Class of 1880 Exam

(Math competition ONLY for Univ. of Pennsylvania freshmen.)

April 15, 2022

Name: _____

Please show all your work.

Time available: 2 hours!

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. (10 points) You are given positive integers $1, 2, \ldots, 2n$. Out of these you choose $n + 1$ numbers. Show that out of these chosen numbers there is a pair of numbers (a, b) such that either a divides b or b divides a.

2. (10 points) Let a, b, c, be positive real numbers such that $a + b + c = 1$. Show that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9.$$

3. (10 points) Find all the strictly increasing functions $f\colon \mathbb{N} \to \mathbb{N}$ with the property that $\dfrac{f(x)+f(y)}{1+f(x+y)}$ is a nonzero natural number for any $x, y \in \mathbb{N}$.

4. (10 points) Given a square in the plane. Consider all line segments that cut across the square and divide it into two parts whose areas are of the ratio $1 : 4$ and at least one of the parts is a trapezoid. Suppose that the endpoints of the line segments are either on the boundary or at the corners of the square. How many such line segments do you need to choose to make sure that at least three of them intersect at some point?

5. (10 points) Seven football fans plan to buy tickets for seven different games. Suppose everyone will only buy their own ticket for each game, and that there cannot be two people who attend two different games together. Now, suppose someone has already decided to buy three tickets, what would be the maximum number of tickets the seven people can buy in total?