow[r, "\exists!"] \arrow[d] & \Delta(V) \arrow[d] \\ H\backslash G \arrow[r] & H\backslash G/K \end{tikzcd}$$

$\mathcal{D}$-modules of "coinvariants" on $H\backslash G/K$

⚠ Not always a V.B. $\mathrm{Stab}_p(\mathfrak{g})$ varies in dimension

Localization functor

Fix: replace $\mathfrak{g}$ with $\hat{\mathfrak{g}}$ and let $\mathbb{1} \in \hat{\mathfrak{g}}$ act as $k \cdot \mathrm{Id}$ on V

$$\{(\hat{\mathfrak{g}}, K)\text{-module}\} \longrightarrow \left\{ \begin{array}{l} \mathcal{D}'_k\text{-modules} \\ \text{of "coinvariants"} \\ \text{on } {}_H\backslash G/_K \end{array} \right\}$$

$$V \longmapsto \Delta(V)$$

Why bother?

$$G = G((t)) \qquad K = G[\![t]\!] \qquad H = G_{out}$$

$$\left\{ (\hat{\mathfrak{g}}, G[\![t]\!])\text{-Mod} \right\} \to \left\{ \begin{matrix} \mathcal{D}_k\text{-mod} \\ \text{on } \mathrm{Bun}_G \end{matrix} \right\}$$

$$V \longmapsto \Delta(V)$$

Precisely sheaf of coinvariants coming from CFT.

$$\mathrm{Stab}_P(\mathfrak{g}) = \mathfrak{g}^P_{out} = \Gamma(X - \{x\}, \mathfrak{g}_P)$$

G-bundle

$$a: \mathfrak{g} \times \mathcal{O}_{H\backslash G} \to \Theta_{H\backslash G}$$

$$\Delta(V)_P = V / \mathfrak{g}^P_{out} \cdot V$$

Let's construct the sheaf $\mathcal{D}_k$ (for $k \in \mathbb{Z}$)

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{C}^* \cdot \mathbb{1} \arrow[r] & \hat{G} \arrow[r] \arrow[dr] & G((t)) \arrow[r] & 0 \\
& & & G[[t]] \arrow[u, phantom, sloped, "\subset"] \arrow[ul, dashed, bend left, "\text{Splits!}"] \arrow[ur, bend right] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\tilde{\mathcal{L}} := \hat{G}/G[[t]] \arrow[d, "\mathbb{C}^*\text{-bundle}"] & \rightsquigarrow & \mathcal{L} := G_{out} \backslash \tilde{\mathcal{L}} \arrow[d] \\
Gr = G((t))/G[[t]] & & Bun_G = G_{out} \backslash G((t))/G[[t]]
\end{tikzcd}
$$

$$\boxed{\mathcal{D}_k := \Omega_{\mathcal{L}^{\otimes k}} \in \mathrm{Mod}(Bun_G)}$$

§2. Vacuum Verma Modules & its center

[$C_{-h^\vee}(\mathfrak{g})$ an equivalent way to see space of $\mathfrak{g}$-opers]

The chiral algebra $V_k(\mathfrak{g})$

$k \in \mathbb{C}$;

$$\underbrace{V_k(\mathfrak{g})}_{\text{vacuum verma module}} = \underbrace{U(\overset{\mathfrak{g}((t)) \oplus \mathbb{C}\mathbb{1}}{\hat{\mathfrak{g}}})}_{\text{universal enveloping alg}} \underset{U(\mathfrak{g}[[t]] \oplus \mathbb{C}\mathbb{1})}{\otimes} \mathbb{C}_k \cong U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[[t^{-1}]])$$

$$\ni \quad J^{a_1}_{n_1} \cdots\cdots J^{a_m}_{n_m} \; \underset{\text{vacuum vector}}{v_k}$$

$$J^{a_1}_{n_1} = \underbrace{J^{a_1}}_{\text{basis elt of } \mathfrak{g}} \otimes t^{n_1}$$

$(n_1 \leq n_2 \leq \cdots \leq n_m < 0)$

The chiral algebra $V_k(\mathfrak{g})$

$k \in \mathbb{C}$;

$\mathfrak{g}((t)) \oplus \mathbb{C}\mathbb{1}$

(as v.s.)

$$V_k(\mathfrak{g}) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[[t]] \oplus \mathbb{C}\mathbb{1})} \mathbb{C}_k \cong U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[[t^{-1}]])$$

vacuum verma module

universal enveloping alg

vacuum vector

$$\boxed{J^{a_1}_{n_1} \cdots\cdots J^{a_m}_{n_m} \, v_k}$$

PBW basis for $V_k(\mathfrak{g})$ of $\deg = -(n_1 + \cdots + n_m)$

$(n_1 \leq n_2 \leq \cdots \leq n_m < 0)$

$J^{a_1}_{n_1} = \underbrace{J^{a_1}}_{\text{basis elt of } \mathfrak{g}} \otimes t^{n_1}$

$v_k = \mathrm{Im}(1 \otimes 1)$ of $\deg = 0$.

Segal-Sugawara & Center

$$S = \frac{1}{2} J_{a,-1} J^{a}_{-1} v_k$$ State

$$S(z) = \sum S_n z^{-n-2}$$ Field

J_a = dual basis elt

$$[S_n, J^a_m] = -(k+h^\vee) m J^a_{n+m}$$

for $k = -h^\vee$; $S_n \in C_{-h^\vee}(\mathfrak{g})$

$$C_{-2}(\mathfrak{sl}_2) = \mathbb{C}[S_n]_{n \leq -2}$$

$$C_k(\mathfrak{g}) = \mathbb{C} v_k \qquad (k \neq -2)$$

Next goal: develop a coordinate independent description

§3. $\mathfrak{g}$-opers on X

[Opers = tool for constructing Harish-Chandra Module]

$\mathfrak{sl}_2$ - opers

$$C^\bullet_{-2}(\mathfrak{sl}_2) \cong \mathrm{Fun}(\underbrace{\mathrm{Proj}\, D_x}_{\text{space of projective connections}})$$

D_x = formal disc at x

polynomial functions on $\mathrm{Proj}\, D_x$

$$\partial_z^2 - v(z) : \Omega^{-1/2} \longrightarrow \Omega^{3/2}$$

$$v(z) = \sum_{n \leq -2} v_n z^{-n-2}$$

$$C^\bullet_{-2}(\mathfrak{sl}_2) \cong \mathbb{C}[S_n]_{n\leq -2} \cong \mathbb{C}[v_n]_{n \leq -2} \cong \mathrm{Fun}(\mathrm{Proj}\, D_x)$$

$$\boxed{C^\bullet_{-h^\vee}(\mathfrak{g}) \cong \mathrm{Fun}(\mathcal{O}p_{{}^L\mathfrak{g}}(D_x))}$$

$\mathfrak{g}$-opers

G = lie group of adjoint type corresponding to $\mathfrak{g}$

$\cup$|

B_+ = Borel subgroup

Example: $\mathfrak{g} = \mathfrak{sl}_2$

$SL_2 \to \mathfrak{sl}_2$

$A \mapsto ABA^{-1}$ (Non faithful adjoint map)

$\begin{bmatrix} -1 & \\ & -1 \end{bmatrix} \mapsto id_{\mathfrak{sl}_2}$

Take Image $= SL_2 / Z(SL_2)$
$= PGL(2)$

A $\mathfrak{g}$-oper is a triple $(\mathcal{F}, \nabla, \mathcal{F}_{B_+})$

$\mathcal{F}$: principal G-bundle on X

∇: connection on $\mathcal{F}$

$\mathcal{F}_{B_+}$: B_+ reduction of $\mathcal{F}$

$U \subseteq X$ open with coordinate t

$$\nabla_{\partial_t} = \left[\partial_t + \sum_{i=1}^{\ell} \psi_i(t) f_i + v(t)\right]$$

an equivalence class of $\partial_t + \sum_{i=1}^{\ell} \psi_i(t) f_i + v(t)$

rank of $\mathfrak{g}$

ith generator of nilpotent subalgebra $\mathfrak{n}_-$

$$\nabla_{\partial_t} = \partial_t + \sum_{i=1}^{\ell} \varphi_i(t) f_i + v(t)$$

nowhere vanishing function

$v \in \mathfrak{b}_+ = \mathfrak{h} + \mathfrak{n}_+$

an element in the Borel subalgebra

also where are these coming from

But! But! But!

WHY 1st ORDER OPERTOR ??

Linear algebra

$\mathfrak{sl}_2$-oper $\ni \partial_t^2 - v(t) : \Omega^{-1/2} \longrightarrow \Omega^{3/2}$

$\mathfrak{sl}_n$-oper $\ni \partial_t^n - u_1(t)\,\partial_t^{n-2} - \dots - u_{n-2}(t)\,\partial_t - u_{n-1}(t)$

$$\Omega^{-(n-1)/2} \longrightarrow \Omega^{(n+1)/2}$$

Linearization

$$\partial_t + \begin{pmatrix} 0 & u_1(t) & u_2(t) & \cdots & u_{n-1}(t) \\ 1 & 0 & \cdots & \cdots & 0 \\ & \ddots & & 0 & \vdots \\ & 0 & \ddots & & \vdots \\ & & & 1 & 0 \end{pmatrix}$$

[D-Sokolov 1985]

$+$ = nowhere vanishing fxn

$*$ = any fxn

$$\mathrm{PreOp}_{\mathfrak{sl}_n}(X) = \text{Space of operators } \partial_t + \begin{pmatrix} * & * & \cdots & * \\ + & * & \cdots & * \\ 0 & + & & * \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 \; + & * \end{pmatrix}$$

$$\mathrm{Op}_{\mathfrak{sl}_n}(X) = \mathrm{PreOp}(X) \big/ \text{gauge transformation}$$

$$= \text{Space of operators } \partial_t + \begin{pmatrix} 0 & u_1 & \cdots & u_1 \\ 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{pmatrix}$$

$$\mathrm{PreOp}_{\mathfrak{g}}(X) = \text{Space of operators } \partial_t + \sum_{i=1}^{\ell} \psi_i(t) f_i + v(t)$$

$$\mathrm{PreOp}_{\mathfrak{g}}(X) = \text{Space of operators } \partial_t + \sum_{i=1}^{\ell} \varphi_i(t) f_i + v(t)$$

fxns v_j

$$\mathrm{Op}_{\mathfrak{g}}(X) = \text{Space of operators } \partial_t + p_{-1} + \sum_{j=1}^{\ell} v_j(t) \cdot p_j$$

$$\wr\wr$$

$$\mathrm{Proj}(X) \times \bigoplus_{j=2}^{\ell} H^0\left(X, \Omega^{\otimes(d_j+1)}\right)$$

$t = \varphi(s)$

$$\partial_s + p_{-1} + \sum_{j=1}^{\ell} \bar{v}_j(s)\, p_j$$

v_1 transforms as a projective connection

v_j " " a (d_j+1)-differential [F'05]

$2 \leq j \leq \ell$

$$\mathcal{O}p_{\mathfrak{g}}(X) = \text{Space of operators } \partial_t + p_{-1} + \sum_{j=1}^{\ell} v_j(t) \cdot p_j$$

$$\begin{array}{c} \rotatebox{90}{$\cong$} \\ \mathrm{Proj}(X) \times \bigoplus_{j=2}^{\ell} H^0(X, \Omega^{\otimes(d_j+1)}) \\ \updownarrow \qquad\qquad \updownarrow \\ v_1 \qquad\qquad v_2, \ldots, v_\ell \end{array}$$

fxns v_j characterize $\mathcal{O}p_{\mathfrak{g}}(X)$

$$\mathcal{O}p_{sl_2}(X) = \text{Space of operators } \partial_t + p_{-1} + v_1(t)\, p_1$$

$$\begin{array}{c} \rotatebox{90}{$\cong$} \\ \mathrm{Proj}(X) \end{array}$$

proj. connection

The End