ON THE BREAKDOWN OF STABILITY FOR THE MUSKAT PROBLEM AND THE EPITAXIAL GROWTH EQUATION

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ABSTRACT

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In this work we investigate the question of the well-posedness of the Muskat problem when low regularity initial data is considered. A natural barrier for well-posedness are the spaces that are critical under the scaling, and therefore an interesting question is if the well-posedness can be established for critical spaces and super-critical spaces. For Navier-Stokes this question was answered negatively in [2], [8], [23] and many other works since then for some other fluid equations, by showing that for some critical spaces the solution map is discontinuous at the origin.

The first part of this work introduces the technical tools, approximations and explain the strategy that is used to prove the ill-posedness result for the Muskat equation.

The next two chapters are dedicated to fill some gaps in the well-posedness theory for the Muskat problem by establishing global existence results for the 2D problem in a periodic domain. In Chapter 2 we prove global existence in a periodic domain for small initial data in the critical space $\mathcal{F}^{1,1}$, the analogous result was

previously known for the non-periodic case in [10], [9]. In Chapter 3 we prove the global existence for H^2 initial data with small slope in a periodic domain by extending a result previously known for the non-periodic case [11].

The last part of the work is devoted to study the question of Ill-posedness for the Muskat equation and the Epitaxial Growth problem. We consider a family of approximations of the equation for which we prove the discontinuity of the solution map at the origin in some supercritical spaces. The sequence of spaces approaches a critical one as we consider higher order approximations which suggest that well-posedness in critical spaces is really the best we should hope for.

Contents

List of symbols

- $\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} f(x)dx$: Fourier transform of f.
- $\delta_s f = \delta_s f(x) = f(x) - f(x-s)$
- $(x)_+ = \frac{x+|x|}{2}$: Positive part of x
- $A + B = \{a + b \in X : a \in A, b \in B\}$, where $A, B \subset X$,
- pdf_X: Probability density function of the random variable X
- $\widehat{E(g)}(t,x) = \int_0^t e^{-2\pi(t-\tau)|\xi|} g(\tau,\xi)d\tau$.
- $f^{*k} = f * \cdots * f$, k times, $f^{*1} = f$.
- $\Re z$: Real part of z.
- $\Im z$: Imaginary part of z.
- $\Lambda f = (-\Delta)^{1/2} f$: Square root of the Laplacian of f.

Chapter 1

The Muskat Problem

1.1 The Model

In fluid mechanics, the Muskat equation describes the evolution of a multi-phase fluid in a porous medium. This situation was first observed in the petroleum industry when studying the oil extraction, in which was of particular interest to understand the interaction of oil and water in sand. In this model the velocity of the fluid is given by the Darcy's law

$$\frac{\mu}{\kappa}\vec{v} = -(\nabla p + \rho g \vec{e}_n), \tag{1.1}$$

where $\vec{v}$ is the velocity, p is the pressure, μ the viscosity, κ the permeability, $\rho > 0$ is the density, g is the gravity acceleration constant and $\vec{e}_n$ is a vector in the vertical direction pointing up. When coupled with the conservation of mass and the incompressibility condition for the velocity field, then we can formulate the

Muskat problem, given a initial density ρ_0, to find $\vec{v}$, p, ρ, such that

$$\begin{cases} \frac{\mu}{\kappa}\vec{v} = -(\nabla p - \rho\vec{g}) & , \quad \Omega \times [0,T], \\ div(\vec{v}) = 0 & , \quad \Omega \times [0,T], \\ \partial_t \rho + div(\rho\vec{v}) = 0 & , \quad \Omega \times [0,T]. \end{cases} \tag{1.2}$$

When $\Omega \subset \mathbb{R}^n$ is bounded, boundary conditions need to be imposed, typical choices are no penetration or no slip at the boundary. In this work we focus in the infinitely deep case and no boundary. When we have a multi-phase fluid, the density is discontinuous and so Darcy's law must be understood in the weak sense. Also, in this case we lose the continuity of the velocity, but it is still continuous in the normal direction to the interface due to the incompressibility. We assume that the fluids have the same viscosity and the permeability is uniform in the domain, and therefore by changing variables we can assume that $\mu/\kappa = g = 1$.

1.2 The Hilbert Transform

In 1D the Hilbert transform in is defined by

$$\mathcal{H}f = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{1}{x-\alpha} f(\alpha) d\alpha, \tag{1.3}$$

the importance of the Hilbert transform is that up to more regular terms it is the only singular operator in 1D. The Hilbert transform play a central role in the theory of singular integral operators and the key property that we will use from it are the mapping properties in L^p space

Lemma 1.2.1 (Properties of the Hilbert transform). *The Hilbert transform as defined by* (1.3) *satisfy the following*

- $\mathcal{H}$ *is self-adjoint,*
- *bounded in* $L^p(\mathbb{R})$ *for* $1 < p < \infty$,
- *translation invariant and has a Fourier multiplier given by*

$$\int_{\mathbb{R}} \mathcal{H}f(x)e^{-2\pi i x\xi}dx = -i\mathrm{sgn}(k)\hat{f}(k). \tag{1.4}$$

Proof. This are classical results that can be found for instance in [24]. □

1.2.1 Hilbert transform for periodic function

In the case of a periodic function, there is a different representation of the Hilbert transform that will be useful for us later. Let $f : \mathbb{R} \to \mathbb{C}$ a 2π periodic function, then we can have the following

$$\begin{aligned} \mathcal{H}f &= \frac{1}{\pi}p.v. \int_{\mathbb{R}} \frac{1}{x-\alpha} f(\alpha)d\alpha \\ &= \frac{1}{\pi}p.v. \sum_{k\in\mathbb{Z}} \int_{2\pi k-\pi}^{2\pi k+\pi} \frac{1}{x-\alpha} f(\alpha)d\alpha \\ &= \frac{1}{\pi}p.v. \sum_{k\in\mathbb{Z}} \int_{-\pi}^{\pi} \frac{1}{x-\alpha-2\pi k} f(\alpha+2\pi k)d\alpha \\ &= \frac{1}{\pi}p.v. \int_{-\pi}^{\pi} \left(\sum_{k\in\mathbb{Z}} \frac{1}{x-\alpha-2\pi k}\right) f(\alpha)d\alpha, \end{aligned} \tag{1.5}$$

now we use that for any $z \in \mathbb{C} \setminus \{2\pi k : j \in \mathbb{Z}\}$

$$\sum_{k\in\mathbb{Z}} \frac{1}{z+2\pi k} = \frac{1}{z} + \sum_{k\geq 1} \frac{2z}{z^2-(2\pi k)^2} = \frac{1}{2\tan(z/2)}, \tag{1.6}$$

then the Hilbert transform can be written as

$$\mathcal{H}f(x) = \frac{1}{2\pi} p.v. \int_{-\pi}^{\pi} \frac{f(\alpha)}{\tan\left(\frac{x-\alpha}{2}\right)} d\alpha. \tag{1.7}$$

Remark 1.2.2. Because we are working with periodic functions, integrating over any interval of length 2π give us the same result. Because of this we will write the integral over $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ to denote the integral over any of such intervals.

Using that $p.v. \int_{\mathbb{T}} \frac{1}{\tan(\alpha/2)} d\alpha = 0$, we can add a zero term to obtain

$$\begin{aligned} \mathcal{H}f(x) &= \frac{1}{2\pi} p.v. \left(\int_{\mathbb{T}} \frac{f(\alpha)}{\tan\left((x-\alpha)/2\right)} d\alpha - \int_{\mathbb{T}} \frac{1}{\tan((x-\alpha)/2)} d\alpha f(x) \right) \\ &= \frac{1}{2\pi} p.v. \int_{\mathbb{T}} \frac{f(\alpha) - f(x)}{\tan\left((x-\alpha)/2\right)} d\alpha \\ &= \frac{1}{2\pi} p.v. \int_{\mathbb{T}} \frac{f(x-\alpha) - f(x)}{\tan(\alpha/2)} d\alpha, \end{aligned} \tag{1.8}$$

the advantage of this representation is that it is less singular because of the extra cancellation that we have introduced in the numerator. Another fact that will be useful for us later is the Fourier transform of the Hilbert transform, or in the case of a periodic domain, a multiplier for the Fourier coefficients

$$\frac{1}{2\pi} \int_{\mathbb{T}} \mathcal{H}f(x) e^{-ikx} dx = -i\text{sgn}(k) \hat{f}(k), \tag{1.9}$$

where $\hat{f}(k) = \int_{\mathbb{T}} e^{-ikx} f(x) dx$. As in the case of the real line, using Calderón-Zygmund theory it can be shown that the Hilbert transform in bounded in $L^p(\mathbb{T})$ for $1 < p < \infty$ (see [27] Section 6.17).

One of the main properties of the Hilbert transform that we will use is its relation with the fractional laplacian.

Definition 1.2.3. Let $f \in \mathcal{S}'$, for $0 < \alpha < 1$ the fractional laplacian of order α of f, denoted by $(-\Delta)^\alpha f$ is defined by the Fourier multiplier

$$\frac{1}{2\pi}\int_{\mathbb{T}} (-\Delta)^\alpha f e^{-ikx} = |k|^{2\alpha}\hat{f}(k). \tag{1.10}$$

In the $\alpha = 1/2$ we can also write $\Lambda f = (-\Delta)^{1/2} f$.

For the case $\alpha = 1/2$, a property that we will use later, is that the fractional laplacian can be written in terms of the Hilbert transform as:

Lemma 1.2.4. *Let $f \in \mathcal{S}'(\mathbb{T})$ then $\partial_x \mathcal{H} f = \Lambda f$.*

Proof. For $f \in \mathcal{S}(\mathbb{T})$ we have the following

$$\begin{aligned} \mathcal{F}(\partial_x \mathcal{H} f) &= ik\mathcal{F}(\mathcal{H}f) \\ &= (ik)(-i\text{sgn}(k))\mathcal{F}(f) \\ &= |k|\mathcal{F}(f) \\ &= \mathcal{F}(\Lambda f), \end{aligned} \tag{1.11}$$

and by duality the same is true for $f \in \mathcal{S}'(\mathbb{T})$. □

One more formula for the fractional laplacian that will be useful later is the following

$$\begin{aligned} \Lambda f &= \partial_x \mathcal{H} f = \partial_x \frac{p.v.}{2\pi}\int_{\mathbb{T}} \frac{f(x-\alpha)-f(x)}{\tan(\alpha/2)}d\alpha = \partial_x \frac{p.v.}{2\pi}\int_{\mathbb{T}} \frac{f(\beta)-f(x)}{\tan((x-\beta)/2)}d\beta \\ &= \frac{p.v.}{2\pi}\int_{\mathbb{T}} \frac{-f'(x)}{\tan((x-\beta)/2)}d\beta - \frac{p.v.}{4\pi}\int_{\mathbb{T}} (f(\beta)-f(x))\frac{\sec^2((x-\beta)/2)}{\tan^2((x-\beta)/2)}d\beta \\ &= \frac{p.v.}{4\pi}\int_{\mathbb{T}} (f(x)-f(x-\alpha))\frac{\sec^2(\alpha/2)}{\tan^2(\alpha/2)}d\alpha \end{aligned} \tag{1.12}$$

1.2.2 The Riesz Transform

Let $n \geq 2$, then for each $i \in \{1, \cdots, n\}$ the Riesz tranform R_i in $\mathbb{R}^n$ is defined by

$$R_i f(x) = \frac{1}{\pi \omega_{n-1}} p.v. \int_{\mathbb{R}^n} \frac{(x_i - y_i)}{|x-y|^{n+1}} f(y) dy, \tag{1.13}$$

where ω_{n-1} is the volume of $(n-1)$ ball. The Riesz transform can be seen as a generalization of the Hilbert transform to higher dimensions. The Riesz transform shares many of the same properties as the Hilbert transform as can be seen in the following Lemma.

Lemma 1.2.5 (Properties of the Riesz transform)**.** *Let R_i be the Riesz transform as defined by* (1.13) *then*

- *R_i is a self adjoint operator,*
- *R_i is bounded in L^p for $1 < p < \infty$,*
- *R_i is translation invariant and has the Fourier multiplier representation*

$$\mathcal{F}\left(R_i(f)\right)(\xi) = \frac{-i}{2\pi} \frac{\xi_i}{|\xi|} \hat{f}(\xi), \tag{1.14}$$

- $R_i = \partial_i (-\Delta)^{-1}$.

Proof. These are classical results that can be found for instance in [24]. □

1.3 Derivation of the Equation

In this section we will derive some equations for the interface between two fluids of constant densities for the Muskat problem in the case when it can be represented by a graph. Additionally we assume that both fluids have the same viscosity and we ignore the surface tension. In this section the density function is discontinuous and therefore Darcy's law will be understood in the weak sense and all the derivatives will be taken in the sense of distributions. Note that in the derivation we will not use the equation for the conservation of mass, but it can be shown that the velocity and density function obtained form this derivation satisfy in fact that last condition.

1.3.1 Muskat equation in 3D

In the 3D case the density function can be written as

$$\rho(x,y,z,t) = \rho_1 + (\rho_2 - \rho_1)1_{\Omega_2(t)}(x,y,z), \quad (x,y,z,t) \in \mathbb{R}^3 \times [0,T], \tag{1.15}$$

where $\Omega_2(t)$ denotes the bottom region occupied by the fluid of density ρ_2. Taking curl on the Darcy's law (1.1) we get

$$\mathrm{curl}\vec{v} = -\left(\partial_y \rho, \partial_x \rho, 0\right), \tag{1.16}$$

taking curl again we get curl curl$\vec{v} = \nabla \mathrm{div}(\vec{v}) - \Delta\vec{v}$ then by the incompressibility of $\vec{v}$ we get

$$-\Delta\vec{v} = \left(\partial_x\partial_z\rho, \partial_y\partial_z\rho, -\partial_x^2\rho - \partial_y^2\rho\right), \tag{1.17}$$

taking $(-\Delta)^{-1}$ we obtain

$$\vec{v} = \left(R_1\partial_z\rho, R_2\partial_z\rho, -R_1\partial_x\rho - R_2\partial_y\rho\right), \tag{1.18}$$

where $R_1 = \partial_x(-\Delta)^{-1}, R_2 = \partial_x(-\Delta)^{-1}$ denote the 3D Riesz transform. Now if we assume that the interface between the two fluids is given by a graph, then we can compute the distributional derivatives of the density in the following way, if the point is not at the interface then the gradient is just zero, at a point in the interface $G(x,y) = (x, y, g(x,y))$ then we consider the frame given by

$$\begin{aligned} V_1 &= \partial_x G(x,y) = (1, 0, \partial_x g(x,y)), \\ V_2 &= \partial_y G(x,y) = (0, 1, \partial_y g(x,y)), \\ N &= V_1 \times V_2 = (-\partial_x g(x,y), -\partial_y g(x,y), 1), \end{aligned} \tag{1.19}$$

V_1 and V_2 are tangent to the interface and therefore the gradient of ρ is zero in that direction at the interface. In the normal direction the function behaves like a negative heaviside function so we get

$$\nabla \rho = -(\rho_2 - \rho_1)\delta_{z-g(x,y)} \left(-\partial_x g, -\partial_y g, 1\right), \tag{1.20}$$

substituting (1.20) in (1.18) we obtain

$$\vec{v} = -(\rho_2 - \rho_1) \begin{pmatrix} R_1 \delta_{z-g(x,y)} \\ R_2 \delta_{z-g(x,y)} \\ R_1 \left(\delta_{z-g(x,y)} \partial_x g\right) + R_2 \left(\delta_{z-g(x,y)} \partial_y g\right) \end{pmatrix}. \tag{1.21}$$

Because we are interested in the evolution of the interface we take a point on the interface and observe its flow with the velocity field, now because we are only interested in the shape of the graph and not the particular parameterization, we can always change our flow at the interface by a tangent vector and that will only affect the parameterization of our surface

$$\partial_t \begin{pmatrix} x \\ \\ y \\ \\ g(x,y) \end{pmatrix} = -(\rho_2-\rho_1) \begin{pmatrix} R_1\delta_{z-g(x,y)} \\ \\ R_2\delta_{z-g(x,y)} \\ \\ R_1\left(\delta_{z-g(x,y)}\partial_x g\right) + R_2\left(\delta_{z-g(x,y)}\partial_y g\right) \end{pmatrix} + v_T, \quad (1.22)$$

where v_T is a vector field that is tangent to the interface. We choose v_T in such a way that the first two coordinates do not move, i.e. $\partial_t x = 0$, $\partial_t y = 0$, to do this we write v_T using the same frame as before to get

$$v_T = aV_1 + bV_2 = a(1, 0, \partial_x g(x,y)) + b(0, 1, \partial_y g(x,y)), \quad (1.23)$$

then we choose a and b such that $\partial_t x = 0$ and $\partial_t y = 0$, we get

$$a = (\rho_2 - \rho_1)R_1\delta_{z-g(x,y)} \text{ and } b = (\rho_2 - \rho_1)R_2\delta_{z-g(x,y)}, \quad (1.24)$$

substituting in (1.22) we get

$$\begin{aligned} \partial_t g(x,y) &= -(\rho_2-\rho_1)R_1\delta_{z-g(x,y)}\partial_x g - (\rho_2-\rho_1)R_2\delta_{z-g(x,y)}\partial_y g \\ &\quad +(\rho_2-\rho_1)\partial_x g R_1\delta_{z-g(x,y)} + (\rho_2-\rho_1)\partial_y g R_2\delta_{z-g(x,y)}. \end{aligned} \quad (1.25)$$

Now we compute the Riesz transform

$$\begin{aligned} R_1\left(\delta_{z-g(x,y)}\partial_x g\right) &= \frac{1}{4\pi} p.v. \int_{\mathbb{R}^3} \frac{(x-x_1)\delta_{x_3=g(x_1,x_2)}\partial_x g(x_1,x_2)}{((x-x_1)^2+(y-x_2)^2+(g(x,y)-x_3)^2)^{3/2}} \\ &= \frac{1}{4\pi} p.v. \int_{\mathbb{R}^2} \frac{(x-x_1)\partial_x g(x_1,x_2)dxdy}{((x-x_1)^2+(y-x_2)^2+(g(x,y)-g(x_1,x_2))^2)^{3/2}}, \end{aligned}$$

analogously

$$\begin{aligned} R_2\left(\delta_{z-g(x,y)}\partial_y g\right) &= \frac{1}{4\pi} \\ &\quad \times p.v. \int_{\mathbb{R}^2} \frac{(y-x_2)\partial_y g(x_1,x_2)}{((x-x_1)^2+(y-x_2)^2+(g(x,y)-g(x_1,x_2))^2)^{3/2}} dxdy, \end{aligned}$$

$$\partial_x g(x,y) R_1\left(\delta_{z-g(x,y)}\right) = \frac{1}{4\pi}\partial_x g(x,y)$$
$$\times\, p.v. \int_{\mathbb{R}^2} \frac{(x-x_1)dxdy}{((x-x_1)^2+(y-x_2)^2+(g(x,y)-g(x_1,x_2))^2)^{3/2}},$$

$$\partial_y g(x,y) R_2\left(\delta_{z-g(x,y)}\right) = \frac{1}{4\pi}\partial_y g(x,y)$$
$$\times\, p.v. \int_{\mathbb{R}^2} \frac{(y-x_2)dxdy}{((x-x_1)^2+(y-x_2)^2+(g(x,y)-g(x_1,x_2))^2)^{3/2}},$$

substituting in (1.25) we obtain the interface problem for the Muskat equation in 3D

$$\begin{aligned} \partial_t g &= -\frac{\rho_2-\rho_1}{4\pi} p.v. \int_{\mathbb{R}^2} \frac{(x-x_1)\partial_x g(x_1,x_2)-(x-x_1)\partial_x g(x,y)}{((x-x_1)^2+(y-x_2)^2+(g(x,y)-g(x_1,x_2))^2)^{3/2}} dxdy \\ &\quad -\frac{\rho_2-\rho_1}{4\pi} p.v. \int_{\mathbb{R}^2} \frac{(y-x_2)\partial_y g(x_1,x_2)-(y-x_2)\partial_y g(x,y)}{((x-x_1)^2+(y-x_2)^2+(g(x,y)-g(x_1,x_2))^2)^{3/2}} dxdy \\ &= \frac{\rho_2-\rho_1}{4\pi} p.v. \int_{\mathbb{R}^2} \frac{(x-x_1,y-x_2)\cdot(\nabla g(x,y)-\nabla g(x_1,x_2))}{((x-x_1)^2+(y-x_2)^2+(g(x,y)-g(x_1,x_2))^2)^{3/2}} dxdy. \end{aligned} \tag{1.26}$$

1.3.2 Muskat equation in 2D

The 2D Muskat problem can be seen as taking a slice of a 3D solution of the problem when we have symmetry along the y axis. The derivation is very similar to the 3D, but this time the density only depend on two variables and can be written as $\rho(x,z) = \rho_2 + (\rho_2-\rho_1)1_\Omega$. Taking the curl of the Darcy's Law (1.1) we get

$$\text{curl}\vec{v} = -\left(0, \partial_x \rho, 0\right), \tag{1.27}$$

taking curl again we get curl $\text{curl}\vec{v} = \nabla\text{div}(\vec{v}) - \Delta\vec{v}$ and because $\text{div}(\vec{v}) = 0$ we get

$$-\Delta\vec{v} = \left(\partial_x\partial_z\rho, 0, -\partial_x^2\rho\right). \tag{1.28}$$

Note that this is a 2D laplacian of u in the plane $x-z$. Taking $(-\Delta)^{-1}$ we obtain

$$\vec{v} = \left(R_1\partial_z\rho, 0, -R_1\partial_x\rho\right), \tag{1.29}$$

where $R_1 = \partial_x(-\Delta)^{-1}$ is the 2D Riesz transform. As before we can compute the distributional derivative of the density function at a point $G(x,y) = (x, y, g(x))$ by consider the frame

$$\begin{aligned} V_1 &= \partial_x G(x,y) = (1, 0, \partial_x g(x)), \\ V_2 &= \partial_y G(x,y) = (0, 1, 0), \\ N &= V_1 \times V_2 = (-\partial_x g(x), 0, 1), \end{aligned} \tag{1.30}$$

we obtain that

$$\nabla\rho = -(\rho_2 - \rho_1)\delta_{z-g(x)}\left(-g'(x), 0, 1\right), \tag{1.31}$$

substituting (1.31) in (1.29) we obtain

$$\vec{v} = -(\rho_2 - \rho_1)\begin{pmatrix} R_1\left(\delta_{z-g(x)}\right) \\ 0 \\ R_1\left(\delta_{z-g(x)}\partial_x g\right) \end{pmatrix}. \tag{1.32}$$

Because we are interested in the evolution of the interface, we look at the evolution of $(x, y, g(x))$ by the flow of velocity field. Note we only care about the shape of the graph and not its particular parameterization, therefore we change the vector field in the direction that is tangent to the interface that will only change the parameterization of the surface and not its shape

$$\partial_t \begin{pmatrix} x \\ \\ y \\ \\ g(x) \end{pmatrix} = -(\rho_2-\rho_1) \begin{pmatrix} R_1\left(\delta_{z-g(x,y)}\right) \\ \\ 0 \\ \\ R_1\left(\delta_{z-g(x)}\partial_x g\right) \end{pmatrix} + v_T, \tag{1.33}$$

where v_T is a vector field that is tangent at the interface. We choose v_T in such a way that the first two coordinates do not move, i.e. we impose the conditions that $\partial_t x = 0$ and $\partial_t y = 0$, to achieve this we consider a smooth extension of the vector fields V_1 and V_2 and write v_T in that frame to get that for a point at the interface we can write

$$v_T = aV_1 + bV_2 = a(1,0,\partial_x g(x)) + b(0,1,0), \tag{1.34}$$

then we choose a and b such that $\partial_t x = 0$ and $\partial_t y = 0$, we obtain

$$a = (\rho_2-\rho_1)R_1\left(\delta_{z-g(x)}\right) \text{ and } b = 0, \tag{1.35}$$

substituting in (1.33) we get for the last component

$$\partial_t g(x) = -(\rho_2-\rho_1)R_1\left(\delta_{z=g(x)}\partial_x g\right) + (\rho_2-\rho_1)\partial_x g(x) R_1 \delta_{z-g(x)}. \tag{1.36}$$

Now we compute the 2D Riesz transform

$$\begin{aligned} R_1\left(\delta_{z-g(x)}\partial_x g\right) &= \frac{1}{2\pi} p.v. \int_{\mathbb{R}^2} \frac{(x-x_1)\delta_{x_3=g(x_1)}}{(x-x_1)^2+(g(x)-x_3)^2}\partial_x g(x_1)dx_1 \\ &= \frac{1}{2\pi} p.v. \int_{\mathbb{R}} \frac{(x-x_1)\partial_x g(x_1)dx_1}{(x-x_1)^2+(g(x)-g(x_1))^2}, \end{aligned} \tag{1.37}$$

analogously

$$\partial_x g(x) R_1\left(\delta_{z=g(x)}\right) = \frac{1}{2\pi}\partial_x g(x) p.v. \int_{\mathbb{R}} \frac{(x-x_1)dxdy}{(x-x_1)^2+(g(x)-g(x_1))^2}.$$

Substituting in (1.36) we obtain the equation for the interface of the 2D Muskat equation

$$\begin{aligned}
\partial_t g(x) &= \frac{\rho_2-\rho_1}{2\pi} p.v. \int_{\mathbb{R}} \frac{-(x-x_1)\partial_x g(x_1)+(x-x_1)\partial_x g(x)}{(x-x_1)^2+(g(x)-g(x_1))^2} dx_1 \\
&= \frac{\rho_2-\rho_1}{2\pi} p.v. \int_{\mathbb{R}} \frac{(x-x_1)(\partial_x g(x)-\partial_x g(x_1))}{(x-x_1)^2+(g(x)-g(x_1))^2} dx_1 \\
&= \frac{\rho_2-\rho_1}{2\pi} p.v. \int_{\mathbb{R}} \frac{\alpha\ \partial_x \delta_\alpha g(x)}{\alpha^2+(\delta_\alpha g(x))^2} d\alpha,
\end{aligned} \tag{1.38}$$

where $\delta_\alpha g(x) = g(x) - g(x-\alpha)$.

1.3.3 The 2D Muskat equation in the periodic domain

If we look for periodic solutions of (1.38) it is possible to derive another formulation for the Muskat equation. Let f be a 2π-periodic solution of (1.38), then we can write

$$\begin{aligned}
\partial_t g &= \frac{\rho_2-\rho_1}{2\pi} p.v. \int_{\mathbb{R}} \frac{\alpha(\partial_x g(x)-\partial_x g(x-\alpha))}{\alpha^2+(g(x)-g(x-\alpha))^2} d\alpha \\
&= \frac{\rho_2-\rho_1}{2\pi} p.v. \sum_{k\in\mathbb{Z}} \int_{-\pi}^{\pi} \frac{(2\pi k+\alpha)(\partial_x g(x)-\partial_x g(x-\alpha-2\pi k))}{(\alpha+2\pi k)^2+(g(x)-g(x-\alpha-2\pi k))^2} d\alpha \\
&= \frac{\rho_2-\rho_1}{2\pi} p.v. \int_{-\pi}^{\pi} \sum_{k\in\mathbb{Z}} \frac{(\alpha+2\pi k)(\partial_x g(x)-\partial_x g(x-\alpha))}{(\alpha+2\pi k)^2+(g(x)-g(x-\alpha))^2} d\alpha.
\end{aligned} \tag{1.39}$$

We focus in the sum in k, for this we notice that this sum can be factorized over the complex numbers as

$$
\begin{aligned}
S &= \sum_{k\in\mathbb{Z}} \frac{(\alpha+2\pi k)}{(\alpha+2\pi k)^2+(g(x)-g(x-\alpha))^2} \\
&= \sum_{k\in\mathbb{Z}} \frac{\alpha+2\pi k}{((\alpha+2\pi k)+i(g(x)-g(x-\alpha))((\alpha+2\pi k)-i(g(x)-g(x-\alpha))} \\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}} \Bigg(\frac{1}{(\alpha+2\pi k)+i(g(x)-g(x-\alpha)} \\
&\qquad\qquad + \frac{1}{(\alpha+2\pi k)-i(g(x)-g(x-\alpha)))} \Bigg).
\end{aligned}
\tag{1.40}
$$

Now we use that for any $z\in\mathbb{C}\setminus\{2\pi k : j\in\mathbb{Z}\}$

$$
\sum_{k\in\mathbb{Z}} \frac{1}{z+2\pi k} = \frac{1}{z} + \sum_{k\geq 1} \frac{2z}{z^2-(2\pi k)^2} = \frac{1}{2\tan(z/2)}, \tag{1.41}
$$

using this we get

$$
S = \frac{1}{4}\Big(\frac{1}{\tan((\alpha+i(g(x)-g(x-\alpha))/2)} + \frac{1}{\tan((\alpha-i(g(x)-g(x-\alpha))/2)}\Big) \tag{1.42}
$$

Now we use that

$$
\begin{aligned}
\frac{1}{\tan(a+ib)} + \frac{1}{\tan(a-ib)} &= 2\Re\frac{1}{\tan(a+ib)} \\
&= 2\Re\frac{\cos(a+ib)}{\sin(a+ib)} \\
&= 2\Re\frac{\cos(a)\cos(ib)-\sin(a)\sin(ib)}{\sin(a)\cos(ib)+\cos(a)\sin(ib)} \\
&= 2\Re\frac{\cos(a)\cosh(b)-i\sin(a)\sinh(b)}{\sin(a)\cosh(b)+i\cos(a)\sinh(b)} \\
&= 2\Re\frac{1-i\tan(a)\tanh(b)}{\tan(a)+i\tanh(b)} \\
&= 2\Re\frac{(1-i\tan(a)\tanh(b))(\tan(a)-i\tanh(b))}{\tan^2(a)+\tanh^2(b)} \\
&= 2\frac{\tan(a)-\tan(a)\tanh^2(b)}{\tan^2(a)+\tanh^2(b)},
\end{aligned}
\tag{1.43}
$$

therefore we conclude

$$\partial_t f = \frac{\rho_2-\rho_1}{4\pi}\int_{\mathbb{T}} \partial_x\delta_\alpha f(x)\frac{\tan(\alpha/2)(1-\tanh^2(\frac{\delta_\alpha f(x)}{2}))}{\tan^2(\alpha/2)+\tanh^2(\frac{\delta_\alpha f(x)}{2})}d\alpha, \tag{1.44}$$

where $\delta_s f(x) = f(x)-f(x-s)$. Another useful representation is to separate the term corresponding to the linear part

$$\begin{aligned} T(f) - \frac{1}{\tan(\alpha/2)} &= \frac{\tan(\alpha/2)(1-\tanh^2(\frac{\delta_\alpha f(x)}{2}))}{\tan^2(\alpha/2)+\tanh^2(\frac{\delta_\alpha f(x)}{2})} - \frac{1}{\tan(\alpha/2)} \\ &= \frac{1}{\tan(\alpha/2)}\Big(\tan^2(\alpha/2)-\tan^2(\alpha/2)+\tanh^2(\delta_\alpha f/2) \\ &\qquad -\tan^2(\alpha/2)-\tanh^2(\delta_\alpha f/2)\Big) \\ &= \frac{1}{\tan(\alpha/2)}\sec^2(\alpha/2)\tanh^2(\delta_\alpha f) \end{aligned} \tag{1.45}$$

and using that $-\frac{1}{2\pi}\int_{\mathbb{T}}\frac{\partial_x\delta_\alpha f(x)}{\tan(\alpha/2)}d\alpha = \Lambda$ we obtain

$$\partial_t f(x) + \frac{(\rho_2-\rho_1)}{2}\Lambda = -\frac{\rho_2-\rho_1}{4\pi}p.v.\int_{\mathbb{T}}\frac{\partial_x\delta_s f(x)}{\tan(s/2)}\frac{\sec^2(s/2)\tanh^2(\delta_s f(x)/2)}{\tan(s/2)^2+\tanh^2(\delta_s f(x)/2)}ds. \tag{1.46}$$

This formulation will be used in Chapter 2. Lastly we will prove the equivalence of one additional formulation that will be used in Chapter 3.

$$\begin{aligned} \partial_t f &= \frac{\rho_2-\rho_1}{4\pi}\int_{\mathbb{T}}\partial_x\delta_\alpha f(x)\frac{\tan(\alpha/2)(1-\tanh^2(\frac{\delta_\alpha f(x)}{2}))}{\tan^2(\alpha/2)+\tanh^2(\frac{\delta_\alpha f(x)}{2})}d\alpha \\ &= \frac{\rho_2-\rho_1}{4\pi}\int_{\mathbb{T}}\partial_x\delta_\alpha f(x)\frac{\tan(\alpha/2)\mathrm{sech}^2(\frac{\delta_\alpha f(x)}{2}))}{\tan^2(\alpha/2)+\tanh^2(\frac{\delta_\alpha f(x)}{2})}d\alpha \\ &= \frac{\rho_2-\rho_1}{4\pi}\int_{\mathbb{T}}\partial_x\delta_\alpha f(x)\frac{\frac{\mathrm{sech}^2(\delta_\alpha f(x)/2)}{\tan(\alpha/2)})}{1+\frac{\tanh^2(\frac{\delta_\alpha f(x)}{2})}{\tan^2(\alpha/2)}}d\alpha \\ &= \frac{\rho_2-\rho_1}{4\pi}\int_{\mathbb{T}}2\partial_x\arctan\left(\frac{\tanh(\frac{f(x)-f(x-\alpha)}{2})}{\tan(\alpha/2)}\right)d\alpha, \end{aligned} \tag{1.47}$$

and using the change of variable $\alpha \to x-\alpha$ we get

$$
\begin{aligned}
\partial_t f &= \frac{\rho_2-\rho_1}{4\pi}\int_{\mathbb{T}} 2\partial_x \arctan\left(\frac{\tanh(\frac{f(x)-f(\alpha)}{2})}{\tan((x-\alpha)/2)}\right) d\alpha \\
&= \frac{\rho_2-\rho_1}{4\pi}\int_{\mathbb{T}} \partial_x f(x)\frac{\tan((x-\alpha)/2)\mathrm{sech}^2(\frac{f(x)-f(\alpha)}{2}))}{\tan^2((x-\alpha)/2)+\tanh^2(\frac{f(x)-f(\alpha)}{2})}d\alpha \\
&\quad -\frac{\rho_2-\rho_1}{4\pi}\int_{\mathbb{T}} \frac{\sec^2((x-\alpha)/2)\tanh(\frac{f(x)-f(\alpha)}{2}))}{\tan^2((x-\alpha)/2)+\tanh^2(\frac{f(x)-f(\alpha)}{2})}d\alpha,
\end{aligned}
\tag{1.48}
$$

finally we can write

$$
\partial_t f + v\partial_x f + \frac{\rho_2-\rho_1}{4\pi} p.v.\int_{\mathbb{T}} \frac{\tanh(\delta_s f/2)\sec^2(s/2)}{\tan^2(s/2)+\tanh^2(\delta_s f/2)}ds = 0, \tag{1.49}
$$

where

$$
v = -\frac{1}{2\pi} p.v.\int_{\mathbb{T}} \frac{\tan(s/2)\mathrm{sech}^2(\delta_s f/2)}{\tan^2(s/2)+\tanh^2(\delta_s f/2)}ds. \tag{1.50}
$$

1.4 Besov-type Spaces

Given $f \in \mathcal{S}(\mathbb{R}^n)$ its Littlewood-Paley decomposition is constructed in the following way. First we consider a smooth function supported in the annulus $\{\xi \in \mathbb{R}^n : 3/4 \le |\xi| \le 8/3\}$ such that $\forall \xi \in \mathbb{R}^n \setminus \{0\}$,

$$
\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1. \tag{1.51}
$$

By defining $h = \mathcal{F}^{-1}\varphi$ we can consider the homogeneous dyadic blocks defined by

$$
\dot{\Delta}_j f = 2^{jn}\int_{\mathbb{R}^n} h(2^j y) f(x-y)dy, \tag{1.52}
$$

then we have formally that $\sum_{j\in\mathbb{Z}} \dot{\Delta} = Id$ modulo distributions supported at the origin on the Fourier side. By using the homogeneous dyadic blocks it is possible to define the homogeneous Besov semi norm $\dot{B}^s_{p,r}$ for $s \in \mathbb{R}$, $p \geq 1$, $r \geq 1$ as

$$\|f\|_{\dot{B}^s_{p,r}} = \left(\sum_{j \in \mathbb{Z}} 2^{jsr} \|\dot{\Delta} f\|^r_{L^p} \right)^{1/r}, \tag{1.53}$$

and the corresponding Besov space $\dot{B}^s_{p,r}(\mathbb{R}^n)$ as the closure of $C^\infty_c(\mathbb{R}^n)$ with respect to this semi norm.

Remark 1.4.1. For general tempered distribution $g \in \mathcal{S}'$ the quantity $\|g\|_{\dot{B}^s_{p,r}}$ is only a semi norm because it vanishes at every tempered distribution supported at the origin on the Fourier side, i.e. the Besov semi norm take the value for polynomials. This is not an issue to define the space $\dot{B}^s_{p,r}$ because the difference of two functions in $C^\infty_c(\mathbb{R}^n)$ is never a nonzero polynomial.

Inspired on this norm, we can define a family of Besov-type norms better suited to the analysis of the Muskat equation. For $k \in \mathbb{Z}$, we consider the annulus $C_k = \{x \in \mathbb{R}^n : 2^k \leq |x| \leq 2^{k+1}\}$ and for $s \in \mathbb{R}$, $p \geq 1$, $q \geq 1$ we consider the norm

$$\|f\|_{\mathcal{F}^{m,p}_q} = \left(\sum_{k \in \mathbb{Z}} \left(\int_{C_k} |\xi|^{mp} |\hat{f}|^p d\xi \right)^{q/p} \right)^{1/q}, \quad f \in C^\infty_c(\mathbb{R}^n), \tag{1.54}$$

where $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \xi} f(x) dx$. In the periodic case we consider the annulus $C_k = \{j \in \mathbb{Z}^n : 2^k \leq |j| \leq 2^{k+1}\}$ and define

$$\|f\|_{\mathcal{F}^{m,p}_q} = \left(\sum_{k \in \mathbb{Z}^n} \left(\sum_{j \in C_k} |j|^{mp} |\hat{f}(j)|^p \right)^{q/p} \right)^{1/q}, \quad f \in C^\infty_c(\mathbb{T}^n), \tag{1.55}$$

where $\hat{f}(k) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} e^{-ik\cdot x} f(x) dx$. Finally we define the spaces $\mathcal{F}^{s,p}_q(\Omega)$ as the closure of $C^\infty_c(\Omega)$ with respect to the $\mathcal{F}^{s,p}_q(\Omega)$ norm. For simplicity sometimes we will use the shorthand $\mathcal{F}^{s,p}$ for $\mathcal{F}^{s,p}_1$.

1.4.1 Scaling and criticality on the Muskat equation

The Muskat equation (1.38) satisfy the following scaling property that: For any $\lambda > 0$ if f is a solution of (1.38), then $f_\lambda = \frac{1}{\lambda} f(\lambda t, \lambda x)$ is also a solution of (1.38). We say that a norm X is critical if for all $\lambda > 0$ $\|f_\lambda\|_X = \|f\|_X$. We call spaces that are more regular than critical spaces subcritical, and the ones that are less regular are called supercritical. In particular for the families of spaces $\mathcal{F}_q^{s,p}$ and $W^{k,p}$ we have the following

$$
\begin{aligned}
\|f_\lambda\|_{\mathcal{F}_q^{s,p}} &= \left(\sum_{j\in\mathbb{Z}} \left(\int_{C_j} |\xi|^{sp} |\hat{f}_\lambda|^p d\xi \right)^{q/p} \right)^{1/q} \\
&= \left(\sum_{j\in\mathbb{Z}} \left(\int_{C_j} |\xi|^{sp} |\frac{1}{\lambda^{1+d}} \hat{f}(\xi/\lambda)|^p d\xi \right)^{q/p} \right)^{1/q} \\
&= \left(\sum_{j\in\mathbb{Z}} \left(\int_{C_j} |\lambda\eta|^{sp} |\frac{1}{\lambda^2} \hat{f}(\eta)|^p \lambda^d d\eta \right)^{q/p} \right)^{1/q} \\
&= \lambda^{(s-1-n)+n/p} \left(\sum_{j\in\mathbb{Z}} \left(\int_{C_j} |\eta|^{sp} |\hat{f}(\eta)|^p d\eta \right)^{q/p} \right)^{1/q} \\
&= \lambda^{s-\left(1+n\frac{p-1}{p}\right)} \|f\|_{\mathcal{F}_q^{s,p}},
\end{aligned}
\tag{1.56}
$$

for λ power of 2. We obtain that for $s = 1 + n\left(\frac{p-1}{p}\right)$ the norm is invariant under the scaling, so we conclude that the spaces $\mathcal{F}_q^{1+n\left(\frac{p-1}{p}\right),p}$ are critical under the scaling of the Muskat equation. For the case of the $\dot{W}^{s,p}$ spaces, for$\alpha \in \mathbb{R}$ we define

$\Lambda^\alpha = (-\Delta)^{\alpha/2}$, then we have

$$\begin{aligned} \|f_\lambda\|_{\dot{W}^{s,p}} &= \|\Lambda^s f_\lambda\|_{L^p} \\ &= \left(\int_{\mathbb{R}^n} \left|\lambda^{s-1}\Lambda^s f(\lambda x)\right|^p dx\right)^{1/p} \\ &= \left(\lambda^{(s-1)p}\int_{\mathbb{R}^n} |\Lambda^s f(y)|^p \frac{1}{\lambda^d} dy\right)^{1/p} \\ &= \lambda^{s-1-n/p}\|f\|_{\dot{W}^{s,p}}, \end{aligned} \tag{1.57}$$

we conclude that for $s = 1+n/p$ the space $W^{s,p}$ is invariant under the scaling. This allows to conclude in particular that for the 2D Muskat problem the spaces $\mathcal{F}_q^{1,1}(\mathbb{R})$ $q \geq 1$, $\dot{H}^{3/2}(\mathbb{R}) = \dot{W}^{3/2,2}(\mathbb{R})$ and $\dot{W}^{1,\infty}(\mathbb{R})$ are critical under the scaling. Note that boundedness in some of the critical spaces for the equation are closely related with the boundedness of the slope, to see this we note that

$$\|g\|_{\dot{W}^{1,\infty}} = \operatorname*{esssup}_{x\in\Omega} |g'(x)|, \tag{1.58}$$

and

$$\begin{aligned} |g'(x)| &= \left|\int_{\mathbb{R}} (2\pi\xi) e^{-2\pi i x\xi}\hat{g}(\xi) d\xi\right| \\ &\leq (2\pi)\int_{\mathbb{R}} |\xi||\hat{g}(\xi)| d\xi \\ &\leq (2\pi)\|g\|_{\mathcal{F}_1^{1,1}}. \end{aligned} \tag{1.59}$$

For the periodic case we use the the same critical spaces by analogy with the non-periodic case.

1.5 Iterative solutions for the Muskat problem

In the study on non-linear partial differential equations finding explicit solutions is usually a very difficult task, that is why having iterative methods to approximate solutions from practical and theoretical points of views. In this section we introduce two of such methods that can be used to study the Muskat problem.

We have two goals in this section, The goal of this section is to study the convergence of an iterative solution for the Muskat problem. For this purpose we consider a family of solutions of the Muskat equation that depend on a parameter $\varepsilon > 0$, then $f = \sum_{\ell\geq 1} \varepsilon^\ell f_\ell$ and the initial condition $f_0 = \varepsilon\varphi$.

1.5.1 The Picard iteration

Consider the equation for the interface in the Muskat problem given by

$$\begin{cases} \partial_t f = G(f) \quad , \quad \text{in } \Omega \times [0,T] \\ f(0) = \varphi \quad , \quad \text{on } \Omega. \end{cases} \tag{1.60}$$

Up to linear level the $G(f)$ behaves like $G(f) \approx -\Lambda f = -(-\Delta)^{1/2} f$, then we can write

$$\begin{cases} \partial_t f + \Lambda f = T(f) \quad , \quad \text{in } \Omega \times [0,T], \\ f(0) = \varphi \quad , \quad \text{on } \Omega. \end{cases} \tag{1.61}$$

Now by setting $f_0 = 0$, and for $k \geq 1$ we define the Picard's iteration of the Muskat equation as

$$\begin{cases} \partial_t f_k + \Lambda f_k = T(f_{k-1}) \quad , \quad \text{in } \Omega \times [0,T], \\ f_k(0) = \varphi \quad , \quad \text{on } \Omega, \end{cases} \tag{1.62}$$

by using the Duhamel's principle the iteration can be written as a fixed point problem

$$f_k = e^{-t\Lambda}\varphi + \int_0^t e^{-(t-\tau)\Lambda} T(f_{k-1}) d\tau, \tag{1.63}$$

by doing this we can see that by the Banach's Fixed point theorem the convergence of the Picard's iteration can be studied by looking at the mapping properties of the operator

$$L(f) = \int_0^t e^{-(t-\tau)\Lambda} T(f). \tag{1.64}$$

1.5.2 A Small Parameter Iterative Solution to the Muskat Equation

This time we will consider an iterative solutions that can be seen as a Taylor expansion of the equation depending on a small parameter on the initial condition, for this purpose, given $\varepsilon > 0$ and some initial data φ, we consider the equation for the interface in the 2D Muskat problem as

$$\begin{cases} \partial_t f + \Lambda f = Tf \quad , \quad \text{in } \Omega \times (0,T) \\ f(0) = \varepsilon\varphi \quad , \quad \text{on } \mathbb{R}. \end{cases} \tag{1.65}$$

In order to find an expansion we look for solutions of the form $f = \sum_{n\geq 1} \varepsilon^n f_n$ and we try to find what are the equations that each one of the f_ℓ satisfy. For this purpose we use the Taylor expansion of the nonlinear term to obtain

$$\begin{aligned} Tf &= -\frac{1}{\pi}\int_{\mathbb{R}} \frac{\partial_x \delta_\alpha f(x)}{\alpha} \frac{(\delta_\alpha f(x))^2}{\alpha^2 + (\delta_\alpha f(x))^2} d\alpha \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} \frac{1}{\pi} \int_{\mathbb{R}} \partial_x \left(\Delta_\alpha f(x)\right)^{2k+1} d\alpha, \end{aligned} \tag{1.66}$$

where $\delta_\alpha f = f(x) - f(x-\alpha)$ and $\Delta_\alpha g = \delta_\alpha g/\alpha$. To obtain an equation for f_n we write $f = \sum_{n\geq 1} \varepsilon^n f_n$ and consider the following expansion

$$(\Delta_\alpha f)^{2k+1} = \sum_{j=2k+1}^{\infty} \varepsilon^j \sum_{i_1+\cdots+i_{2k+1}=j} (\Delta_\alpha f_{i_1})\cdots(\Delta_\alpha f_{i_{2k+1}}), \tag{1.67}$$

using this on (1.66) we obtain

$$\begin{aligned} Tf &= \frac{1}{\pi}\int_{\mathbb{R}} \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} \sum_{j=2k+1}^{\infty} \varepsilon^j \sum_{i_1+\cdots+i_{2k+1}=j} \partial_x(\Delta_\alpha f_{i_1})(\Delta_\alpha f_{i_2})\cdots(\Delta_\alpha f_{i_{2k+1}})d\alpha \\ &= \frac{1}{\pi}\int_{\mathbb{R}} \sum_{j=3}^{\infty} \sum_{k=1}^{\lfloor\frac{j-1}{2}\rfloor} \frac{(-1)^k}{2k+1}\varepsilon^j \sum_{i_1+\cdots+i_{2k+1}=j} \partial_x(\Delta_\alpha f_{i_1})(\Delta_\alpha f_{i_2})\cdots(\Delta_\alpha f_{i_{2k+1}})d\alpha, \end{aligned} \tag{1.68}$$

by matching the coefficients of the terms with the same power of ε we get an the equation for f_n

$$\begin{aligned} \partial_t f_n + \Lambda f_n &= \frac{1}{\pi} \sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor} \frac{(-1)^k}{2k+1} \int_{\mathbb{R}} \sum_{\substack{i_1+\cdots+i_{2k+1}\\=n}} \partial_x(\Delta_\alpha f_{i_1})\,(\Delta_\alpha f_{i_2})\cdots\big(\Delta_\alpha f_{i_{2k+1}}\big)\,d\alpha \\ &= \frac{1}{\pi} \sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor} \frac{(-1)^k}{2k+1} \int_{\mathbb{R}} \sum_{\substack{i_1+\cdots+i_{2k+1}\\=n}} \partial_x(\Delta_\alpha f_{i_1})\,(\Delta_\alpha f_{i_1})\cdots\big(\Delta_\alpha f_{i_{2k+1}}\big)\,d\alpha, \end{aligned} \tag{1.69}$$

and for the initial condition we get $f(0) = \sum_{k\geq 1} \varepsilon^k f_k(0) = \varepsilon\varphi$, therefore $f_1(0) = \varphi$ and $f_k(0) = 0$, $k \geq 2$. Therefore we obtain

$$\begin{cases} \partial_t f_n + \Lambda f_n = \dfrac{1}{\pi} \displaystyle\sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor} \frac{(-1)^k}{2k+1} \int_{\mathbb{R}} \sum_{\substack{i_1+\cdots\\+i_{2k+1}=n}} \partial_x(\Delta_\alpha f_{i_1})\,(\Delta_\alpha f_{i_2})\cdots\big(\Delta_\alpha f_{i_{2k+1}}\big)\,d\alpha \\ \qquad\qquad\quad = G_n(f_1,\cdots,f_{k-1}) \\ f_1(0) = \varphi \quad , \quad f_k(0) = 0, k \geq 2. \end{cases} \tag{1.70}$$

Note that by symmetry of the Muskat equation, we can see that if f is a solution with initial condition $f(0) = f_0$, then $g(x,t) = -f(x,t)$ is also a solution of the Muskat equation with initial condition $g(0) = -f_0$ condition. For our one parameter family of solutions this is equivalent to substitute $\varepsilon \mapsto -\varepsilon$, and therefore we get

$$g(x,t) = \sum_{k\geq 1} (-\varepsilon)^k f_k, \tag{1.71}$$

is a solution of the Muskat equation with initial condition $g(0) = -f_0$, consequently

$$\sum_{k\geq 1} \varepsilon^k f_k = f(x,t) = -g(x,t) = -\sum_{k\geq 1} (-\varepsilon)^k f_k \tag{1.72}$$

we conclude that $\sum_{\ell\geq 1} \varepsilon^{2\ell} f_{2\ell} = 0$ for all ε such that the expansion is valid, which implies that $f_{2\ell} = 0$ for all $\ell \geq 1$.

Note that the equation of each f_n in (1.70) is linear in the previous terms, so under mild assumptions in the initial data we expect that each one of those equations has a solution, for the convergence of this iterative process we need to know something about the size of f_n as $n \to \infty$.

Theorem 1.5.1 (Iterative solution of the Muskat equation)**.** *Consider the 2D Muskat equation in the real line and consider the iterative solution obtained by expanding $f(0) = \varepsilon\varphi$, $f = \sum_{k\geq 1} \varepsilon^k f_k$ as in* (1.70)*. Then there exists $c_0 > 0$ such that if $\|\varphi\|_{\mathcal{F}^{1,1}} < c_0$, for every $k \in \mathbb{N}$ and $T > 0$ there exists a unique solution $f_k \in L^\infty([0,T), \mathcal{F}^{1,1})$ of* (1.70)*. Moreover the sequence $g_M = \sum_{k=1}^{M} \varepsilon^k f_k$ converge in $\mathcal{F}^1(\mathbb{T})$ to a solution $f \in \mathcal{F}^1(\mathbb{T})$ of Muskat problem with initial condition $f(0) = \varepsilon\varphi$.*

Proof of Theorem (1.5.1). The existence of the solutions for (1.70) can be obtained in the following way, consider the problem

$$\begin{cases} \partial_t g + \Lambda g = h \quad , \quad (x,t) \in \mathbb{R} \times (0,T) \\ g(x,0) = 0 \qquad , \quad x \in \mathbb{R}, \end{cases} \tag{1.73}$$

uniqueness in $C([0,T],\mathcal{F}^{1,1}) \cap L^1([0,T]\mathcal{F}^{2,1})$ is obtained by taking taking Fourier transform and integrating. For the existence we suppose that $h \in L^\infty([0,T],\mathcal{F}^{1,1})$ then by the Duhamel principle we can write an explicit solution of (1.73) as

$$g = \int_0^t e^{-(t-\tau)\Lambda} h(x,\tau) d\tau, \tag{1.74}$$

by taking Fourier transform we get

$$\hat{g} = \int_0^t e^{-2\pi(t-\tau)|\xi|} \hat{h}(\xi,\tau) d\tau, \tag{1.75}$$

taking absolute value, multiplying by $|\xi|$ and integrating we get

$$\begin{aligned} \|g\|_{\mathcal{F}^{1,1}} &= \int_{\mathbb{R}} |\xi| \left| \int_0^t e^{-2\pi(t-\tau)|\xi|} \hat{h}(\xi,\tau) d\tau \right| d\xi \\ &\leq \int_{\mathbb{R}} |\xi| \int_0^t \left| \hat{h}(\xi,\tau) \right| d\tau d\xi \\ &\leq \int_0^t \int_{\mathbb{R}} |\xi| \left| \hat{h}(\xi,\tau) \right| d\xi d\tau \\ &= \int_0^t \|h(t)\|_{\mathcal{F}^{1,1}} d\tau \\ &\leq t \sup_{t\in[0,T]} \|h\|_{\mathcal{F}^{1,1}}, \end{aligned} \tag{1.76}$$

by taking supremum we obtain that $\sup_{t\in[0,T]} \|g\|_{\mathcal{F}^{1,1}} \leq T \sup_{t\in[0,T]} \|h\|_{\mathcal{F}^{1,1}}$ which implies that $g \in L^\infty([0,T],\mathcal{F}^{1,1})$. To prove that the right hand side of the equation of each f_n (1.70) belong to $L^\infty([0,T],\mathcal{F}^{1,1})$ we need the following Lemma.

Lemma 1.5.2. *Consider the family of solutions to the Muskat problem obtained considering the initial condition $f(0) = \varepsilon\varphi$ and varying $\varepsilon > 0$ and considering the expansion of the solution as $f = \sum_{\ell\geq 1} \varepsilon^{\ell} f_{\ell}$ with the initial condition $f(0) = \varepsilon\varphi$, then the terms in the expansion satisfy*

$$\sup_{t\in[0,T]} \|f_n\|_{\mathcal{F}^{1,1}} \leq AB^n \|\varphi\|^n_{\mathcal{F}^{1,1}}, \tag{1.77}$$

and

$$2\pi \int_0^T \|f_n\|_{\mathcal{F}^{2,1}} dt \leq AB^n \|\varphi\|^n_{\mathcal{F}^{1,1}}, \tag{1.78}$$

where $B > B_0$ and $A = A(B) > 0$ is large enough.

Remark 1.5.3. The size of the parameter is given by $B_0 = 1/\gamma$ where

$$\gamma = \sup_{z\in[0,1]} \frac{z(1-5z^2-2z^4)}{(1+z^2)^2} \approx 0.151388, \tag{1.79}$$

and therefore $B_0 \approx 6.60118$.

Proof of Lemma 1.5.2. By taking Fourier transform of (1.70) we get for $\xi \in \mathbb{R}$

$$\begin{aligned} \partial \hat{f}_n + 2\pi|\xi|\hat{f}_n = \frac{1}{\pi} \sum_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(-1)^j}{j} \\ \times \int_{\mathbb{R}_\alpha} \sum_{i_0+\cdots+i_{2j}=n} (2\pi i\xi)(m_\alpha \hat{f}_{i_0}) * (m_\alpha \hat{f}_{i_1}) * \cdots * (m_\alpha \hat{f}_{i_{2j}}) d\alpha, \end{aligned} \tag{1.80}$$

where $\mathcal{F}(\Delta_\alpha f)(\xi) = m_\alpha(\xi)\hat{f}(\xi) = \frac{1-e^{-2\pi i\alpha\xi}}{\alpha}\hat{f}(\xi)$. Then by taking

$$\int_{\mathbb{R}} \frac{1}{2} \left((1.80) \frac{\bar{\hat{f}}_n}{|\hat{f}_n|} |\xi|^s + \overline{(1.80)} \frac{\hat{f}_n}{|\hat{f}_n|} |\xi|^s \right) d\xi, \tag{1.81}$$

we obtain

$$\partial_t \|f_n\|_{\mathcal{F}^s} + (2\pi)\|f_n\|_{\mathcal{F}^{s+1}} \leq \frac{1}{\pi} \sum_{j=1}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{i_0+\cdots+i_{2j}=n} \frac{1}{j} \int_{\mathbb{R}} |\xi|^s$$

$$\times \left| \int_{\mathbb{R}_\alpha} (2\pi i\xi)(m_\alpha \hat{f}_{i_0}) * (m_\alpha \hat{f}_{i_1}) * \cdots * (m_\alpha \hat{f}_{i_{2k}}) d\alpha \right| \quad (1.82)$$

By the computations in [10], section 3 we get that

$$\sum_{i_0+\cdots+i_{2k}=n} \sum_{k\in\mathbb{Z}} |\xi|^s \left| \int_{\mathbb{R}_\alpha} (2\pi i\xi)(m_\alpha \hat{f}_{i_0}) * (m_\alpha \hat{f}_{i_1}) * \cdots * (m_\alpha \hat{f}_{i_{2k}}) d\alpha \right| d\xi$$

$$\leq 4\pi(2k+1)^s \sum_{i_0+\cdots+i_{2k}=n} \|f_{i_0}\|_{\mathcal{F}^{s+1,1}} \|f_{i_1}\|_{\mathcal{F}^{s,1}} \cdots \|f_{i_{2k}}\|_{\mathcal{F}^{s,1}}, \quad (1.83)$$

applying this inequality we obtain the estimate

$$\partial_t \|f_n\|_{\mathcal{F}^{s,1}} + 2\pi \|f_n\|_{\mathcal{F}^{s+1,1}}$$

$$\leq 4\pi \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (2k+1)^s \sum_{i_0+\cdots+i_{2k}=n} \|f_{i_0}\|_{\mathcal{F}^{s+1,1}} \|f_{i_1}\|_{\mathcal{F}^{1,1}} \cdots \|f_{i_{2k}}\|_{\mathcal{F}^{1,1}}. \quad (1.84)$$

Integrating the estimate in time and taking supremum between $[0,T]$ we get

$$\max\{\sup_{t\in[0,T]} \|f_n\|_s, (2\pi)\int_0^T \|f_n\|_{s+1}\} \leq 2 \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (2k+1)^s$$

$$\times \sum_{i_0+\cdots+i_{2k}=n} (2\pi)\int_0^T \|f_{i_0}\|_{s+1} dt \sup_{t\in[0,T]} \|f_{i_1}\|_1 \cdots \sup_{t\in[0,T]} \|f_{i_{2k}}\|_1 + \|f_n(0)\|_s. \quad (1.85)$$

Notice that $f_n(0) = 0$ for $n > 1$. For $n = 1$ we have that $\partial_t f_1 + \Lambda f_1 = 0, f_1(0) = \varphi$ and so

$$\max\{\sup_{t\in[0,T]} \|f_1\|_s, (2\pi)\int_0^T \|f_1\|_{s+1}\} \leq \|\varphi\|_s, \quad (1.86)$$

and for $n = 2$ we have that $\partial_t f_2 + \Lambda f_2 = 0,\ f_2(0) = 0$ and so $f_2(x,t) = 0$

$$\max\{\sup_{t\in[0,T]} \|f_2\|_s, (2\pi)\int_0^T \|f_2\|_{s+1}\} = 0 \quad (1.87)$$

Using (1.85) we will prove by induction for $s = 1$ that

$$\max\left\{\sup_{t\in[0,T]} \|f_n\|_1, (2\pi)\int_0^T \|f_n\|_2\right\} \leq C_n \|\varphi\|_1^n. \tag{1.88}$$

from (1.86) and (1.87) we know that this is true for $n = 1$ and $n = 2$ with $C_1 = 1$ and $C_2 = 0$. Now suppose that (1.88) is true for all $j < n$ we want to show that it is also true for $j = n$

$$\begin{aligned} &\max\left\{\sup_{t\in[0,T]} \|f_n\|_1, (2\pi)\int_0^T \|f_n\|_2 dt\right\} \\ &\leq 2\sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor}(2k+1)\sum_{i_0+\ldots+i_{2k}=n}(2\pi)\int_0^T \|f_{i_0}\|_{s+1}dt \sup_{t\in[0,T]}\|f_{i_1}\|_1 \ldots \sup_{t\in[0,T]}\|f_{i_{2k}}\|_1 \\ &\leq 2\sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor}(2k+1)\sum_{i_0+\ldots+i_{2k}=n} C_{i_0}\|\varphi\|_1^{i_0} C_{i_1}\|\varphi\|_1^{i_1}\ldots C_{i_{2k}}\|\varphi\|_1^{i_{2k}} = C_n\|\varphi\|_1^n \end{aligned} \tag{1.89}$$

where

$$C_n = 2\sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor}(2k+1)\sum_{i_0+\ldots+i_{2k}=n} C_{i_0}C_{i_1}\ldots C_{i_{2k}}. \tag{1.90}$$

We conclude that (1.88) is valid for all n. Now we focus our attention to estimate the growth rate of the coefficients C_n.

Lemma 1.5.4. *Consider the sequence* $\{C_n\}_{n\geq 1}$ *as defined by* (1.90) *then for* $B > B_0$ *there exists* $A(B) > 0$ *such that for all* $n \geq 1$

$$C_n \leq AB^n, \tag{1.91}$$

where $B_0 = \frac{1}{\tilde{\gamma}}$ *and* $\tilde{\gamma} = \sup_{t\in[0,1]} \frac{t(1-5t^2-2t^4)}{(1+t^2)^2}$.

Proof of Lemma 1.5.4. To estimate the growth of this sequence we can use the inverse function theorem, to do this we we consider the formal power series given by

$$F = \sum_{k\geq 1} C_k x^k \tag{1.92}$$

then we have that

$$F^{2k+1} = \sum_{n\geq 2k+1} x^n \sum_{i_0+\ldots+i_{2k}=n} C_{i_0} C_{i_1} \ldots C_{i_{2k}}. \tag{1.93}$$

Using this we get that the recurrence (1.90) can be written as

$$F = 2\sum_{k\geq 1}(2k+1)F^{2k+1} + xC_1. \tag{1.94}$$

Notice that the series can be rewritten as

$$\begin{aligned}
\sum_{k\geq 1}(2k+1)F^{2k+1} &= F\sum_{k\geq 1}(2k+1)F^{2k} \\
&= F\frac{\partial}{\partial F}\sum_{k\geq 1}F^{2k+1} \\
&= F\frac{\partial}{\partial F}F^3\sum_{k\geq 0}F^{2k} \\
&= F\frac{\partial}{\partial F}F^3\frac{1}{1-F^2} \\
&= F\frac{(3F^2(1-F^2)+2F^4)}{(1-F^2)^2} \\
&= F\frac{(3F^2-F^4)}{(1-F^2)^2},
\end{aligned} \tag{1.95}$$

and therefore we get

$$G(F) = F\left(1 - \frac{(3F^2-F^4)}{(1-F^2)^2}\right) = x. \tag{1.96}$$

Now we observe that $G(z)$ is holomorphic near $z = 0$ and $F(0) = 0$, $G'(0) = 1$ therefore by the inverse function theorem we get that there exists some neighborhood from zero U such that $G : U \to G(U)$ is biholomorphic and so we get that there exists some holomorphic function $F : G(U) \to U$, which has a nonzero radius of convergence around zero, which implies that the growth of the coefficients in the power series expansion of F have an at most exponential growth and so there exists A, B such that

$$C_n \le AB^n. \tag{1.97}$$

Therefore we conclude that

$$\max\left\{\sup_{t\in[0,T]} \|f_n\|_1, \int_0^T \|f_n\|_2 dt\right\} \le AB^n\|\varphi\|_1^n. \tag{1.98}$$

We can make an explicit estimate for B by using the Rouche's theorem and the following lemma

Lemma 1.5.5. *Let U be an open set of $\mathbb{C}$ and f be a univalent function on U. Then $f' \neq 0$ on U and $f : U \to f(U)$ is biholomorphic.*

We will estimate the size of the region U with the help of the Rouche's theorem. The equation (1.96) for F can be written as

$$F(1 - 5F^2 + 2F^4) = x(1 - F^2)^2 \tag{1.99}$$

When $x = 0$ it is easy to see that there is only one there is only one solution for F in the disk $\{|z| \le \beta\}$ where β is given by

$$1 - 5\beta^2 - 2\beta^4 = 0 \Rightarrow \beta^2 = \frac{5 \pm \sqrt{25+8}}{-4} \tag{1.100}$$

and so

$$\beta = \sqrt{\frac{\sqrt{33}-5}{4}} \approx 0.43144 \tag{1.101}$$

Now we want to use Rouche's theorem to find a region $V \subset \mathbb{C}$ of values of x for which the equation only has one solution. To apply Rouche's theorem use that

$$|F(1-5F^2+2F^4)| \geq |F|(1-5|F|^2-2|F|^4)$$

and

$$|x(1-F^2)^2| \leq |x|(1+|F|^2)^2 \tag{1.102}$$

and therefore it is enough to find a circle where

$$|x|(1+|F|^2)^2 < |F|(1-5|F|^2-2|F|^4), \tag{1.103}$$

$$|x| < \frac{|F|(1-5|F|^2-2|F|^4)}{(1+|F|^2)^2} \leq \tilde{\gamma} = \sup_{t\in[0,1]} \frac{t(1-5t^2-2t^4)}{(1+t^2)^2}. \tag{1.104}$$

We can compute the maximum of the right hand side and we get $|F| = \tilde{t} \approx 0.233893$ and $|x| < \tilde{\gamma} \approx 0.151488$, therefore we can apply Rouche's theorem for $|F| = \tilde{t}$ to get that for $|x| < \tilde{\gamma}$ the equation has a single solution and so $G(F)$ is univalent there and by the lemma $F(x)$ is holomorphic for $|x| < \tilde{\gamma}$ and therefore it radius of convergence around 0 is at least $R > \tilde{\gamma}$. Using this we obtain that

$$\limsup_{n\to\infty} |C_n|^{1/n} = \frac{1}{R} < \frac{1}{\tilde{\gamma}}. \tag{1.105}$$

So we get that for any $\delta > 0$ exists N large enough such that for $n > N$

$$|C_n|^{1/n} \leq \frac{1}{R} + \delta \Rightarrow |C_n| \leq \left(\frac{1}{R} + \delta\right)^n \tag{1.106}$$

and so for any $B > \frac{1}{\tilde{\gamma}}$ we can take by taking $A > 0$ large enough we get that

$$|C_n| \leq A \cdot B^n. \tag{1.107}$$

This concludes the proof of Lemma 1.5.4. □

Continuation of proof of Lemma 1.5.2. By applying Lemma 1.5.4 to (1.88) we get that for $n \in \mathbb{N}$

$$\max\left\{\sup_{t\in[0,T]} \|f_n\|_{\mathcal{F}^{1,1}}, (2\pi)\int_0^T \|f_n\|_{\mathcal{F}^{2,1}} dt\right\} \leq AB^n \|\varphi\|^n_{\mathcal{F}^{1,1}}, \tag{1.108}$$

which concludes the proof of Lemma (1.5.2). □

Continuation of Proof of Theorem 1.5.1. To prove the existence of solutions for the entire family $\{f_n\}$ we proceed by induction, the base case we use that $f_1 = e^{-t\Lambda}\varphi$ and therefore $\sup_{0,T} \|f_1\|_{\mathcal{F}^{1,1}} \leq \|\varphi\|_{\mathcal{F}^{1,1}}$. For the induction step, we assume that we have

$$\max\left\{\sup_{t\in[0,T]} \|f_n\|_{\mathcal{F}^{1,1}}, 2\pi\int_0^T \|f_n\|_{\mathcal{F}^{2,1}} dt\right\} \leq AB^n \|\varphi\|^n_1, \tag{1.109}$$

for $n = 1, \cdots, k$ then by (1.84) we know that the right hand side of (1.70) belongs to $L^1([0,T], \mathcal{F}^{1,1})$ and therefore by our previous computation we obtain that we can solve for f_n and $f_n \in L^1([0,T], \mathcal{F}^{1,1})$. Finally by applying Lemma 1.5.2 we get the existence for all f_n and the growth estimate for the norms. Finally by taking $c_0 > 0$ such that $c_0 B < 1$ where $B > 0$ is the value obtained from Lemma 1.5.2 we get that the sequence $g_n = \sum_{k=1}^n f_k$ is convergent in $L^1([0,T], \mathcal{F}^{1,1})$ and each term f_n satisfy the estimates given by Lemma (1.5.2). This concludes the proof of the Theorem 1.5.1. □

Note that the solution constructed by the Theorem 1.5.1 is not necessarily a solution of (1.38) because it was constructed under the a priori assumption that the Taylor expansion (1.66) converges, to show that the solutions that we just constructed is in fact a solution of (1.38) we notice that

$$|\Delta_\alpha f(x)| \leq \sup_x |f'(x)| \leq \|f\|_{\mathcal{F}^{1,1}}, \tag{1.110}$$

consequently we get that the function given by the theorem will be in fact a solution of (1.38) if $\|f\|_{\mathcal{F}^{1,1}} < 1$, to get this we use that

$$\begin{aligned} \sup_{t\in[0,T]} \|f\|_{\mathcal{F}^{1,1}} &\leq \sum_{k\geq 1} \sup_{t\in[0,T]} \|f_k\|_{\mathcal{F}^{1,1}} \\ &\leq \sum_{k\geq 1} AB^k \|\varphi\|^k_{\mathcal{F}^{1,1}} \\ &\leq \frac{A}{1-B\|\varphi\|_{\mathcal{F}^{1,1}}} \\ &< 1, \end{aligned} \tag{1.111}$$

and therefore by taking $\|\varphi\|_{\mathcal{F}^{1,1}}$ small enough such that

$$\frac{A}{1-B\|\varphi\|_{\mathcal{F}^{1,1}}} < 1, \tag{1.112}$$

we get that the solution given by Theorem 1.5.1 is in fact a solution of (1.38).

1.6 Strategy for Ill-posedness

When studying a differential equation, the usual approach is to understand under which assumptions the problem is well posed in the Hadamard's sense. This analysis is usually done by taking a space that is very regular and study the well posedness there and then try to weaken the assumptions to study obtain well posedness in a less regular space.

The natural barrier to study well posedness are the so called critical spaces. As a general rule it is expected that when you are in a space that is more regular than the critical one, also known as the subcritical case, the problem should be well posed, at least to suitable small data. For spaces that are less regular than the critical ones, also known as supercritical, the analysis is usually harder and less tools are available to study the problem in this regime, but it is expected that bad behaving solutions could exists in this context. The critical situation is typically very delicate and must be studied case to case.

We say that a problem is Hadamard's well posed in a certain space X if the we have the following

(i) There exits a solution in X,

(ii) the solution is unique,

(iii) the solution depends continuously on the data.

This means that in order to study the ill-Posedness we need to study the failure of at least one of those conditions. From now on we focus on the last one. For initial value problems, there are a few properties of the equations that we can look for to obtain an ill-posedness result in a given space X.

(i) Discontinuity of the solution map at the origin: to find a sequence of times and initial conditions $\{(t_k, \varphi_k)\}_{k\in\mathbb{N}}$ with $t_k \to 0$ and $\|\varphi_k\|_X \to 0$ as $k \to \infty$

such that if f_k is a solution of the equation with initial data φ_k then

$$\limsup_{k\to\infty} \|f_k(t_k)\|_X \neq 0. \tag{1.113}$$

(ii) Norm inflation: this is a stronger notion of discontinuity at the origin in which we show that solutions with arbitrarily small norm can become arbitrarily large in a arbitrarily short time, i.e. for any $R > 0$ and $T > 0$ there exists a initial condition $\varphi \in X$ with $\|\varphi\|_X \leq \frac{1}{R}$ and $0 < \tilde{t} < T$ such that if f is a solution of the equation with initial data φ then $\|f(\tilde{t})\|_X \geq R$.

(iii) Strong norm inflation: Given any $\varphi \in X$, $\varepsilon > 0$ and $T > 0$ there exists $\varphi_\varepsilon \in X$ and $0 < \tilde{t} < T$ such that $\|\varphi - \varphi_\varepsilon\|_X < \varepsilon$ and if f is a solution of the initial value problem with initial data φ_ε then

$$\|f(\tilde{t}) - \varphi_\varepsilon\|_X > \frac{1}{\varepsilon}. \tag{1.114}$$

All three of this notions have been used to study the ill posedness of fluid equations. In the case of Muskat we want to study the norm inflation phenomenon in some supercritical spaces. The strategy that we will use is based on studying the an expansion of the solution in terms of the Picard's iteration. First we consider the Taylor expansion of the nonlinearity as in (1.66) and then the equation obtained by truncating the expansion the the first ℓ terms

$$\begin{cases} \partial_t f + \Lambda f = \displaystyle\sum_{k=1}^{\ell} T_k e^{-t\Lambda}\varphi \quad , \quad (x,t) \in \Omega \times [0,T], \\ f(x,0) = \varphi(x) \qquad\qquad\quad , \quad x \in \Omega, \end{cases} \tag{1.115}$$

where in the case $\Omega = \mathbb{R}$, T_k is given by

$$T_k f = (-1)^k \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{\partial_x \delta_\alpha f(x)}{\alpha} \left(\frac{\delta_\alpha f(x)}{\alpha} \right)^{2k} d\alpha, \tag{1.116}$$

and for $\Omega = \mathbb{T}$,

$$T_k f = (-1)^k \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\partial_x \delta_\alpha f(x)}{\tan(\alpha/2)} \left(\frac{\tanh^2\left(\delta_\alpha f(x)/2\right)}{\tan^2(\alpha/2)} \right)^{k} \sec^2(\alpha/2) d\alpha. \tag{1.117}$$

next we consider the Picard's iteration of the equation (1.115), by setting $f_0 = 0$, and for $k \geq 1$

$$\begin{cases} \partial_t f_k + \Lambda f_k = \displaystyle\sum_{n=1}^{\ell} T_n(f_{k-1}) & , \quad \text{in } \Omega \times [0,T], \\ f_k(0) = \varphi & , \quad \text{on } \Omega, \end{cases} \tag{1.118}$$

by using the Duhamel's principle f_k can be written as

$$f_k = e^{-t\Lambda}\varphi + \int_0^t e^{-(t-\tau)\Lambda} T(f_{k-1}) d\tau, \tag{1.119}$$

then assuming that the sequence converges we can write $f = \sum_{k\geq 1}(f_k - f_{k-1})$

$$f = e^{-t\Lambda}\varphi + \int_0^t e^{-(t-\tau)\Lambda} \sum_{n=1}^{\ell} T_n(e^{-\tau\Lambda}\varphi) + R(x,t). \tag{1.120}$$

To get an inflation result the idea is to look at this expansion and identify a large term. The first term regular in general because is the evolution of a heat flow with with Λ instead of the Laplacian. The second will be studied carefully on Chapter 4 to study its inflation properties on the space $\mathcal{F}_q^{\frac{2\ell-1}{2\ell+1},p}$, for $p > 1$ and $q > 2\ell + 1$. For the last term $R(x.t) = \sum_{k=3}(f_k - f_{k-1})$ we need some kind of bound in some supercritical space for the kind of initial data that we are using.

In many situations the Picard's iteration is expected to converge to a solution the problem, but in the case of supercritical spaces this is a hard question in general, especially because we are using highly oscillatory initial data.

1.7 Summary of the known results

For the Muskat problem, in the Rayleigh-Taylor unstable case $\rho_1 > \rho_2$ the problem is known for to be ill-posed in the Sobolev spaces H^s for $s > 3/2$ and $d = 2, 3$ in [15],[16], this is done by scaling a fixed solution and showing that for arbitrarily small initial data the solution blow up after an arbitrarily shot time.

When $\rho_1 < \rho_2$ short time existence [15], [14] in 2D for H^s $s \geq 3$, and in 3D for H^s $s \geq 4$ in the case of a graph interface. [13] in 2D for the non graph case H^k, $k \geq 3$ under the chord-arc condition. [6] in $2D$ for $H^2(\mathbb{R})$ initial data with small $H^{3/2+\varepsilon}$ norm. [32] in $2D$ local existence and uniqueness for H^s $s \in (3/2, 2)$ data for the case without surface tension and for H^s, $s \in (2, 3)$ for the 2D Muskat with surface tension.

For global in time existence in the Muskat problem, in 2D [11] for $W^{2,p}(\mathbb{R})$ data with small slope. [34] for $f_0 \in H^\ell$ $\ell \geq 3$ initial data with small $\|f_0\|_{\mathcal{F}^{1,1}} < k_0$ large time decay in the $\mathcal{F}^{\nu,1}$ norms. [6] in 2D global existence in the periodic case for data with small H^2 norm and in the real line for H^2 initial data with small $H^{3/2+\varepsilon}$ norm. [9] for $d = 2, 3$, global existence for H^s $s \geq 4$ initial data with $\|\nabla f_0\|_{L^\infty} < 1/3$, in 3D global existence for L^∞ initial data and small slope. [10], [9] in 2D and 3D global in time weak solutions for initial data $f_0 \in L^2$ with $\|f_0\|_{\mathcal{F}^{1,1}} < k_0$ and classical solutions if additionally the initial data belongs to H^ℓ for $\ell \geq 2$ in 2D and $\ell \geq 3$ in 3D. [31] in 2D proves global existence for small $H^{3/2+\varepsilon}$ data. [20] for $d = 2, 3$ the viscosity jump case for L^2 data with small $\mathcal{F}^{1,1}$ norm. [19] for Muskat Bubbles with appropriate small $\mathcal{F}^{1,1}(\mathbb{T})$ data in the appropriate parameterization for the problem. [4] in 3D, global existence for unbounded initial data with medium size slope and slow growth at infinity.

For global existence without small slope assumption. In [18] global solutions with monotone initial data with finite limits at infinity. [3] in 2D, [4] in 3D C^1 global solutions when $(\sup f_0')(\sup -f_0') < 1$. [17] in 2D for initial data in $H^{5/2}(\mathbb{R}) \cap H^{3/2}(\mathbb{R})$ with small $\dot{H}^{3/2}(\mathbb{R})$ norm, where the required size depend on the maximum size of the slope. [21] in 3D with $W^{1,\infty} \cap \dot{H}^2$ initial data with small $\dot{H}^2$ where the required size depend on the maximum size of the slope.

For other fluid problems there have been several results on the Ill-posedness in the last few years. In [2] for the 3D Navier-Stokes the norm inflation in the critical space $\dot{B}_{\infty}^{-1,\infty}$ and [35] for the $\dot{B}_{q}^{-1,\infty}$ case. Both results are obtained by studying the mapping properties of the second Picard's iteration of the problem as described in Section 1.6. In [23] discontinuity at the origin for the second Picard's iterate in $B_{q}^{-1,\infty}$ for $q > 2$ and $d \geq 2$. [8] discontinuity of the solution map in a periodic domain for Euler and $d \geq 2$ in $B_{\infty}^{\frac{3}{r}-1,r}$. [7] for discontinuity of the solution map for the Navier-Stokes equation with fractional diffusion. [1] for Euler and $d \geq 2$

show that a small perturbation of a H^{s_c} initial data shows norm inflation where $s_c = d/2 + 1$ is the critical exponent for the equation. [26] for a Drift Diffusion system in 2D show inflation by analyzing the second iterate and using modulation spaces to study the higher iterations. [33] for 2D Euler the discontinuity of the solution map in $C^1(\mathbb{R})$ and $B_1^{1,\infty}(\mathbb{R})$ is obtained.

1.8 Main Results

The first two results concern the stability of the 2D periodic Muskat equation (1.44). The objective is to close some gaps in the well posedness theory for the 2D Muskat equation in a periodic domain.

The first result deal with the question of global existence on a periodic domain in a critical space. Short time existence was known from [15] and the well posedness in critical space $\mathcal{F}^{1,1}$ was proven for the case of the real line in [10]. The next result extend the global existence result in [10] to a periodic domain.

Theorem 1.8.1 (Global existence for small initial data critical space)**.**
Let $f_0 \in H^3(\mathbb{T}) \cap \mathcal{F}^{1,1}(\mathbb{T})$ such that $\|f_0\|_{\mathcal{F}^{1,1}} \leq c_0$. Consider the Muskat problem (1.44) *with initial data f_0 and $\frac{\rho_2-\rho_1}{2} = 1$. Then there exists a unique $f \in C([0,\infty), \mathcal{F}^{1,1}) \cap L^\infty([0,\infty), \mathcal{F}^{1,1}) \cap L^1([0,\infty), \mathcal{F}^{2,1})$ solution of* (1.44)*. Also f satisfies the estimate*

$$\|f\|_{\mathcal{F}^{1,1}} + \sigma \int_0^T \|f\|_{\mathcal{F}^{2,1}} dt \leq \|f_0\|_{\mathcal{F}^{1,1}}, \tag{1.121}$$

for some $\sigma = \sigma(\|f_0\|_{\mathcal{F}^{1,1}}) < 1$.

The proof of Theorem 1.8.1 can be found on Chapter 2. Theorem 1.8.1 and its proof are useful for us for two reasons, first the result itself extend a well posedness result known for the case of the real line to the periodic case, and second, it illustrates some of the principles used in [15] on how to apply techniques from $\mathbb{R}^d$ to obtain a result in $\mathbb{T}^d$. In [15] they use a expansion of the kernel of the Riesz transform in $\mathbb{T}^d$ in terms of the kernel in $\mathbb{R}^d$ up to some terms that needs to be estimated, in the proof we use a more explicit approach that give a more precise estimate on the size of the constant c_0 that tell us how big the data can be for the result to be valid.

Under stronger regularity assumptions and using different techniques it is possible to extend the results from [11], to the periodic setting.

Theorem 1.8.2 (Global existence in H^2 for data with small slope)**.** *Suppose that the initial data* $f_0 \in L^2(\mathbb{T})$ *satisfy* $\int_{\mathbb{T}} f_0 = 0$ *and*

$$\|f_0'\|_{L^\infty} < k_0, \tag{1.122}$$

for a small constant k_0*. If we additionally have that* $f_0'' \in L^2(\mathbb{T})$*, then there exists a unique global in time solution of* (1.44) *with initial data* f_0*. Moreover the solution satisfy*

$$\|f''(t)\|_{L^2} \leq \max\{\|f_0''\|_{L^2}, (2\pi)^{1/3}\}, \tag{1.123}$$

for all $t > 0$.

The proof of Theorem 1.8.2 can be found in Chapter 3 in Theorem 3.1.3. The last result that we prove with respect to the Muskat problem has to do with the question of the Ill-posedness. This result is a intermediate step on proving the existence of norm inflation for the Muskat problem. By following the strategy presented in Section 1.6 we consider the expansion obtain by taking Taylor expansion of the nonlinear term, truncate it to finitely many terms and use the Picard's iteration to obtain the decomposition (1.120).

Theorem 1.8.3 (Norm inflation for truncated system)**.** *Let $\ell \in \mathbb{N}$ and consider the second Picard's iteration of truncation of the Muskat problem of order ℓ given by* (1.115) *for $\Omega = \mathbb{R}$ or $\mathbb{T}$. Then given $T > 0, R > 0$, there exists some $0 < \tilde{t} < T$, and an initial condition $f_0 \in \dot{\mathcal{F}}_q^{\frac{2\ell-1}{2\ell+1},p}(\Omega)$, $p \geq 1$, $q > 2\ell + 1$ such that*

$$\|f_0\|_{\mathcal{F}_q^{\frac{2\ell-1}{2\ell+1},p}} < 1/R \quad \text{and} \quad \|f(\tilde{t})\|_{\dot{\mathcal{F}}_q^{\frac{2\ell-1}{2\ell+1},p}} > R. \tag{1.124}$$

This result is proven in Chapter 4. In order to understand the purpose of this result, we can consider the map

$$L : \mathcal{F}_q^{\frac{2\ell-1}{2\ell+1},p} \to C([0,T]; \mathcal{F}_q^{\frac{2\ell-1}{2\ell+1},p}), \tag{1.125}$$

that takes a function $\varphi \in \dot{\mathcal{F}}_q^{\frac{2\ell-1}{2\ell+1},p}$ and return the solution f of the second Picard's iteration of the truncated Muskat problem of order ℓ with initial condition φ given by

$$\begin{cases} \partial_t f + \Lambda f = \displaystyle\sum_{k=1}^{\ell} T_k e^{-t\Lambda}\varphi \quad , \quad (x,t) \in \Omega \times [0,T], \\ f(x,0) = \varphi(x) \qquad\qquad\qquad , \quad x \in \Omega. \end{cases} \tag{1.126}$$

Now from Theorem 1.8.3 we can conclude that for arbitrarily small time $T > 0$ it is possible to find a sequence of times and initial data $\{(t_N, \varphi_N)\}_{N=1}^{\infty}$ such that if $f_N = L\varphi_N$ satisfy

$$\|\varphi_N\|_{\dot{\mathcal{F}}_q^{\frac{2\ell-1}{2\ell+1},p}} \leq \frac{1}{N} \text{ and } \|f_N(t_N)\|_{\dot{\mathcal{F}}_q^{\frac{2\ell-1}{2\ell+1},p}} > N. \tag{1.127}$$

This implies that the solution map $L : \dot{\mathcal{F}}_q^{\frac{2\ell-1}{2\ell+1},p} \to C([0,T]; \mathcal{F}_q^{\frac{2\ell-1}{2\ell+1},p})$ is discontinuous at the origin.

Chapter 2

Global existence for 2D Muskat problem in a periodic domain

Abstract

In this chapter we establish global existence for solutions of the periodic 2D Muskat problem for small data in the critical space $\mathcal{F}^{1,1}$. This is done by obtaining a priori estimates for the $\mathcal{F}^{1,1}$ norm and adapting the general strategy established in [10], [9] for the non periodic case. A key ingredient required for the a priori estimate is a bound in the $\mathcal{F}^{1,1}$ norm for the nonlinear term. The main contribution is a new estimate for the Fourier transform of the nonlinear term obtained by careful analysis of the size of the coefficients of its Taylor series expansion, which allow us to establish an estimate on the $\mathcal{F}^{1,1}$ norm required for global existence result.

2.1 Introduction

2.1.1 Description of the model

The Muskat problem describe the evolution of an interface between two immersible fluids of different constant densities in a porous media with velocity given by the Darcy's law. We consider the case in which we have two fluid one on top of the other, the fluids are infinitely deep so we can ignore the boundary effects, we neglect surface tension, and assume that the fluids have the same viscosity and therefore no shear effects. The density function is given by

$$\rho(x,y,t) = \begin{cases} \rho_1 \quad , \quad \text{in } \Omega_1(t) = \{y > f(x,t)\} \\ \rho_2 \quad , \quad \text{in } \Omega_2(t) = \mathbb{R}^2 \setminus \Omega_1(t). \end{cases} \tag{2.1}$$

Under these assumptions it is known that a necessary condition for stability is the Rayleigh-Taylor condition $\rho_1 < \rho_2$ [15], [16] otherwise the problem is known to be ill posed. In what follows we only deal with the case in which $\rho_2 > \rho_1$. When the interface can be described as a graph $f(x,t)$, its evolution can be described by using the equation (see Section 1.3.2),

$$\begin{cases} \partial_t f(x) = \dfrac{\rho_2 - \rho_1}{2\pi} p.v. \displaystyle\int_{\mathbb{R}} \frac{\partial_x \delta_\beta f(x) \beta}{\beta^2 + (\delta_\beta f(x))^2} d\beta \quad , \quad (t,x) \in (0,T) \times \mathbb{R}, \\ f(x,0) = f_0(x) \quad , \quad x \in \mathbb{R}, \end{cases} \tag{2.2}$$

where $\delta_s f = f(x) - f(x-s)$.

If we look for periodic solutions of the problem, say $f(x,t) = f(x+2\pi,t)$, we can study the integral in the principal value sense, as in Section 1.3.3 the equation may be rewritten as

$$
\begin{aligned}
\partial_t f(x) &= \frac{\rho_2-\rho_1}{4\pi} p.v. \int_{\mathbb{T}} \frac{\partial_x \delta_s f(x)}{\tan(s/2)} ds \\
&\quad -\frac{\rho_2-\rho_1}{4\pi} p.v. \int_{\mathbb{T}} \frac{\partial_x \delta_s f(x)}{\tan(s/2)} \frac{\sec^2(s/2)\tanh^2(\delta_s f(x)/2)}{\tan(s/2)^2+\tanh^2(\delta_s f(x)/2)} ds \\
&= -\frac{\rho_2-\rho_1}{2}\Lambda f \\
&\quad -\frac{\rho_2-\rho_1}{4\pi} p.v. \int_{\mathbb{T}} \frac{\partial_x \delta_s f(x)}{\tan(s/2)} \frac{\sec^2(s/2)\tanh^2(\delta_s f(x)/2)}{\tan(s/2)^2+\tanh^2(\delta_s f(x)/2)} ds \\
&= -\frac{\rho_2-\rho_1}{2}\Lambda f + \frac{\rho_2-\rho_1}{2} T(f) \\
f(x,0) &= f_0(x), \qquad x \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}),
\end{aligned} \tag{2.3}
$$

by noticing that we can always add a constant to the solution and still have a solution, we can assume that $\int f_0 dx = 0$, and because

$$\partial_t f = \frac{\rho_2-\rho_1}{2\pi} p.v. \int_{\mathbb{T}} \partial_x \arctan\left(\frac{\tanh(\delta_s f(x)/2)}{\tan(s/2)}\right) ds \tag{2.4}$$

we get that the quantity $\int_{\mathbb{T}} f dx$ is preserved over time. Here $\Lambda = (-\Delta)^{1/2}$, or in terms of Fourier series

$$\Lambda f(x) = \sum_{k\in\mathbb{Z}} |k| \hat{f}(k) e^{ikx}, \qquad \hat{f}(k) = \frac{1}{2\pi}\int_{\mathbb{T}} f(x) e^{-ikx} dx. \tag{2.5}$$

We say that a function $f \in L^2$ is a weak solution of (2.3) if for every $g \in C^\infty([0,T]; C^\infty(\mathbb{T}))$

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{T}} \partial_t f(x,t) g(x,t) dx dt \\
&\qquad + \frac{\rho_2-\rho_1}{2\pi} \int_0^T \int_{\mathbb{T}} \int_{\mathbb{T}} \arctan\left(\frac{\tanh(\delta_s f(x)/2)}{\tan(s/2)}\right) ds \partial_x g(x,t) dx dt = 0.
\end{aligned} \tag{2.6}
$$

The goal of this chapter is to extend the results of [9] and [20] for the 2D Muskat equation for a periodic domain.

Definition 2.1.1. Let $s \in \mathbb{R}$ and $f \in C^{\infty}(\mathbb{T})$, we define $\|\cdot\|_{\mathcal{F}^{s,1}}$ by

$$\|f\|_{\mathcal{F}^{s,1}} = \sum_{k\in\mathbb{Z}} |k|^s |\hat{f}(k)|, \quad \text{where } \hat{f}(k) = \mathcal{F}(f)(k) = \frac{1}{2\pi}\int_{\mathbb{T}} f(x)e^{-ikx}dx. \tag{2.7}$$

We also define the spaces $\mathcal{F}^{s,1}(\mathbb{T})$ as the closure of $C^{\infty}(\mathbb{T})$ with respect to $\|\cdot\|_{\mathcal{F}^{s,1}}$.

2.1.2 Main Results

Theorem 2.1.2 (Global existence for small initial data)**.** *Let $f_0 \in H^3(\mathbb{T}) \cap \mathcal{F}^{1,1}(\mathbb{T})$ such that $\|f_0\|_{\mathcal{F}^{1,1}} \leq c_0$. Consider the Muskat problem* (2.3) *with initial data f_0 and $\frac{\rho_2-\rho_1}{2} = 1$. Then there exists a unique $f \in C([0,\infty), \mathcal{F}^{1,1}) \cap L^{\infty}([0,\infty), \mathcal{F}^{1,1}) \cap L^1([0,\infty), \mathcal{F}^{2,1})$ weak solution of* (2.3)*. Also f satisfies the estimate*

$$\|f\|_{\mathcal{F}^{1,1}} + \sigma \int_0^T \|f\|_{\mathcal{F}^{2,1}} dt \leq \|f_0\|_{\mathcal{F}^{1,1}}, \tag{2.8}$$

for some $\sigma = \sigma(\|f_0\|_{\mathcal{F}^{1,1}}) < 1$.

Remark 2.1.3. The hypothesis of $H^3(\mathbb{T})$ initial data ensure that solutions given by Theorem 2.1.2 are in fact classical solution, this hypothesis can be relaxed by following a regularization strategy similar to the one used in [9] for the 3D Muskat problem, to obtain the existence of weak solutions for the problem under the assumption that the initial data belongs to $L^2(\mathbb{T})$ and is small in $\mathcal{F}^{1,1}$.

2.2 Proof of Theorem 2.1.2

The goal of this section is to prove the following estimate

Theorem 2.2.1. *Let $f_0 \in \mathcal{F}^{1,1}(\mathbb{T}) \cap H^3(\mathbb{T})$ and consider the initial value problem for the 2D Muskat equation in a periodic domain* (2.3) *with initial condition f_0, then there exists $\bar{t}(\|f_0\|_1) \in \mathbb{R}$ such that there is a unique solution $f \in C([0,\bar{t}], \mathcal{F}^1)$ of* (2.3) *that satisfy*

$$\|T(f)\|_{\mathcal{F}^{1,1}} \leq \|f\|_{\mathcal{F}^{2,1}} M_1(\|f\|_{\mathcal{F}^{1,1}}), \tag{2.9}$$

and

$$\|T(f)\|_{\mathcal{F}^{2+\delta,1}} \leq \|f\|_{\mathcal{F}^{3+\delta,1}} M_2(\|f\|_{\mathcal{F}^{1,1}}), \tag{2.10}$$

for monotone increasing functions $0 \leq M_1(x) \leq M_2(x)$ that satisfy $M_1(0) = M_2(0) = 0$.

Theorem 2.2.2. *Let $f_0 \in \mathcal{F}^{1,1}(\mathbb{T}) \cap H^3(\mathbb{T})$ such that $\|f_0\|_{\mathcal{F}^{1,1}} < c_0$. Let f be the unique solution of* (2.3) *with initial data $f(0) = f_0$ then*

$$\|f\|_{\mathcal{F}^{1,1}} + \sigma \int_0^t \|f\|_{\mathcal{F}^{2,1}} dt \leq \|f_0\|_{\mathcal{F}^{1,1}} \tag{2.11}$$

for some $\sigma = \sigma(\|f_0\|_{\mathcal{F}^{1,1}}) \in (0,1)$.

Remark 2.2.3. The size of the constant c_0 is chosen such that for some $\delta > 0$

$$H_3(c_0) + H_4(c_0) < 1, \tag{2.12}$$

where

$$H_3(x) = \frac{1}{2\pi} \sum_{k \geq 1} (2k+1)^{2+\delta} \left(6 + \frac{\pi}{1 - (\pi/4)^{2k+1}} + \frac{1}{2k} \left(\frac{\pi}{2}\right)^{2k+1} \right) x^{2k}, \tag{2.13}$$

and

$$H_4(x) = 4 \sum_{k \geq 1} \frac{1}{2k} \sum_{\ell \geq 2k+1} (\ell+1)^{2+\delta} (2x)^{\ell}. \tag{2.14}$$

First we will prove Theorem 2.2.2 by assuming that Theorem 2.2.1.

Proof Theorem 2.2.2. Let f be the solution of equation (2.3) given by Theorem 2.2.1. Let $Lf = \partial_t f + \Lambda f$ and consider

$$\begin{aligned}
\frac{1}{2}\sum_{n\in\mathbb{Z}} |n|^\eta \left(\mathcal{F}(Lf)\frac{\bar{\hat{f}}}{|\hat{f}|} + \overline{\mathcal{F}(Lf)}\frac{\hat{f}}{|\hat{f}|}\right) &= \partial_t \|f\|_{\mathcal{F}^{\eta,1}} + \|f\|_{\mathcal{F}^{\eta+1,1}} \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}} |n|^\eta \left(\mathcal{F}(Tf)\frac{\hat{f}}{|\hat{f}|} + \overline{\mathcal{F}(Tf)}\frac{\bar{\hat{f}}}{|\hat{f}|}\right). \\
&\leq \|Tf\|_{\mathcal{F}^{\eta,1}}.
\end{aligned} \tag{2.15}$$

By applying Theorem 2.2.1 with $\eta = 1$ and $\eta = 2+\delta$, $\delta \in (0, 1/2)$, we can find c_0 small enough such that $M_1(x) < 1$ and $M_2(x) < 1$ for $|x| < c_0$. By the short time existence result in [15] we know that because $f_0 \in H^3(\mathbb{T})$ there is a time $\bar{t} = \bar{t}(\|f_0\|_{H^3})$ such that the solution exist in $[0, \bar{t}]$. For such solution we have that

$$\partial_t \|f\|_{\mathcal{F}^{1,1}} + \|f\|_{\mathcal{F}^{2,1}} \leq \|Tf\|_1 \leq M_1(\|f\|_{\mathcal{F}^{1,1}})\|f\|_{\mathcal{F}^{2,1}} \tag{2.16}$$

$$\partial_t \|f\|_{\mathcal{F}^{1,1}} + (1 - M_1(\|f\|_{\mathcal{F}^{1,1}}))\|f\|_{\mathcal{F}^{2,1}} \leq 0. \tag{2.17}$$

Let $\sigma = 1 - M(c_0)$ and take c_0 small enough so that $\sigma < 1$. Let $\|f_0\|_{\mathcal{F}^{1,1}} < c_0$, by Gronwall inequality we know from (2.16) that if initially $\|f(0)\|_{\mathcal{F}^{1,1}} < c_0$ then the solution still continues to satisfy that condition for a shot time, then we can use that (2.17) to conclude that in fact the $\|f\|_{\mathcal{F}^{1,1}}$ do not increase, and consequently we can bootstrap the same argument for the entire interval of existence of the solution to conclude that

$$\frac{d}{dt}\|f\|_{\mathcal{F}^{1,1}}(t) \leq 0, \quad t \in [0, \bar{t}]. \tag{2.18}$$

By an analogous argument we get that for $\|f_0\|_{\mathcal{F}^{1,1}} < c_0$

$$\frac{d}{dt}\|f\|_{\mathcal{F}^{2+\delta,1}}(t) \leq 0, \quad t \in [0,\bar{t}]. \tag{2.19}$$

Now from [15] we know that if $\|f\|_{C^{2,\delta}}$ remains bounded then we can extend the solution to belong to $C([0,T];H^3(\mathbb{T}))$ for any $T > 0$. The boundedness of the $C^{2,\delta}(\mathbb{T})$ norm is obtained from [10] by using that

$$\|f\|_{C^{2,\delta}} \leq C\left(\|f\|_{L^\infty} + \|f\|_{\mathcal{F}^{1,1}} + \|f\|_{\mathcal{F}^{2+\delta,1}}\right), \tag{2.20}$$

therefore the solution can be continued for all time if $\|f_0\|_{\mathcal{F}^{1,1}} < c_0$ and initially $\|f_0\|_{\mathcal{F}^{2,\delta}}$ is finite, which is the case by Sobolev embedding. □

Now we proceed to prove the main estimate of the chapter.

Proof of Theorem 2.2.1. Consider the Muskat equation in a periodic domain (2.3) with $\frac{\rho_2-\rho_1}{2} = 1$

$$\partial_t f + \Lambda f = T(f), \tag{2.21}$$

by expanding the geometric series we get

$$T(f) = \frac{1}{2\pi}\sum_{k\geq 1}(-1)^k \int_{\mathbb{T}} \partial_x\delta_s f(x)\left(\frac{\tanh\left(\delta_s f(x)/2\right)}{\tan(s/2)}\right)^{2k} \frac{\sec^2(s/2)}{\tan(s/2)}ds. \tag{2.22}$$

In order to estimate this quantity, we want to find an expansion in terms of $\delta_s f(x)$, for this purpose we need information about the size of the coefficients in the Taylor expansion of $\tanh^{2m}(y)$. For this purpose we use the the following Lemma.

Lemma 2.2.4 (Taylor expansion of $\tanh(x)$)**.** *Let $z \in \mathbb{C}$ s.t. $|z| < \pi/2$ then the Taylor expansion of* $\tanh^m(z)$ *can be written as*

$$\tanh^m(z) = z^m + \sum_{k>m} a_k^{(m)} z^k, \tag{2.23}$$

where the coefficients $a_k^{(m)}$ satisfy

$$a_k^{(m)} \leq \left(\frac{4}{\pi}\right)^k. \tag{2.24}$$

Proof of Lemma 2.2.4. The first part of the Lemma is obtained by using the exact values of the first two coefficients of the Taylor expansion $\tanh(0) = 0$, $\frac{d}{dx}\tanh(0) = 1$, then we can write

$$\tanh(x) = x + \sum_{k\geq 2} a_k x^k = x\left(1 + \sum_{k\geq 2} a_k x^{k-1}\right), \tag{2.25}$$

and by taking the m-th power we obtain

$$\tanh^m(x) = x^m\left(1 + \sum_{k\geq 1} b_k x^k\right). \tag{2.26}$$

To estimate the size of the coefficients we will estimate the size of the derivatives at the origin using the Cauchy integral formula, let $f(z) = \tanh(z)$ then

$$D^k(f^m)(0) = \frac{k!}{2\pi i}\int_\gamma \frac{f(w)^m}{(w-0)^k}dw, \tag{2.27}$$

where $\gamma = \{z : |z| = c\}$. To estimate this integral we need to estimate the size of the hyperbolic tangent in a circle, to do this we look for the radius of the circle in which the hyperbolic tangent can be bounded by 1

$$|\tanh(z)|^2 = \left|\frac{e^z - e^{-z}}{e^z + e^{-}z}\right|^2 \leq 1 \tag{2.28}$$

$$
\begin{aligned}
0 &\leq \left|e^z + e^{-z}\right|^2 - \left|e^z - e^{-z}\right|^2 \\
&= (e^z + e^{-z})(e^{\bar{z}} + e^{-\bar{z}}) - (e^z - e^{-z})(e^{\bar{z}} - e^{-\bar{z}}) \\
&= \left(e^z e^{\bar{z}} + e^{-z}e^{-\bar{z}} + e^z e^{-\bar{z}} + e^{-z}e^{\bar{z}}\right) - \left(e^z e^{\bar{z}} + e^{-z}e^{-\bar{z}} - e^z e^{-\bar{z}} - e^{-z}e^{\bar{z}}\right) \\
&= e^z e^{-\bar{z}} + e^{-z}e^{\bar{z}} \\
&= \Re\left(e^{z-\bar{z}}\right) = \Re\left(e^{2i\Im(z)}\right) = \cos(2\Im z),
\end{aligned}
\tag{2.29}
$$

we conclude that $|\tanh(z)| \leq 1$ for $|2\Im z| \leq \pi/2 \Rightarrow |\Im z| \leq \pi/4$. By taking the curve γ to be a circle of radius $c = \pi/4$ centered at the origin we get

$$
\begin{aligned}
|D^k(f^m)(0)| &\leq \frac{k!}{2\pi}\int_\gamma \frac{|f(w)^m|}{|w|^{k+1}}dw \\
&\leq \frac{k!}{2\pi}(\sup_\gamma |f|)^m 2\pi(\pi/4)\frac{1}{(\pi/4)^{k+1}} \\
&= k!\left(\frac{4}{\pi}\right)^k,
\end{aligned}
\tag{2.30}
$$

this concludes the proof of Lemma 2.2.4. □

Continuation of proof of Theorem 2.2.1. By applying Lemma 2.2.4 to (2.22) we get

$$
T(f) = J_1 + J_2, \tag{2.31}
$$

where

$$
\begin{aligned}
J_1 &= \frac{1}{2\pi}\sum_{k\geq 1}(-1)^k \int_{\mathbb{T}} \partial_x \delta_s f(x) \left(\frac{\delta_s f(x)}{2\tan(s/2)}\right)^{2k} \frac{\sec^2(s/2)}{\tan(s/2)}ds, \\
J_2 &= \frac{1}{2\pi}\sum_{k\geq 1}\sum_{\ell > 2k}(-1)^k a_\ell^{(2k)} \\
&\quad \times \int_{\mathbb{T}} \partial_x \delta_s f(x) \left(\frac{\delta_s f(x)}{2\tan(s/2)}\right)^{2k} \left(\frac{\delta_s f(x)}{2}\right)^{\ell-2k} \frac{\sec^2(s/2)}{\tan(s/2)}ds.
\end{aligned}
\tag{2.32}
$$

Now we take the $\mathcal{F}^{1,1}$ norm of J_1 to obtain

$$\|J_1\|_{\mathcal{F}^{1,1}} \leq \frac{1}{2\pi} \sum_{k\geq 1} \sum_{n\in\mathbb{Z}} |n| \left| \int_{\mathbb{T}} (im_s(\cdot)\widehat{\partial_x f}) * \left(\frac{m_s(\cdot)}{2\tan(s/2)} \hat{f} \right)^{*2k} (n) \frac{\sec^2(s/2)}{\tan(s/2)} ds \right|, \tag{2.33}$$

where $m_s(n) = 1-e^{-isn}$. To estimate the integral in s we need the following Lemma.

Lemma 2.2.5. *Let $m_s(n) = 1 - e^{-isn}$ and $n, k_1, \cdots, k_\ell \in \mathbb{Z}$, and $\ell \geq m$ then,*

$$C_{\ell,m} = \int_{\mathbb{T}} m_s(n-k_1)m_s(k_1-k_2)\cdots m_s(k_{\ell-1}-k_\ell)m_s(k_\ell) \frac{\sec^2(s/2)}{\tan^{m+1}(s/2)} ds$$

$$\leq |k_1 - k_2| \cdots |k_m| B_{\ell,m}, \tag{2.34}$$

where

$$|B_{m,m}| \leq 4 + \frac{\pi}{2}\left(\frac{2}{1-(\pi/4)^{m+1}} + (4/\pi) \right) + \frac{1}{m}\left(\frac{\pi}{2}\right)^{m+1} \tag{2.35}$$

and for $\ell > m$

$$B_{\ell,m} \leq \frac{16}{m}\left(\frac{\pi}{2}\right)^{\ell+1}, \tag{2.36}$$

Proof of Lemma 2.2.5. For $s \in \mathbb{R}$ and $n \in \mathbb{Z}$ we consider

$$\begin{aligned} K(s) &= m_s(n-k_1)m_s(k_1-k_2)\cdots m_s(k_{m-1}-k_m)m_s(k_m) \\ &= (-i)^m s^m (k_1-k_2)(k_2-k_3)\cdots(k_{m-1}-k_m)k_m \\ &\qquad \times(1-e^{-is(n-k_1)}) \int_0^1 e^{-is(k_1-k_2)(1-t_1)} dt_1 \cdots \int_0^1 e^{-isk_m(1-t_m)} dt_m \\ &= (-i)^m s^m (k_1-k_2)(k_2-k_3)\cdots(k_{m-1}-k_m)k_m \\ &\qquad \times \int_0^1 \cdots \int_0^1 (\exp(-isA) - \exp(-isB))\, dt_1 \cdots dt_m, \end{aligned} \tag{2.37}$$

where

$$A = (k_1-k_2)(1-t_1) + (k_2-k_3)(1-t_2) + \cdots + k_m(1-t_m), \tag{2.38}$$

$$B = (n-k_1) + (k_1-k_2)(1-t_1) + (k_2-k_3)(1-t_2) + \cdots + k_m(1-t_m), \quad (2.39)$$

then $C_{\ell,m}$ can be written as

$$\begin{aligned} C_{\ell,m} &= \int_{-\pi}^{\pi} K(s)\frac{\sec^2(s/2)}{\tan^{m+1}(s/2)}ds \\ &= (-i)^\ell (k_1-k_2)(k_2-k_3)\cdots(k_{\ell-1}-k_\ell)k_\ell B_{\ell,m}, \end{aligned} \quad (2.40)$$

where

$$B_{\ell,m} = \int_{-\pi}^{\pi}\int_0^1 \cdots \int_0^1 \frac{(\exp(-isA)-\exp(-isB))}{s}\frac{s^{\ell+1}\sec^2(s/2)}{\tan^{m+1}(s/2)}dt_1\cdots dt_\ell ds, \quad (2.41)$$

to estimate $B_{\ell,m}$ we separate the computation in two cases, $\ell = m$ and $\ell > m$.

Case $\ell = m$

We estimate $B_{m,m}$ by using that

$$\begin{aligned} B_{m,m} &= 2\int_0^1 \cdots \int_0^1 \int_{-\pi/2}^{\pi/2} \frac{\left(e^{-2iuA}-e^{-2iuB}\right)}{u}\frac{u^{m+1}\sec^2(u)}{\tan(u)^{m+1}}dudt_1\cdots dt_\ell \\ &= 2\int_0^1 \cdots \int_0^1 \int_{-\pi/4}^{\pi/4} \frac{\left(e^{-2iuA}-e^{-2iuB}\right)}{u}\frac{u^{m+1}\sec^2(u)}{\tan^{m+1}(u)}dudt_1\cdots dt_\ell \\ &\quad +2\int_0^1 \cdots \int_0^1 \int_{\pi/4\le|u|\le\pi/2} \frac{\left(e^{-2iuA}-e^{-2iuB}\right)}{u} \\ &\qquad \times\frac{u^{m+1}\sec^2(u)}{\tan^{m+1}(u)}dudt_1\cdots dt_\ell \\ &= B^{(1)}_{m,m} + B^{(2)}_{m,m}. \end{aligned} \quad (2.42)$$

Now for $x \in [-\pi/4, \pi/4]$ we can write $\frac{x^{m+1}\sec^2(x)}{\tan^{m+1}(x)} = 1 + h_m(x)$, and to estimate $h_m(x)$ we use that $\frac{x}{\tan(x)}$ is even and decreasing in $[0, \pi/4]$, so we can bound

$$\left|\left(\frac{x}{\tan(x)}\right)^{m+1} - 1\right| \le \frac{1}{1-(\pi/4)^m}|x|, \quad (2.43)$$

and also $\sec^2(x) = 1 + x^2 \frac{\tan^2(x)}{x^2}$ and $\frac{\tan^2(x)}{x^2} \leq \left(\frac{4}{\pi}\right)^2$ for $x \in [-\pi/4, \pi/4]$, we conclude

$$\begin{aligned} |h_m(x)| &\leq \frac{|x|}{1-(\pi/4)^{m+1}} + (4/\pi)^2 |x|^2 + \frac{(4/\pi)^2}{1-(\pi/4)^{m+1}}|x|^3 \\ &\leq \left(\frac{1}{1-(\pi/4)^{m+1}} + (4/\pi) + \frac{1}{1-(\pi/4)^{m+1}}\right)|x|, \end{aligned} \tag{2.44}$$

using this estimate we get that

$$\begin{aligned} B^{(1)}_{m,m} &= 2\int_0^1 \cdots \int_0^1 \int_{-\pi/4}^{\pi/4} \frac{\left(e^{-2iuA} - e^{-2iuB}\right)}{u} \frac{u^{m+1}\sec^2(u)}{\tan^{m+1}(u)} du\ dt_1 \cdots dt_\ell \\ &= 2\int_0^1 \cdots \int_0^1 \int_{-\pi/4}^{\pi/4} \frac{\left(e^{-2iuA} - e^{-2iuB}\right)}{u} du\ dt_1 \cdots dt_\ell \\ &\quad +2\int_0^1 \cdots \int_0^1 \int_{-\pi/4}^{\pi/4} \frac{\left(e^{-2iuA} - e^{-2iuB}\right)}{u} h_m(u) du\ dt_1 \cdots dt_\ell \\ &= I_1 + I_2, \end{aligned} \tag{2.45}$$

for the first term we use that

$$\begin{aligned} \left| p.v. \int_{-\pi/4}^{\pi/4} \frac{e^{-2iuA}}{u} \right| &= \left| \int_{-\pi/4}^{\pi/4} \frac{-i\sin(2Au)}{u} du \right| \\ &\leq \int_{-\pi/(4A)}^{\pi/(4A)} \frac{\sin(2Au)}{u} du \\ &\leq 2 \end{aligned} \tag{2.46}$$

and therefore $|I_1| \leq 4$. For the second term we use that $|e^{-2iuA} - e^{-2iuB}| \leq 2$, then

$$\begin{aligned} |I_2| &= \left| \int_0^1 \cdots \int_0^1 \int_0^{\pi/4} \frac{\left(e^{-2iuA} - e^{-2iuB}\right)}{u} h_m(u) du dt_1 \cdots dt_\ell \right| \\ &\leq 2\int_0^{\pi/4} \frac{h_m(u)}{u} du \\ &\leq \frac{\pi}{2}\left(\frac{2}{1-(\pi/4)^{m+1}} + (4/\pi)\right). \end{aligned} \tag{2.47}$$

To estimate $B^{(2)}_{m,m}$, we use that $|e^{-2iuA}-e^{-2iuB}|\leq 2$ to get

$$\begin{aligned}
B^{(2)}_{m,m} &= 8\int_{\pi/4}^{\pi/2}\frac{u^m\sec^2(u)}{\tan^{m+1}(u)}du \\
&\leq \left(\frac{\pi}{2}\right)^{m+1}\int_{\pi/4}^{\pi/2}\frac{\sec^2(u)}{\tan^{m+1}(u)}du \\
&\leq \left(\frac{\pi}{2}\right)^{m+1}\int_{\pi/4}^{\pi/2}\frac{\sec^2(u)}{\tan^{m+1}(u)}du \\
&\leq \frac{1}{m}\left(\frac{\pi}{2}\right)^{m+1}\left.\frac{-1}{\tan^m(u)}\right|_{\pi/4}^{\pi/2} \\
&= \frac{1}{m}\left(\frac{\pi}{2}\right)^{m+1},
\end{aligned} \tag{2.48}$$

we conclude that

$$|B_{m,m}|\leq 4+\frac{\pi}{2}\left(\frac{2}{1-(\pi/4)^{m+1}}+(4/\pi)\right)+\frac{1}{m}\left(\frac{\pi}{2}\right)^{m+1}. \tag{2.49}$$

Case $\ell > m$

In the case $\ell > m$ the integral is less singular and therefore the bound $|e^{-2iuA}-e^{-2iuB}|\leq 2$ is enough for the oscillatory terms, then we obtain for $2\leq m<\ell$

$$\begin{aligned}
|B_{\ell,m}| &= 2\left|\int_{-\pi/2}^{\pi/2}\int_0^1\cdots\int_0^1\frac{(\exp(-2iuA)-\exp(-2iuB))}{u}\frac{u^{\ell+1}\sec^2(u)}{\tan(u)^{m+1}}du\right| \\
&= 4\int_{-\pi/2}^{\pi/2}\left|\frac{u^\ell\sec^2(u)}{\tan(u)^{m+1}}\right|du \\
&= 8\left(\int_0^{\pi/4}\frac{u^\ell\sec^2(u)}{\tan^{m+1}(u)}du+\int_{\pi/4}^{\pi/2}\frac{u^\ell\sec^2(u)}{\tan^{m+1}(u)}du\right).
\end{aligned} \tag{2.50}$$

Because of the powers, is easy to see that the integral is indeed finite, so now we proceed to bound it. First because $x/\tan(x)\leq 1$ for $0<x<\pi/2$ we get

$$\begin{aligned}
\int_0^{\pi/4}\frac{u^\ell\sec^2(u)}{\tan^{m+1}(u)}du &\leq \int_0^{\pi/4}u^{\ell-m-1}\sec^2(u)du \\
&\leq \left(\frac{\pi}{4}\right)^{\ell-m-1}\tan(u)|_0^{\pi/4} \\
&= \left(\frac{\pi}{4}\right)^{\ell-m-1}.
\end{aligned} \tag{2.51}$$

the second part can be estimated in the same way as (2.48).Putting this together we obtain

$$
\begin{aligned}
B_{\ell,m} &\leq 8\left(\left(\frac{\pi}{4}\right)^{\ell-m}+\frac{1}{m}\left(\frac{\pi}{2}\right)^{\ell+1}\right) \\
&\leq 8\left(\frac{\pi}{2}\right)^{\ell+1}\left(\frac{1}{2^{\ell+1}}\left(\frac{\pi}{4}\right)^{-m-1}+\frac{1}{m}\right) \\
&\leq \frac{16}{m}\left(\frac{\pi}{2}\right)^{\ell+1},
\end{aligned}
\tag{2.52}
$$

therefore we conclude

$$
\int_{\mathbb{T}} m_s(n-k_1)m_s(k_1-k_2)\cdots m_s(k_{\ell-1}-k_\ell)m_s(k_\ell)\frac{\sec^2(s/2)}{\tan^{m+1}(s/2)}ds
$$

$$
\leq (k_1-k_2)(k_2-k_3)\cdots(k_{m-1}-k_m)k_m B_{\ell,m}, \tag{2.53}
$$

where $B_{\ell,m}$ is given by (2.49) or (2.52). This concludes the proof of Lemma 2.2.5. □

Continuation of proof of Theorem 2.2.1. By applying Lemma 2.2.5 to equation (2.33) we get

$$
\|J_1\|_{\mathcal{F}^{1,1}} \leq \frac{1}{2\pi}\sum_{k\geq 1}\left(\sum_{n\in\mathbb{Z}}|n|(|\cdot||\hat{f}|)*\left(|\cdot||\hat{f}|\right)^{*2k}(n)\right)B_{2k,2k}, \tag{2.54}
$$

using that $|n| \leq |n-k_1|+|k_1-k_2|+\cdots+|k_m|$ we can apply the Hausdorff-Young inequality to obtain

$$
\|J_1\|_{\mathcal{F}^{1,1}} \leq \frac{1}{2\pi}\sum_{k=1}^{\infty}(2k+1)\|f\|_{\mathcal{F}^{2,1}}\|f\|_{\mathcal{F}^{1,1}}^{2k}B_{2k,2k}.. \tag{2.55}
$$

Analogously for J_2 we get

$$
\|J_2\|_{\mathcal{F}^{1,1}} \leq \frac{1}{2\pi}\sum_{k\geq 1}\sum_{\ell\geq 2k+1}|a_\ell^{(2k)}|\left(\sum_{n\in\mathbb{Z}}|n|(|\cdot||\hat{f}|)*\left(|\cdot||\hat{f}|\right)^{*\ell}(n)\right)B_{\ell,2k} \tag{2.56}
$$

and by applying the Hausdorff-Young inequality we get

$$
\|J_2\|_{\mathcal{F}^{1,1}} \leq \frac{1}{2\pi}\sum_{k\geq 1}\sum_{\ell\geq 2k+1}|a_\ell^{(2k)}|(\ell+1)\|f\|_{\mathcal{F}^{2,1}}\|f\|_{\mathcal{F}^{1,1}}^{\ell}C_{\ell,2k}. \tag{2.57}
$$

Finally using the estimates for $B_{\ell,m}$ in Lemma (2.2.5) we conclude for J_1

$$\begin{aligned}\|J_1\|_{\mathcal{F}^{1,1}} &\leq \frac{1}{2\pi}\sum_{k\geq 1}(2k+1)\left(6+\frac{\pi}{1-(\pi/4)^{2k+1}}+\frac{1}{2k}\left(\frac{\pi}{2}\right)^{2k+1}\right)\|f\|_{\mathcal{F}^{2,1}}\|f\|_{\mathcal{F}^{1,1}}^{2k}\\ &= \|f\|_{\mathcal{F}^{2,1}}H_1\left(\|f\|_{\mathcal{F}^{1,1}}\right),\end{aligned} \tag{2.58}$$

where

$$H_1(x)=\frac{1}{2\pi}\sum_{k\geq 1}(2k+1)\left(6+\frac{\pi}{1-(\pi/4)^{2k+1}}+\frac{1}{2k}\left(\frac{\pi}{2}\right)^{2k+1}\right)x^{2k}, \tag{2.59}$$

and for J_2 we get

$$\begin{aligned}\|J_2\|_{\mathcal{F}^{1,1}} &\leq \frac{1}{2\pi}\|f\|_{\mathcal{F}^{2,1}}\sum_{k\geq 1}\sum_{\ell>2k+1}(\ell+1)\left(\frac{4}{\pi}\right)^{\ell}\frac{16}{2k}\left(\frac{\pi}{2}\right)^{\ell+1}\|f\|_{\mathcal{F}^{1,1}}^{\ell}\\ &= \frac{1}{2\pi}\frac{16\pi}{2}\|f\|_{\mathcal{F}^{2,1}}\sum_{k\geq 1}\frac{1}{2k}\sum_{\ell\geq 2k+1}(\ell+1)\left(2\|f\|_{\mathcal{F}^{2,1}}\right)^{\ell}\\ &= 4\|f\|_{\mathcal{F}^{2,1}}\sum_{k\geq 1}\frac{1}{2k}\left((2k+2)\frac{(2\|f\|_{\mathcal{F}^{1,1}})^{2k+1}}{1-2\|f\|_{\mathcal{F}^{1,1}}}+\frac{(2\|f\|_{\mathcal{F}^{1,1}})^{2k+2}}{(1-2\|f\|_{\mathcal{F}^{1,1}})^2}\right)\\ &= \|f\|_{\mathcal{F}^{2,1}}H_2(\|f\|_{\mathcal{F}^{1,1}}),\end{aligned} \tag{2.60}$$

where

$$H_2(x)=4\sum_{k\geq 1}\frac{1}{2k}\left((2k+2)\frac{(2x)^{2k+1}}{1-2x}+\frac{(2x)^{2k+2}}{(1-2x)^2}\right) \tag{2.61}$$

and therefore we obtain estimate (2.9) with $M_1(x)$ given by

$$M_1(x)=H_1\left(x\right)+H_2(x). \tag{2.62}$$

Estimate in the $(2+\delta)$-norm

For the second part of the theorem we need to estimate $\|J_1\|_{\mathcal{F}^{2+\delta,1}}$, $\|J_2\|_{\mathcal{F}^{2+\delta,1}}$ as defined in (2.32). In the case of J_1 the main change is in equation (2.54), because this time we have

$$\|J_1\|_{\mathcal{F}^{2+\delta,1}} \leq \frac{1}{2\pi}\sum_{k\geq 1}\left(\sum_{n\in\mathbb{Z}}|n|^{2+\delta}(|\cdot||\hat{f}|)*\left(|\cdot||\hat{f}|\right)^{*2k}(n)\right)B_{2k,2k}, \tag{2.63}$$

using that

$$|n|^{2+\delta} \leq (m+1)^{1+\delta}\Big(|n-k_1|^{2+\delta}+|k_1-k_2|^{2+\delta} + \cdots + |k_{m-1}-k_m|^{2+\delta}+|k_m|^{2+\delta}\Big), \tag{2.64}$$

we can apply the Haussdorf-Young inequality to obtain

$$\|J_1\|_{\mathcal{F}^{2+\delta,1}} \leq \frac{1}{2\pi}\sum_{k=1}^{\infty}(2k+1)^{2+\delta}\|f\|_2\|f\|_1^{2k}B_{2k,2k} \tag{2.65}$$

Analogously for J_2 we get

$$\|J_2\|_{\mathcal{F}^{2+\delta,1}} \leq \frac{1}{2\pi}\sum_{k\geq 1}\sum_{\ell\geq 2k+1}|a_\ell^{(2k)}|(\ell+1)^{2+\delta}\|f\|_{\mathcal{F}^{2,1}}\|f\|_{\mathcal{F}^{1,1}}^{\ell}B_{\ell,2k}, \tag{2.66}$$

therefore we obtain estimate (2.10) with $M_2(x)$ given by

$$M_2(x) = H_3(x)+H_4(x), \tag{2.67}$$

where

$$H_3(x) = \frac{1}{2\pi}\sum_{k\geq 1}(2k+1)^{2+\delta}\left(6+\frac{\pi}{1-(\pi/4)^{2k+1}}+\frac{1}{2k}\left(\frac{\pi}{2}\right)^{2k+1}\right)x^{2k}, \tag{2.68}$$

and

$$H_4(x) = 4\sum_{k\geq 1}\frac{1}{2k}\sum_{\ell\geq 2k+1}(\ell+1)^{2+\delta}(2x)^{\ell} \tag{2.69}$$

This concludes the proof of Theorem 2.2.1. □

Chapter 3

Global existence for the 2D periodic Muskat in H^2 for initial data with small slope

Abstract:

We consider the periodic 2D Muskat equation for the interface between two media of different densities, with velocity given by the Darcy's law. In this section we study the global existence for H^2 initial data with small slope. We extend some of the results know for the non periodic case to the periodic case by following the strategy in [11]. The main contribution are new estimates for the second derivative and pointwise lower bounds for nonlocal operators by using the compactness of the domain.

3.1 Introduction

3.1.1 Description of the model

The Muskat problem in 2D describe the evolution of an interface between two immiscible fluids of different constant densities in a porous media, one in top of the other. In the case that we are studying it is also assumed that the fluids are infinitely deep, which means that the effects of the boundary where the fluids are contained are neglected. When the interface can be described as a graph, the equation for the interface $f(x,t)$, $x \in \mathbb{R}$, $t \in (0,T)$ can be written as (see Section 1.3.3)

$$\begin{cases} \partial_t f(x) & = & \dfrac{\rho_2-\rho_1}{2\pi} p.v. \displaystyle\int_{\mathbb{R}} \frac{\partial_x \delta_\alpha f(x) \alpha}{\alpha^2+(f(x,t)-f(x-\alpha,t))^2} d\alpha \\ f(x,0) & = & f_0(x), x \in \mathbb{R}, \end{cases} \tag{3.1}$$

where $\rho_1 > 0$ is the density of the top fluid and $\rho_2 > 0$ the density of the bottom fluid. In this configuration a necessary condition for stability is the Rayleigh-Taylor condition, which in our case says that the heavier fluid must be at the bottom, i.e. $\rho_2 > \rho_1$ [5]. If we look for periodic solutions of the problem, say $f(x,t) = f(x+2\pi,t)$, the equation may be rewritten as

$$\begin{cases} f_t(x) & = & \dfrac{\rho_2-\rho_1}{4\pi} p.v. \displaystyle\int_{\mathbb{T}} \frac{\partial_x \delta_s f(x) \tan(s/2)}{\tan^2(s/2)+\tanh^2\left(\delta_s f(x)/2\right)} ds \\ & & -\dfrac{\rho_2-\rho_1}{4\pi} p.v. \displaystyle\int_{\mathbb{T}} \partial_x \delta_s f(x) \frac{\tan(s/2)\tanh^2\left(\delta_s f(x)/2\right)}{\tan(s/2)^2+\tanh^2\left(\delta_s f(x)/2\right)} ds, \\ f(x,0) & = & f_0(x), \quad x \in \mathbb{T} = \mathbb{R}/\{2\pi\mathbb{Z}\}. \end{cases} \tag{3.2}$$

Because for $\rho_1 > \rho_2$ the problem is known to be ill posed in H^s for $s > 0$ [16], [20], in this chapter we only deal with the case in which $\rho_2 > \rho_1$, therefore after a time reparameterization, we can assume that $\frac{\rho_2-\rho_1}{2} = 1$.

3.1.2 Main results

The strategy used in this work is based in [11] where the global existence for $H^2(\mathbb{R})$ initial data with small slope is studied. The key difference with that work is that in this case we do not have decay at infinity and therefore estimates have to be adapter to use compactness instead. Our first result give short time existence for $H^2(\mathbb{T})$ initial data.

Theorem 3.1.1 (Local existence in H^2)**.** *Let $f_0 \in W^{2,2}(\mathbb{T})$ with $\int_{\mathbb{T}} f_0 = 0$. Then there exists $T = T(\|f_0\|_{W^{2,2}(\mathbb{T})}) > 0$ such that the problem (3.2) with datum $f(x,0) = f_0$ has a unique solution*

$$f \in L^\infty([0,T), W^{2,2}(\mathbb{T})) \cap C([0,T); L^2(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})). \tag{3.3}$$

The next result give us more information about the shape of the interface by showing that if the slope is small enough initially then it satisfies a maximum principle.

Lemma 3.1.2 (Maximum principle for the slope)**.** *Let $f \in L^\infty((0,T); H^s(\mathbb{T}))$, $s \geq 2$ be a solution of (3.2) with initial data $f_0 \in H^s(\mathbb{T})$ such that $\|f_0'\|_{L^\infty} \leq \frac{2}{\sqrt{5}}$, then for $t \in (0,T)$*

$$\|f'(t)\|_{L^\infty} \leq \|f_0'\|_{L^\infty}. \tag{3.4}$$

The next Theorem is the main result in this chapter and give us the global existence for $H^2(\mathbb{T})$ initial data with small slope.

Theorem 3.1.3 (Global existence for data with small slope). *Consider the problem* (3.2) *with initial data* $f_0 \in H^2(\mathbb{T})$ *satisfying* $\int_{\mathbb{T}} f_0 = 0$ *and*

$$\|f_0'\|_{L^\infty} < k_0, \tag{3.5}$$

for a small constant $k_0 > 1$. *Then the local in time solution of* (3.2) *given by Theorem 3.1.1 is in fact global, and* $f''(t)$ *satisfy*

$$\|f''(t)\|_{L^2} \leq \max\{\|f_0''\|_{L^2}, (2\pi)^{1/3}\}, \tag{3.6}$$

for all $t > 0$.

The proof of the global existence uses energy method, for this purpose we consider the energy

$$E(t) = 1 + \|f'\|_{L^\infty} + \|f''\|_{L^2}, \tag{3.7}$$

then we study the evolution of this quantity by studying the evolution of the equation of the second derivative of the equation to obtain that if the slope stay small, then the energy cannot blow up.

Theorem 3.1.4 (Uniqueness of $C^1(\mathbb{T})$ solutions). *Let* $f_1, f_2 \in C^0([0,T], C^1(\mathbb{T}))$ *two solutions of* (3.2) *that are Lipschitz continuous in time with the same initial data, then the* $f_1 = f_2$ *for all* $t \in [0,T]$.

If additionally we assume that $f_1, f_2 \in C^0([0,T], H^2(\mathbb{T}))$ *and there exists* $B, M > 0$ *such that* $\sup_{t\in[0,T]} \|f_i'\|_{L^\infty} \leq B$ *for* $i = 1, 2$, $\sup_{t\in[0,T]} \|f_i''\|_{L^2} \leq M$ *then*

$$\sup_{t\in[0,T]} \|f_1(t) - f_2(t)\|_{L^\infty} \leq \|f_1(0) - f_2(0)\|_{L^\infty} \exp(T\ C(B, M)) \tag{3.8}$$

for some constant $C(B, M) > 0$.

Remark 3.1.5. By comparing the result of Theorem 3.1.3 with [11], we notice that by Sobolev embedding we know that $\int_{\mathbb{T}} f_0 = 0$ and $f_0 \in H^2(\mathbb{T})$ imply that f_0 has finite energy and finite slope, that is $f_0 \in L^2(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$. Also we note that the condition $\int_{\mathbb{T}} f(t) = 0$ is preserved in time, to see this it is enough to write the equation as

$$\partial_t f = \frac{1}{2\pi} \partial_x p.v. \int_{\mathbb{T}} \arctan\left(\frac{\tanh\left(\frac{f(x)-f(s)}{2}\right)}{\tan\left(\frac{x-s}{2}\right)} \right) ds, \tag{3.9}$$

and conclude by integrating. Also because the equation is invariant when adding constants to f, we are not losing generality when assuming that $\int f_0 = 0$ and therefore the result is a direct extension of the global existence result in [11] for the case of the real line.

Remark 3.1.6. The result obtained in Theorem 3.1.3 can be also be compared with the global existence result in [6] for small initial data in H^2. By Sobolev embedding, small H^2 norm imply small C^1 norm and because $\mathbb{T}$ is compact, it also imply that it has small $W^{1,\infty}$ norm, and consequently under a small H^2 initial data condition we can still apply Theorem 3.1.3. To see that this result is strictly more general we will construct a function that has small slope but has large, but finite, H^2 norm.

Consider $g \in C^2(\mathbb{T})$ defined by

$$g(x) = \varepsilon \left(\frac{\tan^2(x/2)}{1+\tan^2(x/2)} \right)^a \sin \left(\frac{1}{(\tan^2(x/2))^b} \right). \tag{3.10}$$

Note that for $a > 0$, $g(x)$ is bounded. Its the first derivative satisfy

$$g'(x) \sim x^{2a-2b-1} \text{ at } 0 \text{ and } g'(x) \sim (x-\pi)^{2b-1} \text{ at } \pi$$

and those are the only point in which we may have singularities, we get that $g'(x)$ is bounded if $2b-1 \geq 0$, and $2a-2b-1 \geq 0$. And so we want $b \geq 1/2$ and $a-b \geq 1/2$. For the second derivative we have want it to be unbounded, integrable, but with large norm. For this we use that

$$g''(x) \sim x^{2(a-2b-1)} \text{ at } 0 \text{ and } g''(x) \sim (x-\pi)^{2b-2} \text{ at } \pi$$

And so for $2b - 2 \geq 0$ it is bounded at π. Also, at 0 because we want it to be unbounded but p-integrable. we want that

$$-1 < 2p(a - 2b - 1) < 0$$

$$1 - \frac{1}{2p} < a - 2b < 1$$

Now if we choose $a = 3$, $b = 1 + \frac{1}{4p}(1 - \frac{1}{k})$ we have that the $g(x)$ and $g'(x)$ we get uniform bounds in k, and by choosing ε small enough we can get an arbitrarily small $W^{1,\infty}$ norm but as $k \to \infty$ we have that $\|g''\|_{L^p} \to \infty$, and so by choosing k we can get an arbitrarily large $W^{2,p}$ norm, and so taking $p = 2$, we get the example.

Remark 3.1.7. As a subproduct of our estimates, using the equation for the second derivative and the estimates in Lemma 3.2.7 and Lemma 3.2.6, It is possible to obtain the following result by following the same proof as in [11] Section 5.

Lemma 3.1.8 (Blow-up criteria for the curvature)**.** *Let* $f \in H^k(\mathbb{T})$ *for* $k \geq 3$ *be a solution of* (3.2) *such that* f' *is bounded in* $[0,T]$*, i.e.*

$$\sup_{t\in[0,T]} \|f'(t)\|_{L^\infty} \leq B < \infty \tag{3.11}$$

Assume that f' *is uniformly continuous in* $\mathbb{T}\times[0,T]$*, that is, there exists a function* $\rho : [0,\infty) \to [0,\infty)$*, that is non-decreasing, bounded, with* $\rho(0) = 0$ *such that* f' *obeys the modulus of continuity* ρ*, i.e. that*

$$|\delta_s f'(x,t)| \leq \rho(|s|), \tag{3.12}$$

for any $x \in \mathbb{T}$*,* $s \in \mathbb{R}$ *and* $t \in [0,T]$*. Then*

$$\sup_{t\in[0,T]} \|f''(t)\|_{L^\infty} < C(\|f_0''\|_{L^\infty}, B, \rho). \tag{3.13}$$

3.1.3 Outline of the work

In Section 3.2 we derive equations for the first and second derivatives and prove some estimates of some of the terms that appear in the equations.

In Section 3.3 we prove Theorem 3.1.1 by using the equations for the first and second derivative to get an estimate for the evolution of $E(t) = 1 + \|f'(t)\|^2_{L^\infty} + \|f''(t)\|^2_{L^2}$ of the form $\frac{d}{dt}E(t) \leq p(E(t))$ for a polynomial $p(x)$ which implies that the $E(t)$ is finite for short time and then we conclude by a standard approximation procedure.

In Section 3.4 we prove a maximum principle for the first derivative given by Lemma 3.1.2 by using the structure of the equation to conclude that is $\|f_0'\|_{L^\infty}$ is small, then it is small for all times.

In Section 3.5 we prove Theorem 3.1.3. For that we use that if $\|f_0'\|_{L^\infty}$ is small, then by a maximum principle it is small for all times. Then we study the evolution of L^2 norm the second derivative and we conclude that for large values, it must decay and so we conclude that it must be bounded for all times. Finally by the local existence criteria we get that the solution must be global.

In Section 3.6 we prove Theorem 3.1.4 by studying the evolution of the L^∞ when we assume uniform continuity.

3.2 Preliminaries

Consider the equivalent formulation of the Muskat problem given by (1.49)

$$f_t + v\partial_x f + \frac{1}{2\pi} p.v. \int_{\mathbb{T}} \frac{\tanh(\delta_s f/2)\sec^2(s/2)}{\tan^2(s/2) + \tanh^2(\delta_s f/2)} ds = 0, \tag{3.14}$$

where $\delta_s f(x) = f(x) - f(x-s)$, $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ and

$$v = -\frac{1}{2\pi} p.v. \int_{\mathbb{T}} \frac{\tan(s/2)\mathrm{sech}^2(\delta_s f/2)}{\tan^2(s/2) + \tanh^2(\delta_s f/2)} ds. \tag{3.15}$$

From formulation (3.14), now we can derive equations for the first of the second derivatives and use those to obtain a priori estimates for the solutions.

3.2.1 Equation for the first derivative

Taking derivative in x to equation (3.14) we get

$$f_t' + v\partial_x f' + \frac{1}{4\pi} p.v. \int \frac{\mathrm{sech}^2(\delta_s f/2)\delta_s f'}{\tan^2(s/2) + \tanh^2(\delta_s f/2)} \sec^2(s/2) ds = RHS$$

$$
\begin{aligned}
RHS \;=\;& \frac{1}{2\pi} f'(x)\ p.v.\int \frac{-\tanh(\delta_s f/2)\tan(s/2)}{\left(\tan^2(s/2)+\tanh^2(\delta_s f/2)\right)^2}\operatorname{sech}^2(\delta_s f/2) \\
&\qquad \times \left(1-\tanh^2(\delta_s f/2)\right)\delta_s f' ds \\
&+\frac{1}{2\pi} f'(x)\ p.v.\int \frac{-\tanh(\delta_s f/2)\tan(s/2)}{\tan^2(s/2)+\tanh^2(\delta_s f/2)}\operatorname{sech}^2(\delta_s f/2)\delta_s f' ds \\
&-\frac{1}{2\pi} p.v.\int \frac{-\tanh^2(\delta_s f/2)}{\left(\tan^2(s/2)+\tanh^2(\delta_s f/2)\right)^2}\operatorname{sech}^2(\delta_s f/2)\sec^2(s/2)\delta f' ds \\
=\;& \frac{1}{2\pi} p.v.\int \frac{\operatorname{sech}^2(\delta_s f/2)\tanh(\delta_s f/2)\delta_s f'}{\left(\tan^2(s/2)+\tanh^2(\delta_s f/2)\right)^2}\left[\tanh(\delta_s f/2)(1+\tan^2(s/2))\right. \\
&\qquad -\tan(s/2)(1-\tanh^2(\delta_s f/2))f'(x) \\
&\qquad \left.-\tan(s/2)(\tan^2(s/2)+\tanh^2(\delta_s f/2))f'(x)\right] ds \\
=\;& \frac{1}{2\pi} p.v.\int \frac{\operatorname{sech}^2(\delta_s f/2)\tanh(\delta_s f/2)\delta_s f'}{\left(\tan^2(s/2)+\tanh^2(\delta_s f/2)\right)^2}\left[\tanh(\delta_s f/2)-\tan(s/2)f'(x)\right. \\
&\qquad \left.+\tanh(\delta_s f/2)\tan^2(s/2)-\tan^3(s/2)f'(x)\right] ds \\
=\;& \frac{1}{2\pi} p.v.\int \frac{\operatorname{sech}^2(\delta_s f/2)\tanh(\delta_s f/2)\delta_s f'}{\left(\tan^2(s/2)+\tanh^2(\delta_s f/2)\right)^2}\left[\tanh(\delta_s f/2)-\tan(s/2)f'(x)\right. \\
&\qquad \left.+\tan^2(s/2)\left[\tanh(\delta_s f/2)-\tan(s/2)f'(x)\right]\right] ds \\
=\;& \frac{1}{2\pi} p.v.\int \frac{\operatorname{sech}^2(\delta_s f/2)\tanh(\delta_s f/2)\sec^2(s/2)\delta_s f'}{\left(\tan^2(s/2)+\tanh^2(\delta_s f/2)\right)^2} \\
&\qquad \times\left[\tanh(\delta_s f/2)-\tan(s/2)f'(x)\right] ds.
\end{aligned}
$$

Multiplying the equation by $f'(x,t)$ we can write

$$
(\partial_t + v\partial_x + \tilde{\mathcal{L}}_f)|f'|^2 + \tilde{D}_f[f'] = T_0, \tag{3.16}
$$

where

$$
\begin{aligned}
T_0 = 2f'(x,t)\frac{1}{2\pi}\ p.v.\int_{\mathbb{T}} & \frac{\operatorname{sech}^2(\delta_s f/2)\tanh(\delta_s f/2)\sec^2(s/2)\delta_s f'}{\left(\tan^2(s/2)+\tanh^2(\delta_s f/2)\right)^2} \\
& \times\left[\tanh(\delta_s f/2)-\tan(s/2)f'(x)\right] ds,
\end{aligned} \tag{3.17}
$$

and

$$\begin{aligned}\tilde{\mathcal{L}}_f g &= \frac{1}{4\pi}\int \frac{\text{sech}^2(\delta_s f/2)\delta_s g}{\tan^2(s/2)+\tanh^2(\delta_s f/2)}\sec^2(s/2)ds,\\ \tilde{D}_f[g] &= \frac{1}{4\pi}\int \frac{\text{sech}^2(\delta_s f/2)(\delta_s g)^2}{\tan^2(s/2)+\tanh^2(\delta_s f/2)}\sec^2(s/2)ds.\end{aligned} \tag{3.18}$$

3.2.2 Equation for the second derivative

Now for the second derivative we obtain

$$\begin{aligned} f_t'' + v\partial_x f'' + \frac{1}{4\pi}\int \frac{\sec^2(s/2)\delta f''}{\tan^2(s/2)+\tanh^2(\delta_s f/2)} &= RHS_2 \\ &= T_1+T_2+T_3+T_4+T_5+T_6+T_7, \end{aligned} \tag{3.19}$$

where

$$T_1 = p.v.\frac{1}{2\pi}\int \frac{\text{sech}^2(\delta_s f/2)\tanh^2(\delta_s f/2)\sec^2(s/2)\,(\delta_s f')^2}{\left(\tan^2(s/2)+\tanh^2(\delta_s f/2)\right)^2} \times [\tanh(\delta_s f/2) - \tan(s/2)\partial f]\,ds$$

$$T_2 = \frac{1}{2}p.v.\frac{1}{2\pi}\int \frac{\text{sech}^4(\delta_s f/2)\sec^2(s/2)\,(\delta_s f')^2}{\left(\tan^2(s/2)+\tanh^2(\delta_s f/2)\right)^2}[\tanh(\delta_s f/2) - \tan(s/2)\partial f]\,ds$$

$$T_3 = p.v.\frac{1}{2\pi}\int \frac{\text{sech}^2(\delta_s f/2)\tanh(\delta_s f/2)\sec^2(s/2)\delta_s f''}{\left(\tan^2(s/2)+\tanh^2(\delta_s f/2)\right)^2} \times [\tanh(\delta_s f/2) - \tan(s/2)\partial f]\,ds$$

$$T_4 = -2p.v.\frac{1}{2\pi}\int \frac{\text{sech}^4(\delta_s f/2)\tanh^2(\delta_s f/2)\sec^2(s/2)\,(\delta_s f')^2}{\left(\tan^2(s/2)+\tanh^2(\delta_s f/2)\right)^3} \times [\tanh(\delta_s f/2) - \tan(s/2)\partial f]\,ds$$

$$T_5 = p.v.\frac{1}{2\pi}\int \frac{\text{sech}^2(\delta_s f/2)\tanh(\delta_s f/2)\sec^2(s/2)\,(\delta_s f')}{\left(\tan^2(s/2)+\tanh^2(\delta_s f/2)\right)^2} \times \left[\text{sech}^2(\delta_s f/2)\frac{\delta_s f'}{2} - \tan(s/2)\partial f'\right] ds$$

$$
\begin{aligned}
T_6 &= p.v.\frac{1}{2\pi}\int \frac{\text{sech}^2(\delta_s f/2)\text{tanh}(\delta_s f/2)\delta_s f'}{\left(\tan^2(s/2)+\tanh^2(\delta_s f/2)\right)^2} \\
&\qquad \times \Big[-f''(x)\tan(s/2)(1-\tanh^2(\delta_s f/2)) \\
&\qquad -f''(x)\tan(s/2)(\tan^2(s/2)+\tanh^2(\delta_s f/2)) \\
&\qquad -\frac{1}{2}(\delta_s f')(1+\tan^2(s/2))(\tan^2(s/2)+\tanh^2(\delta_s f/2)) \\
&\qquad +\frac{1}{2}(\delta_s f')\,\text{sech}^2(\delta_s f/2)(1+\tan^2(s/2))\Big]\,ds \\
&= -p.v.\frac{1}{2\pi}\int \frac{\text{sech}^2(\delta_s f/2)\text{tanh}(\delta_s f/2)\delta_s f'}{\left(\tan^2(s/2)+\tanh^2(\delta_s f/2)\right)^2}(1+\tan^2(s/2)) \\
&\quad \times \left[f''(x)\tan(s/2)+(\delta_s f')\tanh^2(\delta_s f/2)-\frac{1}{2}(\delta_s f')(1-\tan^2(s/2))\right]ds \\
&= -p.v.\frac{1}{2\pi}\int \frac{\text{sech}^2(\delta_s f/2)\text{tanh}(\delta_s f/2)\sec^2(s/2)\delta_s f'}{\left(\tan^2(s/2)+\tanh^2(\delta_s f/2)\right)^2}\Big[f''(x)\tan(s/2) \\
&\quad +(\delta_s f')\tanh^2(\delta_s f/2)-\frac{1}{2}(\delta_s f')(1-\tan^2(s/2))\Big]\,ds, \\
T_7 &= \frac{1}{4\pi}\int \frac{\tanh^2(\delta_s f/2)\sec^2(s/2)\delta f''}{\tan^2(s/2)+\tanh^2(\delta_s f/2)}ds. \qquad (3.20)
\end{aligned}
$$

Multiplying (3.19) by $f''(x,t)$ we get

$$
(\partial_t + v\partial_x + \mathcal{L}_f)\,|f''(x,t)|^2 + D_f[f''] = 2f''(x)(T_1+T_2+T_3+T_4+T_5+T_6+T_7), \quad (3.21)
$$

where

$$
\begin{aligned}
\mathcal{L}_f[g](x) &= \frac{1}{2}p.v.\frac{1}{2\pi}\int \frac{\sec^2(s/2)\delta_s g}{\tan^2(s/2)+\tanh^2(\delta_s f/2)}ds, \\
D_f[g](x) &= \frac{1}{2}p.v.\frac{1}{2\pi}\int \frac{\sec^2(s/2)\,(\delta_s g)^2}{\tan^2(s/2)+\tanh^2(\delta_s f/2)}ds.
\end{aligned} \qquad (3.22)
$$

In particular when f is constant we get

$$
\begin{aligned}
\mathcal{L}_c[g](x) &= \frac{1}{2}p.v.\frac{1}{2\pi}\int \frac{\sec^2(s/2)\delta_s g}{\tan^2(s/2)}ds, \\
D[g] := D_c[g](x) &= \frac{1}{2}p.v.\frac{1}{2\pi}\int \frac{\sec^2(s/2)\,(\delta_s g)^2}{\tan^2(s/2)}ds.
\end{aligned} \qquad (3.23)
$$

Alternatively we can write the equation with the transport term in divergence form as

$$
\begin{aligned}
(\partial_t + \mathcal{L}_f)\,|f''(x,t)|^2 + \partial_x(v|f''|^2) + D_f[f''] \\
= 2f''(x)(T_1+T_2+T_3+T_4+T_5+T_6+T_7) + |f''|^2 T_8, \quad (3.24)
\end{aligned}
$$

where

$$
T_8 = \partial_x v = \frac{1}{2\pi}\int_{\mathbb{T}} \operatorname{sech}^2(\delta_s f/2)\frac{\frac{\tanh(\delta_s f/2)}{\tan(s/2)}}{\left(1+\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)}\right)^2}(\delta_s f')\frac{\sec^2(s/2)}{\tan^2(s/2)}ds \qquad (3.25)
$$

3.2.3 Estimates for Approximate Derivatives

In this subsection we obtain explicit estimates for the error of approximating a difference by a derivative like the ones that appear in the right hand side of equation (3.21). With this in mind we consider the following

$$
\begin{aligned}
R_1[f''](x,s) &:= \delta_s f'(x) - s f''(x), \\
R_1[f'](x,s) &:= \tanh(\delta_s f/2) - \tan(s/2) f'(x),
\end{aligned} \qquad (3.26)
$$

from the Taylor expansion we expect this quantities to be small, but for our estimates we want to give more precise control on how big they are depending on s. then we have the following estimate:

Lemma 3.2.1 (First order estimate)**.** *Let $f \in W^{1,\infty}(\mathbb{T}) \cap W^{2,p}(\mathbb{T})$ a Lipschitz continuous function with Lipschitz constant B and $p > 1$. Let $x \in \mathbb{T}$, $s \in (-\pi, \pi)$, then*

(a) $|R_1[f'']| \le \frac{1}{2\sqrt{3}}\left(D[f'']\right)^{1/2}|s|^{3/2}$,

(b) $|R_1[f']| \leq 2B|\tan(s/2)|$,

(c) $|R_1[f']| \leq C(1+B)\|f''\|_{L^p(\mathbb{T})}|s|^{(p-1)/p}|\tan(s/2)|(1+|\tan(s/2)|)$, $p>1$.

Next we want to take a look to higher order approximation of derivatives, and in this case because we have more terms we expect to get higher powers of s that correspond with the better approximations. For this pupose we consider the following second order approximation of the derivative

$$\begin{aligned} \tilde{R}_2[f''](x) &:= \tanh\left(\frac{\delta_s f}{2}\right) - s/2 f'(x) + h_1(s) f''(x), \\ R_2[f''](x) &:= \tanh\left(\frac{\delta_s f}{2}\right) - \tan(s/2) f'(x) + h(s) f''(x), \end{aligned} \tag{3.27}$$

where $h(s) = h_1(s) - h_2(s) + h_3(s)$,

$$\begin{aligned} h_1(s) &= \frac{1}{2}\int_0^s \int_0^z \operatorname{sech}^2(\delta_w f/2) dw dz, \\ h_2(s) &= \frac{1}{2}\int_0^s \int_0^z \operatorname{sech}^2(\delta_w f/2)\sec^2(s/2) dw dz, \\ h_3(s) &= \frac{1}{2}\int_0^s \sec^2(z/2) \int_0^z \operatorname{sech}^2(\delta_w f/2) dw dz. \end{aligned} \tag{3.28}$$

Using that $\operatorname{sech}(x) \leq 1$ it is easy to see that $h_1(s)$ and $h(s)$ satisfy

$$|h_1(s)| \leq \frac{s^2}{4}, \qquad |h(s)| \leq 3s\tan(s/2). \tag{3.29}$$

Then we have the following estimate:

Lemma 3.2.2 (Second order estimate)**.** *Let $f \in W^{1,\infty}(\mathbb{T}) \cap W^{2,p}(\mathbb{T})$ a Lipschitz continuous function with Lipschitz constant B and $p>1$. Let $x \in \mathbb{T}$, $s \in (-\pi,\pi)$, then*

(a) $|\tilde{R}_2[f'']| \leq C(1+B^2)s^{5/2}\left((D[f''])^{1/2} + |f''(x)|\right)$ *and* $|h_1(s)| \leq \frac{1}{4}s^2$,

(b) $|R_2[f'']| \leq C(1+B^2)s\tan^{3/2}(s/2)\left((D[f''])^{1/2} + |f''(x)|\right)$ *and*

$|h(s)| \leq 3s\tan(s/2)$.

Now we finally proceed to prove the estimates.

Proof of Lemma 3.2.1. The main idea of the proof is to find a integral formula for the difference that allow us to compare it with the quantities we are interested in.

Part (a):

$$\begin{aligned}
R_1[f''] &= \delta_s f'(x) - sf''(x) = \int_0^s (f''(x-z) - f''(x))\,dz \\
|R_1[f'']| &\leq \int_0^s \frac{|\delta_s f''||\sec(z/2)|}{|\tan(z/2)|}\frac{|\tan(z/2)|}{|\sec(z/2)|}dz \\
&\leq \left(\int_0^s \frac{(\delta_z f'')^2}{\tan^2(z/2)}\sec^2(z/2)dz\right)^{1/2}\left(\int_0^s \frac{\tan^2(z/2)}{\sec^2(z/2)}dz\right)^{1/2} \\
&\leq \left(\int_0^s \frac{(\delta_z f'')^2}{\tan^2(z/2)}\sec^2(z/2)dz\right)^{1/2}\left(\int_0^s \frac{z^2}{4}dz\right)^{1/2} \\
&\leq \frac{\pi}{\sqrt{3}}\left(D[f'']\right)^{1/2}s^{3/2},
\end{aligned}$$

here used that $\frac{\tan^2(z/2)}{\sec^2(z/2)} \leq \frac{z^2}{4}$.

Part (b):

$$\begin{aligned}
|R_1[f']| &= |\tanh(\delta_s f/2) - \tan(s/2)f'(x)| \\
&\leq |\tanh(\delta_s f/2)| + |\tan(s/2)||f'(x)| \\
&\leq |\delta_s f(x)/2| + |\tan(s/2)|B \qquad (3.30) \\
&\leq B|s| + |\tan(s/2)|B \\
&\leq B|\tan(s/2)|.
\end{aligned}$$

Part (c):

We can write

$$
\begin{aligned}
R_1[f'] &= \tanh(\delta_s f/2) - \tan(s/2) f'(x) \\
&= \frac{1}{2}\int_0^s \operatorname{sech}^2(\delta_z f/2) f'(x-z)dz - \frac{1}{2}\int_0^s f'(x)dz \\
&\quad + \frac{1}{2}\int_0^s f'(x)dz - \frac{1}{2}\int_0^s \operatorname{sech}^2(\delta_z f/2)\sec^2(z/2) f'(x-z)dz \\
&\quad + \frac{1}{2}\int_0^s \operatorname{sech}^2(\delta_z f/2)\sec^2(z/2) f'(x-z) - \frac{1}{2}\int_0^s \sec^2(z/2) f'(x)dz.
\end{aligned}
$$

Now the key observation is that we can group the integrals in pairs by noting that they can be seen as the integral of the same function up to a translation, for the first one the function is $g_1(z) = \operatorname{sech}^2(\delta_z f/2) f'(x-z)$, for the second one, $g_2(z) = \operatorname{sech}^2(\delta_z f/2)\sec^2(z/2) f'(x-z)$, and the last one we factor the term $\sec^2(z/2)$ and we look at the difference between two points of the function $g_3(z) = \operatorname{sech}^2(\delta_z f/2) f'(x-z)$, then we get

$$
\begin{aligned}
R_1[f'] &= -\frac{1}{2}\int_0^s \int_0^z \operatorname{sech}^2(\delta_w f/2)\tanh(\delta_w f/2)|f'(x-w)|^2 dw dz \\
&\quad -\frac{1}{2}\int_0^s \int_0^z \operatorname{sech}^2(\delta_w f/2) f''(x-w) dw dz \\
&\quad +\frac{1}{2}\int_0^s \int_0^z \operatorname{sech}^2(\delta_w f/2)\tanh(\delta_w f/2)\sec^2(w/2)|f'(x-w)|^2 dw dz \\
&\quad -\frac{1}{2}\int_0^s \int_0^z \operatorname{sech}^2(\delta_w f/2)\sec^2(w/2)\tan(w/2) f'(x-w) dw dz \\
&\quad +\frac{1}{2}\int_0^s \int_0^z \operatorname{sech}^2(\delta_w f/2)\sec^2(w/2) f''(x-w) dw dz \\
&\quad -\frac{1}{2}\int_0^s \sec^2(z/2)\int_0^z \operatorname{sech}^2(\delta_w f/2)\tanh(\delta_w f/2)|f'(x-w)|^2 dw dz \\
&\quad -\frac{1}{2}\int_0^s \sec^2(z/2)\int_0^z \operatorname{sech}^2(\delta_w f/2) f''(x-w) dw dz.
\end{aligned}
$$

Now because $\mathbb{T}$ is compact, there exists $a \in \mathbb{T}$ s.t. $f'(a) = 0$ and therefore we can write

$$\begin{aligned}|f'(x-z)| &= |f'(x-z)-f'(a)| \\ &= \left|\int_a^{x-z}(-f''(x-w))dw\right| \\ &\le \left(\int_a^{x-z}1dw\right)^{\frac{p-1}{p}}\|f''\|_{L^p} \\ &\le \pi^{\frac{p-1}{p}}\|f''\|_{L^p}.\end{aligned}$$

Finally by using that $|\text{sech}(x)| \le 1$, $|\tanh(x)| \le 1$, the previous estimate, and integrating we get

$$|R_1[f']| \le C(1+B)\|f''\|_{L^p}s^{\frac{p-1}{p}}\tan(s/2)(1+\tan(s/2)). \tag{3.31}$$

□

Proof of Lemma 3.2.2.

$$\begin{aligned}R_2[f''] &= \tanh\left(\frac{\delta_s f}{2}\right) - \tan(s/2)f'(x) + h(s)f''(x) \\ &= \frac{1}{2}\int_0^s \text{sech}^2\left(\frac{\delta_z f}{2}\right)f'(x-z) - \frac{1}{2}\int_0^s \sec^2(z/2)dzf'(x) + h(s)f''(x) \\ &= A_1 + A_2 + A_3,\end{aligned} \tag{3.32}$$

where

$$\begin{aligned}A_1 &= \frac{1}{2}\int_0^s \text{sech}^2\left(\frac{\delta_z f}{2}\right)f'(x-z) - \frac{1}{2}\int_0^s dzf'(x) + h_1(s)f''(x) \\ A_2 &= +\frac{1}{2}\int_0^s dzf'(x) - \frac{1}{2}\int_0^s \sec^2(z/2)\text{sech}^2\left(\frac{\delta_z f}{2}\right)f'(x-z)dz - h_2(s)f''(x) \\ A_3 &= +\frac{1}{2}\int_0^s \sec^2(z/2)\text{sech}^2\left(\frac{\delta_z f}{2}\right)f'(x-z)dz \\ &\quad -\frac{1}{2}\int_0^s \sec^2(z/2)dzf'(x) + h_3(s)f''(x)\end{aligned} \tag{3.33}$$

Notice that $A_1 = \tilde{R}_2[f'']$ so the estimate for A_1 also proves part a) of the lemma.

For the first term we have:

$$
\begin{aligned}
A_1 &= \frac{1}{2}\int_0^s \left(\operatorname{sech}^2\left(\frac{\delta_s f}{2}\right)\partial_x f(x-z) - \partial_x f(x)\right) dz + h_1(s)f''(x) \\
&= \frac{-1}{2}\int_0^s\int_0^z \operatorname{sech}^2\left(\frac{\delta_w f}{2}\right)\tanh\left(\frac{\delta_w}{2}\right)|\partial_x f(x-w)|^2 dwdz \\
&\quad -\frac{1}{2}\int_0^s\int_0^z \operatorname{sech}^2\left(\frac{\delta_w f}{2}\right)\partial_x^2 f(x-w)dwdz + h_1(s)f''(x) \\
&= -\frac{1}{2}\int_0^s\int_0^z \operatorname{sech}^2\left(\frac{\delta_w f}{2}\right)\tanh\left(\frac{\delta_w f}{2}\right)|\partial_x f(x-w)|^2 dwdz \\
&\quad -\frac{1}{2}\int_0^s\int_0^z \operatorname{sech}^2\left(\frac{\delta_w f}{2}\right)\left(f''(x-w) - f''(x)\right) dwdz \\
&= I_1 + I_2.
\end{aligned} \tag{3.34}
$$

To estimate I_1 we use the following Lemma:

Lemma 3.2.3. *Let f as before and $z \in [0,\pi)$, then*

$$
\int_0^z |f'(x-w)|^2 h(w)dw \leq 2\pi\int_0^z h(w)dwD[f''] + 2\pi^2\int_0^z h(w)dw|f''(x)|^2, \tag{3.35}
$$

and

$$
\begin{aligned}
&\left(\int_0^z |f'(x-w)|^2 h(w)dw\right)^{1/2} \\
&\qquad \leq \left(\int_0^z h(w)dw\right)^{1/2}\left(\sqrt{2\pi}(D[f''])^{1/2} + \sqrt{2}\pi|f''(x)|\right).
\end{aligned} \tag{3.36}
$$

Proof of Lemma 3.2.3. Because $\mathbb{T}$ is compact, then f reaches its maximum at some point $a \in \mathbb{T}$, and so $f'(a) = 0$, then we can write

$$
\begin{aligned}
f'(x-w) &= f'(x-w) - f'(a) \\
&= \int_a^{x-w} (-f''(x-t))dt \\
&= \int_a^{x-w} \left(\delta_t f''(x) - f''(x)\right) dt
\end{aligned}
$$

$$
\begin{aligned}
|f'(x-w)| &\leq \int_a^{x-w} |\delta_t f''| \frac{\sec(t/2)}{\tan(t/2)} \frac{\tan(t/2)}{\sec(t/2)} dt + \int_a^{x-w} |f''(x)| dt \\
&\leq \sqrt{4\pi} \left(\int_a^{x-w} \frac{\tan^2(t/2)}{\sec^2(t/2)} dt \right)^{1/2} \left(\frac{1}{4\pi} \int_a^{x-w} \frac{(\delta_t f'')^2 \sec^2(t/2)}{\tan^2(t/2)} dt \right)^{1/2} \\
&\qquad + \pi |f''(x)|.
\end{aligned}
$$

Here we are using that the distance between any two points in $\mathbb{T}$ is at most π. Taking squares and integrating we get:

$$
\begin{aligned}
\int_0^z |f'(x-w)|^2 h(w) dw &\leq 2\pi \int_0^z h(w) dw \int_{\mathbb{T}} \frac{(\delta_t f'')^2 \sec^2(t/2)}{\tan^2(t/2)} dt \\
&\qquad + 2\pi^2 \int_0^z h(w) dw |f''(x)|^2 \\
&= 2\pi \int_0^z h(w) dw D[f''] + 2\pi^2 \int_0^z h(w) dw |f''(x)|^2
\end{aligned}
$$

For the second inequality we just complete the square in the right hand side and take square root. $\square$

Continuation of proof of Lemma 3.2.2. By Applying Lemma 3.2.3 and because $|f'| \leq B$ then $|\tanh(\delta_s f/2)| \leq |s| B/2$, then we get

$$
\begin{aligned}
|I_1| &\leq \frac{1}{2} \int_0^s \int_0^z \frac{|\tanh(\delta_w f/2)|}{w} w |\partial_x f(x-w)|^2 dw dz \\
&\leq \frac{B^2}{4} \int_0^s \int_0^z w |\partial_x f(x-w)| dw dz \\
&\leq \frac{B^2}{4\sqrt{3}} \int_0^s z^{3/2} \left(\int_0^z |\partial_x f(x-w)|^2 dw \right)^{1/2} dz \\
&\leq \frac{B^2}{4\sqrt{3}} \int_0^s z^{3/2} \left(\sqrt{2\pi} z^{1/2} D[f'']^{1/2} + \sqrt{2} z^{1/2} \pi |f''(x)| \right) dz \\
&\leq \frac{B^2}{4\sqrt{3}} \int_0^s z^2 dz \left(\sqrt{2\pi} D[f'']^{1/2} + \sqrt{2}\pi |f''(x)| \right) \\
&\leq \frac{B^2}{12\sqrt{3}} \sqrt{2\pi} s^3 D[f'']^{1/2} + \frac{B^2}{12\sqrt{3}} \sqrt{2}\pi s^3 |f''(x)|,
\end{aligned}
$$

for the second term we get

$$\begin{aligned}
|I_2| &= \tfrac{1}{2}\int_0^s\int_0^z \operatorname{sech}^2\left(\tfrac{\delta_w f}{2}\right)\tfrac{|\delta_w f''(x)|\sec(w/2)}{\tan(w/2)}\cdot\tfrac{\tan(w/2)}{\sec(w/2)}dwdz\\
&\le \tfrac{\sqrt{4\pi}}{2}\int_0^s\left(\tfrac{1}{4\pi}\int_0^z\tfrac{(\delta_w f'')^2\sec^2(w/2)}{\tan^2(w/2)}dw\right)^{1/2}\left(\int_0^z\tfrac{\tan^2(w/2)}{\sec^2(w/2)}dw\right)^{1/2}dz\\
&\le \sqrt{\pi}(D[f''])^{1/2}\int_0^s\left(\int_0^z\tfrac{w^2}{4}dw\right)^{1/2}dz\\
&\le \tfrac{\sqrt{\pi}}{5\sqrt{3}}(D[f''])^{1/2}s^{5/2}
\end{aligned}$$

And therefore we obtain

$$\begin{aligned}
|A_1| &\le (D[f''])^{1/2}\left(\tfrac{B^2}{12\sqrt{3}}\sqrt{2\pi}s^3+\tfrac{\sqrt{\pi}}{5\sqrt{3}}s^{5/2}\right)+\tfrac{B^2}{12\sqrt{3}}\sqrt{2}\pi s^3|f''(x)|\\
&\le C(1+B^2)s^{5/2}\left((D[f''])^{1/2}+|f''(x)|\right).
\end{aligned}\tag{3.37}$$

This finishes the proof of part (a). Now we proceed to estimate A_2

$$\begin{aligned}
A_2 &= \frac{1}{2}\int_0^s dzf'(x)-\frac{1}{2}\int_0^s\sec^2(z/2)\operatorname{sech}^2\left(\frac{\delta_z f}{2}\right)f'(x-z)dz-h_2(s)f''(x)\\
&= \frac{-1}{2}\int_0^s\int_0^z\sec^2(w/2)\tan(w/2)\operatorname{sech}^2\left(\frac{\delta_w f}{2}\right)f'(x-w)dz\\
&\quad+\frac{1}{2}\int_0^s\int_0^z\sec^2(w/2)\operatorname{sech}^2\left(\frac{\delta_w f}{2}\right)\tanh\left(\frac{\delta_w f}{2}\right)|f'(x-w)|^2dwdz\\
&\quad+\frac{1}{2}\int_0^s\int_0^z\sec^2(w/2)\operatorname{sech}^2\left(\frac{\delta_w f}{2}\right)(f''(x-w)-f''(x))dz\\
&= K_1+K_2+K_3
\end{aligned}\tag{3.38}$$

For K_1 by Lemma 3.2.3 we get

$$\begin{aligned}
|K_1| &= \left|\frac{-1}{2}\int_0^s\int_0^z\sec^2(w/2)\tan(w/2)\operatorname{sech}^2\left(\frac{\delta_w f}{2}\right)f'(x-w)dz\right|\\
&\le \frac{1}{2}\int_0^s\left(\int_0^z\sec^2(w/2)|f'(x-w)|^2dw\right)^{1/2}\\
&\qquad\qquad\times\left(\int_0^z\tan^2(w/2)\sec^2(w/2)dw\right)^{1/2}dz\\
&\le C\int_0^s\tan^2(z/2)\left[\sqrt{2\pi}(D[f''])^{1/2}+\sqrt{2}\pi|f''(x)|\right]\\
&\le C_1 s\tan^2(s/2)\left((D[f''])^{1/2}+|f''(x)|\right)
\end{aligned}$$

We can also find a different estimate using

$$\begin{aligned}
|K_1| &\leq \frac{1}{2}\int_0^s \left(\int_0^z |f'(x-w)|^2 dw\right)^{1/2} \left(\int_0^z \tan^2(w/2)\sec^4(w/2)dw\right)^{1/2} \\
&\leq C\int_0^s z^{1/2}\left[(D[f''])^{1/2} + |f''(x)|\right]\tan^{1/2}(z/2)\sec^2(z/2)dz \\
&\leq Cs^{1/2}\tan^{3/2}(s/2)\left((D[f''])^{1/2} + |f''(x)|\right) \\
&\leq C_2 s^{1/2}\tan^{3/2}(s/2)\left((D[f''])^{1/2} + |f''(x)|\right),
\end{aligned}$$

here we used that

$$\begin{aligned}
\textstyle\int_0^s \tan^2(w/2)\sec^4(w/2)dw &= \textstyle\int_0^s \tan^2(w/2)(1+\tan^2(w/2))\sec^2(w/2)dw \\
&= \tfrac{2}{3}\tan^3(s/2) + \tfrac{2}{5}\tan^5(s/2) \\
&\leq \tfrac{2}{3}\tan^3(s/2)\sec^2(s/2) \\
&\leq \tfrac{2}{3}\tan(s/2)\sec^4(s/2)
\end{aligned}$$

Now we can combine this two estimates to get

$$\begin{aligned}
|K_1| &\leq \min\{C_1 s\tan^2(s/2), C_2 s^{1/2}\tan^{3/2}(s/2)\}\left((D[f''])^{1/2} + |f''(x)|\right) \\
&= s^{1/2}\tan^{3/2}(s/2)\min\{C_1 s^{1/2}\tan^{1/2}(s/2), C_2\}\left((D[f''])^{1/2} + |f''(x)|\right),
\end{aligned}$$

and because $s \leq \pi$, $\min\{C_1 s^{1/2}\tan^{1/2}(s/2), C_2\} \leq C_3 s$ and therefore

$$\begin{aligned}
|K_1| &\leq s^{3/2}\tan^{3/2}(s/2)\max\{C_1\sqrt{\frac{2}{\pi}}, C_2\frac{2}{\pi}\}\left((D[f''])^{1/2} + |f''(x)|\right) \\
&= C_3 s^{3/2}\tan^{3/2}(s/2)\left((D[f''])^{1/2} + |f''(x)|\right).
\end{aligned} \tag{3.39}$$

For K_2 we get

$$\begin{aligned}
|K_2| &= \left|\frac{1}{2}\int_0^s\int_0^z \sec^2(w/2)\mathrm{sech}^2\left(\frac{\delta_w f}{2}\right)\tanh\left(\frac{\delta_w f}{2}\right)|f'(x-w)|^2 dwdz\right| \\
&\leq \frac{1}{2}\int_0^s\int_0^z \sec^2(w/2)|\tanh(\delta_w f/2)||f'(x-w)|^2 dwdz \\
&\leq \frac{B}{2}\int_0^s\int_0^z w\sec^2(w/2)\frac{|\tanh(\delta_w f/2)|}{w}|f'(x-w)|dwdz \\
&\leq CB^2\int_0^s\left(\int_0^z w^2\sec^2(w/2)dw\right)^{1/2} \\
&\qquad \times\Big(\tan^{1/2}(z/2)(D[f''])^{1/2}+\tan^{1/2}(z/2)|f''(x)|\Big)dz \\
&\leq CB^2\int_0^s z\tan(z/2)\left(D[f'']^{1/2}+|f''(x)|\right) \\
&\leq CB^2 s^2\tan(s/2)\left(D[f'']^{1/2}+|f''(x)|\right),
\end{aligned}$$

and finally for K_3

$$\begin{aligned}
|K_3| &= \left|\frac{1}{2}\int_0^s\int_0^z \sec^2(w/2)\mathrm{sech}^2\left(\frac{\delta_w f}{2}\right)(f''(x-w)-f''(x))dz\right| \\
&\leq C\int_0^s\left(\frac{1}{4\pi}\int_0^z\frac{\sec^2(w/2)(\delta_w f'')^2}{\tan^2(w/2)}dw\right)^{1/2} \\
&\qquad \times\left(\int_0^z \sec^2(w/2)\tan^2(w/2)dw\right)^{1/2}dz \qquad (3.40)\\
&\leq C\left(D[f'']\right)^{1/2}\int_0^s \tan^{3/2}(z/2)dz \\
&\leq C\left(D[f'']\right)^{1/2}s\tan^{3/2}(s/2)
\end{aligned}$$

Therefore we get that A_2 can be bounded by

$$\begin{aligned}
|A_2| &\leq Cs^{3/2}\tan^{3/2}(s/2)\left((D[f''])^{1/2}+|f''(x)|\right) \\
&\quad +CB^2s^2\tan(s/2)\left(D[f'']^{1/2}+|f''(x)|\right) \\
&\quad +C\left(D[f'']\right)^{1/2}s\tan^{3/2}(s/2) \qquad (3.41)\\
&\leq C(D[f''])^{1/2}\left(s^{3/2}\tan^{3/2}(s/2)+B^2s^2\tan(s/2)+s\tan^{3/2}(s/2)\right) \\
&\quad +C|f''(x)|\left(s^{3/2}\tan^{3/2}(s/2)+B^2s^2\tan(s/2)\right) \\
&\leq C(1+B^2)s\tan^{3/2}(s/2)\left((D[f''])^{1/2}+|f''(x)|\right).
\end{aligned}$$

Now we proceed to estimate A_3

$$
\begin{aligned}
A_3 &= \frac{1}{2}\int_0^s \sec^2(z/2)\mathrm{sech}^2\left(\frac{\delta_z f}{2}\right) f'(x-z)dz \\
&\qquad -\frac{1}{2}\int_0^s \sec^2(z/2)dz f'(x) + h_3(s)f''(x) \\
&= \frac{-1}{2}\int_0^s \sec^2(z/2)\int_0^z \mathrm{sech}^2\left(\frac{\delta_w f}{2}\right)\tanh\left(\frac{\delta_w f}{2}\right)|f'(x-w)|^2 dwdz \\
&\quad +\frac{1}{2}\int_0^s \sec^2(z/2)\int_0^z \mathrm{sech}^2\left(\frac{\delta_w f}{2}\right)(f''(x)-f''(x-w))\,dwdz \\
|A_3| &\le C\int_0^s \sec^2(z/2)\int_0^z |\tanh(\delta_w f/2)||f'(x-w)|^2 dwdz \\
&\quad +C\int_0^s \sec^2(z/2)\left(\frac{1}{4\pi}\int_0^z \frac{(\delta_w f)^2\sec^2(w/2)}{\tan^2(w/2)}dw\right)^{1/2} \\
&\qquad\qquad \times\left(\int_0^z \frac{\tan^2(w/2)}{\sec^2(w/2)}dw\right)^{1/2} dz \\
&\le CB\int_0^s \sec^2(z/2)\int_0^z \frac{|\tanh(\delta_w f/2)|}{w}w|f'(x-w)|dwdz \\
&\quad +C\int_0^s \sec^2(z/2)\,(D[f''])^{1/2}\left(\int_0^z \frac{w^2}{4}dw\right)^{1/2} \\
&\le CB^2\int_0^s \sec^2(z/2)\int_0^z w|f'(x-w)|dwdz \\
&\quad +C(D[f''])^{1/2}\int_0^s z^{3/2}\sec^2(z/2)dz
\end{aligned}
\tag{3.42}
$$

And by applying Lemma 3.2.3

$$
\begin{aligned}
|A_3| &\le CB^2\int_0^s \sec^2(z/2)z^{3/2}\left(z^{1/2}(D[f''])^{1/2}+z^{1/2}|f''(x)|\right)dz \\
&\quad +C(D[f''])^{1/2}s^{3/2}\int_0^s \sec^2(z/2)dz \\
&\le CB^2s^2\tan(s/2)(D[f''])^{1/2}+CB^2s^2\tan(s/2)|f''(x)| \\
&\quad +C(D[f''])^{1/2}s^{3/2}\tan(s/2)
\end{aligned}
\tag{3.43}
$$

$$\leq \; C(1+B^2)\Big(D[f'']^{1/2}s^{3/2}\tan(s/2)$$
$$+(D[f''])^{1/2}s^2\tan(s/2)+|f''(x)|s^2\tan(s/2)\Big) \qquad (3.44)$$
$$\leq \; C(1+B^2)s\tan^{3/2}(s/2)\left((D[f''])^{1/2}+|f''(x)|\right).$$

Finally putting all together we conclude

$$\begin{aligned} |sR_2[f'']| &\leq A_1+A_2+A_3 \\ &\leq C(1+B^2)s\tan^{3/2}(s/2)\left((D[f''])^{1/2}+|f''(x)|\right), \end{aligned} \qquad (3.45)$$

which is the estimate in part (b) we were looking for and concludes the proof of Lemma 3.2.2. □

3.2.4 Non linear lower bound

The main goal of this section is to obtain pointwise lower bound of the nonlinear terms appearing in the equation for the second derivative of the equation. The Lemmas in this section are analogous to the ones in Section 3 in [11] but the proofs must be redone for our situation.

Lemma 3.2.4. *Let $f \in W^{1,\infty}(\mathbb{T}) \cap W^{2,p}(\mathbb{T})$ a Lipschitz continuous function with Lipschitz constant B. Then for any $x \in \mathbb{T}$ either*

$$|f''(x)| < \frac{256B}{\pi}, \qquad (3.46)$$

or

$$D_f[f''] \geq \frac{1}{32B(1+B^2)}|f''(x)|^3. \qquad (3.47)$$

Lemma 3.2.5. *Let $f \in W^{1,\infty}(\mathbb{T}) \cap W^{2,p}(\mathbb{T})$ a Lipschitz continuous with Lipschitz constant $B > 0$ then for any $x \in \mathbb{T}$ either*

$$\frac{|f''(x)|}{\|f''\|_{L^p}} < \frac{1}{C_p^{1/p}} \left(\frac{2}{\pi}\right)^{1/p} \tag{3.48}$$

or

$$D_f[f''] \geq \frac{4C_p}{1+B^2} \frac{|f''(x)|^{p+2}}{\|f''\|_{L^p}^p} \tag{3.49}$$

where $C_p = \left(8\left((1 - \frac{1}{2^{(p+1)/(p-1)}})\frac{p-1}{p+1}\right)^{(p-1)/p}\right)^{-p}$

Lemma 3.2.6. *Let $f \in W^{1,\infty}(\mathbb{T}) \cap W^{2,p}(\mathbb{T})$ a Lipschitz continuous with Lipschitz constant B. Assume also that f' obeys a modulus of continuity ρ. Then there exist a continuous function $L_B : [0, \infty) \to [0, \infty)$ such that for any $x \in \mathbb{T}$ we have that either*

$$|f''(x)| < \frac{240B}{\pi} \tag{3.50}$$

or

$$D_f[f''] \geq L_B(|f''(x)|), \tag{3.51}$$

where

$$\lim_{y \to \infty} \frac{L_B(y)}{y^3} = \infty, \tag{3.52}$$

at a rate that depends on how fast $\lim_{r \to 0^+} \rho(r) = 0$.

Proof of Lemma 3.2.4. First notice that because f is Lipschitz, then

$$\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)} \leq \frac{|\delta_s f/2|^2}{\tan^2(s/2)} \leq \frac{|Bs/2|^2}{\tan^2(s/2)} \leq B^2, \tag{3.53}$$

and so we get the estimate

$$
\begin{aligned}
D_f[f''] &= \frac{1}{2}p.v.\frac{1}{2\pi}\int \frac{(\delta_s f'')^2 \sec^2(s/2)}{\tanh^2(\delta_s f/2)+\tan^2(s/2)}ds \\
&\geq \frac{1}{2(1+B^2)}\frac{1}{2\pi}\int \frac{(\delta_s f'')^2 \sec^2(s/2)}{\tan^2(s/2)}ds.
\end{aligned}
\tag{3.54}
$$

Our goal is to bound the term $D_f[f'']$ following a strategy similar to the one used in [12] for a lower bound for the fractional Laplacian in a the periodic domain. For this purpose we use the following identity for the cotangent, which can be obtained by using use that $\csc^2(x) = -\frac{d}{dx}\frac{\frac{d}{dx}\sin(x)}{\sin(x)}$ and the formula for the infinite product of $\sin(x)$

$$
\frac{\sec^2(s/2)}{\tan^2(s/2)} = \csc^2(s/2) = \sum_{k\in\mathbb{Z}} \frac{1}{(s/2-k\pi)^2}, \tag{3.55}
$$

using this expansion on (3.54) we get

$$
D_f[f''] \geq \frac{1}{4\pi(1+B^2)}\sum_{k\in\mathbb{Z}}\int_{-\pi}^{\pi}\frac{(\delta_s f'')^2}{(s/2-k\pi)^2}ds \geq \frac{1}{\pi(1+B^2)}\int_{-\pi}^{\pi}\frac{(\delta_s f'')^2}{s^2}ds. \tag{3.56}
$$

The idea of taking the term with $k=0$ for the lower bound is that because is the only singular term in the expansion, we expect that the main contribution in the integral to come from that term. Here $\eta(x)$ is a smooth cutoff such that $0 \leq \eta(x) \leq 1$, $\eta(x) = 1$ for $|x| \geq 1$ and $\eta(x) = 0$ for $|x| \leq 1/2$, $|\chi'| \leq 4$, then we can bound using this bound we obtain

$$
\begin{aligned}
D_f[f'']ds &\geq \frac{1}{\pi(1+B^2)}\int_{-\pi}^{\pi}\frac{(\delta_s f'')^2}{s^2}ds \\
&\geq \frac{1}{\pi(1+B^2)}\int_{-\pi}^{\pi}\frac{(\delta_s f'')^2\eta(s/r)}{s^2}ds \\
&\geq \frac{1}{\pi(1+B^2)}\int_{-\pi}^{\pi}\frac{|f''(x)|^2-2f''(x)f''(x-s)}{s^2}\eta(s/r)ds \\
&= \frac{1}{\pi(1+B^2)}\left(M_1|f''(x)|^2 - M_2 f''(x)\right).
\end{aligned}
$$

To estimate M_1 we use that for $r \leq \frac{\pi}{2}$,

$$M_1 = \int_{-\pi}^{\pi} \frac{\eta(s/r)}{s^2} ds \geq \int_{r<|s|<\pi} \frac{1}{s^2} ds = 2\left(\frac{1}{r} - \frac{1}{\pi}\right) \geq \frac{1}{r}, \tag{3.57}$$

for M_2 we get

$$M_2 = 2\left|\int_{-\pi}^{\pi} \frac{f''(x-s)\eta(s/r)}{s^2} ds\right| = 2\left|\int_{-\pi}^{\pi} \partial_s \delta_s f'(x) \frac{1}{s^2} \eta(s/r) ds\right|,$$

integrating by parts we get

$$\begin{aligned} M_2 &= 2\left|\int_{-\pi}^{\pi} \delta_s f'(x) \frac{-2}{s^3} \eta(s/r) ds + \frac{1}{r} \int_{-\pi}^{\pi} \delta_s f'(x) \frac{1}{s^2} \eta'(s/r) ds\right| \\ &\leq 2 \int_{r/2<|s|<\pi} |\delta_s f'(x)| \frac{2}{s^3} ds + \frac{4}{r} \int_{r/2<|s|<r} |\delta_s f'(x)| \frac{1}{s^2} ds \\ &\leq 8B \int_{r/2<|s|<\pi} \frac{1}{|s|^3} dx + 8B \int_{r/2<|s|<r} \frac{1}{|s|^3} ds \\ &\leq 16B \left.\frac{(-1)}{s^2}\right|_{r/2}^{\pi} = 16B\left(\frac{4}{r^2} - \frac{1}{\pi^2}\right) \leq \frac{64B}{r^2} \end{aligned}$$

Now we want to choose r such that

$$\frac{|f''(x)|}{2r} = \frac{64B}{r^2}, \Rightarrow \qquad r = \frac{128B}{|f''(x)|}, \tag{3.58}$$

and this can be done if $\frac{128B}{|f''(x)|} \leq \frac{\pi}{2}$, so we get the condition that either

$$|f''(x)| < \frac{256B}{\pi}, \text{ or } D_f[f''] \geq \frac{4}{1+B^2} \frac{|f''(x)|^3}{128B} \tag{3.59}$$

□

Proof of Lemma 3.2.5. From equation 3.56

$$D_f[f''] \geq \frac{1}{\pi(1+B^2)} \int_{-\pi}^{\pi} \frac{(\delta_s f'')^2}{s^2} ds$$

This time we use Hölder inequality instead of integrating by parts, so we get

$$
\begin{aligned}
D_f[f''] &\geq \frac{1}{\pi(1+B^2)} \int_{|s|>r} \frac{(\delta_s f'')^2}{s^2} ds \\
&\geq \frac{1}{\pi(1+B^2)} \int_{|s|>r} \frac{|f''(x)|^2 - 2f''(x)f''(x-s) + |f''(x-s)|^2}{s^2} ds \\
&\geq \frac{1}{\pi(1+B^2)} \Bigg(2|f''(x)|^2 \left(\frac{1}{r} - \frac{1}{\pi}\right) \\
&\qquad -4|f''(x)| \|f''\|_{L^p} \left(\int_r^\pi \frac{1}{s^{\frac{2p}{p-1}}} ds \right)^{(p-1)/p} \Bigg) \\
&\geq \frac{|f''(x)|}{\pi(1+B^2)} \Bigg(2|f''(x)| \left(\frac{1}{r} - \frac{1}{\pi}\right) \\
&\qquad -4\|f''\|_{L^p} \left(\frac{p-1}{p+1}\right)^{(p-1)/p)} \left(\frac{1}{r^{\frac{p+1}{p-1}}} - \frac{1}{\pi^{\frac{p+1}{p-1}}} \right)^{\frac{p-1}{p}} \Bigg).
\end{aligned}
$$

Now notice that for $r \leq \pi/2$, $\frac{1}{r} - \frac{1}{\pi} \geq \frac{1}{2r}$ and

$$
\begin{aligned}
\frac{1}{r^{(p+1)/(p-1)}} - \frac{1}{\pi^{(p+1)/(p-1)}} &= \frac{1-\alpha}{r^{(p+1)/(p-1)}} + \alpha \frac{1}{r^{(p+1)/(p-1)}} - \frac{1}{\pi^{(p+1)/(p-1)}} \\
&\geq \frac{1-\alpha}{r^{(p+1)/(p-1)}} + \alpha \frac{2^{(p+1)/(p-1)}}{\pi^{(p+1)/(p-1)}} - \frac{1}{\pi^{(p+1)/(p-1)}} \\
&\geq \frac{1-\alpha}{r^{(p+1)/(p-1)}},
\end{aligned}
$$

for $\alpha = \frac{1}{2^{(p+1)/(p-1)}}$, applying this to our estimate for $D_f[f'']$ we get

$$
\begin{aligned}
D_f[f''] \geq \frac{|f''(x)|}{\pi(1+B^2)} \Bigg(&\frac{|f''(x)|}{r} \\
&- 4\|f''\|_{L^p} \left((1 - \frac{1}{2^{(p+1)/(p-1)}}) \frac{p-1}{p+1} \right)^{(p-1)/p} \frac{1}{r^{(p+1)/p}} \Bigg). \quad (3.60)
\end{aligned}
$$

Now we want to choose $r \leq \pi/2$ so that

$$
\frac{|f''(x)|^2}{2r} = 4|f''(x)| \|f''\|_{L^p} \left(\left(1 - \frac{1}{2^{(p+1)/(p-1)}}\right) \frac{p-1}{p+1} \right)^{(p-1)/p} \frac{1}{r^{(p+1)/p}}, \quad (3.61)
$$

multiplying by $2r/\|f\|_{L^p}$ we get

$$
\frac{|f''(x)|}{\|f''\|_{L^p}} = 8 \left(\left(1 - \frac{1}{2^{(p+1)/(p-1)}}\right) \frac{p-1}{p+1} \right)^{(p-1)/p} \frac{1}{r^{1/p}}, \quad (3.62)
$$

finally by taking the p power we obtain

$$\frac{1}{r} = C_p \frac{|f''(x)|^p}{\|f''\|_{L^p}^p}. \tag{3.63}$$

This choice or r can be done if

$$C_p \frac{|f''(x)|^p}{\|f''\|_{L^p}^p} \geq \frac{2}{\pi}. \tag{3.64}$$

Therefore we get the condition that either $\frac{|f''(x)|}{\|f''\|_{L^p}} < \frac{1}{C_p^{1/p}} \left(\frac{2}{\pi}\right)^{1/p}$ or

$$D_f[f''] \geq \frac{1}{\pi(1+B^2)} C_p \frac{|f''(x)|^{p+2}}{\|f''\|_{L^p}^p}, \tag{3.65}$$

where $C_p = \left(8\left(\left(1 - \frac{1}{2^{(p+1)/(p-1)}}\right) \frac{p-1}{p+1}\right)^{(p-1)/p}\right)^{-p}$. □

Proof of Lemma 3.2.6. From equation (3.56) we know

$$D_f[f''] \geq \frac{1}{\pi(1+B^2)} \int_{-\pi}^{\pi} \frac{(\delta_s f'')^2}{s^2} ds. \tag{3.66}$$

Let η a cutoff function, such that $\chi(x) = 0$ for $|x| \leq \frac{1}{2}$, $\chi(x) = 1$ for $|t| \geq 1$ and $\eta'(t) = 2$ for $t \in (1/2, 1)$.

$$\begin{aligned} D_f[f''] &\geq \frac{1}{\pi(1+B^2)} \int_{-\pi}^{\pi} \frac{(\delta_s f'')^2 \chi(s/r)}{s^2} ds \\ &\geq \frac{1}{\pi(1+B^2)} \Bigg(|f''(x)|^2 \int_{r<|s|<\pi} \frac{1}{s^2} ds \\ &\qquad -2|f''(x)| \left| \int_{-\pi}^{\pi} \frac{f''(x-s)\chi(s/r)}{s^2} ds \right| \Bigg), \end{aligned}$$

integrating by parts we get

$$\begin{aligned}
&\geq \frac{1}{\pi(1+B^2)}\Bigg(|f''(x)|^2\left(\frac{2}{r}-\frac{2}{\pi}\right)\\
&\qquad -2|f''(x)|\left|\int_{|s|>r/2}\frac{\delta_s f'(-2)\chi(s/r)}{s^3}ds\right|\\
&\qquad -\frac{2}{r}|f''(x)|\left|\int_{r/2<|s|<r}\frac{\delta_s f'\chi'(s/r)}{s^2}ds\right|\Bigg)\\
&\geq \frac{1}{\pi(1+B^2)}\left(|f''(x)|^2\left(\frac{2}{r}-\frac{2}{\pi}\right)-4|f''(x)|\left|\int_{|s|>r/2}\frac{\delta_s f'\chi(s/r)}{s^3}ds\right|\right.\\
&\qquad \left.-4|f''(x)|\int_{r/2<|s|<r}\frac{|\delta_s f'|}{|s|^3}ds\right).
\end{aligned}$$

Now notice that for $r \leq \pi/2$ we have that $\frac{2}{r}-\frac{2}{\pi}\geq\frac{1}{r}$ and therefore

$$\begin{aligned}
D_f[f''] &\geq \frac{1}{\pi(1+B^2)}\left(\frac{|f''(x)|^2}{r}-16|f''(x)|\int_{r/2}^{\pi}\frac{\rho(s)}{s^3}ds\right)\\
&= \frac{|f''(x)|}{\pi(1+B^2)r}\left(|f''(x)|-16r\int_{r/2}^{\pi}\frac{\rho(s)}{s^3}ds\right),
\end{aligned}$$

where $\rho(s)$ is the modulus of continuity of f'. Notice that we can assume that $\lim_{s\to0^+}\rho(s)/s=\infty$ by taking if necessary a function $\rho(s)$ that is larger than the original one and decay slower at 0. We want to choose $r\leq\pi/2$ such that

$$\frac{|f''(x)|}{32} = r\int_{r/2}^{\pi}\frac{\rho(s)}{s^3}ds, \tag{3.67}$$

this can be done if $\frac{|f''(x)|}{32}\geq\frac{15B}{2\pi}$. To see this we use the intermediate value theorem and that

$$\lim_{r\to0} r\int_{r/2}^{\pi}\frac{\rho(s)}{s^3}ds=\infty, \tag{3.68}$$

this is obtained by applying the L'Hospital rule and that $\lim_{s\to0^+}\rho(s)/s=\infty$. At $r\to\pi/2$ we get for the limit

$$\lim_{r\to\pi/2} r\int_{r/2}^{\pi}\frac{\rho(s)}{s^3}ds=\frac{\pi}{2}\int_{\pi/4}^{\pi}\frac{\rho(s)}{s^3}ds\leq B\pi\int_{\pi/4}^{\pi}\frac{1}{s^3}ds=\frac{15B}{2\pi}. \tag{3.69}$$

Which implies that (3.67) can always be satisfied as long as $|f''(x)| \geq \frac{240B}{\pi}$. So far we have that for almost every $x \in \mathbb{T}$ we have that either

$$|f''(x)| < \frac{240B}{\pi} \tag{3.70}$$

or

$$D_f[f''] \geq L_B(|f''(x)|), \tag{3.71}$$

where $L_B(t) = \frac{t^2}{2\pi(1+B^2)r(t)}$ and $r(t)$ satisfies

$$\frac{t}{32} = r(t) \int_{r(t)/2}^{\pi} \frac{\rho(s)}{s^3} ds. \tag{3.72}$$

It is easy to see that $r(t)$ can be chosen to depend continuously on t, To prove that

$$\lim_{y \to \infty} \frac{L_B(y)}{y^3} = \infty, \tag{3.73}$$

we notice that this can be written as

$$\frac{L_B(t)^3}{t} = \frac{1}{2\pi(1+B^2)tr(t)} \tag{3.74}$$

and therefore it is enough to show that the following quantity go to 0 as $y \to \infty$

$$\frac{tr(t)}{32} = r(t)^2 \int_{r(t)/2}^{\pi} \frac{\rho(s)}{s^3} ds. \tag{3.75}$$

Notice that from (3.68) we know that as $t \to \infty$, $r(t) \to 0$. To show that the limit as $t \to \infty$ of (3.75) is zero first we split the integral between $[r(t), \sqrt{r(t)}]$ and $[\sqrt{r(t)}, \pi]$ and notice that

$$r(t)^2 \int_{\sqrt{r(t)}}^{\pi} \frac{\rho(s)}{s^3} ds \leq 2Br(t)^2 \int_{\sqrt{r(t)}}^{\pi} \frac{1}{s^3} ds \leq Br(t) \to 0, \tag{3.76}$$

as $t \to \infty$, next we look at

$$r(t)^2 \int_{\sqrt{r(t)}}^{r(t)} \frac{\rho(s)}{s^3} ds \leq \rho(\sqrt{t}) r(t)^2 \int_{r(t)}^{\infty} \frac{1}{s^3} ds = \frac{\rho(\sqrt{t})}{2} \to 0, \tag{3.77}$$

which proves (3.52) and complete the proof of Lemma 3.2.6. $\square$

3.2.5 Bounds of the Right hand side

In this section we want to find upper bounds for the terms in the right hand side of the equation for the second derivative defined in Subsection 3.2.2. From now on, we will assume that all the integrals are taken in the principal value sense if needed. Also in this section C is a constant that can change on each line that do not depend of B.

Note that is only necessary to estimate $|T_i|$ when $f''(x) \neq 0$ because those terms are multiplied by $f''(x)$ in equation (3.21). The goal of this section is to prove the following estimate.

Lemma 3.2.7. *Let T_i, $i \in \{1, \cdots, 8\}$ as defined in Subsection 3.2.2, with $f \in W^{1,\infty} \cap W^{2,2}$ and Lipschitz constant B. Let $0 < \varepsilon < 1$ then for $x \in \mathbb{T}$ such that $f''(x) \neq 0$ we have*

(a) $|T_1|+|T_2|+|T_3|+|T_4|+|T_5|+|T_6|+|T_7| \leq CB(1+B)^3 \left(\varepsilon^{-2}|f''(x)|^2 + \varepsilon \frac{D[f'']}{|f''(x)|}\right)$,

(b) $|T_8| \leq CB(1+B)^4 \left(\varepsilon^{-2}|f''(x)| + \varepsilon \frac{D[f'']}{|f''(x)|^2} + |Hf''|\right)$.

Proof of Lemma 3.2.7. The key for the proof is a careful application of the Lemma 3.2.4. In what follows $R_1[f'']$ and $R_2[f'']$ are defined as in Lemma 3.2.1 and Lemma

3.2.2. Here $\{|s| \leq \eta\} \stackrel{\text{def}}{:=} \{s \in \mathbb{T} : d(0,s) \leq \eta\}$, and $\{|s| > \eta\} := T \setminus \{s \in \mathbb{T} : d(0,s) \leq \eta\}$.

An important ingredient in the proof is to split the integrals in region where we can apply different estimates, because our estimates are pointwise in x, our choice of such splitting will depend on the point, more specifically given x s.t. $f''(x) \neq 0$ and $\varepsilon \in (0,1)$, we can choose $\eta(x) \in (0,\pi)$ such that

$$\tan\left(\frac{\eta}{2}\right) = \frac{\varepsilon B}{|f''(x)|}. \tag{3.78}$$

Now we proceed to estimate T_i for $i \in \{1, \cdots, 8\}$.

Bound for T1

$$T_1 = \frac{1}{2\pi} \int \frac{\operatorname{sech}^2(\delta_s f/2) \tanh^2(\delta_s f/2) \sec^2(s/2) \, (\delta_s f')^2}{\left(\tan^2(s/2) + \tanh^2(\delta_s f/2)\right)^2} \times \left[\tanh(\delta_s f/2) - \tan(s/2)\partial f\right] ds \tag{3.79}$$

Let $A(s) = \frac{\operatorname{sech}^2(\delta_s f/2) \frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)}}{\left(1 + \frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)}\right)^2}$, we can estimate $A(s)$ by using the following easy facts $|\operatorname{sech}(x)| \leq 1$, $|\tanh(x)| \leq 1$, $|\tanh(\delta_s f/2)| \leq |\delta_s f/2| \leq \frac{B}{2}\pi$ and $\left|\frac{\tanh(\delta_s f/2)}{\tan(s/2)}\right| \leq B$. Then we get

$$|A(s)| \leq \min\{1, B, B^2\}, \quad |A(s)\tan(s/2)| \leq \min\{1, B, B^2 \frac{\pi}{2}\}, \tag{3.80}$$

$$\text{and } |A(s)\tan^2(s/2)| \leq \min\{1, B\frac{\pi}{2}, B^2\frac{\pi^2}{4}\}. \tag{3.81}$$

Consider the following splitting of T_1

$$\begin{aligned}
T_1 &= \frac{1}{2\pi}\int A(s)\frac{\sec^2(s/2)}{\tan^2(s/2)}\left(\delta_s f'\right)^2\left[\tanh(\delta_s f/2)-\tan(s/2)\partial f\right]ds\\
&= \frac{1}{2\pi}\int_{|s|\le\eta} A(s)\frac{\sec^2(s/2)}{\tan^2(s/2)}\left(\delta_s f'\right)^2\left(R_2[f'']-h(s)f''(x)\right)ds\\
&\quad+\frac{1}{2\pi}\int_{|s|>\eta} A(s)\frac{\sec^2(s/2)}{\tan^2(s/2)}\left(\delta_s f'\right)^2\left(R_2[f'']-h(s)f''(x)\right)ds\\
&= \frac{1}{2\pi}\int_{|s|\le\eta} A(s)\frac{\sec^2(s/2)}{\tan^2(s/2)}\left(\delta_s f'\right)\left(sf''(x)+R_1[f'']\right)R_2[f'']ds\\
&\quad-f''(x)\frac{1}{2\pi}\int_{|s|\le\eta} A(s)\frac{\sec^2(s/2)}{\tan^2(s/2)}\left(sf''(x)+R_1[f'']\right)^2h(s)ds\\
&\quad+\frac{1}{2\pi}\int_{|s|>\eta} A(s)\frac{\sec^2(s/2)}{\tan^2(s/2)}\left(\delta_s f'\right)^2\left(R_2[f'']-h(s)f''(x)\right)ds\\
&= I_{in,1}+I_{in,2}+I_{out},
\end{aligned}$$

where $h(s)$ is defined by (3.28). Now we estimate $I_{in,1}$ by using the following splitting

$$\begin{aligned}
I_{in,1} &= f''(x)\frac{1}{2\pi}\int_{|s|\le\eta} A(s)\frac{\sec^2(s/2)}{\tan^2(s/2)}\left(\delta_s f'\right)s(sR_2[f''])ds\\
&\quad+\frac{1}{2\pi}\int_{|s|\le\eta} A(s)\frac{\sec^2(s/2)}{\tan^2(s/2)}\left(\delta_s f'\right)(sR_1[f''])sR_2[f'']ds\\
&= J_1+J_2,
\end{aligned}$$

we can estimate J_1 and J_2 by using Lemma 3.2.1 and Lemma 3.2.2, in the following way

$$
\begin{aligned}
|J_1| &\leq CB(1+B^2)|f''(x)|^2 \int_{s\leq\eta} \left(|A(s)||\tan(s/2)|^{3/2}\right) \frac{\sec^2(s/2)}{\tan^2(s/2)} s^2 ds \\
&\quad +CB(1+B^2)|f''(x)|(D[f''])^{1/2} \int_{s\leq\eta} |A(s)| \frac{\sec^2(s/2)}{\tan^2(s/2)} s^2 |\tan(s/2)|^{3/2} ds \\
&\leq CB^{3/2}(1+B^2)|f''(x)|^2 \int_{s\leq\eta} \frac{\sec^2(s/2)}{\tan^2(s/2)} s^2 ds \\
&\quad CB(1+B^2)|f''(x)|(D[f''])^{1/2} \int_{s\leq\eta} \frac{\sec^2(s/2)}{|\tan(s/2)|^{1/2}} \pi^2 ds \\
&\leq CB^{3/2}(1+B^2)|f''(x)|^2 + CB(1+B^2)|f''(x)|(D[f''])^{1/2} \tan^{1/2}(\frac{\eta}{2}) \\
&= CB^{3/2}(1+B^2)\left(|f''(x)|^2 + |f''(x)|\varepsilon^{1/2}\frac{(D[f''])^{1/2}}{|f''(x)|^{1/2}}\right) \\
&\leq CB(1+B)^{5/2}\left(\varepsilon\frac{D[f'']}{|f''(x)|} + |f''(x)|^2)\right).
\end{aligned}
$$

Here we used that $\int_0^\pi \frac{\sec^2(s/2)}{\tan^2(s/2)} s^2 ds < \infty$ and the definition of our choice of η given by (3.78). For the estimate of J_2 we use

$$
\begin{aligned}
|J_2| &\leq CB(1+B^2)(D[f''])^{1/2}|f''(x)| \int_{|s|\leq\eta} |A(s)| \frac{\sec^2(s/2)}{\tan^2(s/2)} s^{3/2} s \tan^{3/2}(s/2) \\
&\quad +CB(1+B^2)D[f''] \int_{|s|\leq\eta} |A(s)| \frac{\sec^2(s/2)}{\tan^2(s/2)} s^{3/2} s \tan^{3/2}(s/2) \\
&\leq CB^{3/2}(1+B^2)(D[f''])^{1/2}|f''(x)| \int_{|s|\leq\eta} \frac{\sec^2(s/2)}{\tan^{1/2}(s/2)} ds \\
&\quad +CB(1+B^2)D[f''] \int_{|s|\leq\eta} \sec^2(s/2) ds \\
&\leq CB(1+B^2)\left(B^{1/2}D[f'']^{1/2}|f''(x)|\tan^{1/2}(\frac{\eta}{2}) + D[f'']\tan(\frac{\eta}{2})\right) \\
&\leq CB^2(1+B^2)\left(\varepsilon\frac{D[f'']}{|f''(x)|} + |f''(x)|^2\right) \\
&\leq CB(1+B)^3\left(\varepsilon\frac{D[f'']}{|f''(x)|} + |f''(x)|^2\right).
\end{aligned}
$$

For $I_{in,2}$ we have

$$
\begin{aligned}
I_{in,2} &= -f''(x)\frac{1}{2\pi}\int_{|s|\leq\eta} A(s)\frac{\sec^2(s/2)}{\tan^2(s/2)}\left(sf''(x)+R_1[f'']\right)^2 h(s)ds \\
&= -(f''(x))^3\frac{1}{2\pi}\int_{|s|\leq\eta} A(s)\frac{\sec^2(s/2)}{\tan^2(s/2)}s^2h(s)ds \\
&\quad -2(f''(x))^2\frac{1}{2\pi}\int_{|s|\leq\eta} A(s)\frac{\sec^2(s/2)}{\tan^2(s/2)}sh(s)(R_1[f''])ds \\
&\quad -(f''(x))\frac{1}{2\pi}\int_{|s|\leq\eta} A(s)\frac{\sec^2(s/2)}{\tan^2(s/2)}h(s)(R_1[f''])^2ds \\
&= K_1+K_2+K_3.
\end{aligned}
$$

We recall that by Lemma 3.2.2 we know that $h(s)\leq 3s\tan(s/2)$, then

$$
\begin{aligned}
|K_1| &\leq |f''(x)|^3\frac{1}{2\pi}\int_{|s|\leq\eta}|A(s)|\sec^2(s/2)\frac{s^2h(s)}{\tan^2(s/2)}ds \\
&\leq C|f''(x)|^3\frac{1}{2\pi}\int_{|s|\leq\eta}\sec^2(s/2)ds \\
&= C|f''(x)|^3\tan(\eta/2) = CB\varepsilon|f''(x)|^2,
\end{aligned}
$$

$$
\begin{aligned}
|K_2| &\leq CB|f''(x)|^2(D[f''])^{1/2}\frac{1}{2\pi}\int_{|s|\leq\eta}|A(s)|\sec^2(s/2)\frac{sh(s)s^{3/2}}{\tan^2(s/2)}ds \\
&\leq CB|f''(x)|^2(D[f''])^{1/2}\frac{1}{2\pi}\int_{|s|\leq\eta}\sec^2(s/2)\tan^{1/2}(s/2)ds \\
&= CB|f''(x)|^2(D[f''])^{1/2}\tan^{3/2}(\eta/2) \\
&= CB^{5/2}\varepsilon|f''(x)|\varepsilon^{3/2}\frac{(D[f''])^{1/2}}{|f''(x)|^{1/2}} \\
&\leq CB(1+B)^{3/2}\left(\varepsilon^2|f''(x)|^2+\varepsilon\frac{D[f'']}{|f''(x)|}\right),
\end{aligned}
$$

$$\begin{aligned}
|K_3| &\leq C|f''(x)|D[f'']\frac{1}{2\pi}\int_{|s|\leq\eta}|A(s)|\sec^2(s/2)\frac{h(s)s^3}{\tan^2(s/2)}ds \\
&\leq C|f''(x)|D[f'']\frac{1}{2\pi}\int_{|s|\leq\eta}\sec^2(s/2)\tan(s/2)ds \\
&= C|f''(x)|D[f'']\tan^2(\eta/2) \\
&\leq CB^2\varepsilon^2\frac{D[f'']}{|f''(x)|} \\
&\leq CB(1+B)\varepsilon^2\frac{D[f'']}{|f''(x)|}.
\end{aligned}$$

Lastly we estimate I_{out} using

$$\begin{aligned}
I_{out} &= \frac{1}{2\pi}\int_{|s|\geq\eta}A(s)\frac{\sec^2(s/2)}{\tan^2(s/2)}\left(\delta_s f'\right)^2\left[sR_2[f'']-h(s)f''(x)\right]ds \\
|I_{out}| &\leq CB(1+B^2)|f''(x)|\int_{|s|>\eta}|A(s)|s\tan^{3/2}(s/2)\frac{\sec^2(s/2)}{\tan^2(s/2)}ds \\
&\quad +2BD[f'']^{1/2}\int_{|s|>\eta}|A(s)|s\tan^{3/2}(s/2)\frac{\sec^2(s/2)}{\tan^2(s/2)}\tan^{1/2}(s/2)ds \\
&\quad +2B|f''(x)|\int_{|s|>\eta}|A(s)|\tan^2(s/2)\frac{\sec^2(s/2)}{\tan^2(s/2)}\frac{h(s)}{\tan^2(s/2)}ds \\
&\leq B^2(1+B^2)C|f''(x)|\frac{1}{\tan(\eta/2)}+B^{3/2}CD[f'']^{1/2}\frac{1}{\tan^{1/2}(\eta/2)} \\
&\qquad +B^2C|f''(x)|\frac{1}{\tan(\eta/2)} \\
&= \frac{B(1+B^2)C}{\varepsilon}|f''(x)|^2+\frac{BC}{\sqrt{\varepsilon}}|f''(x)|\frac{D[f'']^{1/2}}{|f''(x)|^{1/2}} \\
&\leq B(1+B^2)C\left(\varepsilon^{-1}|f''(x)|^2+\varepsilon^{-2}|f''(x)|^2+\varepsilon\frac{D[f'']}{|f''(x)|}\right) \\
&\leq B(1+B)^2C\left(\varepsilon^{-2}|f''(x)|^2+\varepsilon\frac{D[f'']}{|f''(x)|}\right),
\end{aligned}$$

putting all together we obtain

$$|T_1|\leq CB(1+B)^3\left(\varepsilon^{-2}|f''(x)|^2+\varepsilon\frac{D[f'']}{|f''(x)|}\right). \tag{3.82}$$

Bound for T2

$$T_2 = \frac{1}{4\pi}\int \frac{\operatorname{sech}^4(\delta_s f/2)\sec^2(s/2)\,(\delta_s f')^2}{\left(\tan^2(s/2)+\tanh^2(\delta_s f/2)\right)^2}\left[\tanh(\delta_s f/2)-\tan(s/2)\partial f\right]ds, \quad (3.83)$$

we can split T_2 as

$$\begin{aligned}
T_2 &= \frac{1}{4\pi}\int \frac{\operatorname{sech}^4(\delta_s f/2)\sec^2(s/2)\,(\delta_s f')^2}{\left(\tan^2(s/2)+\tanh^2(\delta_s f/2)\right)^2}\left[\tanh(\delta_s f/2)-\tan(s/2)\partial f\right]ds \\
&= \frac{1}{4\pi}\int \frac{\operatorname{sech}^4(\delta_s f/2)}{(1+\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)})^2}\frac{\sec^2(s/2)}{\tan^4(s/2)}(\delta_s f')^2\left(R_2[f'']-h(s)f''(x)\right)ds \\
&= \frac{1}{4\pi}\int_{|s|\le\eta} \frac{\operatorname{sech}^4(\delta_s f/2)}{(1+\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)})^2}\frac{\sec^2(s/2)}{\tan^4(s/2)}(\delta_s f')(R_1[f'']+sf''(x))(R_2[f''])ds \\
&\quad -f''(x)\frac{1}{2}\frac{1}{2\pi}\int_{|s|\le\eta} \frac{\operatorname{sech}^4(\delta_s f/2)}{(1+\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)})^2}\frac{\sec^2(s/2)}{\tan^4(s/2)}(R_1[f'']+sf''(x))^2 h(s)ds \\
&\quad +\frac{1}{4\pi}\int_{|s|>\eta} \frac{\operatorname{sech}^4(\delta_s f/2)}{(1+\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)})^2}\frac{\sec^2(s/2)}{\tan^4(s/2)}(\delta_s f')^2\left(R_2[f'']-h(s)f''(x)\right)ds \\
&= I_{in,1}+I_{in,2}+I_{out}.
\end{aligned}$$

For $I_{in,1}$ we have:

$$\begin{aligned}
|I_{in,1}| &\le \frac{B}{2}\frac{1}{2\pi}\int_{|s|\le\eta}\frac{\sec^2(s/2)}{\tan^4(s/2)}(R_1[f''])(R_2[f''])ds \\
&\quad +\frac{B}{2}|f''(x)|\frac{1}{2\pi}\int_{|s|\le\eta}\frac{\sec^2(s/2)}{\tan^4(s/2)}s(R_2[f'']) \\
&\le CBD[f'']\int_{|s|\le\eta}\frac{\sec^2(s/2)}{\tan^4(s/2)}|s|^{5/2}|\tan(s/2)|^{3/2}ds \\
&\quad +CB(D[f''])^{1/2}|f''(x)|\int_{|s|\le\eta}\frac{\sec^2(s/2)}{\tan^4(s/2)}s^2|\tan(s/2)|^{3/2}ds \\
&\quad +CB|f''(x)|(D[f''])^{1/2}\int_{|s|\le\eta}\frac{\sec^2(s/2)}{\tan^4(s/2)}s^2|\tan(s/2)|^{3/2}ds \\
&\quad +CB|f''(x)|^2\int_{|s|\le\eta}\frac{\sec^2(s/2)}{\tan^4(s/2)}s^2|\tan(s/2)|^{3/2}ds \\
&\le CBD[f'']\tan(\eta/2)+BC|f''(x)|(D[f''])^{1/2}\tan^{1/2}(\eta/2) \\
&\quad +CB|f''(x)|(D[f''])^{1/2}\tan^{1/2}(\eta/2)+CB|f''(x)|^2 \\
&\le CB(1+B)\left(|f''(x)|^2+\varepsilon\frac{D[f'']}{|f''(x)|}\right).
\end{aligned}$$

Here we used that $\int_0^\pi \frac{s^2 \sec^2(s/2)}{\tan^{5/2}(s/2)} < \infty$. For $I_{in,2}$ we have

$$
\begin{aligned}
|I_{in,2}| &\leq C|f''(x)|D[f''] \int_{|s|\leq\eta} \frac{\sec^2(s/2)}{\tan^4(s/2)}|s|^3|h(s)|ds \\
&\quad +C|f''(x)|^2(D[f''])^{1/2} \int_{|s|\leq\eta} \frac{\sec^2(s/2)}{\tan^4(s/2)}|s|^{5/2}|h(s)|ds \\
&\quad +C|f''(x)|^3 \int_{|s|\leq\eta} \frac{\sec^2(s/2)}{\tan^4(s/2)}s^2|h(s)|ds \\
&\leq C|f''(x)|D[f''] \tan^2(\eta/2) + C|f(x)|^2(D[f''])^{1/2} \tan^{3/2}(\eta/2) \\
&\qquad +C|f''(x)|^3 \tan(\eta/2) \\
&\leq CB(1+B)(|f''(x)|^2 + \varepsilon \frac{D[f'']}{|f''(x)|})
\end{aligned}
$$

And for I_{out},

$$
\begin{aligned}
|I_{out}| &\leq CB^2(D[f''])^{1/2} \frac{1}{2\pi} \int_{|s|>\eta} \frac{\sec^2(s/2)}{\tan^4(s/2)}|s||\tan(s/2)|^{3/2}ds \\
&\quad +CB^2|f''(x)|\frac{1}{2\pi} \int_{|s|>\eta} \frac{\sec^2(s/2)}{\tan^4(s/2)}|s||\tan(s/2)|^{3/2}ds \\
&\quad +CB^2|f''(x)|\frac{1}{2\pi} \int_{|s|>\eta} \frac{\sec^2(s/2)}{\tan^4(s/2)}|h(s)|ds \\
&\leq CB^2(D[f''])^{1/2} \frac{1}{\tan^{1/2}(\eta/2)} \\
&\quad +CB^2|f''(x)|\frac{1}{\tan(\eta/2)} \\
&= CB(1+B)^{1/2} \left(\varepsilon^{-1}|f''(x)|\varepsilon^{1/2} \frac{D[f'']^{1/2}}{|f(x)|^{1/2}} + \varepsilon^{-1}|f''(x)| \right) \\
&\leq CB(1+B)^{1/2} \left(\varepsilon^{-2}|f''(x)|^2 + \varepsilon \frac{D[f'']}{|f''(x)|} \right).
\end{aligned}
$$

Putting all together we get

$$
|T_2| \leq CB(1+B) \left(\varepsilon^{-2}|f''(x)|^2 + \varepsilon \frac{D[f'']}{|f''(x)|} \right). \tag{3.84}
$$

Bound for T3

$$T_3 = \frac{1}{2\pi}\int \frac{\text{sech}^2(\delta_s f/2)\tanh(\delta_s f/2)\sec^2(s/2)\delta_s f''}{\left(\tan^2(s/2)+\tanh^2(\delta_s f/2)\right)^2} \times [\tanh(\delta_s f/2) - \tan(s/2)\partial_x f(x)]\,ds, \quad (3.85)$$

we can split T_3 as

$$\begin{aligned} T_3 &= \frac{1}{2\pi}\int_{|s|\le\eta} \frac{\text{sech}^2(\delta_s f/2)\frac{\tanh(\delta_s f/2)}{\tan(s/2)}}{\left(1+\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)}\right)^2}\frac{\sec^2(s/2)}{\tan^3(s/2)}(\delta_s f'')(R_2[f''])ds \\ &\quad -f''(x)\frac{1}{2\pi}\int_{|s|\le\eta} \frac{\text{sech}^2(\delta_s f/2)\frac{\tanh(\delta_s f/2)}{\tan(s/2)}}{\left(1+\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)}\right)^2}\frac{\sec^2(s/2)}{\tan^3(s/2)}(\delta_s f'')h(s) \\ &\quad +\frac{1}{2\pi}\int_{|s|>\eta} \frac{\text{sech}^2(\delta_s f/2)\tanh(\delta_s f/2)\sec^2(s/2)\delta_s f''}{\left(\tan^2(s/2)+\tanh^2(\delta_s f/2)\right)^2} \\ &\qquad\qquad \times [\tanh(\delta_s f/2)-\tan(s/2)\partial_x f(x)]\,ds \\ &= I_1 + I_2 + I_{out}. \end{aligned}$$

Let $A(s) = \frac{\frac{\tanh(\delta_s f/2)}{\tan(s/2)}}{\left(1+\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)}\right)^2}$, then in a similar way to the estimates for T_1 we get that $|A(s)| \le \min\{1, B\}$, $|A(s)\tan(s/2)| \le \min\{1, B\frac{\pi}{2}\}$.

$$\begin{aligned} |I_1| &\le C(1+B^2)|f''(x)|\int_{|s|\le\eta}\left(|A(s)|\text{sech}^2(\delta_s f/2)\right) \\ &\qquad \times\left(\sec(s/2)\frac{|s||\tan(s/2)|^{3/2}}{\tan^2(s/2)}\right)\left(\frac{\sec(s/2)|\delta_s f''|}{|\tan(s/2)|}\right)ds \\ &\quad +C(1+B^2)(D[f''])^{1/2}\int_{|s|\le\eta}\left(|A(s)|\text{sech}^2(\delta_s f/2)\right) \\ &\qquad \times\left(\sec(s/2)\frac{s|\tan(s/2)|^{3/2}}{\tan^2(s/2)}\right)\left(\frac{\sec(s/2)|\delta_s f''|}{|\tan(s/2)|}\right)ds. \end{aligned}$$

For the first term we use the bound $\left|\frac{|s||\tan(s/2)|^{3/2}}{\tan^2(s/2)}\right| \le \pi$ and for the second one $\left|\frac{|s||\tan(s/2)|^{3/2}}{\tan^2(s/2)}\right| \le 2|\tan(s/2)|^{1/2}$, then we can apply Cauchy-Schwarz and the definition of $D[f'']$ given by (3.23) to get

$$\begin{aligned}
|I_1| &\leq C(1+B^2)|f''(x)|\,(D[f''])^{1/2}\left(\int_{|s|\leq\eta}\sec^2(s/2)ds\right)^{1/2}\\
&\qquad +C(1+B^2)D[f'']\left(\int_{|s|\leq\eta}|\tan(s/2)|\sec^2(s/2)ds\right)^{1/2}\\
&\leq C(1+B)^2|f''(x)|\left((D[f''])^{1/2}\tan^{1/2}(\eta/2)+D[f'']\tan(\eta/2)\right)\\
&\leq CB^{1/2}(1+B)^{5/2}\left(|f''(x)|^2+\varepsilon\frac{D[f'']}{|f''(x)|}\right).
\end{aligned}$$

For I_2 we get

$$\begin{aligned}
|I_2| &\leq \frac{|f''(x)|}{2\pi}\int_{|s|\leq\eta}|A(s)|\mathrm{sech}^2(\delta_s f/2)\left(\sec(s/2)\frac{|h(s)|}{\tan^2(s/2)}\right)\\
&\qquad\times\left(\frac{\sec(s/2)|\delta_s f''(x)|}{|\tan(s/2)|}\right)ds\\
&\leq C|f''(x)|\left(\int_{|s|\leq\eta}\sec^2(s/2)\frac{h(s)^2}{\tan^4(s/2)}ds\right)^{1/2}(D[f''])^{1/2}\\
&\leq C|f''(x)|(D[f''])^{1/2}\tan^{1/2}(\eta/2)\\
&\leq CB^{1/2}\left(|f''(x)|^2+\varepsilon\frac{D[f'']}{|f''(x)|}\right),
\end{aligned}$$

and finally for I_{out}

$$\begin{aligned}
|I_{out}| &= \left|\frac{1}{2\pi}\int_{|s|>\eta}\frac{\mathrm{sech}^2(\delta_s f/2)\tanh(\delta_s f/2)\sec^2(s/2)\delta_s f''}{\left(\tan^2(s/2)+\tanh^2(\delta_s f/2)\right)^2}\right.\\
&\qquad\left.\times\left[\tanh(\delta_s f/2)-\tan(s/2)\partial_x f(x)\right]ds\right|\\
&= \left|\frac{1}{2\pi}\int_{|s|>\eta}A(s)\mathrm{sech}^2(\delta_s f/2)\frac{\sec(s/2)}{\tan(s/2)}\right.\\
&\qquad\left.\times\left(\frac{R_1[f']}{\tan(s/2)}\right)\left(\frac{(\delta_s f'')\sec(s/2)}{\tan(s/2)}\right)ds\right|\\
&\leq CB^2(D[f''])^{1/2}\left(\int_{|s|>\eta}\frac{\sec^2}{\tan^2(s/2)}ds\right)^{1/2}\\
&= CB^2(D[f''])^{1/2}\frac{1}{\tan^{1/2}(\eta/2)}\\
&\leq CB^{3/2}\left(\varepsilon^{-2}|f''(x)|^2+\varepsilon\frac{D[f'']}{|f''(x)|}\right),
\end{aligned}$$

here we used that by Lemma 3.2.1 $\left|\frac{R_1[f']}{\tan(s/2)}\right| \leq 2B$. Finally putting all together we conclude

$$|T_3| \leq CB(1+B)\left(\varepsilon^{-2}|f''(x)|^2 + \varepsilon\frac{D[f'']}{|f''(x)|}\right). \tag{3.86}$$

Bound for T4

$$\begin{aligned}
T_4 &= -\frac{1}{\pi}\int \frac{\mathrm{sech}^4(\delta_s f/2)\tanh^2(\delta_s f/2)\sec^2(s/2)\,(\delta_s f')^2}{\left(\tan^2(s/2)+\tanh^2(\delta_s f/2)\right)^3} \\
&\qquad\qquad\qquad\qquad \times[\tanh(\delta_s f/2) - \tan(s/2)\partial_x f(x)]\,ds \\
&= -\frac{1}{\pi}\int \frac{\mathrm{sech}^4(\delta_s f/2)\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)}}{\left(1+\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)}\right)^3}\frac{\sec^2(s/2)}{\tan^4(s/2)}(\delta_s f')^2[R_2[f''] - h(s)f''(x)]ds
\end{aligned}$$

Let $A(s) = \frac{\mathrm{sech}^4(\delta_s f/2)\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)}}{\left(1+\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)}\right)^3}$, then $|A(s)| \leq \min\{1, B, B^2\}$, $|A(s)\tan(s/2)| \leq \min\{1, B, B^2\frac{\pi}{2}\}$, $|A(s)\tan^2(s/2)| \leq \min\{1, B\frac{\pi}{2}, B^2\frac{\pi^2}{4}\}$. We can split T_4 as

$$\begin{aligned}
T_4 &= -\frac{1}{\pi}\int_{|s|\leq\eta} A(s)\frac{\sec^2(s/2)}{\tan^4(s/2)}(\delta_s f')^2[R_2[f''] - h(s)f''(x)] \\
&\quad -\frac{1}{\pi}\int_{|s|>\eta} A(s)\frac{\sec^2(s/2)}{\tan^4(s/2)}(\delta_s f')^2[R_2[f''] - h(s)f''(x)] \\
&= -\frac{1}{\pi}\int_{|s|\leq\eta} A(s)\frac{\sec^2(s/2)}{\tan^4(s/2)}(\delta_s f')(R_1[f''] + sf''(x))R_2[f'']ds \\
&\quad +f''(x)\frac{1}{\pi}\int_{|s|\leq\eta} A(s)\frac{\sec^2(s/2)}{\tan^4(s/2)}(\delta_s f')(R_1[f''] + sf''(x))h(s)ds \\
&\quad -\frac{1}{\pi}\int_{|s|>\eta} A(s)\frac{\sec^2(s/2)}{\tan^4(s/2)}(\delta_s f')^2[R_2[f''] - h(s)f''(x)]ds \\
&= I_{1,in} + I_{2,in} + I_{out},
\end{aligned}$$

$$
\begin{aligned}
|I_{1,in}| &\leq \frac{1}{\pi}\int_{|s|\leq\eta} |A(s)|\frac{\sec^2(s/2)}{\tan^4(s/2)}|\delta_s f'||R_1[f''] + sf''(x)||R_2[f'']|ds \\
&\leq CB(1+B^2)D[f'']\int_{|s|\leq\eta} |A(s)|\frac{\sec^2(s/2)}{\tan^4(s/2)}|s|^{5/2}|\tan(s/2)|^{3/2}ds \\
&\quad +CB(1+B^2)(D[f''])^{1/2}|f''(x)| \\
&\qquad \times\int_{|s|\leq\eta} |A(s)|\frac{\sec^2(s/2)}{\tan^4(s/2)}|s|^{5/2}|\tan(s/2)|^{3/2}ds \\
&\quad +CB(1+B^2)(D[f''])^{1/2}|f''(x)| \\
&\qquad \times\int_{|s|\leq\eta} |A(s)|\frac{\sec^2(s/2)}{\tan^4(s/2)}s^2|\tan(s/2)|^{3/2}ds \\
&\quad +CB(1+B^2)|f''(x)|^2\int_{|s|\leq\eta} |A(s)|\frac{\sec^2(s/2)}{\tan^4(s/2)}s^2|\tan(s/2)|^{3/2}ds \\
&\leq CB(1+B^2)D[f'']\int_{|s|\leq\eta} \sec^2(s/2)ds \\
&\quad +CB(1+B^2)(D[f''])^{1/2}|f''(x)|\int_{|s|\leq\eta} \frac{\sec^2(s/2)}{|\tan(s/2)|^{1/2}}ds \\
&\quad +CB(1+B^2)(D[f''])^{1/2}|f''(x)|\int_{|s|\leq\eta} \frac{\sec^2(s/2)}{|\tan(s/2)|^{1/2}}ds \\
&\quad +CB(1+B^2)|f''(x)|^2\int_{|s|\leq\eta} \frac{\sec^2(s/2)}{|\tan(s/2)|^{5/2}}s^2ds \\
&\leq CB(1+B^2)\Big(D[f'']\tan(\eta/2) + (D[f''])^{1/2}|f''(x)|\tan^{1/2}(\eta/2) \\
&\qquad +|f''(x)|^2\Big) \\
&\leq CB(1+B)^3\left(\varepsilon\frac{D[f'']}{|f''(x)|} + |f''(x)|^2\right),
\end{aligned}
$$

here we used that $\int_{\mathbb{T}} \frac{\sec^2(s/2)s^2}{|\tan(s/2)|^{5/2}}ds < \infty$. To estimate $I_{2,in}$ we use the following

$$
\begin{aligned}
|I_{2,in}| &= |f''(x)|\left|\frac{1}{\pi}\int_{|s|\leq\eta} A(s)\frac{\sec^2(s/2)}{\tan^4(s/2)}(R_1[f'']+sf''(x))^2h(s)ds\right| \\
&\leq C|f''(x)|\frac{1}{2\pi}\int_{|s|\leq\eta}|A(s)|\frac{\sec^2(s/2)}{\tan^4(s/2)}(R_1[f''])^2|h(s)|ds \\
&\quad +C|f''(x)|^2\int_{|s|\leq\eta}|A(s)|\frac{\sec^2(s/2)}{\tan^4(s/2)}(R_1[f''])|s||h(s)|ds \\
&\quad +C|f''(x)|^3\int_{|s|\leq\eta}|A(s)|\frac{\sec^2(s/2)}{\tan^4(s/2)}s^2|h(s)|ds \\
&\leq C|f''(x)|D[f'']\int_{|s|\leq\eta}\frac{\sec^2(s/2)}{\tan^4(s/2)}s^4|\tan(s/2)|ds \\
&\quad +C|f''(x)|^2D[f'']\int_{|s|\leq\eta}\frac{\sec^2(s/2)}{\tan^4(s/2)}|s|^{7/2}|\tan(s/2)|ds \\
&\quad +C|f''(x)|^3\int_{|s|\leq\eta}\frac{\sec^2(s/2)}{\tan^4(s/2)}|s|^3|\tan(s/2)|ds \\
&\leq C\Big(|f''(x)|D[f'']\tan^2(\eta/2)+|f''(x)|^2(D[f''])^{1/2}\tan^{3/2}(\eta/2) \\
&\qquad\qquad +|f''(x)|^3\tan(\eta/2)\Big) \\
&= CB(1+B)\left(\varepsilon^2\frac{D[f'']}{|f''(x)|}+\varepsilon^{3/2}\frac{(D[f''])^{1/2}}{|f''(x)|^{1/2}}|f''(x)|+\varepsilon|f''(x)|^2\right) \\
&\leq CB(1+B)\left((\varepsilon^2+\varepsilon^3)\frac{D[f'']}{|f''(x)|}+(1+\varepsilon)|f''(x)|\right) \\
&\leq CB(1+B)\left(\varepsilon^2\frac{D[f'']}{|f''(x)|}+|f''(x)|\right).
\end{aligned}
$$

Now we proceed to estimate I_{out}

$$
\begin{aligned}
|I_{out}| &= \left|\frac{1}{\pi}\int_{|s|>\eta}A(s)\frac{\sec^2(s/2)}{\tan^4(s/2)}(\delta_s f')^2[R_2[f'']-h(s)f''(x)]ds\right| \\
&\leq \frac{1}{\pi}\int_{|s|>\eta}|A(s)|\frac{\sec^2(s/2)}{\tan^4(s/2)}(\delta_s f')^2|R_2[f'']|ds \\
&\quad +|f''(x)|\frac{1}{\pi}\int_{|s|>\eta}|A(s)|\frac{\sec^2(s/2)}{\tan^4(s/2)}(\delta_s f')^2|h(s)|ds
\end{aligned}
$$

$$\begin{aligned}
|I_{out}| &\leq CB(1+B^2)(D[f''])^{1/2}\int_{|s|>\eta}\frac{\sec^2(s/2)}{\tan^4(s/2)}|s||\tan(s/2)|^{3/2}ds \\
&\quad +CB(1+B^2)|f''(x)|\int_{|s|>\eta}\frac{\sec^2(s/2)}{\tan^4(s/2)}|s||\tan(s/2)|^{3/2}ds \\
&\quad +CB|f''(x)|\int_{|s|>\eta}\frac{\sec^2(s/2)}{\tan^4(s/2)}|s||\tan(s/2)|ds \\
&\leq CB(1+B)^2\left(D[f'']^{1/2}\frac{1}{\tan^{1/2}(\eta/2)}+|f''(x)|\frac{1}{\tan(\eta/2)}\right) \\
&\leq C(1+B)^{5/2}\left(\varepsilon^{1/2}\frac{D[f'']^{1/2}}{|f''(x)|^{1/2}}\varepsilon^{-1}|f''(x)|+\varepsilon^{-1}|f''(x)|^2\right) \\
&\leq C(1+B)^{5/2}\left(\varepsilon\frac{D[f'']}{|f''(x)|}+\varepsilon^{-2}|f''(x)|^2\right).
\end{aligned}$$

Putting all together we conclude

$$|T_4| \leq CB(1+B)^3\left(\varepsilon\frac{D[f'']}{|f''(x)|}+\varepsilon^{-2}|f''(x)|^2\right). \tag{3.87}$$

Bound for T5

$$\begin{aligned}
T_5 = \frac{1}{2\pi}\int &\frac{\operatorname{sech}^2(\delta_s f/2)\tanh(\delta_s f/2)\sec^2(s/2)\,(\delta_s f')}{\left(\tan^2(s/2)+\tanh^2(\delta_s f/2)\right)^3} \\
&\times\left[\operatorname{sech}^2(\delta_s f/2)\frac{\delta_s f'}{2}-\tan(s/2)f''(x)\right]ds
\end{aligned} \tag{3.88}$$

To bound this term we first focus on the term

$$\begin{aligned}
R_3[f''] &= \operatorname{sech}^2(\delta_s f/2)\frac{\delta_s f'}{2}-\tan(s/2)f''(x) \\
&= \frac{1}{2}\operatorname{sech}^2(\delta_s f/2)\int_0^s f''(x-w)dw-\frac{1}{2}\operatorname{sech}^2(\delta_s f/2)\int_0^s dwf''(x) \\
&\quad +\left(\operatorname{sech}^2(\delta_s f/2)\frac{s}{2}-\tan(s/2)\right)f''(x),
\end{aligned} \tag{3.89}$$

$$\begin{aligned}
|R_3[f'']| &\leq \frac{1}{2}\int_0^s \operatorname{sech}^2(\delta_s f/2)\left(|\delta_w f''|\frac{\sec(s/2)}{\tan(s/2)}\right)\left(\frac{\tan(s/2)}{\sec(s/2)}\right)ds \\
&\quad +\left|\frac{s}{2}-\tan(s/2)-\frac{\tanh^2(\delta_s f/2)}{s^2}s^2\frac{s}{2}\right|\cdot|f''(x)|
\end{aligned} \tag{3.90}$$

To bound the first term we use Cauchy-Schwartz and that $\frac{\tan^2(z/2)}{\sec^2(z/2)} \leq \frac{z^2}{4}$. For the second one we use the following

$$\begin{aligned}\left|\frac{s}{2} - \tan(s/2)\right| &= \left|\frac{1}{2}\int_0^s dt - \frac{1}{2}\int_0^s \sec^2(t/2)dt\right| \\ &\leq \frac{1}{2}\left|\int_0^s (1-\sec^2(t/2))dt\right| \\ &\leq \frac{1}{2}\int_0^s \tan^2(t/2)dt \\ &\leq \frac{1}{2}|\tan(s/2)|^2\int_0^s dt \\ &\leq \frac{s}{2}\tan^2(s/2).\end{aligned} \tag{3.91}$$

Applying this to $R_3[f'']$ we get

$$\begin{aligned}R_3[f''] &\leq (D[f''])^{1/2}\left(\int_0^s \frac{w^2}{4}ds\right)^{1/2} \\ &\quad + \frac{|s|}{2}\tan^2(s/2)|f''(x)| + \frac{B^2}{4}\frac{s^3}{2}|f''(x)| \\ &\leq \frac{1}{2\sqrt{3}}(D[f''])^{1/2}|s|^{3/2} + |f''(x)|\left(\frac{|s|}{2}\tan^2(s/2) + \frac{B^2}{8}|s|^3\right) \\ &\leq C(D[f''])^{1/2}|s|^{3/2} + C(1+B^2)|f''(x)||s|\tan^2(s/2) \\ &\leq C(1+B^2)\left((D[f''])^{1/2}s^{3/2} + |f''(x)||s|\tan^2(s/2)\right).\end{aligned} \tag{3.92}$$

Now define

$$A(s) = \frac{\operatorname{sech}^2(\delta_s f/2)\frac{\tanh(\delta_s f/2)}{\tan(s/2)}}{\left(1 + \frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)}\right)^2}, \tag{3.93}$$

then is easy to check that $A(s)$ satisfy

$$|A(s)| \leq \min\{1, B\}, \quad |A(s)\tan(s/2)| \leq \min\{1, B\pi\}. \tag{3.94}$$

We consider the splitting of T_5 as

$$\begin{aligned}
T_5 &= \frac{1}{2\pi}\int A(s)\frac{\sec^2(s/2)}{\tan^3(s/2)}(R_1[f'']+sf''(x))(R_3[f''])ds \\
&= \frac{1}{2\pi}\int_{|s|\leq\eta} A(s)\frac{\sec^2(s/2)}{\tan^3(s/2)}(R_1[f'']+sf''(x))(R_3[f''])ds \\
&\quad + \frac{1}{2\pi}\int_{|s|>\eta} A(s)\frac{\sec^2(s/2)}{\tan^3(s/2)}(\delta f')(R_3[f''])ds \\
&= I_{in}+I_{out}.
\end{aligned}$$

First we estimate I_{in} using

$$\begin{aligned}
|I_{in}| &\leq C(D[f''])^{1/2}\frac{1}{2\pi}\int_{|s|\leq\eta}|A(s)|\frac{\sec^2(s/2)}{|\tan(s/2)|^3}|s|^{3/2}|R_3[f'']|ds \\
&\quad +|f''(x)|\frac{1}{2\pi}\int_{|s|\leq\eta}|A(s)|\frac{\sec^2(s/2)}{|\tan(s/2)|^3}|s|(R_3[f''])ds \\
&\leq C(1+B^2)D[f'']\int_{|s|\leq\eta}|A(s)|\frac{\sec^2(s/2)}{|\tan(s/2)|^3}|s|^3ds \\
&\quad +C(1+B^2)(D[f''])^{1/2}|f''(x)|\int_{|s|\leq\eta}|A(s)|\frac{\sec^2(s/2)}{|\tan(s/2)|^3}|s|^{5/2}|\tan(s/2)|^2ds \\
&\quad +C(1+B^2)(D[f''])^{1/2}|f''(x)|\int_{|s|\leq\eta}|A(s)|\frac{\sec^2(s/2)}{|\tan(s/2)|^3}|s|^{5/2}ds \\
&\quad +C(1+B^2)|f''(x)|^2\int_{|s|\leq\eta}(|A(s)|\tan(s/2))\,\frac{\sec^2(s/2)}{\tan^4(s/2)}|s|^2|\tan(s/2)|^2ds \\
&\leq C(1+B^2)D[f'']\int_{|s|\leq\eta}\sec^2(s/2)ds \\
&\quad +C(1+B^2)(D[f''])^{1/2}|f''(x)|\int_{|s|\leq\eta}\frac{\sec^2(s/2)}{|\tan(s/2)|^{1/2}}ds \\
&\quad +C(1+B^2)(D[f''])^{1/2}|f''(x)|\int_{|s|\leq\eta}\frac{\sec^2(s/2)}{|\tan(s/2)|^{1/2}}ds \\
&\quad +C(1+B^2)|f''(x)|^2\int_{|s|\leq\eta}\frac{\sec^2(s/2)}{\tan^2(s/2)}s^2ds \\
&\leq C(1+B^2)\left(D[f'']\tan(\eta/2)+(D[f''])^{1/2}|f''(x)|\tan^{1/2}(\eta/2)+|f''(x)|^2\right) \\
&\leq C(1+B)^2\left(\varepsilon\frac{D[f'']}{|f''(x)|}+|f''(x)|^2\right).
\end{aligned}$$

Now for I_{out} we get

$$\begin{aligned}
|I_{out}| &= \tfrac{1}{2\pi}\int_{|s|>\eta}|A(s)|\tfrac{\sec^2(s/2)}{|\tan(s/2)|^3}|\delta f'||R_3[f'']|ds\\
&\leq C(1+B^2)(D[f''])^{1/2}\int_{|s|>\eta}|A(s)|\tfrac{\sec^2(s/2)}{\tan^3(s/2)}|s|^{3/2}ds\\
&\quad +C(1+B^2)|f''(x)|\int_{|s|>\eta}(|A(s)||\tan(s/2)|)\,\tfrac{\sec^2(s/2)}{\tan^4(s/2)}|s||\tan(s/2)|^2ds\\
&\leq CB(1+B^2)(D[f''])^{1/2}\int_{|s|>\eta}\tfrac{\sec^2(s/2)}{|\tan(s/2)|^{3/2}}ds\\
&\quad +CB(1+B^2)|f''(x)|\int_{|s|>\eta}\tfrac{\sec^2(s/2)}{\tan^2(s/2)}ds\\
&\leq (1+B)^{5/2}\left(\tfrac{(D[f''])^{1/2}}{\tan^{1/2}(\eta/2)}+\tfrac{|f''(x)|}{\tan(\eta/2)}\right)\\
&= C(1+B)^{5/2}\left(\varepsilon^{-1}|f''(x)|\varepsilon^{1/2}\tfrac{(D[f''])^{1/2}}{|f''(x)|^{1/2}}+\varepsilon^{-1}|f''(x)|^2\right)\\
&\leq C(1+B)^{5/2}\left(\varepsilon\tfrac{D[f'']}{|f''(x)|}+\varepsilon^{-2}|f''(x)|^2\right),
\end{aligned}$$

and therefore we obtain

$$|T_5|\leq C(1+B)^{5/2}\left(\varepsilon\frac{D[f'']}{|f''(x)|}+\varepsilon^{-2}|f''(x)|^2\right). \tag{3.95}$$

Bound for T6

$$\begin{aligned}
T_6 = -\frac{1}{2\pi}\int & \frac{\mathrm{sech}^2(\delta_s f/2)\tanh(\delta_s f/2)\sec^2(s/2)\delta_s f'}{\left(\tan^2(s/2)+\tanh^2(\delta_s f/2)\right)^2}\\
&\times\left[f''(x)\tan(s/2)+(\delta_s f')\tanh^2(\delta_s f/2)-\frac{1}{2}(\delta_s f')\left(1-\tan^2(s/2)\right)\right]ds
\end{aligned} \tag{3.96}$$

Similar to the estimate for $R_3[f'']$ in the bound for T_5 we can bound

$$\begin{aligned}
K(s) &= f''(x)\tan(s/2)+(\delta_s f')\tanh^2(\delta_s f/2)-\frac{1}{2}(\delta_s f')(1-\tan^2(s/2))\\
&= \left(f''(x)\frac{s}{2}-\frac{1}{2}(\delta_s f')\right)+f''(x)\left(\tan(s/2)-\frac{s}{2}\right)\\
&\quad +\delta_s f'\left(\tanh^2(\delta_s f/2)+\frac{1}{2}\tan^2(s/2)\right)\\
&= K_1(s)+f''(x)K_2(s)+\delta_s f' K_3(s),
\end{aligned}$$

where we can bound $K_i(s)$ $i = 1, 2, 3$ by using

$$\begin{aligned}
|K_1(s)| &\leq C|s|^{3/2}(D[f''])^{1/2}, \\
|K_2(s)| &\leq C|s||\tan(s/2)|^2, \\
|K_3(s)| &\leq \frac{B^2}{4}s^2 + \frac{1}{2}\tan^2(s/2) \\
&\leq C(1+B)^2\left(s^2 + \tan^2(s/2)\right).
\end{aligned} \tag{3.97}$$

Define $A(s) = \frac{\text{sech}^2(\delta_s f/2)\frac{\tanh(\delta_s f/2)}{\tan(s/2)}}{\left(1+\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)}\right)^2}$, then it is easy to see that

$$|A(s)| \leq \min\{1, B\}, |A(s)\tan(s/2)| \leq \min\{1, B\pi\}. \tag{3.98}$$

Now we consider the following splitting of T_6

$$\begin{aligned}
T_6 &= -\frac{1}{2\pi}\int A(s)\frac{\sec^2(s/2)}{\tan^3(s/2)}(\delta_s f')K(s)ds \\
&= -\frac{1}{2\pi}\int_{|s|\leq\eta} A(s)\frac{\sec^2(s/2)}{\tan^3(s/2)}(\delta_s f')K(s)ds \\
&\quad -\frac{1}{2\pi}\int_{|s|>\eta} A(s)\frac{\sec^2(s/2)}{\tan^3(s/2)}(\delta_s f')K(s)ds \\
&= I_{in} + I_{out}.
\end{aligned}$$

First we estimate I_{in} using

$$\begin{aligned}
I_{in} &= -\frac{1}{2\pi}\int_{|s|\leq\eta} A(s)\frac{\sec^2(s/2)}{\tan^3(s/2)}(R_1 + sf''(x))K_1(s)ds \\
&\quad -\frac{1}{2\pi}\int_{|s|\leq\eta} A(s)\frac{\sec^2(s/2)}{\tan^3(s/2)}(R_1 + sf''(x))K_2(s)f''(x)ds \\
&\quad -\frac{1}{2\pi}\int_{|s|\leq\eta} A(s)\frac{\sec^2(s/2)}{\tan^3(s/2)}(R_1 + sf''(x))^2K_3(s)ds
\end{aligned}$$

$$
\begin{aligned}
|I_{in}| \leq\ & CD[f''] \int_{|s|\leq\eta} |A(s)| \frac{\sec^2(s/2)}{|\tan(s/2)|^3} |s|^3 ds \\
& + C|f''(x)|(D[f''])^{1/2} \int_{|s|\leq\eta} |A(s)| \frac{\sec^2(s/2)}{|\tan(s/2)|^3} |s|^{5/2} ds \\
& + C|f''(x)|(D[f''])^{1/2} \int_{|s|\leq\eta} |A(s)| \frac{\sec^2(s/2)}{|\tan(s/2)|^3} |s|^{5/2} |\tan(s/2)|^2 ds \\
& + C|f''(x)|^2 \int_{|s|\leq\eta} |A(s)| \frac{\sec^2(s/2)}{|\tan(s/2)|^3} |s|^2 |\tan(s/2)|^2 ds \\
& + C(1+B)^2 D[f''] \int_{|s|\leq\eta} |A(s)| \frac{\sec^2(s/2)}{|\tan(s/2)|^3} |s|^3 (s^2 + \tan^2(s/2)) ds \\
& + C(1+B)^2 (D[f''])^{1/2} |f''(x)| \\
& \qquad \times \int_{|s|\leq\eta} |A(s)| \frac{\sec^2(s/2)}{|\tan(s/2)|^3} |s|^{5/2} (s^2 + \tan^2(s/2)) ds \\
& + C(1+B)^2 |f''(x)|^2 \int_{|s|\leq\eta} |A(s)\tan(s/2)| \frac{\sec^2(s/2)}{\tan^4(s/2)} s^2 (s^2 + \tan^2(s/2)) ds
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & C(1+B)^2 \left(D[f''] \tan(\eta/2) + |f''(x)|(D[f''])^{1/2} \tan^{1/2}(\eta/2) + |f''(x)|^2 \right) \\
\leq\ & C(1+B)^3 \left(\varepsilon \frac{D[f'']}{|f''(x)|} + |f''(x)|^2 \right).
\end{aligned}
$$

Finally we estimate I_{out} using

$$
\begin{aligned}
I_{out} &= -\frac{1}{2\pi} \int_{|s|>\eta} A(s) \frac{\sec^2(s/2)}{\tan^3(s/2)} (\delta_s f') K(s) ds \\
&= -\frac{1}{2\pi} \int_{|s|>\eta} A(s) \frac{\sec^2(s/2)}{\tan^3(s/2)} (\delta_s f') K_1(s) ds \\
&\quad -\frac{1}{2\pi} \int_{|s|>\eta} A(s) \frac{\sec^2(s/2)}{\tan^3(s/2)} (\delta_s f') f''(x) K_2(s) ds \\
&\quad -\frac{1}{2\pi} \int_{|s|>\eta} A(s) \frac{\sec^2(s/2)}{\tan^3(s/2)} (\delta_s f') (sR_1[f''] + sf''(x)) K_3(s) ds
\end{aligned}
$$

Now using the bounds (3.97) we obtain

$$
\begin{aligned}
|I_{out}| \leq\ & CD[f'']^{1/2}\int_{|s|>\eta}|A(s)|\frac{\sec^2(s/2)}{\tan^3(s/2)}|\delta_s f'||s|^{3/2}ds \\
&+C|f''(x)|\int_{|s|>\eta}(|A(s)|\tan(s/2))\,\frac{\sec^2(s/2)}{\tan^4(s/2)}|\delta_s f'||s||\tan(s/2)|^2 ds \\
&+C(1+B^2)D[f'']^{1/2}\int_{|s|>\eta}|A(s)\tan(s/2)|\frac{\sec^2(s/2)}{\tan^4(s/2)}|\delta_s f'||s|^{3/2} \\
&\qquad\times(s^2+\tan^2(s/2))ds \\
&+C(1+B^2)|f''(x)|\int_{|s|>\eta}|A(s)\tan(s/2)|\frac{\sec^2(s/2)}{\tan^4(s/2)}|\delta_s f'|s \\
&\qquad\times(s^2+\tan^2(s/2))ds \\
\leq\ & C(1+B^2)\left(\frac{(D[f''])^{1/2}}{\tan^{1/2}(\eta/2)}+\frac{|f''(x)|}{\tan(\eta/2)}\right) \\
\leq\ & C(1+B)^{5/2}\left(\varepsilon\frac{D[f'']}{|f''(x)|}+\varepsilon^{-2}|f''(x)|^2\right),
\end{aligned}
$$

and therefore

$$
|T_6| \leq C(1+B)^3\left(\varepsilon\frac{D[f'']}{|f''(x)|}+\varepsilon^{-2}|f''(x)|^2\right). \tag{3.99}
$$

Bound for T7

$$
T_7 = \frac{1}{2\pi}\int\frac{\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)}}{1+\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)}}\sec^2(s/2)(\delta_s f'')ds. \tag{3.100}
$$

Consider the splitting of T_7 given by

$$
\begin{aligned}
T_7 &= \frac{1}{2\pi}\int\frac{\tanh^2(\delta_s f/2)}{1+\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)}}\frac{\sec^2(s/2)}{\tan^2(s/2)}(\delta_s f'')ds \\
&= \frac{1}{2\pi}\int_{|s|\leq\eta}\frac{\tanh^2(\delta_s f/2)}{1+\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)}}\frac{\sec^2(s/2)}{\tan^2(s/2)}(\delta_s f'')ds \\
&\quad+\frac{1}{2\pi}\int_{|s|>\eta}\frac{\tanh^2(\delta_s f/2)}{1+\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)}}\frac{\sec^2(s/2)}{\tan^2(s/2)}(\delta_s f'')ds \\
&= I_{in}+I_{out},
\end{aligned} \tag{3.101}
$$

for I_{in} we can write

$$
\begin{aligned}
I_{in} \;=\; & \frac{1}{2\pi}\int_{|s|\le\eta}\frac{\tanh(\delta_s f/2)}{1+\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)}}\Big(\tanh(\delta_s f/2)-\frac{s}{2}f'(x)+h_1(s)f''(x)\Big)\,\delta_s f'' \\
& -f''(x)\frac{1}{2\pi}\int_{|s|\le\eta}\frac{\tanh(\delta_s f/2)}{1+\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)}}h_1(s)\frac{\sec^2(s/2)}{\tan^2(s/2)}\delta_s f'' ds \\
& +\frac{1}{2\pi}\int_{|s|\le\eta}\frac{\tanh(\delta_s f/2)}{1+\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)}}\frac{s}{2}f'(x-s)\frac{\sec^2(s/2)}{\tan^2(s/2)}\delta_s f'' ds \\
& +\frac{1}{2\pi}\int_{|s|\le\eta}\frac{\tanh(\delta_s f/2)}{1+\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)}}\frac{\sec^2(s/2)}{\tan^2(s/2)}\frac{s}{2}(R_1[f''])\delta_s f'' ds \\
& +f''(x)\frac{1}{2\pi}\int_{|s|\le\eta}\frac{\tanh(\delta_s f/2)}{1+\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)}}\frac{\sec^2(s/2)}{\tan^2(s/2)}\frac{s^2}{2}\delta_s f'' ds
\end{aligned}
$$

Using Lemma 3.2.2 we can bound $\tilde{R}_2[f''] = \tanh(\delta_s f/2) - \frac{s}{2}f'(x) + h_1(s)f''(x)$ and obtain

$$
\begin{aligned}
|I_{in}| \;\le\; & C(1+B^2)((D[f''])^{1/2}+|f''(x)|) \\
& \qquad\times\int_{|s|\le\eta}\left|\frac{\sec(s/2)}{\tan(s/2)}\right||s|^{5/2}\left(\frac{\sec(s/2)}{\tan(s/2)}\delta_s f''\right)ds \\
& +C|f''(x)|\int_{|s|\le\eta}\left|\frac{\sec(s/2)}{\tan(s/2)}\right|s^2\left(\frac{\sec(s/2)}{\tan(s/2)}\delta_s f''\right)ds \\
& +C\int_{|s|\le\eta}\left|\frac{\sec(s/2)}{\tan(s/2)}\right||sf'(x-s)|\left|\frac{\sec(s/2)}{\tan(s/2)}\delta_s f''\right|ds \\
& +CD[f'']^{1/2}\int_{|s|\le\eta}\left|\frac{\sec(s/2)}{\tan(s/2)}\right||s|^{3/2}\left|\frac{\sec(s/2)}{\tan(s/2)}\delta_s f''\right| \\
& +C|f''(x)|\int_{|s|\le\eta}\left|\frac{\sec(s/2)}{\tan(s/2)}\right||s|\left|\frac{\sec(s/2)}{\tan(s/2)}\delta_s f''\right|ds,
\end{aligned}
$$

using Cauchy-Schwarz and Lemma 3.2.3 we get

$$\begin{aligned}
|I_{in}| \;\leq\;& C(1+B^2)(D[f'']^{1/2}+|f''(x)|)D[f'']^{1/2}\left(\int_{|s|\leq\eta}\frac{s^5\sec^2(s/2)}{\tan^2(s/2)}ds\right)^{1/2} \\
&+C|f''(x)|D[f'']^{1/2}\left(\int_{|s|\leq\eta}\frac{s^4\sec^2(s/2)}{\tan^2(s/2)}ds\right)^{1/2} \\
&+CD[f'']^{1/2}\left(\int_{|s|\leq\eta}\frac{s^2\sec^2(s/2)}{\tan^2(s/2)}|f'(x-s)|^2ds\right)^{1/2} \\
&+CD[f'']\left(\int_{|s|\leq\eta}\frac{s^3\sec^2(s/2)}{\tan^2(s/2)}ds\right)^{1/2} \\
&+C|f''(x)|D[f'']^{1/2}\left(\int_{|s|^2\leq\eta}\frac{s^5\sec^2(s/2)}{\tan^2(s/2)}ds\right)^{1/2} \\
\leq\;& C(1+B^2)D[f'']\tan(\eta/2) \\
&+C(1+B^2)D[f'']^{1/2}|f''(x)|\tan^{1/2}(\eta/2) \\
&+CD[f'']^{1/2}|f''(x)|\tan^{1/2}(\eta/2) \\
&+CD[f'']^{1/2}\left(\int_{|s|\leq\eta}\frac{s^2\sec^2(s/2)}{\tan^2(s/2)}ds\right)^{1/2}\left(D[f'']^{1/2}+|f''(x)|\right) \\
&+CD[f'']\tan(\eta/2) \\
&+CD[f'']^{1/2}|f''(x)|\tan^{1/2}(\eta/2) \\
\leq\;& CB(1+B^2)\left(\varepsilon\frac{D[f'']}{|f''(x)|}+\frac{1}{\varepsilon^2}|f''(x)|^2\right)
\end{aligned}$$

For I_{out} we get

$$\begin{aligned}
|I_{out}| \;\leq\;& C\int_{|s|>\eta}\frac{\tanh^2(\delta_s f/2)}{1+\frac{\tanh^2(\delta_s f/2)}{\tan^2(s/2)}}\frac{|\sec(s/2)|}{|\tan(s/2)|}\left(\frac{|\sec(s/2)|}{|\tan(s/2)|}|\delta_s f''|\right)ds \\
\leq\;& C\left(\int_{|s|>\eta}\frac{\sec^2(s/2)}{\tan^2(s/2)}\right)^{1/2}D[f'']^{1/2} \\
\leq\;& C\frac{1}{\tan^{1/2}(\eta/2)}D[f'']^{1/2} \\
\leq\;& C\frac{|f''(x)|}{\varepsilon}\cdot\varepsilon^{1/2}\frac{D[f'']^{1/2}}{|f''(x)|^{1/2}} \\
\leq\;& C\left(\varepsilon\frac{D[f'']}{|f''(x)|}+\varepsilon^{-2}|f''(x)|^2\right)
\end{aligned}$$

And therefore we get

$$|T_7| \leq CB(1+B)\left(\varepsilon\frac{D[f'']}{|f''(x)|}+\varepsilon^{-2}|f''(x)|^2\right) \tag{3.102}$$

Bound for T8

$$T_8 = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\frac{\tanh(\delta_s f/2)}{\tan(s/2)}}{\left(1+\frac{\tanh^2(\delta_s f/2}{\tan(s/2)}\right)^2}\operatorname{sech}^2(\delta_s f/2)(\delta_s f')\frac{\sec^2(s/2)}{\tan^2(s/2)}ds \tag{3.103}$$

To bound T_8 we want to add and subtract a few terms in order to use the bound that we already know. Define

$$T_{8,PV} = \frac{f'(x)}{(1+(f'(x))^2)^2}\frac{1}{2\pi}\int_{\mathbb{T}}\operatorname{sech}^2(\delta_s f/2)(\delta_s f')\frac{\sec^2(s/2)}{\tan^2(s/2)}ds \tag{3.104}$$

Now using that

$$\begin{aligned}\frac{A}{(1+A^2)^2}-\frac{B}{(1+B^2)^2} &= \frac{(A-B)\left(A^3B+A^2B^2+AB^3+2AB-1\right)}{\left(A^2+1\right)^2\left(B^2+1\right)^2}\\ &= (A-B)G(A,B)\end{aligned} \tag{3.105}$$

And that $|G(A,B)| = \left|\frac{\left(A^3B+A^2B^2+AB^3+2AB-1\right)}{\left(A^2+1\right)^2\left(B^2+1\right)^2}\right| \leq 2$ for all $A,B\in\mathbb{R}$. Using this we can estimate the difference of this two terms as

$$T_8 - T_{8,PV} = \int_T K(s)\left(\tfrac{\tanh(\delta_s f/2)}{\tan(s/2)} - f'(x)\right)(\delta_s f')\tfrac{\sec^2(s/2)}{\tan^2(s/2)}ds \tag{3.106}$$

with $|K(s)| = |G\left(\frac{\tanh(\delta_s f/2)}{\tan(s/2)}, f'(x)\right)| \leq 2$. Then we can bound this using

$$
\begin{aligned}
T_8 - T_{8,PV} &= \frac{1}{2\pi}\int_T K(s)(\tanh(\delta_s f/2) - \tan(s/2) f'(x))(\delta_s f')\frac{\sec^2(s/2)}{\tan^3(s/2)}ds \\
&= \frac{1}{2\pi}\int_{|s|\leq\eta} K(s)(R_2[f''])(R_1[f''])\frac{\sec^2(s/2)}{\tan^3(s/2)}ds \\
&\quad + f''(x)\frac{1}{2\pi}\int_{|s|\leq\eta} K(s)(R_2[f''])s\frac{\sec^2(s/2)}{\tan^3(s/2)}ds \\
&\quad - f''(x)\frac{1}{2\pi}\int_{|s|\leq\eta} K(s)h(s)(R_1[f''])\frac{\sec^2(s/2)}{\tan^3(s/2)}ds \\
&\quad - |f''(x)|^2\frac{1}{2\pi}\int_{|s|\leq\eta} K(s)h(s)s\frac{\sec^2(s/2)}{\tan^3(s/2)}ds \\
&\quad + \frac{1}{2\pi}\int_{|s|>\eta} K(s)(R_1[f'])(\delta_s f')\frac{\sec^2(s/2)}{\tan^3(s/2)}ds,
\end{aligned}
$$

using this decomposition we can estimate

$$
\begin{aligned}
|T_8 - T_{8,PV}| &\leq C(1+B^2)(D[f''] + (D[f''])^{1/2}|f''(x)|) \\
&\qquad \times \int_{|s|\leq\eta} |s|^{5/2}|\tan(s/2)|^{3/2}\frac{\sec^2(s/2)}{\tan^3(s/2)}ds \\
&\quad + C(1+B^2)((D[f''])^{1/2}|f''(x)| + |f''(x)|^2) \\
&\qquad \times \int_{|s|\leq\eta} s^2|\tan(s/2)|^{3/2}\frac{\sec^2(s/2)}{|\tan(s/2)|^3}ds \\
&\quad + C(D[f''])^{1/2}|f''(x)|\int_{|s|\leq\eta} |s|^{5/2}|\tan(s/2)|\frac{\sec^2(s/2)}{|\tan(s/2)|^3}ds \\
&\quad + C|f''(x)|^2\int_{|s|\leq\eta} s^2|\tan(s/2)|\frac{\sec^2(s/2)}{|\tan(s/2)|^3}ds \\
&\quad + CB^2\int_{|s|>\eta} |\tan(s/2)|\frac{\sec^2(s/2)}{|\tan(s/2)|^3}ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq C(1+B^2)(D[f''] \tan^2(\eta/2) + (D[f''])^{1/2}|f''(x)|| \tan(\eta/2)|^{3/2}) \\
&\quad +C(1+B^2)((D[f''])^{1/2}|f''(x)|| \tan(s/2)|^{3/2} + |f''(x)|^2 \tan(\eta/2)) \\
&\quad +C(D[f''])^{1/2}|f''(x)| \tan^{3/2}(\eta/2) \\
&\quad +C|f''(x)|^2 \tan(\eta/2) \\
&\quad +CB^2 \frac{1}{\tan(\eta/2)} \\
&\leq CB^2(1+B)^2 \left(\varepsilon \frac{D[f'']}{|f''(x)|^2} + \frac{1}{\varepsilon}|f''(x)| \right),
\end{aligned}
$$

and therefore we get

$$
|T_8 - T_{8,PV}| \leq CB^2(1+B)^2 \left(\varepsilon \frac{D[f'']}{|f''(x)|^2} + \varepsilon^{-1}|f''(x)| \right). \tag{3.107}
$$

Now notice that the term $T_{8,PV}$ is almost a fractional laplacian $\Lambda = (-\Delta)^{1/2}$, which from equation (1.12) can be written as

$$
2\Lambda f' = \frac{1}{2\pi} \int_{\mathbb{T}} (\delta_s f') \frac{\sec^2(s/2)}{\tan^2(s/2)} ds. \tag{3.108}
$$

therefore the only difference inside the integral is the extra $\text{sech}^2(\delta_s f/2)$ inside $T_{8,PV}$, then we can consider the difference between this two operators to get

$$
\begin{aligned}
T_{8,PV} - \frac{2f'(x)}{(1+(f'(x))^2)^2}\Lambda f' &= -\frac{1}{2\pi} \int_{\mathbb{T}} \tanh^2(\delta_s f/2)(\delta_s f') \frac{\sec^2(s/2)}{\tan^2(s/2)} ds \\
&= I_1 + I_2 + I_3 + I_{out},
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= -\frac{1}{2\pi}\int_{|s|\le\eta}\tanh(\delta_s f/2) \\
&\qquad \times\left(\tanh(\delta_s f/2) - sf'(x) + h_1(s)f''(x)\right) \\
&\qquad \times(\delta_s f')\frac{\sec^2(s/2)}{\tan^2(s/2)}ds \\
I_2 &= -\frac{1}{2\pi}\int_{|s|\le\eta}\tanh(\delta_s f/2)sf'(x)(\delta_s f')\frac{\sec^2(s/2)}{\tan^2(s/2)}ds \\
I_3 &= +f''(x)\frac{1}{2\pi}\int_{|s|\le\eta}\tanh(\delta_s f/2)h_1(s)(\delta_s f')\frac{\sec^2(s/2)}{\tan^2(s/2)}ds \\
I_{out} &= -\frac{1}{2\pi}\int_{|s|>\eta}\tanh^2(\delta_s f/2)(\delta_s f')\frac{\sec^2(s/2)}{\tan^2(s/2)}ds
\end{aligned}
$$

To estimate I_1 we use the following

$$
\begin{aligned}
|I_1| &= \left|\frac{1}{2\pi}\int_T \tanh(\delta_s f/2)(\tilde{R}_2)(\delta_s f')\frac{\sec^2(s/2)}{\tan^2(s/2)}ds\right| \\
&\le CB(1+B^2)\Big((D[f''])^{1/2}\int_{|s|\le\eta}|s|^{5/2}\frac{\sec^2(s/2)}{\tan^2(s/2)}ds \\
&\qquad +|f''(x)|\int_{|s|\le\eta}|s|^{5/2}\frac{\sec^2(s/2)}{\tan^2(s/2)}ds\Big) \\
&\le CB(1+B^2)\left((D[f''])^{1/2}\tan^{1/2}(\eta/2)+|f''(x)|\right) \\
&= CB(1+B)^{5/2}\left(\varepsilon^{1/2}\frac{(D[f''])^{1/2}}{|f''(x)|}|f''(x)|^{1/2}+|f''(x)|\right) \\
&= CB(1+B)^{5/2}\left(\varepsilon\frac{D[f'']}{|f''(x)|^2}+|f''(x)|\right).
\end{aligned}
$$

Now we proceed to estimate I_2

$$
\begin{aligned}
|I_2| &= \left|\frac{1}{2\pi}\int_{|s|\le\eta}\tanh(\delta_s f/2)sf'(x)(\delta_s f' - sf''(x)+sf''(x))\frac{\sec^2(s/2)}{\tan^2(s/2)}ds\right| \\
&\le CB\int_{|s|\le\eta}|s||R_1[f'']|\frac{\sec^2(s/2)}{\tan^2(s/2)}ds + CB|f''(x)|\int_{|s|\le\eta}s^2\frac{\sec^2(s/2)}{\tan^2(s/2)}ds \\
&\le CB(D[f''])^{1/2}\int_{|s|\le\eta}|s|^{3/2}\frac{\sec^2(s/2)}{\tan^2(s/2)}ds + CB|f''(x)| \\
&\le CB\left((D[f''])^{1/2}\tan^{1/2}(\eta/2)+|f''(x)|\right) \\
&\le CB(1+B)^{1/2}\left(\varepsilon\frac{D[f'']}{|f''(x)|^2}+|f''(x)|\right).
\end{aligned}
$$

Next we estimate I_3 with

$$\begin{aligned} |I_3| &\leq CB|f''(x)| \int_{|s|\leq\eta} |sh_1(s)| \frac{\sec^2(s/2)}{\tan^2(s/2)} ds \\ &\leq CB|f''(x)| \int_{|s|\leq\eta} |s|^3 \frac{\sec^2(s/2)}{\tan^2(s/2)} ds \\ &\leq CB|f''(x)|. \end{aligned}$$

And finally for I_{out}

$$\begin{aligned} |I_{out}| &= \left| \int_{|s|>\eta} \tanh^2(\delta_s f/2)(\delta_s f') \tfrac{\sec^2(s/2)}{\tan^2(s/2)} ds \right| \\ &\leq CB \int_{|s|>\eta} \tfrac{\sec^2(s/2)}{\tan^2(s/2)} ds \\ &= CB \tfrac{1}{\tan(\eta)} \\ &= C \tfrac{|f''(x)|}{\varepsilon}. \end{aligned}$$

By putting all together we get

$$|T_{8,PV} - \frac{f'(x)}{(1+(f'(x))^2)^2} \Lambda f'| \leq CB(1+B)^{5/2} \left(\varepsilon \frac{D[f'']}{|f''(x)|^2} + \frac{1}{\varepsilon} |f''(x)| \right). \quad (3.109)$$

Now notice that

$$\left| \frac{2f'(x)}{(1+(f'(x))^2)^2} \Lambda f' \right| \leq CB|\Lambda f'| = CB|Hf''|, \quad (3.110)$$

where $\mathcal{H}$ denotes the Hilbert transform and we use that $\Lambda f = \partial_x \mathcal{H} f$. Finally using (3.107), (3.109), (3.110) we conclude that

$$|T_8| \leq CB^{1/2}(1+B)^{5/2} \left(\varepsilon \frac{D[f'']}{|f''(x)|^2} + \varepsilon^{-2}|f(x)| + |Hf''| \right). \quad (3.111)$$

□

3.3 Proof of Theorem 3.1.1: Local existence

Proof of Theorem 3.1.1. The proof of the local existence will be done using classic energy method, in particular we will show that the energy given by

$$E(t) = 1 + \|f'(t)\|_{L^\infty}^2 + \|f''\|_{L^p}^2. \tag{3.112}$$

For this purpose we will study the equations of the first and second derivative of the equation, equations (3.16) and (3.24), to establish appropriate energy estimates that allow us to obtain that there exists $T = T(E(0)) > 0$ such that $E(t)$ is finite in $[0, T)$.

Most of the proof will be written depending on p, even though we have only proved the required lemmas for $p = 2$, this part of proof still work in the general case of $W^{2,p}$ instead of $W^{2,2}$.

Evolution of the Maximal Slope

The goal of this section is to study the evolution of the equation for the first derivative and use it to get information about the evolution of the maximum of the slope. For this purpose we consider equation (3.16)

$$(\partial_t + v\partial_x + \tilde{\mathcal{L}}_f)|f'|^2 + \tilde{D}_f[f'] = T_0,$$

where

$$T_0 = 2f'(\bar{x}, t)\frac{1}{2\pi}\int \frac{\tanh(\delta_s f/2)\mathrm{sech}^2(\delta_s f/2)\sec^2(s/2)(\delta_s f')}{(\tan(s/2) + \tanh^2(\delta_s f/2))^2} \times (\tanh(\delta_s f/2) - \tan(s/2)f'(x))\,ds.$$

Let $B(t) = \max_{x\in\mathbb{T}} |f'(x,t)|$ be the Lipschitz constant of f at time t and let $\bar{x}(t)$ be a point s.t $|f'(\bar{x}(t),t)| = B(t)$. Then by the Radamacher theorem (See Appendix in [11]) we can describe the evolution of $B(t)$ by

$$\frac{d}{dt}B(t)^2 = \partial|f'(\bar{x},t)|^2 = T_0 - \tilde{D}_f[f'] - \tilde{\mathcal{L}}|f'(\bar{x},t)|^2. \tag{3.113}$$

To estimate the right hand side we consider the following identity

$$\begin{aligned}(\delta_s g)^2 + \delta_s|g|^2 &= g(x)^2 - 2g(x)g(x-s) + g(x-s)^2 + g(x)^2 - g(x-s)^2 \\ &= 2g(x)^2 - 2g(x)g(x-s) \\ &= 2g(x)(g(x) - g(x-s)),\end{aligned}$$

we can use this to write the following splitting of T_0

$$\begin{aligned}T_0 &= \frac{1}{2\pi}\int_{|s|\le\varepsilon} A(s)\frac{\operatorname{sech}^2(\delta_s f/2)\sec^2(s/2)}{\tan^2(s/2)+\tanh^2(\delta_s f)}(\delta_s f')^2\frac{R_1[f']}{\tan(s/2)}ds \\ &\quad+\frac{1}{2\pi}\int_{|s|\le\varepsilon} A(s)\frac{\operatorname{sech}^2(\delta_s f/2)\sec^2(s/2)}{\tan^2(s/2)+\tanh^2(\delta_s f)}\delta_s|f'|^2\frac{R_1[f']}{\tan(s/2)}ds \\ &\quad+2f'(x,t)\frac{1}{2\pi}\int_{|s|>\varepsilon} A(s)\frac{\operatorname{sech}^2(\delta_s f/2)\sec^2(s/2)}{\tan^2(s/2)+\tanh^2(\delta_s f)}(\delta_s f')\frac{R_1[f']}{\tan(s/2)}ds \\ &= I_{1,in} + I_{2,in} + I_{out},\end{aligned}$$

where $A(s) = \frac{\frac{\tanh(\delta_s f/2)}{\tan(s/2)}}{\left(1+\frac{\tanh^2(\delta_s f/2}{\tan^2(s/2)}\right)}$, then $|A(s)| \le 1$. By using Lemma 3.2.1 parts (b) and (c), and noting that because $\bar{x}(t)$ is a point where the maximum of $|f'(x)|$ is attained, then $\delta_s|f'|^2(\bar{x}(t)) \ge 0$, then we get that

$$\begin{aligned}|I_{1,in}| &\le C(1+B)\|f''\|_{L^p}\varepsilon^{\frac{p-1}{p}}(1+\tan(\varepsilon/2))\tilde{D}_f[f'] \\ |I_{2,in}| &\le C(1+B)\|f''\|_{L^p}\varepsilon^{\frac{p-1}{p}}(1+\tan(\varepsilon/2))\tilde{\mathcal{L}}_f|f'|^2 \\ |I_{out}| &\le \frac{32B^3}{2\pi}\frac{1}{\tan(\varepsilon/2)}.\end{aligned}$$

We get that the T_0 can be bounded by

$$|T_0| \leq 2C(1+B)\|f''\|_{L^p}\varepsilon^{(p-1)/p}(1+\tan(\varepsilon/2))\left(\tilde{D}_f[f'](\bar{x})+\tilde{\mathcal{L}}_f|f'|^2\right) \\ +\frac{32B^3}{2\pi}\frac{1}{\tan(\varepsilon/2)}. \quad (3.114)$$

Now because $|x| \leq |\tan(x)|$ for $|x| < \pi/2$ we can bound

$$\varepsilon^{(p-1)/p}(1+\tan(\varepsilon/2)) \leq \begin{cases} 2^{(2p-1)/p}\tan^{(p-1)/p}(\varepsilon/2) & \text{if} \quad \tan(\varepsilon/2) \leq 1 \\ 2^{(2p-1)/p}\tan^{(2p-1)/p}(\varepsilon/2) & \text{if} \quad \tan(\varepsilon/2) > 1 \end{cases}, \quad (3.115)$$

Not that the right hand side is a continuous monotone function in ε. Now, if we call $g(\varepsilon)$ such upper bound, we want is to choose ε such that

$$g(\varepsilon)2C(1+B)\|f''\|_{L^p} = g(\varepsilon)A = 1, \quad (3.116)$$

and we can do that by taking

$$\tan(\varepsilon/2) = \begin{cases} \dfrac{2^{-(2p-1)/(p-1)}}{A^{p/(p-1)}} & , \quad \dfrac{1}{A} \leq 2^{2p-1} \\ \dfrac{1}{2A^{p/(2p-1)}} & , \quad \dfrac{1}{A} > 2^{2p-1} \end{cases}. \quad (3.117)$$

we get that

$$|T_0(\bar{x})| \leq (\tilde{D}_f[f'](\bar{x})+\tilde{\mathcal{L}}_f|f'(\bar{x})|^2)+\frac{32B^3}{2\pi}\frac{1}{\tan(\varepsilon/2)}. \quad (3.118)$$

Now using that

$$\frac{1}{\tan(\varepsilon/2)} \leq 2^{-(2p-1)/(p-1)}A^{p/(p-1)}+2^{-1}A^{p/(2p-1)}, \quad (3.119)$$

we conclude that

$$
\begin{aligned}
\frac{d}{dt}B(t)^2 &\leq \frac{32}{2\pi}2^{-\frac{2p-1}{p-1}}B(t)^3(C2^{(3p-1)/p}(1+B(t)^2)\|f''\|_{L^p})^{p/(p-1)} \\
&\quad +\frac{16}{2\pi}B(t)^3(C2^{(3p-1)/p}(1+B(t)^2)\|f''\|_{L^p})^{p/(2p-1)} \\
&\leq CB(t)^3(1+B(t)^2)^{p/(p-1)}(1+B(t)^2+M_p(t)^2)^{p/(2p-2)} \\
&\quad +CB(t)^3(1+B(t)^2)^{p/(2p-1)}(1+B(t)^2+M_p(t)^2)^{p/(4p-2)} \\
&\leq C(1+B^2+M_p(t)^2)^{3+\frac{3p}{2p-2}} \\
&= C(1+B^2+M_p(t)^2)^{\frac{5p-2}{2p-2}},
\end{aligned}
\tag{3.120}
$$

where $M_p(t) = \|f''\|_{L^p}$.

Evolution of the norm of the second derivative

Consider the equation for the evolution of $|f''|^2$ in divergence form given by equation (3.24), apply the upper bound given by Lemma 3.2.7 for the terms $T_1, \cdots, T_8$ on the right hand side of (3.24), and the lower bound given by equation (3.54) to get that the following equation is valid for $p = 2$

$$
\begin{aligned}
&(\partial_t + \mathcal{L}_f)|f''(x,t)|^p + \partial_x(v(x,t)|f''(x,t)|^p) \\
&\qquad + \frac{1}{2(1+B(t)^2)}|f''|^{p-2}D[f''](x,t) + \frac{1}{2}|f''|^{p-2}D_f[f''](x,t) \\
&\qquad \leq C_1 B(t)(1+B(t)^2)^2\Big(|f''(x,t)|^p|Hf''| + \frac{1}{\varepsilon(t)^2}|f''(x,t)|^{p+1} \\
&\qquad\qquad + \varepsilon(t)|f''(x,t)|^{p-2}D[f''](x,t)\Big).
\end{aligned}
\tag{3.121}
$$

Now choose $\varepsilon(t) = \min\{\frac{1}{4C_1B(t)(1+B(t)^2)^3}, 1\}$ to get

$$
\begin{aligned}
&(\partial_t + \mathcal{L}_f)|f''(x,t)|^p + \partial_x(v(x,t)|f''(x,t)|^p) \\
&\qquad + \frac{1}{4(1+B(t)^2)}|f''|^{p-2}D[f''](x,t) + \frac{1}{2}|f''|^{p-2}D_f[f''](x,t)\cdot 1_A(x) \\
&\qquad\qquad \leq C_1B(t)(1+B(t)^2)^2\Big(|f''(x,t)|^p|Hf''| \\
&\qquad\qquad\qquad + 16C_1^2B(t)^2(1+B(t)^2)^6|f''(x,t)|^{p+1}\Big), \quad (3.122)
\end{aligned}
$$

where $A = \left\{x : \frac{|f''(x)|}{\|f''\|_{L^p}} \geq \left(\frac{2}{\pi C_p}\right)^{1/p}\right\}$. Applying Lemma 3.2.5 and defining $M_p(t) = \|f''(t)\|_{L^p}$, we obtain by integrating (3.122)

$$
\begin{aligned}
&\frac{d}{dt}M_p(t)^p + \frac{1}{4(1+B(t)^2)}\int_{\mathbb{T}}|f''|^{p-2}D[f''](x,t) \\
&\qquad + \frac{C_p}{2}\frac{4}{1+B(t)^2}\frac{1}{M_p(t)^p}\int_A|f''(x,t)|^{2p}dx \\
&\qquad \leq C_1B(t)(1+B(t)^2)^2\Big(\int_{\mathbb{T}}|f''(x,t)|^p|Hf''|dx \\
&\qquad\qquad + 16C_1^2B(t)^2(1+B(t)^2)^6M_{p+1}(t)^{p+1}\Big), \quad (3.123)
\end{aligned}
$$

here we used that

$$
\int_{\mathbb{T}}\mathcal{L}_f[|f''|^p](x)dx = p.v.\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{\sec^2(s/2)(|f''(x)|^p - |f''(x-s)|^p)}{\tan^2(s/2)+\tanh^2(\delta_s f/2)}dsdx = 0, \quad (3.124)
$$

to see this is is enough to make the change of variables $(x,s) \to (y-t, -t)$. Now in $\mathbb{T}\setminus A$ we know that

$$
\frac{|f''(x,t)|}{\|f''\|_{L^p}} \leq \frac{1}{C_p^{1/p}}\left(\frac{2}{\pi}\right)^{1/p} \quad (3.125)
$$

and therefore

$$
\frac{\int_{\mathbb{T}\setminus A}|f''(x,t)|^{2p}}{\|f''\|_{L^p}^p} \leq CM_p(t)^p \leq C_2M_{p+1}(t)^p \quad (3.126)
$$

now we can add inequalities (3.123) and (3.126) to get

$$\begin{aligned}\frac{d}{dt}M_p(t)^p + \frac{2C_p}{1+B(t)^2}\frac{M_{2p}(t)^{2p}}{M_p(t)^p} &\leq C_1B(t)(1+B(t)^2)^2\int_{\mathbb{T}}|f''(x,t)|^p|Hf''|dx \\ &\quad +16C_1^3B(t)^3(1+B(t)^2)^8M_{p+1}(t)^{p+1} \\ &\quad +\frac{C_p}{2}\frac{4}{1+B(t)^2}C_2M_{p+1}(t)^p.\end{aligned} \tag{3.127}$$

Now using Hölder inequality and that the boundedness of the Hilbert transform in $L^p(\mathbb{T})$ we get that

$$\int_{\mathbb{T}}|f''|^p|Hf''|dx \leq C\|f''\|_{L^{p+1}}^{p+1} = CM_{p+1}(t)^{p+1}, \tag{3.128}$$

and so we can bound the right hand side of (3.127) as

$$\begin{aligned}\frac{d}{dt}M_p(t)^p + \frac{2C_p}{1+B(t)^2}\frac{M_{2p}(t)^{2p}}{M_p(t)^p} &\leq CB(t)^3(1+B(t)^2)^8M_{p+1}(t)^{p+1} \\ &\quad +\frac{C}{1+B(t)^2}M_{p+1}(t)^p \qquad (3.129) \\ &= C(I_1+I_2).\end{aligned}$$

Now since $p>1$ we have that $p+1<2p$ and so we may interpolate

$$M_{p+1}(t) \leq M_p(t)^{(p-1)/(p+1)}M_{2p}(t)^{2/(p+1)}, \tag{3.130}$$

then we get

$$I_1 = B(t)^3\left(1+B(t)^2\right)^8M_{p+1}(t)^{(p+1)} \leq g(B)M_p(t)^p\left(\frac{M_{2p}(t)^2}{M_p(t)}\right). \tag{3.131}$$

Now by the Young's Inequality we get know that

$$ab \leq \frac{p-1}{p}\frac{1}{\varepsilon^{1/(p-1)}}a^{p/(p-1)} + \frac{1}{p}\varepsilon b^p, \tag{3.132}$$

using this we get

$$\begin{aligned} I_1 &\le \frac{p-1}{p}\frac{g(B)^{p/(p-1)}}{\left(\frac{pC_p}{1+B^2}\right)^{1/(p-1)}} M_p(t)^{p^2/(p-1)} + \frac{C_p}{1+B^2}\left(\frac{M_{2p}(t)^2}{M_p(t)}\right)^p \\ &= \frac{p-1}{p^{p/(p-1)}C_p^{1/(p-1)}} B^{3p/(p-1)}(1+B^2)^{(8p+1)/(p-1)} M_p(t)^{p^2/(p-1)} \\ &\qquad + \frac{C_p}{1+B^2}\left(\frac{M_{2p}(t)^2}{M_p(t)}\right)^p. \end{aligned} \tag{3.133}$$

Analogously for I_2

$$I_2 = \frac{1}{1+B^2}M_{p+1}(t)^p \le h(B)M_p(t)^{p^2(p+1)}\left(\frac{M_{2p}(t)^2}{M_p(t)}\right)^{p/(p+1)}, \tag{3.134}$$

as before we apply the Young's Inequality

$$ab \le \frac{p}{p+1}\frac{1}{\varepsilon^{1/p}}a^{(p+1)/p} + \frac{1}{p+1}\varepsilon b^{(p+1)}, \tag{3.135}$$

to conclude

$$\begin{aligned} I_2 &\le \frac{p}{p+1}\frac{h(B)^{(p+1)/p}}{\left(\frac{(p+1)C_p}{1+B^2}\right)^{1/p}} M_p(t)^p + \frac{C_p}{1+B^2}\left(\frac{M_{2p}(t)^2}{M_p(t)}\right)^p \\ &= \frac{p}{(p+1)^{(p+1)/p}C_p^{1/p}}\frac{1}{1+B^2}M_p(t)^p + \frac{C_p}{1+B^2}\left(\frac{M_{2p}(t)^2}{M_p(t)}\right)^p. \end{aligned} \tag{3.136}$$

Replacing this (3.133) and (3.136) in (3.129) we get

$$\frac{d}{dt}M_p(t)^p \le CB^{3p/(p-1)}(1+B^2)^{(8p+1)/(p-1)}M_p(t)^{p^2/(p-1)} + CM_p(t)^p.$$

Now using that $\frac{d}{dt}M_p(t)^p = \frac{p}{2}M_p(t)^{p-2}\frac{d}{dt}M_p(t)^2$ we get

$$\frac{d}{dt}M_p(t)^2 \le CB^{3p/(p-1)}(1+B^2)^{(8p+1)/(p-1)}(M_p(t)^2)^{(3p-2)/(2p-2)} + CM_p(t)^2.$$

Finally by bounding $B^2 \le (1+B^2+M_p^2)$, $M_p^2 \le (1+B^2+M_p^2)$ we get

$$\frac{d}{dt}M_p(t)^2 \le C(1+B(t)^2+M_p(t)^2)^{11p/(p-1)}$$

which together with (3.120)

$$\frac{d}{dt}(B(t)^2 + M_p(t)^2) \leq C(1 + B(t)^2 + M_p(t)^2)^{11p/(p-1)}, \tag{3.137}$$

for some positive constant C. Integrating, we obtain that there exists $T = T(\|f_0'\|_{L^\infty}, \|f_0''\|_{L^p}) > 0$ for which the energy

$$E(t) = 1 + B(t) + M_p(t)^2, \tag{3.138}$$

stays finite and therefire by energy methods it can be show that a solution for (3.14) with finite $W^{2,p}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$ norm. This concludes the proof of Theorem 3.1.1. □

3.4 A Maximum principle for first derivative: Proof of Lemma 3.1.2

The goal of this section is to prove one of the key ingredients in the proof of global existence result, which is a bound that is uniform in time for the slope of the solution. In the proof of the global existence result, we need to show that under appropriate conditions the energy of the equation remain bounded for all time, and therefore the solution can be extended for all time, and a key ingredient for that estimate is that if the initial maximum slope is small enough, then that condition is preserved for all time.

Proof of Lemma 3.1.2. The strategy to prove that the maximum slope is decreasing will be the following. For fixed $t \in [0,T]$ we consider a point $\bar{x}(t)$ at which the first derivative achieves a maximum or minimum, by the Radamacher's theorem if $M(t)$ is the value of the maximum at time t, then $M(t)$ satisfy

$$\frac{d}{dt}M(t) = \partial_t f'(\bar{x}(t), t). \tag{3.139}$$

Because of this our goal is to show that in the time direction that value can only decrease (respectively increase) if the size of the initial slope was small enough initially. For this purpose consider equation (3.2) and the change of variables $s \to x - s$ in equation (3.14)

$$f_t = \frac{1}{2\pi} p.v. \int_{\mathbb{T}} \frac{f'(x)\tan\left(\frac{x-s}{2}\right)\operatorname{sech}^2\left(\frac{f(x)-f(s)}{2}\right) - \tanh\left(\frac{f(x)-f(s)}{2}\right)\sec^2\left(\frac{x-s}{2}\right)}{\tan^2\left(\frac{x-s}{2}\right) + \tanh^2\left(\frac{f(x)-f(s)}{2}\right)} ds. \tag{3.140}$$

Taking the spatial derivative of the equation and returning to the original variables in the integral

$$\begin{aligned} f_t' \;=\; & \frac{1}{2\pi}\int \frac{f''(x)\tan(s/2)\operatorname{sech}^2\left(\frac{\delta_s f}{2}\right)}{\tan^2(s/2) + \tanh^2\left(\frac{\delta_s f}{2}\right)} ds \\ & + \frac{1}{2\pi}\int \frac{-f'(x)^2\tan(s/2)\tanh\left(\frac{\delta_s f}{2}\right)\operatorname{sech}^2\left(\frac{\delta_s f}{2}\right)}{\tan^2(s/2) + \tan^2\left(\frac{\delta_s f}{2}\right)} ds \\ & + \frac{1}{2\pi}\int \frac{-\tanh\left(\frac{\delta_s f}{2}\right)\tan(s/2)\sec^2(s/2)}{\tan^2(s/2) + \tan^2\left(\frac{\delta_s f}{2}\right)} ds \\ & - \frac{1}{2\pi}\int \frac{\left(f'(x)\tan(s/2)\operatorname{sech}^2\left(\frac{\delta_s f}{2}\right) - \tanh\left(\frac{\delta_s f}{2}\right)\sec^2(s/2)\right)}{\left(\tan^2(s/2) + \tan^2\left(\frac{\delta_s f}{2}\right)\right)^2} \\ & \quad \times \left(\tan(s/2)\sec^2(s/2) + \tanh\left(\frac{\delta_s f}{2}\right)\operatorname{sech}^2\left(\frac{\delta_s f}{2}\right) f'(x)\right) ds. \end{aligned} \tag{3.141}$$

Because at a maximum or a minimum of the first derivative $f''(x) = 0$, the second term has an appropriate sign, so we only need to show that the last two terms are non positive (respectively. non negative) when $f'(x)$ is small. We write

$$\begin{aligned} I_1 &= f'(x)\tan(s/2)\operatorname{sech}^2\left(\frac{\delta_s f}{2}\right) - \tanh\left(\frac{\delta_s f}{2}\right)\sec^2(s/2) \\ &= \left(f'(x)\tan(s/2)\operatorname{sech}^2\left(\frac{\delta_s f}{2}\right) - \tanh\left(\frac{\delta_s f}{2}\right)\operatorname{sech}^2\left(\frac{\delta_s f}{2}\right)\right) \\ &\quad + \left(\tanh\left(\frac{\delta_s f}{2}\right)\operatorname{sech}^2\left(\frac{\delta_s f}{2}\right) - \tanh\left(\frac{\delta_s f}{2}\right)\sec^2(s/2)\right) \\ &= \tan(s/2)\operatorname{sech}^2\left(\frac{\delta_s f}{2}\right)\left(f'(x) - \frac{\tanh\left(\frac{\delta_s f}{2}\right)}{\tan(s/2)}\right) \\ &\quad + \left(\tanh\left(\frac{\delta_s f}{2}\right)\operatorname{sech}^2\left(\frac{\delta_s f}{2}\right) - \tanh\left(\frac{\delta_s f}{2}\right)\sec^2(s/2)\right). \end{aligned} \tag{3.142}$$

Now because f is periodic, we know that $\int_{\mathbb{T}} f'(t)dt = 0$ which imply that $\max_{\mathbb{T}} f' \geq 0$, and $\min_{\mathbb{T}} f' \leq 0$. Now by the mean value theorem we can write

$$\tanh\left(\frac{\delta_s f}{2}\right) = \frac{s}{2}\operatorname{sech}^2\left(\frac{\delta_\xi f}{2}\right) f'(x-\xi), \tag{3.143}$$

for some $\xi \in [x, s]$. Now if $\frac{\tanh(\delta_s f/2)}{s/2} \geq 0$ we can bound

$$0 \leq \frac{\tanh(\delta_s f/2)}{\tan(s/2)} \leq \frac{\tanh(\delta_s f/2)}{s/2} = \operatorname{sech}^2\left(\frac{\delta_\xi f}{2}\right) f'(\xi) \leq \max f'. \tag{3.144}$$

Analogously when $\frac{\tanh(\delta_s f/2)}{s/2} \leq 0$ we get

$$\min f' \leq \frac{\tanh(\delta_s f/2)}{s/2} \leq 0. \tag{3.145}$$

Putting this two fact together we get that

- $f'(x) - \frac{\tanh(\delta_s f/2)}{\tan(s/2)} \geq 0$ at the maximum of f' and
- $f'(x) - \frac{\tanh(\delta_s f/2)}{\tan(s/2)} \leq 0$ at the minimum of f'.

Now we can write equation (3.141) as

$$f'_t = N_1 + N_2 + N_3, \tag{3.146}$$

where

$$
\begin{aligned}
N_1 &= \frac{1}{2\pi}\int \frac{f''(x)\tan(s/2)\,\mathrm{sech}^2\left(\frac{\delta_s f}{2}\right)}{\tan^2(s/2)+\tanh^2\left(\frac{\delta_s f}{2}\right)}ds \\
N_2 &= -\frac{1}{2\pi}\int \frac{\tan^2(s/2)\,\mathrm{sech}^2\left(\frac{\delta_s f}{2}\right)\left(f'(x)-\frac{\tanh\left(\frac{\delta_s f}{2}\right)}{\tan(s/2)}\right)}{\left(\tan^2(s/2)+\tan^2\left(\frac{\delta_s f}{2}\right)\right)^2}ds \\
&\quad\times\left(\sec^2(s/2)+\frac{\tanh\left(\frac{\delta_s f}{2}\right)}{\tan(s/2)}\mathrm{sech}^2\left(\frac{\delta_s f}{2}\right)f'(x)\right)ds
\end{aligned}
\tag{3.147}
$$

$$
\begin{aligned}
N_3 &= \frac{1}{2\pi}\int \frac{-f'(x)^2\tan(s/2)\tanh\left(\frac{\delta_s f}{2}\right)\mathrm{sech}^2\left(\frac{\delta_s f}{2}\right)}{\tan^2(s/2)+\tanh^2\left(\frac{\delta_s f}{2}\right)}ds \\
&\quad -\frac{1}{2\pi}\int \frac{\tanh\left(\frac{\delta_s f}{2}\right)\tan(s/2)\sec^2(s/2)}{\tan^2(s/2)+\tanh^2\left(\frac{\delta_s f}{2}\right)}ds \\
&\quad -\frac{1}{2\pi}\int \frac{\left(\tanh\left(\frac{\delta_s f}{2}\right)\mathrm{sech}^2\left(\frac{\delta_s f}{2}\right)-\tanh\left(\frac{\delta_s f}{2}\right)\sec^2(s/2)\right)}{\left(\tan^2(s/2)+\tanh^2\left(\frac{\delta_s f}{2}\right)\right)^2} \\
&\quad\quad\times\left(\tan(s/2)\sec^2(s/2)+\tanh\left(\frac{\delta_s f}{2}\right)\mathrm{sech}^2\left(\frac{\delta_s f}{2}\right)f'(x)\right)ds.
\end{aligned}
\tag{3.148}
$$

We know that at the point where the first derivative reaches its maximum or minimum $N_1 = 0$. Now we want to show that for $\|f'\|_{L^\infty}$ small enough, then $N_2 + N_3$ we can tell its sign. To do this first notice that

$$
\mathrm{sech}^2\left(\frac{\delta_s f}{2}\right)-\sec^2(s/2) = -\tanh^2\left(\frac{\delta_s f}{2}\right)-\tan^2(s/2), \tag{3.149}
$$

using this identity we get that N_3 can be written as

$$\begin{aligned}
N_3 &= \frac{1}{2\pi}\int \frac{-f'(x)^2\tan(s/2)\tanh\left(\frac{\delta_s f}{2}\right)\operatorname{sech}^2\left(\frac{\delta_s f}{2}\right)}{\tan^2(s/2)+\tan^2\left(\frac{\delta_s f}{2}\right)}ds \\
&\quad -\frac{1}{2\pi}\int \frac{\tanh\left(\frac{\delta_s f}{2}\right)\tan(s/2)\sec^2(s/2)}{\tan^2(s/2)+\tan^2\left(\frac{\delta_s f}{2}\right)}ds \\
&\quad +\frac{1}{2\pi}\int \frac{\tanh\left(\frac{\delta_s f}{2}\right)}{\tan^2(s/2)+\tanh^2\left(\frac{\delta_s f}{2}\right)} \\
&\qquad \times\left(\tan(s/2)\sec^2(s/2)+\tanh\left(\frac{\delta_s f}{2}\right)\operatorname{sech}^2\left(\frac{\delta_s f}{2}\right)f'(x)\right)ds \\
&= \frac{1}{2\pi}\int \frac{\tanh\left(\frac{\delta_s f}{2}\right)\tan(s/2)\operatorname{sech}^2\left(\frac{\delta_s f}{2}\right)f'(x)\left(-f'(x)+\frac{\tanh\left(\frac{\delta_s f}{2}\right)}{\tan(s/2)}\right)}{\tan^2(s/2)+\tanh^2\left(\frac{\delta_s f}{2}\right)}ds
\end{aligned} \tag{3.150}$$

using this we obtain that N_2+N_3 can be written as

$$N_2+N_3=-\frac{1}{2\pi}\int \frac{\tan^2(s/2)\operatorname{sech}^2\left(\frac{\delta_s f}{2}\right)\left(f'(x)-\frac{\tanh(s/2)}{\tan(s/2)}\right)A(s)}{\left(\tan^2(s/2)+\tanh^2\left(\frac{\delta_s f}{2}\right)\right)^2}ds, \tag{3.151}$$

where

$$\begin{aligned}
A(s) &= \sec^2(s/2)+\frac{\tanh\left(\frac{\delta_s f}{2}\right)}{\tan(s/2)}\operatorname{sech}^2\left(\frac{\delta_s f}{2}\right)f'(x) \\
&\quad +f'(x)\frac{\tanh\left(\frac{\delta_s f}{2}\right)}{\tan(s/2)}\left(\tan^2(s/2)+\tanh^2\left(\frac{\delta_s f}{2}\right)\right) \\
&= \sec^2(s/2)+f'(x)\frac{\tanh\left(\frac{\delta_s f}{2}\right)}{\tan(s/2)}+f'(x)\tan(s/2)\tanh\left(\frac{\delta_s f}{2}\right) \\
&\geq \sec^2(s/2)-\|f'(x)\|^2_{L^\infty}-\|f'(x)\|_{L^\infty}\tan(s/2),
\end{aligned} \tag{3.152}$$

then by writing

$$\begin{aligned}
\sec^2(s/2)-y^2-y\tan(s/2) &= 1+\tan^2(s/2)-y^2-y\tan(s/2) \\
&= (\tan(s/2)-y/2)^2-\tfrac{5}{4}y^2+1\geq 0.
\end{aligned} \tag{3.153}$$

We see that this quantity is positive for all $s\in(-\pi,\pi)$ if $\|f'\|_{L^\infty}\leq\frac{2\sqrt{5}}{5}$. Therefore we obtain that at the maximum $x=\bar{x}(t)$

$$f'_t(\bar{x}(t),t)=N_2+N_3\leq 0, \tag{3.154}$$

analogously at the minimum $\tilde{x}(t)$ we obtain

$$f'_t(\tilde{x}(t), t) \geq 0, \tag{3.155}$$

and therefore

$$\|f'(t)\|_{L^\infty} \leq \|f'_0\|_{L^\infty}. \tag{3.156}$$

This concludes the proof of Lemma 3.1.2. □

3.5 Proof of Theorem 3.1.3: Global existence

The basic idea of this proof is very similar to the proof of the local existence in Section 3.3. The idea is to show that the energy give by (3.112) is bounded for all time. For this purpose we use the maximum principle for the derivative to conclude that $\|f'\|_{L^\infty}$ will be bounded for all time. Then we use the equation for the second derivative to get that if the slope is small enough, then the equation cannot blow up and conclude using energy methods.

Proof of Theorem 3.1.3. First by the maximum principle Lemma 3.1.2, we know that if $\|f'_0\|_{L^\infty} < \frac{2\sqrt{5}}{5}$ then

$$B(t) = \|f'(t)\|_{L^\infty} \leq \|f'_0\|_{L\infty} \tag{3.157}$$

for all $t > 0$, and therefore if the slope is small initially then we can control the slope for all time. Consider set $A = \{x : |f''(x)| \geq 256B/\pi\}$. Then from the equation for the second derivative in divergence form equation (3.24), the bounds in Lemma

3.2.7, and the nonlinear lower bound in Lemma 3.2.4 we get

$$(\partial_t + \mathcal{L}_f)\,|f''(x,t)|^2 + \partial_x(v|f''|^2) + \frac{1}{2(1+B^2)}D[f''] + \frac{1}{64(1+B^2)}|f''(x)|^3 \cdot 1_A(x)$$
$$\leq CB(1+B^2)^2\varepsilon^{-2}|f''(x)|^3 + CB(1+B^2)^2\varepsilon D[f'']$$
$$+ CB(1+B^2)^2|f''(x)|^2|Hf''|. \quad (3.158)$$

Now we choose $\varepsilon = \min\{\frac{1}{4CB(1+B^2)^3}, 1\}$, then we have

$$(\partial_t + \mathcal{L}_f)\,|f''(x,t)|^2 + \partial_x(v|f''|^2)$$
$$+ \frac{1}{2(1+B^2)}D[f''] + \frac{1}{64(1+B^2)}|f''(x)|^3 \cdot 1_A(x)$$
$$\leq 16C^3B^3(1+B^2)^8|f''(x)|^3 + CB(1+B^2)^2|Hf''||f''(x)|^2, \quad (3.159)$$

where 1_A is the characteristic function of the set A. Now because in A^c we have that $|f''(x)| < \frac{256B}{\pi}$, then we can bound

$$\frac{1}{1+B^2}|f''(x)|^3 \leq \left(\frac{256B}{\pi}\right)^\alpha |f''(x)|^{3-\alpha}. \quad (3.160)$$

Adding equations (3.159) and (3.160) we get

$$(\partial_t + \mathcal{L}_f)\,|f''(x,t)|^2 + \partial_x(v|f''|^2) + \frac{1}{2(1+B^2)}D[f''] + \frac{1}{64(1+B^2)}|f''(x)|^3$$
$$\leq 16C^3B^3(1+B^2)^8|f''(x)|^3 + CB(1+B^2)^2|Hf''||f''(x)|^2$$
$$+ \left(\frac{256B}{\pi}\right)^\alpha |f''(x)|^{3-\alpha}. \quad (3.161)$$

For the integral of $\mathcal{L}_f|f''|^2$ we use (3.124). Now we integrate and use the boundedness of the Hilbert transform in L^p to get

$$\frac{d}{dt}M_2(t)^2 + \frac{1}{4(1+B^2)}\int_{\mathbb{T}} D[f''] + \frac{1}{64(1+B^2)}M_3(t)^3 \\ \leq C(B^3(1+B^2)^8 + B(1+B^2)^2)M_3(t)^3 + \left(\frac{256B}{\pi}\right)^{\alpha} M_{3-\alpha}(t)^{3-\alpha}, \quad (3.162)$$

where $M_p(t) = \|f''(t)\|_{L^p}$. We choose B small enough such that

$$C(B^3(1+B^2)^8 + B(1+B^2)^2) \leq \frac{1}{128(1+B^2)}, \quad (3.163)$$

and so we get

$$\frac{d}{dt}M_2(t)^2 + \frac{1}{4(1+B^2)}\int_{\mathbb{T}} D[f''] + \frac{1}{128(1+B^2)}M_3(t)^3 \\ \leq \left(\frac{256B}{\pi}\right)^{\alpha} M_{3-\alpha}(t)^{3-\alpha}. \quad (3.164)$$

By taking $\alpha = 1$, and using that $M_3(t) \geq \frac{1}{(2\pi)^{1/3}}M_2(t)$ we get

$$\frac{d}{dt}M_2(t)^2 + C_B M_2(t)^3 \leq \epsilon_B M_2(t)^2, \quad (3.165)$$

where $C_B = \frac{1}{128(1+B^2)(2\pi)^{1/3}}$ and $\epsilon_B = \frac{256B}{\pi}$. From this we get that if $M(0) = 0$, then $\frac{d}{dt}M(0) \leq 0$ and therefore $M(t)$ is constant equal to zero, which imply that f is constant, and because f has zero mean, we conclude that $f = 0$. Because of this in what follows we can assume that $M(t) > 0$. Equation (3.165) is a differential inequality that looks like the Riccati equation. Because we know that $M_2(t) \geq 0$, we can consider the change of variable $N(t) = \frac{1}{M(t)}$, replacing this we get

$$\frac{d}{dt}\left(\frac{1}{N(t)}\right) + C_B\frac{1}{N(t)^2} - \epsilon\frac{1}{N(t)} \leq 0$$

$$-\frac{1}{N(t)^2}\frac{d}{dt}N(t) + C_B\frac{1}{N(t)^2} - \epsilon\frac{1}{N(t)} \leq 0$$
$$\frac{1}{N(t)}\left(-\frac{d}{dt}N(t) + C_B - \epsilon N(t)\right) \leq 0$$
$$C_B \leq \frac{d}{dt}N(t) + \epsilon_B N(t)$$
$$C_B \leq e^{-\epsilon_B t}\frac{d}{dt}e^{\epsilon_B t}N(t)$$
$$C_B e^{\epsilon_B t} \leq \frac{d}{dt}e^{\epsilon_B t}N(t),$$

integrating we obtain

$$\frac{1}{\epsilon_B}C_B\left(e^{\epsilon_B t} - 1\right) \leq e^{\epsilon_B}N(t) - N(0).$$

Returning to $M(t)$

$$\frac{C_B}{\epsilon_B}\left(e^{\epsilon_B t} - 1\right) + \frac{1}{M(0)} \leq e^{\epsilon_B t}\frac{1}{M(t)} \tag{3.166}$$
$$M(t) \leq M(0)\frac{e^{\epsilon_B t}}{M(0)\frac{C_B}{\epsilon_B}(e^{\epsilon_B t} - 1) + 1} \tag{3.167}$$

To understand the right hand side we compute the derivative to get

$$\frac{d}{dt}\frac{e^{\epsilon_B t}}{M(0)\frac{C_B}{\epsilon_B}(e^{\epsilon_B t} - 1) + 1} = \frac{e^{\epsilon_B t}(\epsilon_B - C_B M(0))}{\left(M(0)\frac{C_B}{\epsilon_B}(e^{\epsilon_B t} - 1) + 1\right)^2} \tag{3.168}$$

Therefore the right hand side is increasing when $C_B M(0) \leq \epsilon_B$ decreasing otherwise and the limit value as $t \to \infty$ is $\frac{\epsilon}{C_B}$, therefore we can conclude that

$$M(t) \leq \max\{M(0), \frac{\epsilon}{C_B}\}. \tag{3.169}$$

Finally we obtain that if the slope satisfies (3.163) then the energy $E(t) = 1 + \|f'\|^2_{L^\infty} + \|f''\|^2_{L^2}$ is finite for all time which implies that the local solution given by Theorem 3.1.1 can be extended for all time, which concludes the proof of the Theorem 3.1.3. □

3.6 Uniqueness for C^1 solutions: Proof of Theorem 3.1.4

Proof of Theorem 3.1.4. We want to show that if $f_1, f_2 \in C^0([0,T]; C^1(\mathbb{T}))$ are two solutions of such that $\partial_t f_i$, $i = 1,2$ exists for all $(x,t) \in [0,T] \times \mathbb{T}$ and $\|f_i'\|_{L^\infty} \leq B$, $i = 1,2$ such that if they agree initially, then they must agree for all time. From equation (3.14) we know that f_i, $i = 1,2$ satisfy

$$\partial_t f_i + v_i \partial_x f_i + \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\tanh(\delta_s f_i/2) \sec^2(s/2)}{\tan^2(s/2) + \tanh^2(\delta_s f_i/2)} ds = 0, \tag{3.170}$$

where

$$v_i = -\frac{1}{2\pi} p.v. \int_{\mathbb{T}} \frac{\tan(s/2) \mathrm{sech}^2(\delta_s f_i/2)}{\tan^2(s/2) + \tanh^2(\delta_s f_i/2)} ds. \tag{3.171}$$

We get an equation for $g = f_1 - f_2$ by subtracting the equations for f_1 and f_2

$$\begin{aligned} Lg &= \partial_t g + v_1 \partial_x g + \frac{1}{4\pi} \int_{\mathbb{T}} \frac{\delta_s g \sec^2(s/2)}{\tan^2(s/2) + \tanh^2(\delta_s f_1/2)} \\ &= -(v_1 - v_2)\partial_x f_2 \\ &\quad - \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\tanh(\delta_s f_2/2) \sec^2(s/2) \left(\tanh^2(\delta_s f_1/2) - \tanh^2(\delta_s f_2/2)\right)}{(\tan^2(s/2) + \tanh^2(\delta_s f_1/2))(\tan^2(s/2) + \tanh^2(\delta_s f_2/2))} ds \\ &\quad - \frac{1}{2\pi} \int_{\mathbb{T}} \frac{(\tanh(\delta_s f_1/2) - \tanh(\delta_s f_2/2) - \tanh(\delta_s g/2)) \sec^2(s/2)}{\tan^2(s/2) + \tanh^2(\delta_s f_1/2)} ds \\ &\quad - \frac{1}{2\pi} \int_{\mathbb{T}} \frac{(\tanh(\delta_s g/2) - \frac{\delta_s g}{2}) \sec^2(s/2)}{\tan^2(s/2) + \tanh^2(\delta_s f_1/2)} \end{aligned} \tag{3.172}$$

$$\begin{aligned} &= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\sec^2(s/2)(\tanh^2(\delta_s f_2/2)-\tanh^2(\delta_s f_1/2))}{(\tan^2(s/2)+\tanh^2(\delta_s f_1/2))(\tan^2(s/2)+\tanh^2(\delta_s f_2/2))} \\ &\qquad \times(\tanh(\delta_s f_2/2)-\tan(s/2)\partial_x f_2)ds \\ &\quad -\frac{1}{2\pi}\int_{\mathbb{T}} \frac{\sec^2(s/2)(\tanh(\delta_s f_1/2)-\tanh(\delta_s f_2/2)-\tanh(\delta_s g/2))}{\tan^2(s/2)+\tanh^2(\delta_s f_1/2)}ds \qquad (3.173) \\ &\quad -\frac{1}{2\pi}\int_{\mathbb{T}} \frac{(\tanh(\delta_s g/2)-\frac{\delta_s g}{2})\sec^2(s/2)}{\tan^2(s/2)+\tanh^2(\delta_s f_1/2)} \\ &= I_1+I_2+I_3. \end{aligned}$$

Multiplying by $g(x)$ we get

$$\left(\partial_t + v_1\partial_x + \bar{\mathcal{L}}_{f_1}\right)|g|^2 + \bar{D}_{f_1}[g] = 2g(x)(I_1+I_2+I_3), \qquad (3.174)$$

where $\mathcal{L}_f[g]$ and $D_f[g]$ are defined by (3.22). Fix $\delta > 0$ such that

$$|\tanh(\delta_s f_2(x)/2) - \tan(s/2)\partial_x f_2(x)| \leq \frac{1}{2(1+B^2)}|\tan(s/2)| \qquad (3.175)$$

for all $x \in \mathbb{T}$ and $|s| \leq \delta$, which can be done because f_2 is uniformly continuous. Now we write $I_1 = I_{1,in} + I_{1,out}$, where

$$\begin{aligned} I_{1,in} &= \frac{1}{2\pi}\int_{|s|\leq\delta} \frac{\sec^2(s/2)(\tanh^2(\delta_s f_2/2)-\tanh^2(\delta_s f_1/2))}{(\tan^2(s/2)+\tanh^2(\delta_s f_1/2))(\tan^2(s/2)+\tanh^2(\delta_s f_2/2))} \\ &\qquad \times(\tanh(\delta_s f_2/2)-\tan(s/2)\partial_x f_2)ds \\ &= \frac{1}{2\pi}\int_{|s|\leq\delta} \frac{\sec^2(s/2)\tanh(\delta_s g/2)K(s)}{\tan^2(s/2)}ds \end{aligned} \qquad (3.176)$$

and

$$\begin{aligned} I_{1,out} &= \frac{1}{2\pi}\int_{|s|>\delta} \frac{\sec^2(s/2)(\tanh^2(\delta_s f_2/2)-\tanh^2(\delta_s f_1/2))}{(\tan^2(s/2)+\tanh^2(\delta_s f_1/2))(\tan^2(s/2)+\tanh^2(\delta_s f_2/2))} \\ &\qquad \times(\tanh(\delta_s f_2/2)-\tan(s/2)\partial_x f_2)ds. \end{aligned} \qquad (3.177)$$

Now we focus on estimating $K(s)$

$$K(s) = \frac{(1-\tanh(\delta_s f_1/2)\tanh(\delta_s f_2/2))}{(1+\frac{\tanh^2(\delta_s f_1/2)}{\tan^2(s/2)})(1+\frac{\tanh^2(\delta_s f_2/2)}{\tan^2(s/2)})}\left(\frac{\tanh(\delta_s f_1/2)}{\tan(s/2)}+\frac{\tanh(\delta_s f_2/2)}{\tan(s/2)}\right) \times \frac{(\tanh(\delta_s f_2/2)-\tan(s/2)\partial_x f_2)}{\tan(s/2)} \quad (3.178)$$

then from (3.175) we know that $K(s) \leq \frac{1}{1+B^2}$ for $|s| \leq \delta$, and $K(s) \leq 4B$ for all s. Now using that that

$$\tanh(\delta_s g/2)g = \frac{1}{4}(\delta_s|g|^2 + (\delta_s g)^2) + (\tanh(\delta_s g/2) - \delta_s g/2)g, \quad (3.179)$$

we get for $I_{1,in}g(x)$

$$\begin{aligned} I_{1,in}g(x) &= \frac{1}{8\pi}\int_{|s|\leq\delta}\frac{(\delta_s|g|^2)\sec^2(s/2)K_{1,2}}{\tan^2(s/2)}ds \\ &+\frac{1}{8\pi}\int_{|s|\leq\delta}\frac{(\delta_s g)^2\sec^2(s/2)K_{1,2}}{\tan^2(s/2)}ds \\ &+\frac{g(x)}{2\pi}\int_{|s|\leq\delta}\frac{(\tanh(\delta_s g/2)-\delta_s g/2)\sec^2(s/2)K_{1,2}}{\tan^2(s/2)}ds. \end{aligned} \quad (3.180)$$

By the mean value theorem we know that $|\tanh(\delta_s g/2) - \delta_s g/2| \leq (\delta_s g/2)^3$, and so

$$|\tanh(\delta_s g/2) - \delta_s g/2| \leq (\delta_s g/2)^3 \leq |\delta_s g|\frac{s^2}{8}\|g'\|^2_{L^\infty}, \quad (3.181)$$

Now for a fixed t let $\bar{x}(t)$ be a point where we reach the maximum of $|g(x,t)|$ is attained, then we get that $\delta_s|g|^2(\bar{x}) \geq 0$ and therefore we can bound

$$\begin{aligned}
|I_{1,in}(\bar{x})g(\bar{x})| &\leq \frac{1}{2}\mathcal{L}_{f_1}|g|^2 + \frac{1}{2}D_{f_1}[g] + \frac{|g(x)|\|g'\|^2_{L^\infty}}{16\pi(1+B^2)}\int_{|s|\leq\delta}\frac{|\delta_s g|s^2\sec^2(s/2)}{\tan^2(s/2)}ds \\
&\leq \frac{1}{2}\mathcal{L}_{f_1}|g|^2 + \frac{1}{2}D_{f_1}[g] \\
&\quad + \frac{|g(x)|\|g'\|^2_{L^\infty}}{4}\left(\frac{1}{4\pi}\int_{|s|\leq\delta}\frac{s^4\sec^2(s/2)}{\tan^2(s/2)}ds\right)^{1/2}(D_{f_1}[g])^{1/2} \\
&\leq \frac{1}{2}\mathcal{L}_{f_1}|g|^2 + \frac{1}{2}D_{f_1}[g] \\
&\quad + \frac{1}{8}D_{f_1}[g] + |g(x)|^2\frac{\|g'\|^4_{L^\infty}}{8}\frac{1}{4\pi}\int_{|s|\leq\delta}\frac{s^4\sec^2(s/2)}{\tan^2(s/2)}ds.
\end{aligned} \tag{3.182}$$

For $I_{1,out}$ we get

$$\begin{aligned}
|I_{1,out}g(x)| &\leq \frac{|g(x)|}{2\pi}\left|\int_{|s|>\delta}\frac{\tanh(\delta_s g)\sec^2(s/2)K_{1,1}}{\tan^2(s/2)}ds\right| \\
&\leq 8B(1+B^2)^{1/2}|g(x)|\left(\frac{1}{4\pi}\int_{|s|>\delta}\frac{\sec^2(s/2)}{\tan^2(s/2)}ds\right)^{1/2}(D_{f_1}[g])^{1/2} \\
&\leq \frac{8B(1+B^2)^{1/2}}{\sqrt{\pi}\tan^{1/2}(\delta/2)}|g(x)|(D_{f_1}[g])^{1/2} \\
&\leq \frac{1}{8}D_{f_1}[g] + \frac{128B^2(1+B^2)}{\pi\tan(\delta/2)}|g(x)|^2.
\end{aligned} \tag{3.183}$$

Therefore

$$|I_1(\bar{x})g(\bar{x})| \leq \frac{5}{8}D_{f_1}[f] + \frac{1}{2}\mathcal{L}_{f_1}|g|^2 + |g(x)|^2 h(B,\delta), \tag{3.184}$$

where

$$h(B,\delta) = \frac{\|g'\|^4_{L^\infty}}{32\pi}\int_{|s|\leq\delta}\frac{s^4\sec^2(s/2)}{\tan^2(s/2)}ds + \frac{128B^2(1+B^2)}{\pi\tan(\delta/2)}. \tag{3.185}$$

To estimate I_2 we need the following trigonometric identity

$$\tanh(a) - \tanh(b) - \tanh(a-b) = -\tanh(a-b)\tanh(a)\tanh(b), \tag{3.186}$$

then by using $a = \delta_s f_1$, $b = \delta_s f_2$ we get for I_2

$$
\begin{aligned}
|I_2| &= \frac{1}{2\pi}\left|\int_{\mathbb{T}} \frac{\sec^2(s/2)(\tanh(\delta_s f_1/2) - \tanh(\delta_s f_2/2) - \tanh(\delta_s g/2))}{\tan^2(s/2) + \tanh^2(\delta_s f_1/2)} ds\right| \\
&\leq \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\sec^2(s/2)|\tanh(\delta_s g/2)\tanh(\delta_s f_1/2)\tanh(\delta_s f_2/2)|}{\tan^2(s/2) + \tanh^2(\delta_s f_1/2)} ds \\
&\leq \frac{B}{2\pi}\int_{\mathbb{T}} \frac{\sec^2(s/2)|\delta_s g/2||s|}{\tan^2(s/2) + \tanh^2(\delta_s f_1/2)} ds \\
&\leq 2B\left(\frac{1}{4\pi}\int_{\mathbb{T}} \frac{s^2\sec^2(s/2)}{\tan^2(s/2)} ds\right)^{1/2} (D_{f_1}[g])^{1/2},
\end{aligned}
\tag{3.187}
$$

and therefore we get

$$
|I_2(x)g(x)| \leq \frac{1}{8} D_{f_1}[g] + |g(x)|^2 \frac{2B^2}{\pi}\left(\int_{\mathbb{T}} \frac{s^2\sec^2(s/2)}{\tan^2(s/2)} ds\right). \tag{3.188}
$$

To estimate I_3, we use the mean value theorem to show that given $x \in (-\pi, \pi)$, there exists $t \in [0, x]$ such that

$$
|\tanh(x) - x| \leq |\tanh^2(t)||x| \leq |\tanh^2(x)||x| \leq |x|^3. \tag{3.189}
$$

Using this we can estimate I_3 in the following way

$$
\begin{aligned}
|I_3| &= \frac{1}{2\pi}\left|\int_{\mathbb{T}} \frac{(\tanh(\delta_s g/2) - \delta_s g/2)\sec^2(s/2)}{\tan^2(s/2) + \tanh^2(\delta_s f_1/2)}\right| \\
&\leq \frac{1}{16\pi}\int_{\mathbb{T}} \frac{|\delta_s g|^3\sec^2(s/2)}{\tan^2(s/2) + \tanh^2(\delta_s f_1/2)} ds \\
&\leq \frac{\|g'\|_{L^\infty}^2}{4}\left(\frac{1}{4\pi}\int_{\mathbb{T}} \frac{s^4\sec^2(s/2)}{\tan^2(s/2)} ds\right)^{1/2} (D_{f_1}[g])^{1/2},
\end{aligned}
\tag{3.190}
$$

and therefore we get

$$
|I_3(x)g(x)| \leq \frac{1}{8} D_{f_1}[g] + |g(x)|^2 \frac{\|g'\|_{L^\infty}^4}{32\pi}\left(\int \frac{s^4\sec^2(s/2)}{\tan^2(s/2)} ds\right). \tag{3.191}
$$

Finally using (3.184), (3.188) and (3.191) we get from equation (3.174) at a maximum $\bar{x}(t)$ of $|g|$

$$
(\partial_t + v_1\partial_x + \mathcal{L}_{f_1})|g|^2 + D_{f_1}[g] \leq \frac{7}{8} D_{f_1}[g] + \frac{1}{2}\mathcal{L}_{f_1}|g|^2 + |g(\bar{x})|^2 h_2(B, \delta), \tag{3.192}
$$

$$\partial_t |g(\bar{x})|^2 \leq |g(\bar{x})|^2 h_2(B,\delta), \tag{3.193}$$

where

$$\begin{aligned} h_2(B,\delta) &= \frac{\|g'\|_{L^\infty}^4}{32\pi}\int_{\mathbb{T}} \frac{s^4 \sec^2(s/2)}{\tan^2(s/2)} ds + \frac{128B^2(1+B^2)}{\pi \tan(\delta/2)} \\ &\quad + \frac{2B^2}{\pi}\int_{\mathbb{T}} \frac{s^2 \sec^2(s/2)}{\tan^2(s/2)} ds + \frac{\|g'\|_{L^\infty}^4}{32\pi}\int_{\mathbb{T}} \frac{s^4 \sec^2(s/2)}{\tan^2(s/2)} ds \\ &\leq CB^2(1+B^2)\left(1 + \frac{1}{\tan(\delta/2)}\right). \end{aligned} \tag{3.194}$$

Finally by the Radamacher theorem we get

$$\partial_t \|g\|_{L^\infty}^2 \leq \|g(\bar{x})\|_{L^\infty}^2 h_2(B,\delta), \tag{3.195}$$

and by integrating we get

$$\|g(t)\|_{L^\infty}^2 \leq \|g(0)\|_{L^\infty}^2 \exp(t h_2(B,\delta)), \tag{3.196}$$

which concludes the proof of the uniqueness of C^1 solutions. If we additionally assume that $f_i(t) \in H^2(\mathbb{T})$ then using Lemma 3.2.1 we can estimate the required size of δ in (3.175) in terms of the H^2 norm and so we get the existence of a constant $C(B,M)$ that depend only in an upper bound for the slope B for all (t,x) and the bound M for the L^2 norm of the second derivative, such that

$$\|g(t)\|_{L^\infty}^2 \leq \|g(0)\|_{L^\infty}^2 \exp(tC(B,M)), \tag{3.197}$$

this concludes the proof of Lemma 3.1.4. □

Chapter 4

Norm Inflation for a truncated 2D Muskat problem in supercritical spaces

Abstract

In this chapter we study the question of the continuity of the solution map if the Muskat problem in supercritical spaces, for this purpose we consider a sequence of approximations of the Muskat problem obtained by a Taylor expansion and then considering the second Picard iteration. For such systems the same stability results as for Muskat apply, in particular the stability in the critical space $\mathcal{F}^{1,1}_1$. The main result of this chapter is that for such approximate problems, we prove the existence of a sequence of solutions in some supercritical space $\mathcal{F}^{m,p}_q$ with

$m < 1$ such that for arbitrarily small time t^* there exists an initial condition arbitrarily small such that the solution of the approximate problem with such initial data become arbitrarily large, before time t^* which implies that the solution map is not continuous at the origin.

4.1 Introduction

4.1.1 Description of the model

The Muskat equation describes the interface between two immersible fluids with different densities in a porous media, ignoring the effect of surface tension the evolution of the fluids can be described by the system

$$\begin{cases} \rho_t + \vec{u} \cdot \nabla \rho = 0 & , \quad x \in \Omega \times (0, T) \\ \dfrac{\mu}{\kappa} \vec{u} = -\nabla p - \rho g \vec{e}_n & , \quad x \in \Omega \times (0, T), \end{cases} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^d$, μ the viscosity, κ the permeability of the media, ρ is the density, $\vec{u}$ the velocity, p is the pressure and g is the gravity acceleration constant. The first equation corresponds to the conservation of mass and the second one describes the evolution of velocity of the fluid, which in the case of a porous media, is given by the Darcy's law.

In this chapter we focus our attention in the situation in which we have two immiscible fluids with same viscosity and the denser fluid is at the bottom and we ignore the surface tension. By changing variables we can assume for simplicity that

$\mu/\kappa = 1$ and $g = 1$. In what follows we assume that we are in the regime where the interface between the two fluids can be described by a graph $x_n = f(x_1, \cdots, x_{n-1})$ and consequently the density can be written as

$$\rho(x,t) = \begin{cases} \rho^1 & , \quad x \in \Omega_1(t) = \{x \in \Omega : x_d > f(x_1, \cdots, x_{n-1})\} \\ \rho^2 & , \quad x \in \Omega_2(t) = \Omega \setminus \Omega_1(t) \end{cases} . \tag{4.2}$$

Here we consider Ω to be either $\mathbb{R}^{n-1}$ or $\mathbb{T}^{n-1}$. In 2D (with a 1D interface) when $\Omega = \mathbb{R}$ the initial value problem for the evolution of the interface is given by

$$\begin{cases} \partial_t f + \Lambda f = -\frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{\partial_x \delta_\alpha f(x)}{\alpha} \frac{(\delta_\alpha f(x))^2}{\alpha^2 + (\delta_\alpha f(x))^2} d\alpha & , \quad (x,t) \in \mathbb{R} \times (0,T), \\ f(0) = f_0 & , \quad x \in \mathbb{R}, \end{cases} \tag{4.3}$$

where $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x) dx$, $\mathcal{F}(\Lambda f) = 2\pi |\xi| \hat{f}$ and $\delta_\alpha f(x) = f(x) - f(x - \alpha)$. In the periodic case we can use the compactness to get rid of principal value to obtain (see [16])

$$\begin{cases} \partial_t f + \Lambda f = -\frac{1}{2\pi} \int_{\mathbb{T}} \frac{\partial_x \delta_\alpha f(x)}{\tan(\alpha/2)} \frac{\sec^2(\alpha/2) \tanh^2(\delta_\alpha f(x)/2)}{\tan^2(\alpha/2) + \tanh^2(\delta_\alpha f(x)/2)} d\alpha & , \quad \mathbb{T} \times (0,T), \\ f(x,0) = f_0(x) & , \quad x \in \mathbb{T}, \end{cases} \tag{4.4}$$

where $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, $\mathcal{F}(\Lambda f)(k) = |k| \hat{f}(k)$ and $\hat{f}(k) = \int_{\mathbb{T}} e^{-ikx} f(x) dx$.

4.1.2 Main results

Suppose that f is a Lipschitz continuous solution of the Muskat equation (4.3) or (4.4) with Lipschitz constant less than 1, then it is possible to use the Taylor expansion to expand the nonlinear term as,

$$\partial_t f + \Lambda f = Tf = \sum_{k\geq 1} T_k f, \tag{4.5}$$

where in the case $\Omega = \mathbb{R}$, T_k is given by

$$T_k f = (-1)^k \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{\partial_x \delta_\alpha f(x)}{\alpha} \left(\frac{\delta_\alpha f(x)}{\alpha} \right)^{2k} d\alpha, \tag{4.6}$$

and for $\Omega = \mathbb{T}$,

$$T_k f = (-1)^k \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\partial_x \delta_\alpha f(x)}{\tan(\alpha/2)} \left(\frac{\tanh^2 (\delta_\alpha f(x)/2)}{\tan^2(\alpha/2)} \right)^{k} \sec^2(\alpha/2) d\alpha. \tag{4.7}$$

The main result in this chapter considers a finite truncation of equation (4.5) for which we can find initial data that illustrate a norm inflation phenomenon. We say that f is the solution of the truncation of the Muskat problem of order ℓ if

$$\begin{cases} \partial_t f + \Lambda f = \displaystyle\sum_{k=1}^{\ell} T_k f & , \quad (x,t) \in \Omega \times [0,T] \\ f(0) = f_0 & , \quad x \in \Omega \end{cases} \tag{4.8}$$

where T_k is given by (4.6) or (4.7) depending on the domain of the problem. Now we consider the Picard's iteration of the problem. Define $f^{(0)} = 0$ and consider the sequence

$$\partial_t f^{(n)} + \Lambda f^{(n)} = \sum_{k=1}^{\ell} T_k f^{(n-1)}, \tag{4.9}$$

with this definition we obtain that the first two Picard's iterations are given by

$$\partial_t f^{(1)} + \Lambda f^{(1)} = 0, \quad f^{(1)}(0) f_0 \Rightarrow \ f^{(1)} = e^{-t\Lambda} f_0, \tag{4.10}$$

$$\partial_t f^{(2)} + \Lambda f^{(2)} = \sum_{k=1}^{\ell} T_k e^{-t\Lambda} f_0, \quad x \in \Omega. \tag{4.11}$$

In many situations the Picard's iteration it is expected to converge to a solution the problem, but in the case of supercritical spaces this is a hard question in general. In this chapter we focus our attention to the evolution of the second Picard's iteration for some highly oscillatory initial data. For this purpose we study the following problem, given $\varphi \in \mathcal{F}_q^{\frac{2\ell-1}{2\ell+1},p}$ consider the solution $f \in C([0,T];\mathcal{F}_q^{\frac{2\ell-1}{2\ell+1},p})$ of

$$\begin{cases} \partial_t f + \Lambda f = \displaystyle\sum_{k=1}^{\ell} T_k e^{-t\Lambda}\varphi \quad , \quad (x,t) \in \Omega \times [0,T], \\ f(x,0) = \varphi(x) \qquad\qquad\quad , \quad x \in \Omega. \end{cases} \tag{4.12}$$

By linearity we get the uniqueness and by global existence comes from the fact that we have a explicit solution for the problem. The result that we are interested in can be stated as follows.

Theorem 4.1.1 (Norm inflation for truncated system)**.** *Let $\ell \in \mathbb{N}$ and consider the second Picard's iteration of truncation of the Muskat problem of order ℓ given by* (4.12) *for $\Omega = \mathbb{R}$ or $\mathbb{T}$. Then given $T > 0, R > 0$, there exists some $0 < \tilde{t} < T$, and an initial condition $f_0 \in \dot{\mathcal{F}}_q^{\frac{2\ell-1}{2\ell+1},p}(\Omega)$, $p \geq 1$, $q > 2\ell+1$ such that*

$$\|f_0\|_{\mathcal{F}_q^{\frac{2\ell-1}{2\ell+1},p}} < 1/R \quad \text{and} \quad \|f(\tilde{t})\|_{\dot{\mathcal{F}}_q^{\frac{2\ell-1}{2\ell+1},p}} > R \tag{4.13}$$

Remark 4.1.2. If we can consider the map

$$L : \dot{\mathcal{F}}_q^{\frac{2\ell-1}{2\ell+1},p} \to C([0,T];\mathcal{F}_q^{\frac{2\ell-1}{2\ell+1},p}), \tag{4.14}$$

that takes a function $\varphi \in \dot{\mathcal{F}}_q^{\frac{2\ell-1}{2\ell+1},p}$ and return the solution f of the second Picard's iteration of the truncated Muskat problem of order ℓ with initial condition φ given for

$$\begin{cases} \partial_t f + \Lambda f = \displaystyle\sum_{k=1}^{\ell} T_k e^{-t\Lambda}\varphi \quad , \quad (x,t) \in \Omega \times [0,T], \\ f(x,0) = \varphi(x) \qquad\qquad\qquad , \quad x \in \Omega. \end{cases} \tag{4.15}$$

Now from Theorem 4.1.1 we can conclune that for arbitrarily small time $T > 0$ to conclude it is possible to find a sequence of times and initial data $\{(t_N, \varphi_N)\}_{N=1}^{\infty}$ such that if $f_N = L\varphi_N$ satisfy

$$\|\varphi_N\|_{\dot{\mathcal{F}}_q^{\frac{2\ell-1}{2\ell+1},p}} \leq \frac{1}{N} \text{ and } \|f_N(t_N)\|_{\dot{\mathcal{F}}_q^{\frac{2\ell-1}{2\ell+1},p}} > N, \tag{4.16}$$

which implies that the solution map $L : \dot{\mathcal{F}}_q^{\frac{2\ell-1}{2\ell+1},p} \to C([0,T]; \mathcal{F}_q^{\frac{2\ell-1}{2\ell+1},p})$ is discontinous at the origin.

Outline of the chapter: In Section 4.2.1 we discuss the choice of initial that produces the inflation.

4.3 4.4

4.2 Norm inflation for $\ell = 1$

4.2.1 On the choice of initial Data

The initial data considered in this work is inspired by the works of Bourgain-Pavlovic [2] and Iwabuchi-Ogawa [26]. Given $N \in \mathbb{N}$ and $\ell \in \mathbb{N}$, we consider $\varphi : \mathbb{R} \to \mathbb{R}$ of the form

$$\hat{\varphi} = \beta_N \sum_{j \in S(N)} \gamma_j \Big(P_{k_s}(\xi) + P_{2\ell k_s + M}(\xi) \Big), \tag{4.17}$$

where for $A \in \mathbb{R}$ define $P_A(\xi) = \chi(\xi - A) + \chi(\xi + A)$ and $\chi(\xi)$ denotes the characteristic function of the interval $[-1, 1]$. $\{k_s\}_{s \geq 0}$ is a sequence of positive integers that grow very fast, $M > 2\ell$ is fixed and $\{\gamma_j\}_j$ a sequence of positive numbers to be chosen later. N is a parameter that will be large in general, $S(N) = \{j : N \leq j \leq (1+\delta)N\}$, and β_N is a scaling factor that also depend on the parameter N.

An important property of the initial data that we will consider is that they can be made small in appropriate norms with bounds that can be made uniform in N.

Lemma 4.2.1 (Size of the Initial data)**.** *Consider* φ *defined by* (4.17) *then*

$$\|\varphi\|_{\dot{\mathcal{F}}_q^{m,p}} \leq C\beta_N \left(\sum_{j \in S(N)} \gamma_j^q k_j^{qm} \right)^{1/q}. \tag{4.18}$$

Remark 4.2.2. From Lemma 4.2.1 we see that the properties of the right hand side as we change N, depend on the summability of the sequence $\{k_j^m \gamma_j\}_j$ in $\ell^q(\mathbb{N})$. In particular if we take $\gamma_s = k_s^{-m}$, and $q = \infty$ we get that

$$\|\varphi\|_{\mathcal{F}_\infty^{m,p}} \leq C\beta_N, \tag{4.19}$$

therefore β_N can be chosen in such a way that the right hand side tend to 0 as $N \to \infty$. If we want to work with finite values of q, we use that for $1 \leq q \leq \infty$ by taking $\gamma_j = k_j^{-\bar{m}}$ for $\bar{m} > m$ we get

$$\|\varphi\|_{\mathcal{F}_\infty^{m,p}} \leq C\beta_N \left(\sum_{j \in S(N)} \frac{1}{k_j^{(\tilde{m}-m)q}} \right)^{1/q}, \tag{4.20}$$

then if the sequence $\{k_j\}_{j \in \mathbb{Z}}$ grow fast enough so that the series $\sum_{j \in \mathbb{N}} \frac{1}{k_j^{(\tilde{m}-m)q}}$ converges we get that the right hand side go to 0 as $N \to \infty$.

Proof of Lemma 4.2.1. Because the sequence $\{k_s\}$ is growing fast, at most one of them belong to each C_k annulus. Also, because the C_k are dyadic we can ensure that k_j and $2k_j + M$ belong to different annulus. With this observation in mind we get that if $k_{\bar{j}} \in C_k$ then

$$\int_{C_k} |\xi|^{mp} \, |\hat{\varphi}|^p \, d\xi \leq \left(\beta_N \gamma_{\bar{j}}\right)^p 2^{mp+1} |k_{\bar{j}}|^{mp}. \tag{4.21}$$

Similarly if $2k_{\bar{j}} + M \in C_k$

$$\int_{C_k} |\xi|^{mp} \, |\hat{\varphi}|^p \, d\xi \leq \left(\beta_N \gamma_{\bar{j}}\right)^p 2^{2mp+1} \left| k_{\bar{j}} + \frac{M}{2} \right|^{mp} \leq \left(\beta_N \gamma_{\bar{j}}\right)^p 2^{3mp+1} |k_{\bar{j}}|^{mp}, \tag{4.22}$$

taking the q/p power and summing over k we get

$$\begin{aligned} \sum_k \left(\int_{C_k} |\xi|^{mp} |\hat{\varphi}|^p \right)^{q/p} &\leq (\beta_N)^q \sum_{j \in S(N)} \gamma_j^q \left(2^{\frac{q(mp+1)}{p}} |k_j|^{mq} + 2^{\frac{q(3mp+1)}{p}} |k_j|^{mq} \right) \\ &\leq 2 \left(\beta_N 2^{\frac{(3mp+1)}{p}} \right)^q \sum_{j \in S(N)} \gamma_j^q k_j^{mq}, \end{aligned} \tag{4.23}$$

taking the q-th root we obtain

$$\|\varphi\|_{\dot{\mathcal{F}}_q^{m,p}} \leq C \beta_N \left(\sum_{j \in S(N)} \gamma_j^q k_j^{qm} \right)^{1/q}. \tag{4.24}$$

This completes the proof of Lemma 4.2.1.

□

4.2.2 Preliminary Estimates case $\ell = 1$

The main idea of the inflation results to understand the behaviour of

$$G_1 = \int_0^t e^{-(t-\tau)\Lambda} T_1(e^{-\tau\Lambda}\varphi) d\tau, \tag{4.25}$$

where T_1 is given by (4.6) a key ingredient to understand the behaviour of this operator is to study its Fourier transform

$$\hat{G}_1 = \frac{1}{3}\int_0^t e^{-(t-\tau)|\xi|}\int_{\mathbb{R}}(2\pi i\xi)(m_\alpha e^{-\tau|\cdot|}\hat{\varphi}) * (m_\alpha e^{-\tau|\cdot|}\hat{\varphi}) * (m_\alpha e^{-\tau|\cdot|}\hat{\varphi})d\alpha d\tau. \quad (4.26)$$

In order to study its behaviour we want to analize its effect on characteristic functions,

$$\begin{aligned} &I(\chi_A, \chi_B, \chi_C) \\ &= \frac{1}{3}\int_0^t e^{-(t-\tau)|\xi|}\int_{\mathbb{R}}(2\pi i\xi)(m_\alpha e^{-\tau|\cdot|}\chi_A) * (m_\alpha e^{-\tau|\cdot|}\chi_B) * (m_\alpha e^{-\tau|\cdot|}\chi_C)d\alpha d\tau, \end{aligned} \quad (4.27)$$

when A, B, C are large in magnitude so that a characteristic function centered at them is supported away from zero a reasonable approximation is $g(x)\chi_A \approx g(A)\chi_A$, another observation is that a convolution of characteristic functions can be compared with another characteristic function centered at the sum of the center, $\chi_A * \chi_B \approx \chi_{A+B}$ (we will make this notion precise later), with this in mind we get that

$$\begin{aligned} I(\chi_A, \chi_B, \chi_C) &\approx \frac{1}{3}e^{-t(|A+B+C|)}\int_0^t e^{-\tau(|A|+|B|+|C|-|A+B+C|)} \\ &\quad\times\int_{\mathbb{R}}(2\pi i\xi)(m_\alpha\chi_A) * (m_\alpha\chi_B) * (m_\alpha\chi_C)d\alpha d\tau \\ &\approx \frac{2\pi i}{3}\frac{(A+B+C)e^{-t(|A+B+C|)}}{|A|+|B|+|C|-|A+B+C|} \\ &\quad\times\left(1-e^{-t(|A|+|B|+|C|-|A+B+C|)}\right) \\ &\quad\times\int_{\mathbb{R}}(m_\alpha\chi_A) * (m_\alpha\chi_B) * (m_\alpha\chi_C)d\alpha. \end{aligned} \quad (4.28)$$

By our previous remark we know that the integral term is supported near the frequency $A+B+C$ and therefore in the size of this term there are two competing factors. On one hand we have the exponential term $e^{-t|A+B+C|}$ that tell us that high frequency terms decay faster, on the other hand we need to understand the size of $\int_{\mathbb{R}}(m_\alpha\chi_A)*(m_\alpha\chi_B)*(m_\alpha\chi_C)d\alpha$. Our choice of initial condition is made so that we can control precisely the size of $\int_{\mathbb{R}}(m_\alpha\chi_A)*(m_\alpha\chi_B)*(m_\alpha\chi_C)d\alpha$ for the low frequency terms which in appropriate norms we expect to be the largest. With this in mind the goal of this subsection is to provide precise estimates for $I(\chi_A,\chi_B,\chi_C)$.

The idea of Lemma 4.2.3 is to illustrate the basic techniques that we will later use in the inflation estimate. On one hand it provides a precise estimate of the integral in α and provide estimates on the decay that depend on the region where the convolution is supported.

Lemma 4.2.3. *Let $A,B,C\in\mathbb{R}$, $M>4$, $|A|,|B|,|C|\gg M$, $t\leq 1$, $|A+B+C|\geq 2M$ then*

$$\begin{aligned} S &= \int_{\mathbb{R}} 2\pi i\xi(m_\alpha e^{-2\pi t|\cdot|}\chi_A)*(m_\alpha e^{-2\pi t|\cdot|}\chi_B)*(m_\alpha e^{-2\pi t|\cdot|}\chi_C)d\alpha \\ &\sim (A+B+C)e^{-2\pi t(|A|+|B|+|C|)}(\Gamma(A,B,C)+O(|A|+|B|+|C|))g(\xi), \end{aligned} \tag{4.29}$$

where $m_\alpha(\xi)=\frac{1-e^{-i\alpha\xi}}{\alpha}$, $\Gamma(x,y,z)$ is defined by

$$\begin{aligned} \Gamma(x,y,z) &= i\int_{\mathbb{R}} \frac{(1-e^{-2\pi x\alpha})}{\alpha}\frac{(1-e^{-2\pi y\alpha})}{\alpha}\frac{(1-e^{-2\pi z\alpha})}{\alpha}d\alpha \\ &= 2\pi^3\Big(x|x|+y|y|+z|z|-(x+y)|x+y|-(x+z)|x+z| \\ &\qquad -(y+z)|y+z|+(x+y+z)|x+y+z|\Big), \end{aligned} \tag{4.30}$$

and $g(\xi)$ satisfy

$$\chi(\xi-A-B-C)\leq g(\xi)\leq 4\chi\left(\frac{\xi-A-B-C}{3}\right). \tag{4.31}$$

Proof of Lemma 4.2.3. Consider

$$\begin{aligned}
S &= \int_{\mathbb{R}} 2\pi i\xi (m_\alpha e^{-2\pi t|\cdot|}\chi_A) * (m_\alpha e^{-2\pi t|\cdot|}\chi_B) * (m_\alpha e^{-2\pi t|\cdot|}\chi_C) d\alpha \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} 2\pi\xi e^{-2\pi t|\xi-z|} e^{-2\pi t|z-y|} e^{-2\pi t|y|} \\
&\qquad\qquad \times \frac{i}{\alpha^3}(1-e^{-2\pi i\alpha(\xi-z)})(1-e^{-2\pi i\alpha(z-y)})(1-e^{-2\pi i\alpha y})d\alpha \\
&\qquad\qquad \times \chi_A(\xi-z)\chi_B(z-y)\chi_Z(y)dzdy \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} 2\pi\xi e^{-2\pi t|\xi-z|} e^{-2\pi t|z-y|} e^{-2\pi t|y|}\chi_A(\xi-z)\chi_B(z-y)\chi_Z(y) I(\xi,y,z)dzdy.
\end{aligned} \tag{4.32}$$

For the innermost integral we have α

$$\begin{aligned}
I &= \Gamma(\xi-z, z-y, y) \\
&= p.v. \int_{\mathbb{R}} \frac{i}{\alpha^3}(1-e^{-2\pi i\alpha(\xi-z)})(1-e^{-2\pi i\alpha(z-y)})(1-e^{-2\pi i\alpha y})d\alpha \\
&= p.v. \int_{\mathbb{R}} \frac{i}{2!}\frac{d^2}{d\alpha^2}\left(\frac{1}{\alpha}\right)(1-e^{-2\pi i\alpha(\xi-z)})(1-e^{-2\pi i\alpha(z-y)})(1-e^{-2\pi i\alpha y})d\alpha \\
&= p.v. \int_{\mathbb{R}} \frac{i}{2\alpha}\frac{d^2}{d\alpha^2}\Big[(1-e^{-2\pi i\alpha(\xi-z)})(1-e^{-2\pi i\alpha(z-y)})(1-e^{-2\pi i\alpha y})\Big]d\alpha \\
&= \frac{i(2\pi)^2}{2} p.v. \int_{\mathbb{R}} \left(\frac{1}{\alpha}\right)\Big[-(\xi-z)^2 e^{-2\pi i\alpha(\xi-z)} - (z-y)^2 e^{-2\pi i\alpha(z-y)} \\
&\qquad -y^2 e^{-2\pi i\alpha y} + (\xi-y)^2 e^{-2\pi i\alpha(\xi-y)} + (\xi-z+y)^2 e^{-2\pi i\alpha(\xi-z+y)} \\
&\qquad +z^2 e^{-2\pi i\alpha z} - \xi^2 e^{-2\pi i\alpha\xi}\Big]d\alpha \\
&= 2\pi^3\Big((\xi-z)|\xi-z| + (z-y)|z-y| + y|y| - (\xi-y)|\xi-y| \\
&\qquad -(\xi-z+y)|\xi-z+y| - z|z| + \xi|\xi|\Big)
\end{aligned}$$

In the last step we used that

$$p.v. \int_{\mathbb{R}} \frac{1-e^{-2\pi i\alpha w}}{\alpha} = i p.v. \int_{\mathbb{R}} \frac{\sin(2\pi\alpha w)}{\alpha} d\alpha = i\pi\, sgn(w) \tag{4.33}$$

Substituing the computation for $I(\xi, y, z)$ in (4.32) and applying Lemma 4.2.4 we get

$$\begin{aligned} S &= 4\pi^4 \int\int \xi e^{-2\pi t|\xi-z|} e^{-2\pi t|z-y|} e^{-2\pi t|y|} \Big((\xi - z)|\xi - z| + (z-y)|z-y| \\ &\quad + y|y| - (\xi - y)|\xi - y| - (\xi - z + y)|\xi - z + y| - z|z| + \xi|\xi| \Big) \\ &\quad \times \chi_A(\xi - z)\chi_B(z-y)\chi_C(y) dz dy \end{aligned}$$

$$\begin{aligned} &\sim 4\pi^4 \xi e^{-2\pi t(|A|+|B|+|C|)} (A|A| + B|B| + C|C| - (A+B)|A+B| \\ &\quad -(A+C)|A+C| - (B+C)|B+C| \\ &\quad +(A+B+C)|A+B+C| + O(|A|+|B|+|C|)) g(\xi) \\ &= 4\pi^4 \xi e^{-2\pi t(|A|+|B|+|C|)} \left(\Gamma(A,B,C) + O(|A|+|B|+|C|)\right) g(\xi), \end{aligned} \tag{4.34}$$

where

$$g(\xi) = \int\int \chi_A(\xi - z)\chi_B(z-y)\chi_C(y) dz dy. \tag{4.35}$$

Next, using that $|A+B+C| \geq M$ we can estimate

$$\begin{aligned} |\xi g(\xi)| &\geq (|A+B+C| - M) g(\xi) \geq \frac{|A+B+C|}{2} g(\xi), \\ |\xi g(\xi)| &\leq (|A+B+C| + M) g(\xi) \leq 2|A+B+C| g(\xi). \end{aligned} \tag{4.36}$$

Because of your assumption in the size of $A+B+C$ we also know that the sign of $\xi g(\xi)$ is the same as the sign of $A+B+C$, then we conclude

$$\begin{aligned} S &\sim (A+B+C) e^{-2\pi t(|A|+|B|+|C|)} \\ &\qquad \times \Big(\Gamma(A,B,C) + O(|A|+|B|+|C|) \Big) g(\xi). \end{aligned} \tag{4.37}$$

The estimate (4.31) is obtained by applying Lemma 4.2.5 to $g(\xi)$. □

To complete the proof we proceed to prove the following Lemma used in (4.34).

Lemma 4.2.4. *Let $t > 0$, $A, B, C \in \mathbb{R}$, and $\Gamma(x,y,z)$ defined by (4.30) then* *i)* $\Gamma(\xi - z, z - y, y)h(y, z) = (\Gamma(A, B, C) + O(|A| + |B| + |C|))\, h(y, z)$

ii) $e^{-2\pi t|\xi - z|}e^{-2\pi t|z-y|}h(y, z) \sim e^{-2\pi t(|A|+|B|+|C|)}h(y, z)$

where $h(y, z) = \chi_A(\xi - z)\chi_B(z - y)\chi_C(y)$*.*

Proof of Lemma 4.2.4. For part i) we consider $\Gamma(x_1, x_2, x_3)$ as defined in equation (4.30),

$$\Gamma(x_1, x_2, x_3) = 2\pi^3\Big(x_1|x_1| + x_2|x_2| + x_3|x_3| - (x_1 + x_2)|x_1 + x_2| - (x_1 + x_3)|x_1 + x_3| - (x_2 + x_3)|x_2 + x_3| + (x_1 + x_2 + x_3)|x_1 + x_2 + x_3|\Big),$$

notice that in the range of values that we are interested x_1, x_2 and x_3 do not change signs, and so we can estimate directly the derivative of $\Gamma(x_1, x_2, x_3)$ by

$$|\partial_{x_i}\Gamma(x_1, x_2, x_3)| \leq 16\pi^3(|x_1| + |x_2| + |x_3|)..$$

To prove the Lemma we need to estimate $\Gamma(\xi - z, z - y, y)$ in the support of $h(z, y)$. In such set, each entry only takes values on a interval, $\xi - z \in [A - 1, A + 1]$, $z - y \in [B - 1, B + 1]$ and $y \in [C - 1, C + 1]$ therefore we can apply the mean value theorem to obtain

$$\begin{aligned} |\Gamma(\xi - z, z - y, y) - \Gamma(A, B, C)| &\leq \sum_{i=1}^{3} \sup_{(x_1,x_2,x_3)} |\partial_{x_i}\Gamma(x_1, x_2, x_3)| \\ &\leq 48\pi^3(|A| + |B| + |C| + 3). \end{aligned} \tag{4.38}$$

For part ii) we use that $0 \leq t \leq 1$ and therefore on the support of $h(y, z)$

$$\begin{aligned} e^{-2\pi t|\xi - z|}e^{-2\pi t|z-y|}e^{-2\pi t|y|} &\leq e^{-2\pi t(|A|-1)-2\pi t(|B|-1)-2\pi t(|C|-1)} \\ &= e^{6\pi t}e^{-2\pi t(|A|+|B|+|C|)}, \end{aligned} \tag{4.39}$$

in a similar way

$$\begin{aligned} e^{-2\pi t|\xi-z|}e^{-2\pi t|z-y|}e^{-2\pi t|y|} &\geq e^{-2\pi t(|A|+1)-2\pi t(|B|+1)-2\pi t(|C|+1)} \\ &= e^{-6\pi t}e^{-2\pi t(|A|+|B|+|C|)}. \end{aligned} \tag{4.40}$$

Which concludes the proof of Lemma 4.2.4. □

The next Lemma provides a precise notion on how a convolution of characteristic functions can be compared with a single characteristic function.

Lemma 4.2.5. *[Convolutions of characteristic functions] Let* $c_1, \cdots, c_k \in \mathbb{R}$ *and* χ_A *as defied in Subsection 4.2.1, then*

$$\chi\left(\xi - (c_1 + \cdots + c_k)\right) \leq \chi_{c_1} * \chi_{c_2} * \cdots * \chi_{c_k} \leq 2^k \chi\left(\frac{\xi - (c_1 + \cdots + c_k)}{k}\right) \tag{4.41}$$

Proof of Lemma 4.2.5. For the lower bound the key fact is the following

$$(\chi_A * \chi_B)(\xi) = (2 - (\xi - A - B))_+ \geq \chi_{A+B}(\xi). \tag{4.42}$$

By iterating this inequality we obtain the lower bound. For the upper bound we need two observations, the first one is about the size of the support of a convolution. More specifically

$$\operatorname{supp} \chi_A * \chi_B \subset A + B = \{a + b : a \in A, b \in B\} \tag{4.43}$$

the second observation has to do with the maximum value, to do this we notice that

$$\begin{aligned} \chi\left(\frac{\cdot - A}{a}\right) * \chi\left(\frac{\cdot - B}{b}\right) &= \int \chi\left(\frac{\xi - y - A}{a}\right)\chi\left(\frac{y - B}{b}\right) dy \\ &\leq \int \chi\left(\frac{y - B}{b}\right) dy \\ &= 2b \end{aligned} \tag{4.44}$$

And by symmetry $\chi(\frac{\cdot - A}{a}) * \chi(\frac{\cdot - B}{b}) \leq 2\min\{a, b\}$, iterating this result we get that

$$\chi_{c_1} * \chi_{c_2} * \cdots * \chi_{c_k} \leq 2^k \chi\left(\frac{\xi - (c_1 + \cdots + c_k)}{k}\right) \tag{4.45}$$

□

Lemma 4.2.6. *(Properties of Γ) Let $\Gamma(x, y, z)$ as defined by equation (4.30), A, B, C, k, $N \in \mathbb{R}$, $N > 0$ then we have the following*

i) $\Gamma(kA, kB, kC) = k|k|\Gamma(A, B, C)$,

ii) $\Gamma(N, N, N) = 0$,

iii) $\Gamma(N, N, -N) = -2(2\pi^3)N^2$,

iv) The values of $\Gamma(A, B, C)$ do not change if we permute the inputs,

v) $|\Gamma(A, B, C)| \leq 2(2\pi)^2 \min\{|AB|, |BC|, |AC|\}$,

vi) $|\Gamma(A, B, C)| \leq 2(2\pi)^2 |ABC|^{2/3}$,

vii) $\Gamma(0, B, C) = 0$,

viii) If $A, B, C \geq 0$ then $\Gamma(A, B, C) = 0$.

Proof of Lemma 4.2.6. Part i) follows directly from the definition of $\Gamma(x, y, z)$. Using i) to prove ii) it is enough to compute $\Gamma(1, 1, 1)$,

$$\Gamma(1, 1, 1) = 2\pi^3 \left(1 + 1 + 1 - 2^2 - 2^2 - 2^2 + 3^2\right) = 0. \tag{4.46}$$

In the same way for *iii*) it is enough to compute $\Gamma(1,1,-1)$,

$$\Gamma(1,1,1) = 2\pi^3\left(1+1-1-2^2-0^2-0^2+1^2\right) = -2. \tag{4.47}$$

Part *iv*) comes directly from the symmetry of $\Gamma(x,y,z)$.

To prove part *v*) we need to use the integral formula that define $\Gamma(x,y,z)$,

$$\Gamma(x,y,z) \;=\; i\int \frac{1-e^{-2\pi i\alpha x}}{\alpha}\frac{1-e^{-2\pi i\alpha y}}{\alpha}\frac{1-e^{-2\pi i\alpha z}}{\alpha}d\alpha \tag{4.48}$$

Here we observe that

$$\frac{1-e^{-2\pi i\alpha x}}{\alpha} = 2\pi i x\int_0^1 e^{-2\pi i x\alpha(1-t_1)}dt_1 \tag{4.49}$$

Applying this to (4.48) we get

$$\begin{aligned}\Gamma(x,y,z) \;&=\; i(2\pi i)^2 yz\int_{\mathbb{R}}\int_0^1\int_0^1\frac{1}{\alpha}\left(1-e^{-2\pi i\alpha x}\right)e^{-2\pi i\alpha y(1-t_2)} \\ &\qquad \times e^{-2\pi i\alpha z(1-t_3)}dt_2dt_3d\alpha \\ &=\; i(2\pi i)^2 yz\int_0^1\int_0^1\int_{\mathbb{R}}\frac{1}{\alpha}\left(e^{-2\pi i\alpha(y(1-t_2)+z(1-t_3))}-1\right)d\alpha dt_1dt_2 \\ &\qquad -i(2\pi i)^2 yz\int_0^1\int_0^1\int_{\mathbb{R}}\frac{1}{\alpha}\left(1-e^{-2\pi i\alpha x-2\pi i\alpha(y(1-t_2)+z(1-t_3))}\right)d\alpha dt_1dt_2.\end{aligned} \tag{4.50}$$

Now using that $\int\frac{1-e^{-2\pi i\alpha x}}{\alpha}d\alpha = i\pi\mathrm{sgn}(x)$ we get that

$$\begin{aligned}|\Gamma(x,y,z)| \;&\leq\; (2\pi)^2|yz|\int_0^1\int_0^1\Big|-\mathrm{sgn}\left(y(1-t_2)+z(1-t_3)\right) \\ &\qquad +\mathrm{sgn}\left(x+y(1-t_2)+z(1-t_3)\right)\Big|dt_2dt_3 \\ &\leq\; 2(2\pi)^2|yz|.\end{aligned} \tag{4.51}$$

Part *vi*) is obtained from *v*) by taking the geometric average Part *vii*) This is direct consequence of *v*). Part *viii*) can be obtained from (4.51) by noticing that

$y(1-t_2)+z(1-t_3) > 0$ and $x+y(1-t_2)+z(1-t_3) > 0$ and therefore the integrand vanishes.

□

4.2.3 Norm inflation for the First Order Truncation

A useful notation that we will use in the rest of the chapter is the following.

Definition 4.2.7. Given $g : [0,T] \times \mathbb{R} \to \mathbb{C}$ we define $E(g) : \mathbb{R} \to \mathbb{C}$ by

$$\widehat{E(g)}(t,x) = \int_0^t e^{-2\pi(t-\tau)|\xi|} g(\tau,\xi) d\tau. \tag{4.52}$$

We consider the Taylor expansion of the Muskat equation (4.8), truncated up to the first non-zero non-linear term,

$$\partial_t f + \Lambda f = T_1 f, \qquad f(0) = f_0, \tag{4.53}$$

where T_1 is defined by (4.7). We look at its second Picard iteration

$$\partial_t f + \Lambda f = T_1 e^{-t\Lambda} f_0, \qquad f(0) = f_0. \tag{4.54}$$

Then using the Duhamel formula we can write the solution as

$$f(t) = e^{t\Lambda}\varphi + \int_0^t e^{-(t-\tau)\Lambda} T_1 \left(e^{t\Lambda} f_0\right) d\tau = e^{-t\Lambda}\varphi + g_3. \tag{4.55}$$

Our goal is to show that for certain spaces $\mathcal{F}_q^{m,p}$, given $T > 0$ there exists some time $0 < \tilde{t} < T$ and some initial condition such that the term $g_3(\tilde{t})$ becomes large and is the dominant term in the expansion (4.55). More precisely we will prove the following:

Theorem 4.2.8. *Consider the truncation of the Muskat problem given by*

$$\partial_t f + \Lambda f = T_1 e^{-t\Lambda}\varphi, \qquad f(0) = 0, \tag{4.56}$$

where φ *is given by* (4.17)*, and* $t > 0$ *is a time such that* $t(M+1) < 1$ *and* $tk_0 \gg 1$*. Then the solution* f *of* (4.56) *satisfies*

$$\|f\|_{\dot{\mathcal{F}}^{m,p}_q} \geq C_1 \beta_N^3 \sum_{j\in S(N)} \gamma_j^3 k_j - \frac{C_2}{t^2}\beta_N^3 \left(\sum_{j\in S(N)} \left(\gamma_j^3 k_j^m\right)^q\right)^{1/q} - \beta_N^3 \frac{C_3}{t^4 k_N}\left(\sum_{j\in S(N)} \gamma_j\right)^3 \tag{4.57}$$

Where the constants C_1*,* C_2 *and* C_3 *only depend on* M*,*m*,*q*,*p*.*

Corollary 4.2.9. *For any* $T > 0$*,* $R > 0$ *consider the problem* (4.56)*. Then there exists some* $0 < \tilde{T} < T$ *and some initial* φ_R *such that for* $p \geq 1$

$$\|\varphi_R\|_{\dot{\mathcal{F}}^{1/3,p}_\infty} < \frac{1}{R} \tag{4.58}$$

and

$$\|f(\tilde{R})\|_{\dot{\mathcal{F}}^{1/3,p}_\infty} \geq R \tag{4.59}$$

Proof of Corollary 4.2.9. First by Remark 4.2.2 for $m = \frac{1}{3}$, $q = \infty$ define $\gamma_j = \frac{1}{k_j^m}$ then we have $\gamma_j k_j^m = 1$, and then we get that

$$\|\varphi\|_{\dot{\mathcal{F}}^{1/3,p}_\infty} \leq C\frac{Q}{r^\alpha}. \tag{4.60}$$

Notice that this expression tends to zero as $r \to \infty$ for any $\alpha > 0$. Now using Theorem 4.2.8 and the linearity, we can bound the solution of (4.56) with initial condition φ using (4.55)

$$\|f\|_{\dot{\mathcal{F}}^{m,p}_q} \geq \|g_3\|_{\dot{\mathcal{F}}^{m,p}_q} - \|e^{-t\Lambda}\varphi\|_{\dot{\mathcal{F}}^{m,p}_q}. \tag{4.61}$$

By taking $t < \frac{1}{R}$ so that $t(M+1) < 1$ and $tk_0 \gg 1$ then

$$\begin{aligned} \|g_3(t)\|_{\dot{\mathcal{F}}^{m,p}_q} &\geq c_1\beta_N^3 \sum_{s\in S(N)} 1 - \frac{C_2}{t^2}\beta_N^3 - \beta_N^3 \frac{C_3}{t^4 k_{\min j}} \left(\sum_{j\in S(N)} \gamma_j\right)^3 \\ &= c_1\beta_N^3(\#S(N)) - \frac{C_2}{t^2}\beta_N^3 - \beta_N^3\frac{C_3}{t^4 k_{\min j}} \left(\sum_{j\in S(N)} \gamma_j\right)^3. \end{aligned} \tag{4.62}$$

Now because t is fixed, and because γ_j decay very fast, it is easy to see that the last two terms are bounded in N, and the first one is going to grow if $\beta_N^3(\#S(N))$ is increasing in N, then given $R > 0$ there exists some N_0 such that for any $N > N_0$

$$\|g_3(t)\|_{\dot{\mathcal{F}}^{m,p}_q} > 2R \tag{4.63}$$

Finally because $e^{-2\pi t|\xi|}|\hat{\varphi}| \leq |\hat{\varphi}|$ we get that

$$\|e^{-t\Lambda}\varphi\|_{\dot{\mathcal{F}}^{m,p}_q} \leq \|\varphi\|_{\dot{\mathcal{F}}^{m,p}_q} \tag{4.64}$$

therefore we get from (4.61) that

$$\|f\|_{\dot{\mathcal{F}}^{1/3,p}_\infty} \geq 2R - \|\varphi\|_{\dot{\mathcal{F}}^{1/3,p}_\infty} \geq R - \beta_N \tag{4.65}$$

by taking N_0 even larger if needed to ensure that $\|\varphi\|_{\dot{\mathcal{F}}^{1/3,p}_\infty} \leq R$. This can always be done because $\|\varphi\|_{\dot{\mathcal{F}}^{1/3,p}_\infty} \leq \beta_N \to 0$ as $r \to \infty$. Therefore $\beta_N = N^{-\frac{1}{3}+\varepsilon}$, $m = 1/3$, $q = \infty$, $p \geq 1$, $N \geq N_0$, $k_{\hat{j}}$ large ($\hat{j} = \min_{S(N)} j$) so that $tk_{\hat{j}} \gg 1$ we conclude the inflation result of Corollary 4.2.9.

□

The idea of the construction is to get initial data that can concentrate after a short time near frequency M, and then use that the smoothing effect allow us to estimate the decay of the high frequency part to conclude that for a special small time it is possible to observe the norm inflation phenomenon.

Proof of Theorem 4.2.8. Before proceeding to estimate g_3 we look at the following integral

$$\begin{aligned} I(\xi) &= \mathcal{F}\left(T_1 e^{-\tau\Lambda}\varphi\right) \\ &= \frac{1}{3}\int 2\pi i\xi\left(m_\alpha e^{-2\pi\tau|\cdot|}\hat{\varphi}\right) * \left(m_\alpha e^{-2\pi\tau|\cdot|}\hat{\varphi}\right) * \left(m_\alpha e^{-2\pi\tau|\cdot|}\hat{\varphi}\right) d\alpha \end{aligned} \tag{4.66}$$

To evaluate $I(\xi)$ we will expand (4.66) by substituting the initial condition (4.17) and use Lemma 4.2.3. We focus on what happen near frequency $\xi = M$, because the low frequency terms decay slower

$$I(\xi) = \beta_N^3 \sum_{j\in S(N)} \gamma_j^3 (J_1^j + J_2^j + J_3^j + J_4^j) + HF, \tag{4.67}$$

where

$$\begin{aligned} J_1^j &= \frac{(-1)}{3}\int_{\mathbb{R}} 2\pi i\xi (m_\alpha e^{-2\pi\tau|\cdot|} P_{k_j}) * (m_\alpha e^{-2\pi\tau|\cdot|} P_{k_j}) * (m_\alpha e^{-2\pi\tau|\cdot|} P_{k_j}) d\alpha, \\ J_2^j &= -\int_{\mathbb{R}} 2\pi i\xi (m_\alpha e^{-2\pi\tau|\cdot|} P_{k_j}) * (m_\alpha e^{-2\pi\tau|\cdot|} P_{k_j}) * (m_\alpha e^{-2\pi\tau|\cdot|} P_{2k_j+M}) d\alpha, \\ J_3^j &= -\int_{\mathbb{R}} 2\pi i\xi (m_\alpha e^{-2\pi\tau|\cdot|} P_{k_j}) * (m_\alpha e^{-2\pi\tau|\cdot|} P_{2k_j+M}) * (m_\alpha e^{-2\pi\tau|\cdot|} P_{2k_j+M}) d\alpha, \\ J_4^j &= \frac{(-1)}{3}\int_{\mathbb{R}} 2\pi i\xi (m_\alpha e^{-2\pi\tau|\cdot|} P_{2k_j+M}) * (m_\alpha e^{-2\pi\tau|\cdot|} P_{2k_j+M}) \\ &\qquad * (m_\alpha e^{-2\pi\tau|\cdot|} P_{2k_j+M}) d\alpha, \end{aligned} \tag{4.68}$$

and HF correspond to the off-diagonal terms

$$HF = -\frac{1}{3}\beta_N^3 \sum_{(s_1,s_2,s_3)\in S} \sum_{(a,b,c)\in\Lambda(s_1,s_2,s_3)} \gamma_{s_1}\gamma_{s_2}\gamma_{s_3} \\ \times \int_{\mathbb{R}} 2\pi i\xi(m_\alpha e^{-2\pi\tau|\cdot|}P_a) * (m_\alpha e^{-2\pi\tau|\cdot|}P_b) * (m_\alpha e^{-2\pi\tau|\cdot|}P_c)d\alpha, \quad (4.69)$$

where

$$\begin{aligned} S &= \{(s_1,s_2,s_3) \in S(N)^3 : s_1, s_2, s_3 \text{ not all equal }\}, \\ \Lambda(s_1,s_2,s_3) &= \{(a_1,a_2,a_3) : a_i \in \{\pm k_{s_i}, \pm(2k_{s_i}+M)\}, i = 1,2,3\}. \end{aligned} \quad (4.70)$$

Lemma 4.2.10 (Lower bound for J_2)**.** *Let $t > 0$ such that $tk_1 \gg 1$, $t(M+1) < 1$. Then term J_2 satisfies*

$$|\widehat{E(J_2^s)}| = \left|\beta_N^3 \sum_{j\in S(N)} \int_0^t e^{-2\pi(t-\tau)|\xi|} J_2^j d\tau\right| \geq \\ c_1\beta_N^3 \sum_{j\in S(N)} \gamma_j^3 k_j P_M - \frac{c_2}{t^2}\beta_N^3 \sum_{j\in S(N)} \gamma_j^3 \left(P_{2k_j+M} + P_{4k_j+M}\right) \quad (4.71)$$

and consequently

$$\left\|\beta_N^3 \sum_j \gamma_j^3 E(J_2^j)(t)\right\|_{\dot{\mathcal{F}}_q^{m,p}} \geq c_3\beta_N^3 \sum_j \gamma_j^3 k_j - \frac{c_4}{t^2}\beta_N^3 \left(\sum_j \left(\gamma_j^3 k_j^m\right)^q\right)^{1/q} \quad (4.72)$$

Proof of Lemma 4.2.10.

$$J_2^j = 2\pi \sum_{A,B,C\in\Omega(j)} \int\int \xi e^{-2\pi\tau|\xi-\xi_1|} e^{-2\pi\tau|\xi_1-\xi_2|} e^{-2\pi\tau|\xi_2|} G_{ABC}(\xi,\xi_1,\xi_2)d\xi_1 d\xi_2, \quad (4.73)$$

where $\Omega(j) = \{(A,B,C) : A = \pm k_j, B = \pm k_j, C = \pm(2k_j+M)\}$ and

$$G_{ABC}(\xi,\xi_1,\xi_2) = i\int_{\mathbb{R}} m_\alpha(\xi-\xi_1)m_\alpha(\xi_1-\xi_2)m_\alpha(\xi_2)d\alpha \\ \times \chi_A(\xi-\xi_1)\chi_B(\xi_1-\xi_2)\chi_C(\xi_2). \quad (4.74)$$

We already computed $G_{ABC}(\xi,\xi_1,\xi_2)$ in Lemma 4.2.3 and for this particular case we can compute some specific values of $\Gamma(A,B,C)$ directly

$$\Gamma(k_j,k_j,-(2k_j+M)) = -(2\pi^3)4k_j^2, \qquad \Gamma(k_j,k_j,(2k_j+M)) = 0$$

$$\Gamma(k_j,-k_j,(2k_j+M)) = -(2\pi^3)2k_j^2, \qquad \Gamma(k_j,-k_j,-(2k_j+M)) = (2\pi^3)2k_j^2$$

Applying Lemma 4.2.3 we can estimate J_2 using

$$\begin{aligned} J_2^j &= 2\pi \sum_{A,B,C} \xi \int\int e^{-2\pi(t-\tau)(|\xi-\xi_1|+|\xi_1-\xi_2|+|\xi_2|)} G_{ABC}(\xi,\xi_1,\xi_2) d\xi_1 d\xi_2 \\ &\geq G_j(\xi,\tau) + H_j(\xi,\tau), \end{aligned} \tag{4.75}$$

where $G_j(\xi,\tau)$ and $H_j(\xi,\tau)$ are given by

$$\begin{aligned} G_j(\xi,\tau) = 4\pi^4 \xi e^{-2\pi\tau(4k_j+M+3)} \Big(& k_j^2 \chi(\xi - M)(4 + O(1/k_j)) \\ & - k_j^2 \chi(\xi + M)(4 + O(1/k_j)) \Big) \end{aligned} \tag{4.76}$$

$$\begin{aligned} H_j(\xi,\tau) = -4\pi^4 |\xi| e^{-2\pi\tau(4k_j+M-3)} \Big(& 4k_j^2 \chi\left(\frac{\xi - 4k_j - M}{4}\right) O(1/k_j) \\ & + 4k_j^2 \chi\left(\frac{\xi - 2k_j - M}{4}\right)(2 + O(1/k_j)) \\ & + 8k_j^2 \chi\left(\frac{\xi + 2k_j + M}{4}\right)(2 + O(1/k_j)) \\ & + 8k_j^2 \chi\left(\frac{\xi + 4k_j + M}{4}\right)(2 + O(1/k_j)) \Big) \end{aligned} \tag{4.77}$$

Notice supp $G_j \subset [-M-1,-M+1] \cup [M-1,M+1]$ and supp $H_j \subset (-k_j,k_j)^c$.

Now we define

$$\begin{aligned} L(t,\xi) &= \beta_N^3 \sum_{j \in S(N)} \gamma_j^3 \int_0^t e^{-2\pi(t-\tau)|\xi|} J_2^j d\tau \\ &\geq L_{1,1} + L_{1,2} \end{aligned} \tag{4.78}$$

Where

$$\begin{aligned} L_{1,1} &= \beta_N^3 \sum_{j\in S(N)} \gamma_j^3 \int_0^t e^{-2\pi(t-\tau)|\xi|} G_j(\xi,\tau)d\tau \\ L_{1,2} &= \beta_N^3 \sum_{j\in S(N)} \gamma_j^3 \int_0^t e^{-2\pi(t-\tau)|\xi|} H_j(\xi,\tau)d\tau \end{aligned} \tag{4.79}$$

For $L_{1,1}$ we have

$$\begin{aligned} L_{1,1} &:= \beta_N^3 \sum_{j\in S(N)} \gamma_j^3 \int_0^t e^{-2\pi(t-\tau)|\xi|} G_j(\xi,\tau)d\tau \\ &\geq C\beta_N^3 \sum_{j\in S(N)} \gamma_j^3 \int_0^t e^{-2\pi(t-\tau)|\xi|} e^{-\tau(4k_j+M+3)} 2\pi\xi k_j^2 (4+O(1/k_j))d\tau P_M \\ &\geq C\beta_N^3 \sum_{j\in S(N)} \gamma_j^3 e^{-2\pi t(M+1)} \int_0^t e^{-2\pi\tau(4k_j+2)} k_j^2(4+O(1/k_j))d\tau P_M \\ &\geq C\beta_N^3 \sum_{j\in S(N)} \gamma_j^3 e^{-2\pi t(M+1)} (1-e^{-2\pi t(4k_j+2)}) \frac{k_j^2}{4k_j+2}(4+O(1/k_j))P_M. \end{aligned} \tag{4.80}$$

By choosing t such that $tk_j \gg 1$ and $t(M+1) < 1$ we get

$$L_{1,1} \geq C\beta_N^3 \sum_{j\in S(N)} \gamma_j^3 k_j P_M. \tag{4.81}$$

The next term that we need to estimate is $L_{1,2}$

$$\begin{aligned} L_{1,2} &:= \beta_N^3 \sum_{j\in S(N)} \gamma_j^3 \int_0^t e^{-2\pi(t-\tau)|\xi|} H_j(\xi,\tau)d\tau \\ |L_{1,2}| &\leq C\beta_N^3 \sum_{j\in S(N)} \int_0^t \gamma_j^3 e^{-2\pi(t-\tau)|\xi|} e^{-2\pi\tau(4k_j+M-3)}(4k_j+M) \\ &\qquad \times k_j^2(1+1/k_j)h_j(\xi)d\tau \\ &\leq C\beta_N^3 \sum_{j\in S(N)} \int_0^t \gamma_j^3 e^{-2\pi(t-\tau)k_j} e^{-2\pi\tau(4k_j+M-3)} k_j^3 h_j(\xi)d\tau \\ &\leq C\beta_N^3 \sum_{j\in S(N)} \gamma_j^3 h_j(\xi) e^{-2\pi t k_j} \frac{k_j^3}{3k_j+M-3}(1-e^{-2\pi t(3k_j+M-3)}) \\ &\leq C\beta_N^3 \sum_{j\in S(N)} \gamma_j^3 k_j^2 e^{-2\pi t k_j} h_j(\xi), \end{aligned} \tag{4.82}$$

where $h_j(\xi) = P_{2k_j+M} + P_{4k_j+M}$. To complete the estimate of $L_{1,2}$ we need the following observation.

Lemma 4.2.11. *Let* $t, x > 0$, $n \in \mathbb{N}$ *then*

$$x^n e^{-tx} \leq 2^n n! \frac{e^{tx/2}}{t^n}. \tag{4.83}$$

Proof of Lemma 4.2.11.

$$e^{-tx} x^n = e^{-tx} \frac{t^n x^n}{2^n \cdot n!} \frac{2^n \cdot n!}{t^n} \leq 2^n n! \frac{e^{-tx/2}}{t^n} \tag{4.84}$$

□

Remark 4.2.12. The purpose of Lemma 4.2.11 is to make precise the notion that the exponential dominates over powers and it shows the dependence on t of this estimate, which will be important for us as we want to take t to be small.

Using Lemma 4.2.11 we get that $k_j^2 e^{-2\pi t k_j} \leq \frac{C}{t^2} e^{-2\pi t k_j/2}$ and therefore

$$|L_{1,2}| \leq C \frac{1}{t^2} \beta_N^3 \sum_{j \in S(N)} \gamma_j^3 e^{-2\pi t k_j/2} h_j(\xi). \tag{4.85}$$

□

A similar analysis we can be used to estimate J_1, J_3 and J_4 more precisely

Lemma 4.2.13 (Estimate for J_1, J_3 and J_4). *Under the same conditions of Lemma 4.2.10*

$$|\widehat{E(J_i)}| = \left| \beta_N^3 \sum_{j \in S(N)} \gamma_j^3 \int_0^t e^{-2\pi\tau|\xi|} J_i^j \, d\tau \right| \leq \frac{C}{t^2} \beta_N^3 \sum_{j \in S(N)} \gamma_j^3 \tilde{h}_j(\xi), \qquad i = 1, 3, 4 \tag{4.86}$$

where $\text{supp}\,\tilde{h}_j(\xi) \subset [k_j/2, 7k_j]$ *and* $\|\tilde{h}_j\|_{\dot{\mathcal{F}}_q^{m,p}} \leq C k_j^m$ *where* C *is independent of* j*. And by taking the* $\dot{\mathcal{F}}_q^{m,p}$ *norm, we get*

$$\|E(J_i)(t)\|_{\dot{\mathcal{F}}_q^{m,p}} \leq \frac{C}{t^2}\beta_N^3 \left(\sum_j \left(\gamma_j^3 k_j^m\right)^q \right)^{1/q} \tag{4.87}$$

Proof of Lemma 4.2.13. To estimate the terms J_1, J_3, J_4 we use the same idea as for the estimate for the high frequency part of J_2. Consider

$$J_i^j = c_i \sum_{a,b,c\in\Omega_i(j)} \int\int \xi e^{-2\pi\tau|\xi-\xi_1|} e^{-2\pi\tau|\xi_1-\xi_2|} e^{-2\pi\tau|\xi_2|} G_{abc}(\xi,\xi_1,\xi_2) d\xi_1 d\xi_2, \tag{4.88}$$

for $i = 1, 3, 4$ and where $c_1 = c_4 = \frac{-2\pi}{3}$, $c_3 = -2\pi$,

$$\begin{aligned} \Omega_1(j) &= \{(a,b,c) : a = \pm k_j, b = \pm k_j, c = \pm k_j\} \\ \Omega_3(j) &= \{(a,b,c) : a = \pm k_j, b = \pm(2k_j + M), c = \pm(2k_j + M)\} \\ \Omega_4(j) &= \{(a,b,c) : a = \pm(2k_j + M), b = \pm(2k_j + M), c = \pm(2k_j + M)\} \end{aligned} \tag{4.89}$$

The key part of the estimate is to notice that in all the cases G_{abc} given by

$$G_{abc}(\xi,\xi_1,\xi_2) = \frac{1}{i}\int_{\mathbb{R}} m_\alpha(\xi-\xi_1) * m_\alpha(\xi_1-\xi_2) * m_\alpha(\xi_2) d\alpha$$
$$\times\, \chi_a(\xi-\xi_1)\chi_b \xi_1 - \xi_2)\chi_c(\xi_2). \tag{4.90}$$

can be estimated using Lemma 4.2.3 obtaining

$$|G_{abc}| \leq C\left((2k_j + M)^2 + O(k_j)\right) h_j(\xi) \tag{4.91}$$

where $\text{supp}\, h_j(\xi) \subset [k_j/2, 7k_j]$ and $\|h_j\|_{\mathcal{F}_q^{m,p}} \leq C k_j^m$ and therefore we get that

$$\begin{aligned}
\widehat{E(J_i)} &:= \beta_N^3 \sum_{j\in S(N)} \gamma_j^3 \int_0^t e^{-2\pi(t-\tau)|\xi|} J_i^j d\tau \\
|\widehat{E(J_i)}| &\leq C\beta_N^3 \sum_{j\in S(N)} \int_0^t \gamma_j^3 e^{-2\pi(t-\tau)k_j/2} e^{-2\pi\tau(3k_j-3)}(7k_j)^3 h_j(\xi) d\tau \\
&\leq C\beta_N^3 \sum_{j\in S(N)} \gamma_j^3 e^{-2\pi t k_j/2} h_j(\xi) \frac{1}{5/2k_j - 3} k_j^3 (1 - e^{-2\pi t(5/2k_j-3)}) \\
&\leq \frac{C}{t^2}\beta_N^3 \sum_{j\in S(N)} \gamma_j^3 e^{-2\pi t k_j/4} h_j(\xi)
\end{aligned} \tag{4.92}$$

We conclude by the proof of Lemma 4.2.13 taking the $\mathcal{F}_q^{m,p}$ norm and using the triangle inequality.

□

Lemma 4.2.14 (Estimate High frequency part)**.** *Let $t > 0$ such that $tk_1 \gg 1$, $t(M+1) < 1$. Let HF given by* (4.69) *Then*

$$|\widehat{E(HF)}| = \left| \int_0^t e^{-2\pi(t-\tau)|\xi|} HF d\tau \right| \leq$$
$$C\beta_N^3 \sum_{(s_1,s_2,s_3)\in S} \sum_{\substack{a\in\Lambda(s_1), b\in\Lambda(s_2) \\ c\in\Lambda(s_3)}} \gamma_{s_1}\gamma_{s_2}\gamma_{s_3} \frac{1}{t^2} e^{-2\pi t|a|/4} \chi\left(\frac{\xi - (a+b+c)}{3}\right) \tag{4.93}$$

and

$$\|E(HF)\|_{\dot{\mathcal{F}}_q^{m,p}} \leq \beta_N^3 \frac{C(M,m,p,q)}{t^4 k_{\min j}} \left(\sum_{j\in S(N)} \gamma_j \right)^3 \tag{4.94}$$

Proof of Lemma 4.2.14. From equation (4.69) we know that the high frequency part is given by

$$\begin{aligned}
HF &= -\frac{1}{3}\beta_N^3 \sum_{(s_1,s_2,s_3)\in S} \sum_{\substack{a\in\Lambda(s_1), b\in\Lambda(s_2) \\ c\in\Lambda(s_3)}} \gamma_{s_1}\gamma_{s_2}\gamma_{s_3} \\
&\quad \times \int_{\mathbb{R}} 2\pi i\xi (m_\alpha e^{-2\pi\tau|\cdot|} P_a) * (m_\alpha e^{-2\pi\tau|\cdot|} P_b) * (m_\alpha e^{-2\pi\tau|\cdot|} P_c) d\alpha
\end{aligned} \tag{4.95}$$

where

$$\begin{aligned} S &= \{(s_1, s_2, s_3) \in S(N)^3 : s_1, s_2, s_3 \text{ not all equal } \}, \\ \Lambda(s_i) &= \{a : a \in \{\pm k_{s_i}, \pm(2k_{s_i} + M)\}\}, i = 1, 2, 3. \end{aligned} \tag{4.96}$$

then the general term that we have to estimate is

$$R_{ABC} = \frac{1}{3} \int e^{-2\pi(t-\tau)|\xi|} 2\pi i \chi (m_\alpha e^{-2\pi\tau|\cdot|} \chi_A) * (m_\alpha e^{-2\pi\tau|\cdot|} \chi_B) * (m_\alpha e^{-2\pi\tau|\cdot|} \chi_C) d\tau \tag{4.97}$$

Where $A \in \Lambda(s_1)$, $B \in \Lambda(s_2)$, $C \in \Lambda(s_3)$. We split the terms in two groups

i) $\Omega_1 = \{(s_1, s_2, s_3) \in S(N)^3$: one s_i is strictly larger than the other two$\}$,

ii) $\Omega_2 = \{(s_1, s_2, s_3) \in S(N)^3$: two s_i are equal and the third one is smaller$\}$.

Notice that $S = \Omega_1 \cup \Omega_2$. Using this we can split (4.69)

$$HF = HF_1 + HF_2 \tag{4.98}$$

Where

$$HF_i = \sum_{\Omega_i} \sum_{A,B,C} \gamma_1 \gamma_2 \gamma_3 R_{ABC}, \qquad i = 1, 2. \tag{4.99}$$

Esimate for HF_1

To estimate R_{abc} in HF_1 first we notice that by symmetry we can assume that

$|a| \geq |b| \geq |c|$. For the terms in Ω_1 we can estimate (4.97) by using (4.82)

$$
\begin{aligned}
|\widehat{E(R_{abc})}| &= \left| \int_0^t e^{-2\pi(t-\tau)|\xi|} R_{abc} d\tau \right| \\
&\leq C \int_0^t e^{-2\pi(t-\tau)(|a+b+c|-3)} (|a+b+c|+3) e^{-2\pi\tau(|a|+|b|+|c|-3)} \\
&\qquad \times (\Gamma(a,b,c) + O(|a|+|b|+|c|))\, \chi\left(\frac{\xi-(a+b+c)}{3}\right) \\
&\leq C e^{-2\pi t(|a+b+c|-3)} \int_0^t e^{2\pi\tau(|a+b+c|-3)} (|a+b+c|+3) e^{-2\pi\tau(|a|+|b|+|c|-3)} \\
&\qquad \times (\Gamma(a,b,c) + O(|a|+|b|+|c|))\, \chi\left(\frac{\xi-(a+b+c)}{3}\right).
\end{aligned} \tag{4.100}
$$

Here we use that $|a|/2 > |b+c|$, then $|a+b+c|-3 > |a|/2$, $|a|+|b|+|c| > |a|$ and $|a|+|b|+|c|-|a+b+c| \geq |a|/2$. Now by Lemma 4.30 we know that $|\Gamma(a,b,c)| \leq C|abc|^{2/3}$ therefore

$$
\begin{aligned}
|\widehat{E(R_{abc})}| &\leq C(|a+b+c|+3) e^{-2\pi t(|a+b+c|-3)} \\
&\qquad \times \int_0^t e^{-2\pi\tau|a|/2} |abc|^{2/3} \chi\left(\frac{\xi-(a+b+c)}{3}\right) d\tau \\
&\leq \frac{C}{t} e^{-2\pi t(|a+b+c|-3)/2} \frac{|abc|^{2/3}}{|a|/2} (1 - e^{-2\pi t|a|/2}) \chi\left(\frac{\xi-(a+b+c)}{3}\right) \\
&\leq \frac{C}{t} |abc|^{1/3} e^{-2\pi t|a|/4} \chi\left(\frac{\xi-(a+b+c)}{3}\right).
\end{aligned} \tag{4.101}
$$

Now to estimate HF_1 we need to sum over all terms that satisfy this condition, to do so we need to count how many terms satisfy this estimate. In Ω_1, up to permutations, we can assume that $s_1 > s_2$ and $s_1 > s_3$ and therefore all corresponding R_{abc} are supported in the same annulus C_k as a belongs to (we might need a slightly wider annulus C_k , but that is not important).

$$\begin{aligned} \|E(HF_1)\|_{\mathcal{F}_q^{m,p}} &= \left\| \beta_N^3 \sum_{s_1,s_2,s_3} \sum_{a,b,c} \gamma_{s_1}\gamma_{s_2}\gamma_{s_3} R_{abc} \right\|_{\mathcal{F}_q^{m,p}} \\ &\leq C\beta_N^3 \sum_{s_1,s_2,s_3} \gamma_{s_1}\gamma_{s_2}\gamma_{s_3} \|R_{abc}\|_{\mathcal{F}_q^{m,p}} \\ &\leq \frac{C}{t}\beta_N^3 \sum_{s_1,s_2,s_3} \gamma_{s_1}\gamma_{s_2}\gamma_{s_3} (|a||b||c|)^{1/3}(|a+b+c|+3)^m e^{-2\pi t|a|/4} \\ &\leq \frac{C}{t^4 k_{\min j}}\beta_N^3 \sum_{s_1,s_2,s_3} \gamma_{s_1}\gamma_{s_2}\gamma_{s_3} \frac{(|a||b||c|)^{1/3}}{|a|} \\ &\leq \frac{C}{t^3 k_{\min j}}\beta_N^3 \left(\sum_{j\in S(N)} \gamma_j \right)^3 \end{aligned} \tag{4.102}$$

Notice that the dependence on p and q is included in the constant and comes from $\|\chi\left(\frac{\cdot-(a+b+c)}{3}\right)\|_{\mathcal{F}_q^{m,p}} \leq C_{p,q}(|a+b+c|+3)^m$. Therefore we obtain

$$\|E(HF_1)\|_{\mathcal{F}_q^{m,p}} \leq \beta_N^3 \frac{C(M,m,p,q)}{t^4 k_{\min j}} \left(\sum_{j\in S(N)} \gamma_j \right)^3 \tag{4.103}$$

Estimate for HF_2

For HF_2 we proceed in a similar way. Again assuming that (a,b,c) are decreasing in modulus, and so we have that a and b are of comparable sizes so we need to be more careful. In the counting step we get that when we fix the maximum we have (s_1-1) options for the third value. Now we consider two cases and we split

$$HF_2 = HF_2^{(1)} + HF_2^{(2)}. \tag{4.104}$$

Case 1: If $|a+b| > a/2$ or $|a+b| > b/2$ everything works exactly the same and all the estimates for HF_1 are valid, thus

$$\|E(HF_2^{(1)})\|_{\mathcal{F}_q^{m,p}} \leq \beta_N^3 \frac{C(M,m,p,q)}{t^4 k_{\min j}} \left(\sum_{j\in S(N)} \gamma_j \right)^3. \tag{4.105}$$

Case 2: If we are not in the situation of case 1, then necessarily we have that $a+b=0$, and lots of terms simplify. Now we proceed to estimate $HF_2^{(2)}$. In equation (4.100) we get instead

$$\begin{aligned} |\widehat{E(R_{abc})}| &\leq C\int_0^t e^{-2\pi(t-\tau)(|c|-3)}(|c|+3)e^{-2\pi\tau(|a|+|b|+|c|-3)} \\ &\qquad \times (\Gamma(a,b,c)+O(|a|+|b|+|c|))\,\chi\left(\frac{\xi-c}{3}\right) \\ &\leq Ce^{-2\pi t(|c|-3)}(|c|+3)\int_0^t e^{-2\pi\tau(|a|+|b|)} \\ &\qquad \times (\Gamma(a,b,c)+O(|a|+|b|+|c|))\,\chi\left(\frac{\xi-c}{3}\right) \end{aligned} \tag{4.106}$$

By Lemma 4.2.6 we know that

$$|\Gamma(a,b,c)| \leq C|abc|^{2/3} \tag{4.107}$$

then

$$|\widehat{E(R_{abc})}| \leq C\int_0^t e^{-2\pi(t-\tau)|c|/2}e^{-\pi\tau|a|}|a||c|d\tau\chi\left(\frac{\xi-c}{3}\right) \tag{4.108}$$

Then we get

$$\begin{aligned} |\widehat{E(R_{abc})}| &\leq \frac{C}{t}e^{-\pi t|c|}\int_0^t e^{-\pi\tau|a|}|abc|^{2/3}d\tau\chi\left(\frac{\xi-c}{3}\right) \\ &\leq Ce^{-\pi t|c|}\frac{|abc|^{2/3}}{2\pi|a|}(1-e^{-2\pi\tau|a|})\chi\left(\frac{\xi-c}{3}\right) \\ &\leq \frac{C}{t^2|c|}e^{-\pi t|c|/2}\frac{|abc|^{2/3}}{|a|}\chi\left(\frac{\xi-c}{3}\right) \end{aligned} \tag{4.109}$$

Now we need to sum over all the triples that satisfy the estimate,

$$\begin{aligned} |\widehat{E(HF_2^{(2)})}| &= \beta_N^3\sum_{\Omega_2}\sum_{a,b,c}\gamma_{s_1}\gamma_{s_2}\gamma_{s_3}|\widehat{E(R_{abc})}| \\ &\leq \beta_N^3\sum_{\Omega_2}\sum_{a,b,c}\gamma_{s_1}\gamma_{s_2}\gamma_{s_3}\frac{C}{t^3k_{\min j}}\frac{|abc|^{2/3}}{|a||c|}e^{-\pi t|c|/2}\chi\left(\frac{\xi-c}{3}\right) \end{aligned} \tag{4.110}$$

taking the $\dot{\mathcal{F}}_q^{m,p}$ norm of $E(HF_2)$ we obtain

$$\begin{aligned}
\|E(HF_2)\|_{\dot{\mathcal{F}}_q^{m,p}} &\leq C\beta_N^3 \frac{1}{t^3 k_{\min j}} \sum_{\Omega_2} \sum_{a,b,c} \gamma_{s_1}\gamma_{s_2}\gamma_{s_3} e^{-\pi t|c|/2} \left\| \chi\left(\frac{\cdot - c}{3}\right) \right\|_{\dot{\mathcal{F}}_q^{m,p}} \\
&\leq C\beta_N^3 \frac{1}{t^3 k_{\min j}} \sum_{\Omega_2} \sum_{a,b,c} \gamma_{s_1}\gamma_{s_2}\gamma_{s_3} e^{-\pi t|c|/2} C_{pq}|c|^m \\
&\leq C\beta_N^3 \frac{1}{t^4 k_{\min j}} \left(\sum_{j\in S(N)} \gamma_j \right)^3 .
\end{aligned} \tag{4.111}$$

This concludes the proof of Lemma 4.2.14. The idea of this estimate is that by the smoothing effect of the equation we can cancel as many powers of k_j as needed and we only need to powers with powers of t. □

(Continuation of the proof of Theorem 4.2.8)

We can apply Lemma 4.2.10 and 4.2.13 and 4.2.14 to obtain the lower bound for the norm of the evolution of Equation 4.67 to obtain

$$\begin{aligned}
\|g_3\|_{\dot{\mathcal{F}}_q^{m,p}} &= \left\|\beta_N^3 E(I)\right\|_{\dot{\mathcal{F}}_q^{m,p}} \\
&\geq \left\| \beta_N^3 \sum_s \gamma_s E(J_2^s) \right\|_{\dot{\mathcal{F}}_q^{m,p}} - \left\| \beta_N^3 \sum_s \gamma_s E(J_1^s) \right\|_{\dot{\mathcal{F}}_q^{m,p}} \\
&\quad - \left\| \beta_N^3 \sum_s \gamma_s E(J_3^s) \right\|_{\dot{\mathcal{F}}_q^{m,p}} - \left\| \beta_N^3 \sum_s \gamma_s E(J_3^s) \right\|_{\dot{\mathcal{F}}_q^{m,p}} \\
&\quad - \|E(HF)\|_{\dot{\mathcal{F}}_q^{m,p}} \\
&\geq C\beta_N^3 \sum_{s=0}^r \gamma_s^3 k_s - C_2 \frac{1}{t^2} \beta_N^3 \left(\sum_{s=0}^r \left(\gamma_s^3 k_s^m\right)^q \right)^{1/q} \\
&\qquad -C_3 \beta_N^3 \frac{1}{t^5} e^{-2\pi t k_{s_1}/8} - C_4 \beta_N^3 \frac{1}{t^2}.
\end{aligned} \tag{4.112}$$

This concludes the proof of Theorem 4.2.8. □

4.3 Norm Inflation for $\ell \geq 2$

In this section we want to construct initial data such that the truncation of the expansion of the Muskat problem (4.5) at some order ℓ produces norm inflation. For this purpose we consider the initial data $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi \in C^\infty(\mathbb{R})$ defined by

$$\hat{\varphi} = \beta_N \sum_{N \leq j \leq (1+\delta)N} \gamma_j (P_{k_j} + P_{2\ell k_j + M}). \tag{4.113}$$

First we need some preliminary estimates about the size of this initial data

Lemma 4.3.1. *[Size of the Initial data]Let φ given by* (4.113)*, then*

$$\|\varphi\|_{\mathcal{F}_q^{m,p}} \sim \beta_N \ell \left(\sum_{N \leq j \leq (1+\delta)N} \gamma_j^q k_j^{mq} \right)^{1/q} \tag{4.114}$$

Proof of Lemma 4.113. By definition of the $\mathcal{F}_q^{m,p}$ norm

$$\|\varphi\|_{\mathcal{F}_q^{m,p}} \sim \beta_N \left(\sum_{N \leq j \leq (1+\delta)N} \gamma_j^q \left(\left(\int |\xi|^{mp} |P_{k_j}|^p d\xi \right)^{q/p} \right.\right.$$
$$\left.\left. + \left(\int |\xi|^{mp} |P_{2\ell k_j + M}|^p d\xi \right)^{q/p} \right) \right)^{1/q}, \tag{4.115}$$

here we used that each term is supported in a different annulus C_k, then we get

$$\begin{aligned} \|\varphi\|_{\mathcal{F}_q^{m,p}} &\sim \beta_N \left(\sum_{N \leq j \leq (1+\delta)N} \gamma_j^q \Big((k_j+1)^{mq} + (2\ell k_j + M + 1)^{mq} \Big) \right)^{1/q} \\ &\sim \beta_N \ell \left(\sum_{N \leq j \leq (1+\delta)N} \gamma_j^q k_j^{mq} \right)^{1/q} \end{aligned} \tag{4.116}$$

□

Remark 4.3.2. Note that this estimate can be made independent of p. By using that $P_{k_j} + P_{2\ell k_j + M}$ are supported in only two annulus C_k, and they are disjoint for different j, and the easy observation that

$$\frac{1}{2}(|a|+|b|) \leq (|a|^p+|b|^p)^{1/p} \leq |a|+|b|. \tag{4.117}$$

Consider the Taylor expansion of the Muskat problem (4.5) but truncated up to the first ℓ non-zero non-linear

$$\partial_t + \Lambda f = \sum_{k=1}^{\ell} T_k f, \quad f(0) = f_0 \tag{4.118}$$

As before we consider the expansion $f = \sum_k \varepsilon^k f_k$, $f(0) = \varepsilon\varphi$, we get

$$\begin{cases} \partial_t f_1 + \Lambda f_1 = 0 & , \quad f_1(0) = \varphi \\ \partial_t f_{2k+1} + \Lambda f_{2k+1} & \\ \qquad = \displaystyle\sum_{j=1}^{k} \sum_{i_1+\cdots+i_{2j+1}=k} \frac{(-1)^{j-1}}{2j+1} \int_{\mathbb{R}} \partial_x(\Delta_\alpha f_{i_1}) \cdots (\Delta_\alpha f_{i_{2j+1}}) d\alpha & , \quad f_{2k+1}(0) = 0 \\ & \quad k \geq 1 \\ f_{2k} = 0 & , \quad k \geq 1 \end{cases} \tag{4.119}$$

To study this equation we consider a second approximation of this problem, which is obtained by considering the its second Picard iteration, the idea is that if we have good convergence properties for the Picard iteration, then the first few iterations should give a good approximation of the real behaviour of the solution. By doing this we obtain.

$$\partial_t f + \Lambda f = \sum_{k=1}^{\ell} T_k e^{-t\Lambda} f_0, \quad f(0) = f_0 \tag{4.120}$$

In terms of the system (4.119) we obtain

$$\begin{cases} \partial_t f_1 + \Lambda f_1 = 0 & , \quad f_1(0) = \varphi \\ \partial_t f_{2k+1} + \Lambda f_{2k+1} = \dfrac{(-1)^{k-1}}{2k+1} \displaystyle\int_{\mathbb{R}} \partial_x (\Delta_\alpha f_1)^{(2k+1)} d\alpha & , \quad f_{2k+1}(0) = 0, 1 \leq k \leq \ell \\ f_k = 0 & , \quad k \text{ even or } k \geq 2\ell + 2. \end{cases} \tag{4.121}$$

The main result that we will prove in this Section is the following.

Theorem 4.3.3 (Norm inflation for higher order truncations). *Given $\ell \in \mathbb{N}$, $R > 0$, $T > 0$ and $\varepsilon > 0$ there exists $\tilde{T} < T$ and some initial data φ of the form (4.113) such that the unique solution $f \in C(0, T; \mathcal{F}^{m,p}_q(\mathbb{R}))$ of the second Picard's iteration (4.118) of order ℓ with initial data φ satisfy*

$$\|f\|_{\mathcal{F}^{m,p}_q}(\tilde{T}) > R \text{ and } \|\varphi\|_{\mathcal{F}^{m,p}_q} < 1/R, \tag{4.122}$$

for $m = \frac{2\ell-1}{2\ell+1}$, $q > (2\ell+1)(1+\varepsilon)$, and any $p \geq 1$, $\gamma_j = \frac{1}{k_j^m j^{\frac{1+\varepsilon}{q}}}$, $\beta_N = 1$.

Lemma 4.3.4 (Estimate for $k < \ell$). *Let f_{2k+1} as defined by (4.121), t such that $tM < 1$ and $tk_N \gg 1$ then*

$$\begin{aligned} \|f_{2k+1}\|_{\mathcal{F}^{m,p}_q} \quad &\leq \quad \frac{C}{t^{2/q}} \beta_N^{2k+1} \left(\sum_j \gamma_j^{q(2k+1)} k_j^{(2k-2+m)q} \right)^{1/q} \\ &\qquad + \frac{C\beta_N^{2k+1}}{t^2 k_N} \left(\sum_j \gamma_j k_j^{\frac{2k-1}{2k+1}} \right)^{2k+1} \end{aligned} \tag{4.123}$$

Where the constant C depend on m, p, q, k, ℓ.

Lemma 4.3.5 (Estimate for $k = \ell$). *Let $f_{2\ell+1}$ as defined by (4.121), t such that*

$tM < 1$ *and* $tk_N \gg 1$ *then*

$$\|f_{2\ell+1}\|_{\mathcal{F}_q^{m,p}} \geq C \sum_{N\leq j\leq (1+\delta)N} \gamma_j^{2\ell+1} k_j^{2\ell-1} - \beta_N^{2\ell+1} \frac{C}{t^2 k_N} \sum_j \gamma_j^{2k+1} k_j^{2k-1} - \beta_N^{2\ell+1} \frac{C}{t^2 k_N} \left(\sum_j \gamma_j k_j^{\frac{2k-1}{2k+1}} \right)^{2k+1} \quad (4.124)$$

Where the constant C *depend on* m, p, q, ℓ.

Proof of Theorem 4.3.3. We prove theorem 4.3.3 using Lemmas 4.3.4 and 4.3.5. First, by definition of the $\dot{\mathcal{F}}_q^{m,p}$ norm, it is easy to see that

$$\|e^{-t\Lambda}\varphi\|_{\dot{\mathcal{F}}_q^{m,p}} \leq \|\varphi\|_{\dot{\mathcal{F}}_q^{m,p}} \quad (4.125)$$

Now we fix some small time $\tilde{T} < T$ such that $\tilde{T}M < 1$, then for $N > \tilde{N}$ such that $\tilde{T}k_{\tilde{N}} \gg 1$ Lemmas 4.3.4 and 4.3.5 are valid. Now we take consider $f = \sum_k \varepsilon^k f_k$ for $\varepsilon = 1$, then from Lemma 4.3.4 and 4.3.5 we get that

$$\begin{aligned}
\|f\|_{\dot{\mathcal{F}}_q^{m,p}} &\geq \|f_{2\ell+1}\|_{\dot{\mathcal{F}}_q^{m,p}} - \sum_{k<\ell} \|f_{2k+1}\|_{\dot{\mathcal{F}}_q^{m,p}} - \|e^{-t\Lambda}\varphi\|_{\dot{\mathcal{F}}_q^{m,p}} \\
&\geq C_1 \beta_N^{2\ell+1} \sum_{N\leq j\leq(1+\delta)N} \gamma_j^{2\ell+1} k_j^{2\ell-1} - \beta_N^{2\ell+1} \frac{C_2}{t^2 k_N} \sum_j \gamma_j^{2\ell+1} k_j^{2\ell-1} \\
&\quad -\beta_N^{2\ell+1} \frac{C_3}{t^2 k_N} \left(\sum_j \gamma_j k_j^{\frac{2\ell-1}{2\ell+1}} \right)^{2\ell+1} \\
&\quad -\sum_{k=1}^{\ell-1} \frac{C_4}{t^{2/q}} \beta_N^{2k+1} \left(\sum_j \gamma_j^{q(2k+1)} k_j^{(2k-2+m)q} \right)^{1/q} \\
&\quad -\sum_{k=1}^{\ell-1} \frac{C_5 \beta_N^{2k+1}}{t^2 k_N} \left(\sum_j \gamma_j k_j^{\frac{2k-1}{2k+1}} \right)^{2k+1} \\
&\quad -C_6 \beta_N \left(\sum_{N\leq j\leq(1+\delta)N} \gamma_j^q k_j^{mq} \right)^{1/q} \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6
\end{aligned} \quad . \quad (4.126)$$

Next by taking $m = \frac{2\ell-1}{2\ell+1}$, $q > (2\ell+1)(1+\varepsilon)$, and any $p \geq 1$, $\gamma_j = \frac{1}{k_j^m j^{\frac{1+\varepsilon}{q}}}$, $\beta_N = 1$. Then we get the following estimates for each of the terms

i)

$$I_1 = \sum_{N \leq j \leq (1+\delta)N} \gamma_j^{2\ell+1} k_j^{2\ell-1} \sim \sum_{N \leq j \leq (1+\delta)N} \frac{1}{j^{\frac{(1+\varepsilon)(2\ell+1)}{q}}} \to \infty \text{ as } N \to \infty$$

So take N such that $C_1 \beta_N^{2\ell+1} \sum_{N \leq j \leq (1+\delta)N} \gamma_j^{2\ell+1} k_j^{2\ell-1} > 4R$

ii) The coefficient next to the first term is clearly larger than the one next to the second cone, so more precisely we can take N large such that

$$I_2 = \frac{C_2}{\tilde{T}^2 k_N} \leq \frac{1}{2} C_1$$

iii)

$$I_3 = \sum_{N \leq j \leq (1+\delta)N} \gamma_j k_j^{\frac{2\ell-1}{2\ell+1}} \sim \sum_{N \leq j \leq (1+\delta)N} \frac{1}{j^{\frac{2\ell-1}{2\ell+1}\frac{1+\varepsilon}{q}}}$$

Because this sum diverges in order to bound that term, we make use that we have a factor of k_N in the denominator, so we can add the assumption that

$$\sum_{N \leq j \leq (1+\delta)N} \frac{1}{j^{\frac{2\ell-1}{2\ell+1}\frac{1+\varepsilon}{q}}} < \frac{1}{N} k_N^{\frac{1}{2\ell+1}}$$

So we take N so that

$$\beta_N^{2\ell+1} \frac{C_2}{t^2 k_N} \sum_{N \leq j \leq (1+\delta)N} \gamma_j^{2\ell+1} k_j^{2\ell-1} < 1$$

iv)

$$I_4 = \sum_j \gamma_j^{q(2k+1)} k_j^{(2k-2+m)q} \sim \sum_{N \leq j \leq (1+\delta)N} \frac{1}{k_j^{q(2k+1)\frac{2\ell-1}{2\ell+1} - (2k-2+m)q}} \frac{1}{j^{(2k-2+m)(1+\varepsilon)}}$$

Notice that because $m < 1$

$$q(2k+1)\frac{2\ell-1}{2\ell+1} - (2k-2+m)q = q(2k+1)\left(\frac{2\ell-1}{2\ell+1} - \frac{(2k-2+m)}{2k+1}\right) < 0$$

Take N so that

$$\beta_N^{2\ell+1}\frac{C_3}{t^2 k_N}\left(\sum_{N\le j\le(1+\delta)N}\gamma_j k_j^{\frac{2\ell-1}{2\ell+1}}\right)^{2\ell+1} < 1$$

v)

$$I_5 = \sum_{N\le j\le(1+\delta)N}\gamma_j^{q(2k+1)}k_j^{(2k-2+m)q} \to 0 \text{ as } N\to\infty$$

Take N so that

$$\sum_{k=1}^{\ell-1}\frac{C_4}{t^{2/q}}\beta_N^{2k+1}\left(\sum_{N\le j\le(1+\delta)N}\gamma_j^{q(2k+1)}k_j^{(2k-2+m)q}\right)^{1/q} < 1$$

vi)

$$I_6 = \sum_{N\le j\le(1+\delta)N}\gamma_j k_j^{\frac{2k-1}{2k+1}} \sim \sum_{N\le j\le(1+\delta)N}\frac{1}{k_j^{\frac{2\ell-1}{2\ell+1}-\frac{2k-1}{2k+1}}}\frac{1}{j^{\frac{2k-1}{2k+1}\frac{1+\varepsilon}{q}}} \to 0 \text{ as } N\to\infty$$

Take N so that

$$\sum_{k=1}^{\ell-1}\frac{C_5\beta_N^{2k+1}}{t^2 k_N}\left(\sum_{N\le j\le(1+\delta)N}\gamma_j k_j^{\frac{2k-1}{2k+1}}\right)^{2k+1} < 1.$$

vi)

$$\sum_{N\le j\le(1+\delta)N}\gamma_j^q k_j^{mq} \sim \sum_{N\le j\le(1+\delta)N}\frac{1}{j^{1+\varepsilon}} \to 0 \text{ as } N\to\infty$$

Take N so that

$$C_6\beta_N\left(\sum_{N\le j\le(1+\delta)N}\gamma_j^q k_j^{mq}\right)^{1/q} < 1,$$

and consequently we can take $\|\varphi\|_{\dot{\mathcal{F}}_q^{m,p}}$ arbitrarily small say take N large enough so that

$$\|\varphi\|_{\dot{\mathcal{F}}_q^{m,p}} < \frac{1}{R}$$

Then taking N that satisfy all this requirements we conclude that

$$\|f_{2\ell+1}\|_{\dot{\mathcal{F}}_q^{m,p}} \geq 2R - 4 \geq R.$$

For N large enough, and because this can be done for any $T > 0$ and $R > 0$ which completes the proof of Theorem 4.3.3

□

For the proof of Lemma 4.3.4 an important technical tool is an estimate of the integral in α analogous to Lemma 4.2.6. Define

$$\Gamma_{2k+1}(A_1, \cdots, A_{2k+1}) \;:=\; i\int_{\mathbb{R}_\alpha} m_\alpha(A_1)m_\alpha(A_2)...m_\alpha(A_{2k})m_\alpha(A_{2k+1})d\alpha \quad (4.127)$$

Then we have the following Lemma

Lemma 4.3.6 (Properties of Γ_{2k+1})**.** *Let $k \geq 1$, then the function Γ_{2k+1} defined by* (4.127) *satisfy*

i) $\Gamma(A_1, \ldots, A_{2k+1})$ *is given explicitly,*

$$\begin{aligned}\Gamma(A_1, \ldots, A_{2k+1}) \;=\; & \frac{(2\pi)^{2k}\pi}{(2k)!}\big(A_1^{2k-1}|A_1| + A_2^{2k-1}|A_2| + \cdots + A_{2k+1}^{2k-1}|A_{2k+1}| \\ & -(A_1 + A_2)^{2k-1}|A_1 + A_2| - \cdots \\ & +(A_1 + \cdots + A_{2k+1})^{2k-1}|A_1 + \cdots + A_{2k+1}|\big)\end{aligned} \quad (4.128)$$

ii) $\Gamma_{2k+1}(cA_1, ..., cA_{2k+1}) = c^{2k}\mathrm{sgn}(c)\Gamma_{2k+1}(A_1, ..., A_{2k+1})$,

iii) $\Gamma_{2k+1}(A_1, ..., A_{2k+1}) = 0$ *if* $A_\ell > 0$ *for all* ℓ,

iv) Γ_{2k+1} *is symmetric in all variables,*

v) $\Gamma_{2k+1}(A_1, ..., A_{2k+1}) \leq 2(2\pi)^{2k}|A_1||A_2|...|A_{2k}|$. *Notice that there are only* $2k$ *terms in the right hand side and not* $2k+1$.

vi) $|\Gamma(A_1, ..., A_{2\ell+1})| \leq 2(2\pi)^{2k} \min_j |A_1 A_2 ... A_{j-1}\check{A}_j A_{j+1} .... A_{2k+1}|$.

vii) Let $x_i \in [A_i - 1, A - i + 1]$ *then*

$$|\Gamma(x_1, \cdots, x_{2k+1}) - \Gamma(A_1, \cdots, A_{2k+1})| \leq C(|A_1|^{2k-1} + \cdots + |A_{2k+1}|^{2k-1}) \tag{4.129}$$

Proof of Lemma 4.3.6. i) is obtained by integration by parts.

ii) is direct consequence of the explicit formula in part i). iii) is obtained from the integral representation by using that

$$m_\alpha(A) = \frac{1 - e^{-2\pi i\alpha A}}{\alpha} = \frac{2ie^{-\pi i\alpha A}}{\alpha}\sin(\pi\alpha A), \tag{4.130}$$

then you get

$$\begin{aligned} \Gamma(A_1, \cdots, A_{2k+1}) &= i(2i)^{2k+1} \int e^{\pi i(A_1 + \cdots + A_{2k+1})} \\ &\quad \times \frac{\sin(\pi\alpha A_1)}{\alpha} \cdots \frac{\sin(\pi\alpha A_{2k+1})}{\alpha} d\alpha \end{aligned} \tag{4.131}$$

Now the integral can be seen as the Fourier transform at the point $A_1 + \cdots + A_{2k+1}$. And from computations in Section 6.1 we conclude that his integral is equal to zero.

For part iv) is direct from the definition. Part vi) is obtained from v) and the observation that because of iv) the variable that we omit in the estimate can be any variable. For part v) the proof is analogous to the proof in Lemma 4.2.6 for $k = 1$. To prove vii) we use that $\Gamma(x_1, \cdots, x_{2k+1})$ is differentiable and therefore it is enough to estimate the partial derivatives around the point $(A_1, \cdots, A_{2k+1})$.

$$\begin{aligned} \frac{d}{dx_i}\Gamma(x_1, \cdots, x_{2k+1}) &\leq (2^{2k+2}-1)(2k)(|x_1| + \cdots + |x_{2k+1}|)^{2k-1} \\ &\leq C(k)(|x_1|^{2k-1} + \cdots + |x_{2k+1}|^{2k-1}) \end{aligned} \tag{4.132}$$

Then we get that

$$\begin{aligned} J &\leq |\Gamma(x_1, \cdots, x_{2k+1}) - \Gamma(A_1, \cdots, A_{2k+1})| \\ &\leq \sum_i \left| \frac{d}{dx_i}\Gamma(y_1, \cdots, y_{2k+1}) \right| \\ &\leq (2k+1)C(k)(|A_1+1|^{2k-1} + \cdots + |A_{2k+1}+1|^{2k-1}) \\ &\leq C_2(k)(|A_1|^{2k-1} + \cdots + |A_{2k+1}+1|^{2k-1}) \end{aligned} \tag{4.133}$$

□

Proof of Lemma 4.3.4. First taking Fourier transform to (4.121) we get that $\hat{f}_{2k+1}$ can be writen as

$$\hat{f}_{2k+1} = \frac{(-1)^{k-1}}{2k+1} \int_0^t e^{-2\pi|\xi|(t-\tau)} I_{2k+1}(\xi, \tau) d\tau \tag{4.134}$$

for $1 \leq k \leq \ell$, where,

$$I_{2k+1}(\xi, \tau) = (2\pi i \xi) \int_{\mathbb{R}} (m_\alpha \hat{f}_1)^{*(2k+1)} d\alpha. \tag{4.135}$$

Substituting $\hat{f}_1 = \beta_N \sum_j \gamma_j e^{-2\pi t|\xi|}(P_{k_j} + P_{2\ell k_j+M})$ in I_{2k+1}

$$I_{2k+1}(\xi,t) = \beta_N^{2k+1} \sum_{N\le j\le(1+\delta)N} \sum_{\substack{c_i^j\in\Lambda(k_j)\\ i=1,\cdots,2k+1}} \gamma_j^{2k+1} R_{c_1^j\cdots c_{2k+1}^j}(\xi,t) + HF(\xi,t), \quad (4.136)$$

where $\Lambda(k_s) = \{\pm k_s, \pm(2\ell k_s + M)\}$ and

$$R_{c_1^j\cdots c_{2k+1}^j}(\xi,t) = (2\pi i\xi)\int (e^{-2\pi t|\cdot|}m_\alpha \chi_{c_1^j}) * \ldots * (e^{-2\pi t|\cdot|}m_\alpha \chi_{c_{2k+1}^j})d\alpha, \quad (4.137)$$

$$HF = \beta_N^{2k+1} \sum_{\substack{N\le s_i\le(1+\delta)N\\ \text{not all equal}}} \sum_{c_i^{s_i}\in\Lambda(k_{s_i})} \gamma_{s_1}\cdots\gamma_{s_{2k+1}}$$
$$\times (2\pi i\xi)\int (e^{-2\pi t|\cdot|}m_\alpha \chi_{c_1^{s_1}}) * \ldots * (e^{-2\pi t|\cdot|}m_\alpha \chi_{c_{2k+1}^{s_{2k+1}}})d\alpha. \quad (4.138)$$

Here HF represent the off diagonal terms in the sum that we expect to have high frequency and should decay faster, which should make them easier to estimate.

Lemma 4.3.7 (Estimate diagonal terms in Lemma 4.3.4). *Let $k < \ell$ and $0 < t < 1$, then*

$$\left\| \beta_N^{2k+1} \sum_{N\le j\le(1+\delta)N} \sum_{c_i^j\in\Lambda(k_j)} \gamma_j^{2k+1} E(R_{c_1^j\cdots c_{2k+1}^j}) \right\|_{\dot{\mathcal{F}}_q^{m,p}}$$
$$\le \frac{C}{t^{2/q}}\beta_N^{2k+1}\left(\sum_{j=N}^{(1+\delta)N} \gamma_j^{(2k+1)q} k_j^{(2k-2+m)q} e^{-\pi t q k_j/2}\right)^{1/q} \quad (4.139)$$

Lemma 4.3.8 (Estimate off diagonal terms in Lemma 4.3.4). *Let HF as defined by* (4.138)*, then*

$$\|E(HF)\|_{\mathcal{F}_q^{m,p}} \le \frac{C}{t^2}\left(\sum_j \gamma_j k_j^{\frac{2k-1}{2k+1}}\right)^{2k+1} \quad (4.140)$$

Proof of Lemma 4.3.7. First we write

$$\begin{aligned} R_{c_1^j \cdots c_{2k+1}^j} &= (2\pi i\xi)\int (m_\alpha e^{-2\pi t|\cdot|}\chi_{c_1^j}) * \cdots * (m_\alpha e^{-2\pi t|\cdot|}\chi_{c_{2k+1}^j}) d\alpha \\ &= (2\pi\xi)\int d\xi_1 \cdots \int d\xi_{2k}\Gamma_{2k+1}(\xi-\xi_1, \xi_1-\xi_2, \cdots, \xi_{2k}) \\ &\quad \times e^{-2\pi t|\xi-\xi_1|}\chi_{c_1^j}(\xi-\xi_1)e^{-2\pi t|\xi_1-\xi_2|}\chi_{c_2^j}(\xi_1-\xi_2) \\ &\quad \times \cdots e^{-2\pi t|\xi_{2k}|}\chi_{c_{2k+1}^j}(\xi_{2k}) \end{aligned} . \tag{4.141}$$

Notice that we only have to integrate in the region

$$\operatorname{supp}\{\chi_{c_1^j}(\xi-\xi_1)\chi_{c_2^j}(\xi_1-\xi_2)\cdots\chi_{c_{2k+1}^j}(\xi_{2k})\}, \tag{4.142}$$

also notice that $|c_i^j| \le 2\ell k_j + M$. By Lemma 4.3.6 part iii) when all the entries of Γ_{2k+1} are positive or negative then this expression is zero. By using parts v) and vii) of the same Lemma we can estimate

$$\begin{aligned} |\Gamma_{2k+1}(\xi-\xi_1, \xi_1-\xi_2, \cdots, \xi_{2k})| &\le |\Gamma(c_1^j, \cdots, c_{2k+1}^j)| \\ &\quad + O(|c_1^j|^{2k-1} + \cdots + |c_{2k+1}^j|^{2k-1}) \\ &\le C(2\ell k_j + M)^{2k} + O(|k_j|^{2k-1}) \\ &\le Ck_j^{2k} + O(|k_j|^{2k-1}) \end{aligned} \tag{4.143}$$

Multiplying by γ_j^{2k+1} and summing over all c_j we get

$$J_{2k+1}(\xi, t) = \sum_{N \le j \le (1+\delta)N} \sum_{c_j \in \Lambda_j} \gamma_j^{2k+1} R_{c_1^j \cdots c_{2k+1}^j} \tag{4.144}$$

where

$$\Lambda_j = \{\vec{c} \in \{\pm k_j, \pm(2\ell k_j + M)\}^{2k+1} : c_i \text{ not all same sign }\}, \tag{4.145}$$

Then by applying (4.143) we get

$$
\begin{aligned}
|J_{2k+1}| \;\leq\; & C(k) \sum_{N \leq j \leq (1+\delta)N} \gamma_j^{2k+1} \sum_{\vec{c^j} \in \Lambda_j} (2\pi|\xi|)(k_j^{2k} + O(|k_j|^{2k-1})) \\
& \times \int \cdots \int (e^{-2\pi t|\xi-\xi|} \chi_{c_1^j}(\xi - \xi_1))(e^{-2\pi t|\xi_1-\xi_2|} \chi_{c_2^j}(\xi_1 - \xi_2)) \\
& \times \cdots (e^{-2\pi t|\xi_{2k}|} \chi_{c_{2k+1}^j}(\xi_{2k})) d\xi_1 \cdots d\xi_{2k} \\
\leq\; & C(k) \sum_{N \leq j \leq (1+\delta)N} \gamma_j^{2k+1} \sum_{\vec{c^j} \in \Lambda_j} (2\pi|\xi|)(k_j^{2k} + O(|k_j|^{2k-1})) \\
& \times e^{-t(|c_1^j|+\ldots+|c_{2k+1}^j|-(2k+1))} h_{c_1^j \cdots c_{2k+1}^j}(\xi),
\end{aligned}
\tag{4.146}
$$

where

$$
h_{c_1^j \cdots c_{2k+1}^j}(\xi) = \left(\chi_{c_1^j} * \chi_{c_2^j} * \ldots * \chi_{c_{2k+1}^j} \right)(\xi). \tag{4.147}
$$

The estimate of $h_{c_1^j \cdots c_{2k+1}^j}(\xi)$ is a direct application of Lemma 4.2.5. Next we need an estimate about the sums $c_1 + \cdots + c_{2k+1}$

Lemma 4.3.9. *Let $c_1, \cdots c_{2k+1} \in \Lambda(k_j)$ and suppose that not all c_i have the same sign and $(2k+1)M < k_j/2$, then*

i) $|c_1 + \ldots + c_{2k+1}| \geq k_j/2$,

ii) $|c_1| + \cdots + |c_{2k+1}| - |c_1 + \ldots + c_{2k+1}| \geq 2k_j$.

Proof of Lemma 4.3.9. To see that $|c_1^j + \ldots + c_{2k+1}^j| \geq k_j/2$, is enough to notice that it is impossible to write a zero as the sum of $2k+1$ terms using only $\{\pm 1, \pm 2\ell\}$ for $k < \ell$. Now we write $c_i = a_i k_j + \epsilon_i$, where $a_i \in \{\pm 1, \pm 2\ell\}$ and $\epsilon_i \in \{-M, 0, M\}$. Then

$$
c_1 + \cdots + c_{2k+1} = (a_1 + \cdots + a_{2k+1})k_j + (\epsilon_1 + \cdots + \epsilon_{2k+1}).
$$

By the previous observation we see that $|a_1 + \cdots + a_{2k+1}| \geq 1$ and therefore

$$\begin{aligned} |c_1 + \cdots + c_{2k+1}| &\geq k_j - |\epsilon_1 + \cdots + \epsilon_{2k+1}| \\ &\geq k_j - (2k+1)M \\ &\geq k_j/2. \end{aligned} \tag{4.148}$$

For part ii) it is enough to notice that because not all of the c_i have the the same sign then

$$|c_1 + \cdots + c_{2k+1}| \leq |c_1| + \cdots + |c_{2k+1}| - 2|c_j|,$$

for some $1 \leq j \leq 2k+1$, then we get

$$|c_1| + \cdots + |c_{2k+1}| - |c_1 + \cdots + c_{2k+1}| \geq 2|c_j| \geq 2k_j.$$

This concludes the proof of Lemma 4.3.9. □

Continuation of proof Lemma 4.3.7. Using Lemma 4.3.9 part i) we can estimate the integral,

$$\begin{aligned} \widehat{E(J_{2k+1})} &= \int_0^t e^{-2\pi(t-\tau)|\xi|} J_{2k+1}(\xi,\tau)d\tau \\ &\leq C \sum_{N \leq j \leq (1+\delta)N} \sum_{\vec{c^j} \in \Lambda_j} \gamma_j^{2k+1} 2\pi|\xi| \int_0^t e^{-2\pi(t-\tau)|\xi|} \left(k_j^{2k} + O(k_j^{2k-1})\right) \\ &\qquad \times e^{-2\pi\tau(|c_1^j|+\ldots+|c_{2k+1}^j|-(2k+1))} d\tau \chi\left(\frac{\xi - (c_1^j + \ldots + c_{2k+1}^j)}{2k}\right) \end{aligned} \tag{4.149}$$

$$
\begin{aligned}
|\widehat{E(J_{2k+1})}| \;\leq\; & C \sum_{N \leq j \leq (1+\delta)N} \sum_{\vec{c^j} \in \Lambda_j} \gamma_j^{2k+1} 2\pi |\xi| \left(k_j^{2k} + O(k_j^{2k-1})\right) \\
& \times e^{-2\pi t(|c_1^j + \cdots + c_{2k+1}^j| - 2k)} \\
& \times \int_0^t e^{2\pi\tau(|c_1^j + \cdots + c_{2k+1}^j| - 2k)} e^{-2\pi\tau(|c_1^j| + \cdots + |c_{2k+1}^j| - (2k+1))} d\tau \\
& \times \chi\left(\frac{\xi - (c_1^j + \cdots + c_{2k+1}^j)}{2k}\right),
\end{aligned}
\tag{4.150}
$$

now by Lemma 4.3.9 part ii) we get

$$
\begin{aligned}
|\widehat{E(J_{2k+1})}| \;\leq\; & C(k) \sum_{N \leq j \leq (1+\delta)N} \sum_{\vec{c^j} \in \Lambda_j} \gamma_j^{2k+1} 2\pi |\xi| \left(k_j^{2k} + O(k_j^{2k-1})\right) \\
& \times e^{-2\pi t(c_1^j + \ldots + c_{2k+1}^j - 2k)} \\
& \times \int_0^t e^{-2\pi\tau(k_j - 1)} d\tau \, \chi\left(\frac{\xi - (c_1^j + \ldots + c_{2k+1}^j)}{2k}\right) \\
\leq\; & C \sum_j \sum_{c_i^j} \gamma_j^{2k+1} 2\pi |\xi| \left(k_j^{2k} + O(k_j^{2k-1})\right) e^{-2\pi t(|c_1^j + \cdots + c_{2k+1}^j| - 2k)} \\
& \times \int_0^t e^{-2\pi\tau(k_j - 1)} d\tau \, \chi\left(\frac{\xi - (c_1^j + \cdots + c_{2k+1}^j)}{2k}\right) \\
\leq\; & C \sum_j \sum_{c_i^j} \gamma_j^{2k+1} |\xi| \left(k_j^{2k} + O(k_j^{2k-1})\right) e^{-2\pi t(k_j - 2k)} \\
& \times \frac{1}{2\pi(k_j - 1)} (1 - e^{-2\pi t(k_j - 1)}) \chi\left(\frac{\xi - (c_1^j + \cdots + c_{2k+1}^j)}{2k}\right) \\
\leq\; & C \sum_j \sum_{c_i^j} \gamma_j^{2k+1} H_j(t) \chi\left(\frac{\xi - (c_1^j + \cdots + c_{2k+1}^j)}{2k}\right),
\end{aligned}
$$

where $H_j(t) = (2\ell + 2)^2 k_j^{2k+1} \frac{1}{k_j - 1} e^{-2\pi t(k_j - 2k)} (1 - e^{-2\pi t(k_j - 1)})$. Then

$$
|\widehat{E(J_{2k+1})}| \leq \sum_j \gamma_j^{2k+1} H_j(t) B_j(\xi) \tag{4.151}
$$

where

$$B_j(\xi) = \sum_{\substack{c_i^j \in \Lambda(k_j) \\ \text{not all same sign}}} \chi\left(\frac{\xi - (c_1^j + ... + c_{2k+1}^j)}{2k}\right). \tag{4.152}$$

Now we compute the $\mathcal{F}_q^{m,p}$ norm. First we notice that for different values of j the terms $B_j(\xi)$ have disjoint support. Let $R \in \mathbb{N}$ such that $2^R > (2\ell)^2$. Because we can bound the quantity

$$k_j \leq |c_1^j + ... + c_{2k+1}^j| \leq (2k+1)(2\ell k_j + M) \leq (2\ell+1)^2 k_j$$

Then the term $B_j(\xi)$ is supported in at most $R+1$ dyadic annulus C_i,

$$\begin{aligned} I &= \left\| \sum_{N \leq j \leq (1+\delta)N} \sum_{c_i^j \in \Lambda(k_j)} \gamma_j^{2k+1} R_{c_1^j \cdots c_{2k+1}^j} \right\|_{\dot{\mathcal{F}}_q^{m,p}}^q \\ &= \|E(J_{2k+1})\|_{\mathcal{F}_q^{m,p}}^q \\ &= \sum_{r \in \mathbb{Z}} \left(\int_{C_r} |\xi|^{mp} |K_{2k+1}|^p \right)^{q/p} \end{aligned}$$

Let $R_j = [k_j, (2\ell+1)^2 k_j]$ then

$$\begin{aligned} I &= \sum_j \sum_{r \in R_j} \left((\gamma_j^{2k+1} H_j(t))^p \int_{C_r} |\xi|^{mp} |B_j|^p \right)^{q/p} \\ &\leq \sum_j (\gamma_j^{2k+1} H_j(t))^q (R+1) \left(\int_{\tilde{C}_r} |\xi|^{mp} |B_j|^p \right)^{q/p} \end{aligned}$$

where $\tilde{C}_j = \bigcup_{r=log_2 k_j - 1}^{\log_2 k_j + R} C_r$. Now we can estimate the integral of B_j as

$$\begin{aligned}\int |\xi|^{mp}|B_j|^p d\xi &\leq ((2\ell+1)^2 k_j + 2k)^{mp} \int |B_j|^p \\ &\leq ((2\ell+1)^2 k_j + 2k)^{mp} (\#\Lambda_j)^p 2k(\#\Lambda_j) \\ &\leq ((2\ell+1)^2 k_j + 2k)^{mp} (4^{2k+1})^p 2k 4^{2k+1} \ , \\ &\leq (2\ell+1)^{2mp} 4^{(2k+1)(p+1)} 2k (k_j + 2k)^{mp} \\ &\leq C(m,p,\ell,k) k_j^{mp}\end{aligned}$$

where $C(m,p,\ell,k)^{1/p}$ can be bounded independent of p. Using this we get

$$\begin{aligned} I &\leq C \sum_j (\gamma_j^{2k+1} H_j(t))^q k_j^{mq} \\ &\leq C \sum_j \gamma_j^{q(2k+1)} (2\ell+2)^q k_j^{(2k+1)q} \frac{1}{(k_j-1)^q} e^{-tq(k_j-2k)} (1-e^{-2\pi t(k_j-1)})^q k_j^{mq} \\ &\leq C \sum_j \gamma_j^{q(2k+1)} k_j^{(2k+m)q} e^{-2\pi tqk_j/2} \\ &\leq \frac{C}{t^2} \sum_j \gamma_j^{q(2k+1)} k_j^{(2k-2+m)q} e^{-\pi tqk_j/2} \end{aligned}$$

Therefore

$$\left\| \sum_{N \leq j \leq (1+\delta)N} \sum_{c_i^j \in \Lambda(k_j)} \gamma_j^{2k+1} R_{c_1^j \cdots c_{2k+1}^j} \right\|_{\dot{\mathcal{F}}_q^{m,p}} \leq \frac{C}{t^{2/q}} \left(\sum_j \gamma_j^{q(2k+1)} k_j^{(2k-2+m)q} e^{-\pi tqk_j/2} \right)^{1/q} \tag{4.153}$$

This complete the proof of Lemma 4.3.7. □

Proof of Lemma 4.3.8. Now we proceed to estimate the high frequency part. From equation (4.138) we know that HF can be written as

$$HF = \beta_N^{2k+1} \sum_{\substack{N \leq s_i \leq (1+\delta)N \\ \text{not all equal}}} \sum_{c_i^{s_i} \in \Lambda(k_{s_i})} \gamma_{s_1} \cdots \gamma_{s_{2k+1}} R_{c_1^{s_1} \cdots c_{2k+1}^{s_{2k+1}}}, \tag{4.154}$$

where $\Lambda(s_i) = \{\pm k_{s_i}, \pm(2k_{s_i} + M)\}$ and

$$\begin{aligned} R_{c_1^{s_1}\cdots c_{2k+1}^{s_{2k+1}}} &= (2\pi\xi)\int d\xi_1 \cdots \int d\xi_{2k}\Gamma_{2k+1}(\xi-\xi_1, \xi_1-\xi_2, \cdots, \xi_{2k}) \\ &\quad \times e^{-2\pi t|\xi-\xi_1|}\chi_{c_1^j}(\xi-\xi_1)e^{-2\pi t|\xi_1-\xi_2|}\chi_{c_2^j}(\xi_1-\xi_2) \\ &\quad \times \cdots e^{-2\pi t|\xi_{2k}|}\chi_{c_{2k+1}^j}(\xi_{2k}). \end{aligned} \tag{4.155}$$

The idea is to use an estimate similar to the one used in the proof of Lemma 4.2.14. An important estimate concerning the proof has to with the size of the sums $c_1^{s_1} + \cdots + c_{2k+1}^{s_{2k+1}}$.

Lemma 4.3.10. *Suppose that for $k \le \ell$, $M > 2\ell + 2$, $k_N/2 > (2\ell+1)M$, $|c_1| \le |c_2| \le \cdots \le |c_{2k+1}|$, $c_i \in \cup_{j=N}^{(1+\delta)N}\{\pm k_j, \pm(2k_j + M)\}$, not all with the same sign and $c_1 + \cdots + c_{2k+1} \ne \pm M$ then*

i) $|c_1 + \cdots + c_{2k+1}| > |k_{i_1}|/2$ *for some* i_1

ii) $|c_1| + \cdots + |c_{2k+1}| - |c_1 + \cdots + c_{2k+1}| > |k_{i_2}|/2$ *for some* i_2

iii) At least one among $|c_1 + \cdots + c_{2k+1}|$ *and* $|c_1| + \cdots + |c_{2k+1}| - |c_1 + \cdots c_{2k+1}|$ *is at least* $c_{2k+1}/2$

Proof. For part i) we write $c_j = a_j k_{i_j} + \varepsilon_j M$, where $a_j \in \{\pm 1, \pm 2\ell\}$, $\varepsilon \in \{-1, 0, 1\}$. Then we get

$$c_1 + \cdots + c_{2k+1} = (a_1 k_{i_1} + \cdots + a_{2k+1}) + (\varepsilon_1 + \cdots + \varepsilon_{2k+1})M \tag{4.156}$$

Because of the difference in the order of magnitude, in order for for the term $\left(a_1 k_{i_1} + \cdots + a_{2k+1}k_{i_{2k+1}}\right)$ to vanish we need that the coefficients a_i with the same

k_i factor add up to zero, but this is impossible by parity for $k < \ell$ and for the case $k = \ell$ we use that $c_1 + \cdots + c_{2k+1} \neq \pm M$. We conclude that for at least one k_i the sum of the corresponding coefficients is not zero and therefore

$$|a_1 k_{i_1} + \cdots + a_{2k+1} k_{i_{2k+1}}| \geq k_{i_1} \tag{4.157}$$

For some i_1. The second summand satisfy $|\varepsilon_1 + \cdots + \varepsilon_{2k+1}| \leq 2k+1$ and then by the assumption $k_N/2 > (2\ell + 1)M$ we conclude that

$$|c_1 + \cdots + c_{2k+1}| \geq k_{i_1} - k_N/2 \geq \frac{1}{2} k_{i_1} \tag{4.158}$$

Part ii) come from the assumption that not all the c_i have the same sign, and therefore

$$|c_1 + \cdots + c_{2k+1}| \leq |c_1| + \cdots + |c_{2k+1}| - 2|c_m| \tag{4.159}$$

For some i, therefore we conclude that

$$|c_1| + \cdots + |c_{2k+1}| - |c_1 + \cdots + c_{2k+1}| \geq 2|c_m| \geq 2k_{i_m}. \tag{4.160}$$

Part *iii*) come from the observation that

$$\begin{aligned}(|c_1 + \cdots + c_{2k+1}|) + (|c_1| + \cdots + |c_{2k+1}| - |c_1 + \cdots c_{2k+1}|)& \\ = |c_1| + \cdots + |c_{2k+1}| \geq |c_{2k+1}|& \end{aligned} \tag{4.161}$$

and because both terms are positive we get the result. □

Under this assumptions we have the following

$$|\Gamma(c_1^{s_1}, \cdots, c_{2k+1}^{s_{2k+1}})| \leq C |c_1^{s_1} \cdots c_{2k+1}^{s_{2k+1}}|^{\frac{2k}{2k+1}}, \tag{4.162}$$

using this we can estimate (4.155) by

$$|R_{c_1^{s_1}\cdots c_{2k+1}^{s_{2k+1}}}| \leq C(|c_1^{s_1}+\cdots+c_{2k+1}^{s_{2k+1}}|+2k+1) \\ \times |c_1^{s_1}\cdots c_{2k+1}^{s_{2k+1}}|^{\frac{2k}{2k+1}} e^{-2\pi t(|c_1^{s_1}|+\cdots+|c_{2k+1}^{s_{2k+1}}|-2k-1)} h_{c_1^{s_1}\cdots c_{2\ell+1}^{s_{2\ell+1}}}(\xi), \quad (4.163)$$

where

$$\chi_{c_1^{s_1}+\cdots+c_{2k+1}^{s_{2k+1}}}(\xi) \leq h_{c_1^{s_1}\cdots c_{2\ell+1}^{s_{2\ell+1}}}(\xi) \leq 2^{2k}\chi\left(\frac{\xi-(c_1^{s_1}+\cdots+c_{2k+1}^{s_{2k+1}})}{2k}\right). \quad (4.164)$$

Now we look at the evolution of this term

$$\begin{aligned} \left|\mathcal{F}\left(E(R_{c_1^{s_1}\cdots c_{2k+1}^{s_{2k+1}}})\right)\right| &\leq C\int_0^t e^{-2\pi(t-\tau)|\xi|}(|c_1^{s_1}+\cdots+c_{2k+1}^{s_{2k+1}}|+2k+1) \\ &\quad \times|c_1^{s_1}\cdots c_{2k+1}^{s_{2k+1}}|^{\frac{2k}{2k+1}} \\ &\quad \times e^{-2\pi t(|c_1^{s_1}|+\cdots+|c_{2k+1}^{s_{2k+1}}|-2k-1)}h(\xi)d\tau \\ &\leq C\int_0^t e^{-2\pi(t-\tau)(|c_1^{s_1}+\cdots+c_{2k+1}^{s_{2k+1}}|-2k)} \\ &\quad \times(|c_1^{s_1}+\cdots+c_{2k+1}^{s_{2k+1}}|+2k+1)|c_1^{s_1}\cdots c_{2k+1}^{s_{2k+1}}|^{\frac{2k}{2k+1}} \\ &\quad \times e^{-2\pi t(|c_1^{s_1}|+\cdots+|c_{2k+1}^{s_{2k+1}}|-2k-1)}h(\xi)d\tau \\ &\leq Ce^{-2\pi t(|c_1^{s_1}+\cdots+c_{2k+1}^{s_{2k+1}}|-2k)}h(\xi) \\ &\quad \times(|c_1^{s_1}+\cdots+c_{2k+1}^{s_{2k+1}}|+2k+1)|c_1^{s_1}\cdots c_{2k+1}^{s_{2k+1}}|^{\frac{2k}{2k+1}} \\ &\quad \times\int_0^t e^{-2\pi\tau\left(|c_1^{s_1}|+\cdots+|c_{2k+1}^{s_{2k+1}}|-|c_1^{s_1}+\cdots+c_{2k+1}^{s_{2k+1}}|-1\right)}d\tau \end{aligned} \quad (4.165)$$

$$
\begin{aligned}
&\leq \; Ce^{-2\pi t(|c_1^{s_1}+\cdots+c_{2k+1}^{s_{2k+1}}|-2k)}h(\xi) \\
&\qquad \times(|c_1^{s_1}+\cdots+c_{2k+1}^{s_{2k+1}}|+2k+1)|c_1^{s_1}\cdots c_{2k+1}^{s_{2k+1}}|^{\frac{2k}{2k+1}} \\
&\qquad \times\int_0^t e^{-2\pi\tau(k_i/2)}d\tau \qquad (4.166)\\
&\leq \; C\frac{|c_1^{s_1}\cdots c_{2k+1}^{s_{2k+1}}|}{|c_{\hat{i}}^{s_{\hat{i}}}|}h(\xi)e^{-\pi t|c_1^{s_1}+\cdots+c_{2k+1}^{s_{2k+1}}|}\frac{\left(1-e^{-2\pi t k_{i_2}/2}\right)}{2\pi k_{i_2}/2}e \\
&\leq \; \frac{C}{t}|c_1^{s_1}\cdots c_{2k+1}^{s_{2k+1}}|^{\frac{2k}{2k+1}}h(\xi)\frac{1}{k_{i_1}k_{i_2}}e^{-\pi t|c_1^{s_1}+\cdots+c_{2k+1}^{s_{2k+1}}|/2}
\end{aligned}
$$

Now by iii) in Lemma 4.3.10 we know that among k_{i_1} and k_{i_2} at least one of them can be bounded below by $\frac{1}{2}k_{\hat{i}}$, and for the other one we can use the bound k_N then we get

$$
\left|\mathcal{F}\left(E(R_{c_1^{s_1}\cdots c_{2k+1}^{s_{2k+1}}})\right)\right| \;\leq\; \frac{C}{tk_N}\frac{|c_1^{s_1}\cdots c_{2k+1}^{s_{2k+1}}|^{\frac{2k}{2k+1}}}{k_{\hat{i}}}h(\xi)e^{-\pi t|c_1^{s_1}+\cdots+c_{2k+1}^{s_{2k+1}}|/2} \tag{4.167}
$$

Now notice that $k_{\hat{i}} \geq \frac{c_j^{s_j}}{(2\ell+1)}$ for all j, then we can bound

$$
\frac{|c_1^{s_1}\cdots c_{2k+1}^{s_{2k+1}}|^{\frac{2k}{2k+1}}}{k_{\hat{i}}} \leq \frac{1}{2\ell+1}\prod_i |c_i^{s_i}|^{\frac{2k-1}{2k+1}} \tag{4.168}
$$

Now we can bound

$$
\begin{aligned}
\|R_{c_1^{s_1}\cdots c_{2k+1}^{s_{2k+1}}}\|_{\dot{\mathcal{F}}_q^{m,p}} &\leq \frac{C}{t}\prod_i |c_i^{s_i}|^{\frac{2k-1}{2k+1}}\|h(\xi)\|_{\dot{\mathcal{F}}_q^{m,p}}e^{-\pi t|c_1^{s_1}+\cdots+c_{2k+1}^{s_{2k+1}}|/2} \\
&\leq \frac{C}{tk_N}\prod_i |c_i^{s_i}|^{\frac{2k-1}{2k+1}}(|c_1^{s_1}+\cdots+c_{2k+1}^{s_{2k+1}}|+2k)^m \qquad (4.169)\\
&\qquad \times e^{-\pi t|c_1^{s_1}+\cdots+c_{2k+1}^{s_{2k+1}}|/2} \\
&\leq \frac{C_{k,p,q,\ell}}{t^2k_N}\prod_i |c_i^{s_i}|^{\frac{2k-1}{2k+1}}
\end{aligned}
$$

Here we used that because $|c_1^{s_1}+\cdots+c_{2k+1}^{s_{2k+1}}| \geq k_N/2$ then we can find a upper bound for the number of dyadic annulus that the interval

$$
[c_1^{s_1}+\cdots+c_{2k+1}^{s_{2k+1}}-2k, c_1^{s_1}+\cdots+c_{2k+1}^{s_{2k+1}}+2k] \tag{4.170}
$$

interesect, say R, then

$$
\begin{aligned}
\|h(\xi)\|_{\dot{\mathcal{F}}_q^{m,p}} &= \left(\sum_k \left(\int_{C_k} |\xi|^{mp}|\hat{h}(\xi)|^p d\xi\right)^{q/p}\right)^{1/q} \\
&\leq \Big(\sum_k \Big(\int_{C_k} |c_1+\cdots+c_{2k+1}+2k|^{mp} \\
&\qquad\qquad \times|2^{2k}\chi\left(\frac{\xi-(c_1+\cdots+c_{2k+1})}{2k}\right)|^p d\xi\Big)^{q/p}\Big)^{1/q} \\
&\leq 2^{2k}|c_1+\cdots+c_{2k+1}+2k|^m \\
&\qquad\qquad \times\left(R\left(\int_{\mathbb{R}} |\chi(\frac{\xi-(c_1+\cdots+c_{2k+1})}{2k})|^p d\xi\right)^{q/p}\right)^{1/q} \\
&\leq 2^{2k}|c_1+\cdots+c_{2k+1}+2k|^m R^{1/q}(4k)^{1/p}
\end{aligned}
\tag{4.171}
$$

Now we need to sum over all the tuples $(c_1^{s_1}, \cdots, c_{2k+1}^{s_{2k+1}})$, we get

$$
\begin{aligned}
L &= \left\|\sum_{s_i}\sum_{\vec{c_j}\in\Lambda(s_1,\cdots,s_{2k+1})} \gamma_{s_1}\cdots\gamma_{s_{2k+1}} R_{c_1^{s_1}\cdots c_{2k+1}^{s_{2k+1}}}\right\|_{\dot{\mathcal{F}}_q^{m,p}} \\
&\leq \sum_{s_i}\sum_{\vec{c_j}\in\Lambda(s_1,\cdots,s_{2k+1})} \gamma_{s_1}\cdots\gamma_{s_{2k+1}} \left\|R_{c_1^{s_1}\cdots c_{2k+1}^{s_{2k+1}}}\right\|_{\dot{\mathcal{F}}_q^{m,p}} \\
&\leq \sum_{s_i}\sum_{\vec{c_j}\in\Lambda(s_1,\cdots,s_{2k+1})} \gamma_{s_1}\cdots\gamma_{s_{2k+1}} \frac{C}{t^2 k_N}\prod_i |c_i^{s_i}|^{\frac{2k-1}{2k+1}} \qquad (4.172) \\
&\leq \frac{C}{t^2 k_N}\left(\sum_j 4(2\ell+1)^{\frac{2k-1}{2k+1}}\gamma_j k_j^{\frac{2k-1}{2k+1}}\right)^{2k+1} \\
&\leq \frac{C_{k,p,q,\ell}}{t^2 k_N}\left(\sum_j \gamma_j k_j^{\frac{2k-1}{2k+1}}\right)^{2k+1}
\end{aligned}
$$

This completes the proof of Lemma 4.3.8. □

Continuation of proof of Lemma 4.3.4. Using the estimates given by Lemmas 4.3.7

and 4.3.8 we get that

$$
\begin{aligned}
\|f_{2k+1}\|_{\mathcal{F}_q^{m,p}} &\le \frac{\beta_N^{2k+1}}{2k+1}\left\|\sum_j\sum_{\Lambda_j}\gamma_j^{2k+1}E(R_{c_1^s\cdots c_{2k+1}^s})\right\|_{\mathcal{F}_q^{m,p}} \\
&\quad+\frac{\beta_N^{2k+1}}{2k+1}\left\|\sum_{j_1,\cdots,j_{2k+1}}\sum_{\vec{C}\in\Lambda(s_1,\dots s_{2k+1})}\gamma_{s_1}\cdots\gamma_{s_{2k+1}}E(R_{c_1^s\cdots c_{2k+1}^s})\right\|_{\mathcal{F}_q^{m,p}} \\
&\le \frac{C}{t^{2/q}}\beta_N^{2k+1}\left(\sum_j\gamma_j^{q(2k+1)}k_j^{(2k-2+m)q}e^{-\pi tqk_j/2}\right)^{1/q} \\
&\qquad+\frac{C}{t^2k_N}\left(\sum_j\gamma_j k_j^{\frac{2k-1}{2k+1}}\right)^{2k+1}
\end{aligned}
\tag{4.173}
$$

Where the constant C depend on m, q, ℓ,k . (This complete the proof of Lemma 4.3.4) □

Proof of Lemma 4.3.5 . To estimate the term $f_{2\ell+1}$ we use the following decomposition

$$f_{2\ell+1}=J_1+HF_1+HF_2, \tag{4.174}$$

where

$$J_1=\beta_N^{2\ell+1}\sum_j\sum_{\Omega_j}\gamma_j^{2\ell+1}\int_0^t e^{-2\pi(t-\tau)|\xi|}R_{c_1\cdots c_{2\ell+1}}d\tau, \tag{4.175}$$

$$\Omega_j=\{(c_1,\cdots,c_{2\ell+1}):c_i\in\{\pm k_j,\pm(2\ell k_j+M)\}\text{ and }c_1+\cdots+c_{2\ell+1}=\pm M\}, \tag{4.176}$$

and

$$HF_1=\beta_N^{2\ell+1}\sum_j\sum_{\Lambda_j\backslash\Omega_j}\gamma_j^{2\ell+1}\int_0^t e^{-2\pi(t-\tau)|\xi|}R_{c_1\cdots c_{2\ell+1}}d\tau \tag{4.177}$$

$\Lambda_j = \{\pm k_j, \pm(2\ell k_j + M)\}^{2\ell+1}$,

$$HF_2 = \beta_N^{2\ell+1} \sum_{s_1,\cdots,s_{2\ell+1}} \sum_{\Omega(s_1,\cdots,s_{2\ell+1})} \gamma_{s_1}\cdots\gamma_{s_{2\ell+1}} \int_0^t e^{-2\pi(t-\tau)|\xi|} R_{c_1\cdots c_{2\ell+1}} d\tau, \quad (4.178)$$

$\Omega(s_1,\cdots,s_{2\ell+1}) = \Lambda(s_1)\times\cdots\times\Lambda(s_{2\ell+1})$. To estimate these terms we use the following Lemmas

Lemma 4.3.11. *Let $M > 2\ell$, $tM \le 1$, $tk_j \gg 1$, then*

$$\|J_1\|_{\dot{\mathcal{F}}_q^{m,p}} \ge \beta_N^{2\ell+1} C \sum_j \gamma^{2\ell+1} k_j^{2\ell-1} \quad (4.179)$$

Where C depend on p, q, m, M, ℓ.

Lemma 4.3.12. *Let HF_1 and HF_2 as defined by, then*

$$\|HF_1\|_{\dot{\mathcal{F}}_q^{m,p}} \le \beta_N^{2\ell+1} \frac{C}{t^2 k_N} \sum_j \gamma_j^{2k+1} k_j^{2k-1} \quad (4.180)$$

and

$$\|HF_2\|_{\dot{\mathcal{F}}_q^{m,p}} \le \beta_N^{2\ell+1} \frac{C}{t^2 k_N} \left(\sum_j \gamma_j k_j^{\frac{2k-1}{2k+1}}\right)^{2k+1}. \quad (4.181)$$

Continuation of proof Lemma 4.3.5. For now we will just use Lemmas 4.3.11 and 4.3.12. From the decomposition given by equation (4.174) we can bound the norm of $f_{2\ell+1}$ by

$$\|f_{2\ell+1}\|_{\dot{\mathcal{F}}_q^{m,p}} \ge \|J_1\|_{\dot{\mathcal{F}}_q^{m,p}} - \|HF_1\|_{\dot{\mathcal{F}}_q^{m,p}} - \|HF_2\|_{\dot{\mathcal{F}}_q^{m,p}} \quad (4.182)$$

And by Lemmas 4.3.11 and 4.3.12 we can estimate

$$\|f_{2\ell+1}\|_{\dot{\mathcal{F}}_q^{m,p}} \ge C\beta_N^{2\ell+1} \sum_j \gamma_j^{2\ell+1} k_j^{2\ell-1} - \beta_N^{2\ell+1} \frac{C}{t^2 k_N} \sum_j \gamma_j^{2k+1} k_j^{2k-1} - \beta_N^{2\ell+1} \frac{C}{t^2 k_N} \left(\sum_j \gamma_j k_j^{\frac{2k-1}{2k+1}}\right)^{2k+1} \quad (4.183)$$

□

Now we proceed to prove Lemmas 4.3.11 and 4.3.12

Proof of Lemma 4.3.11. The key element for this proof is a lower bound of $R_{c_1^j \cdots c_{2k+1}^j}$. For this purpose we need to estimate the value of $\Gamma_{2\ell+1}(-k_j, \cdots, -k_j, 2\ell k_j + M)$, to do this we use the integral formula for $\Gamma_{2\ell+1}$

$$\begin{aligned} L &= \Gamma_{2\ell+1}(-k_j, \cdots, -k_j, 2\ell k_j + M) \\ &= i \int \frac{(1 - e^{2\pi i k_j \alpha})^{2\ell}(1 - e^{-2\pi i(2k\ell+M)\alpha})}{\alpha^{2\ell+1}} d\alpha \\ &= i \int \frac{(1 - e^{2\pi i k_j \alpha})^{2\ell}(1 - e^{-2\pi i(2\ell k_j)\alpha})}{\alpha^{2\ell+1}} d\alpha \\ &\quad + i \int \frac{(1 - e^{2\pi i k_j \alpha})^{2\ell}(e^{-2\pi i(2\ell k_j)\alpha} - e^{-2\pi i(2\ell k_j+M)\alpha})}{\alpha^{2\ell+1}} d\alpha \\ &= I_1 + I_2 \end{aligned} \tag{4.184}$$

$$\begin{aligned} I_1 &= i \int \frac{(1 - e^{2\pi i k_j \alpha})^{2\ell}(1 - e^{-2\pi i(2k\ell)\alpha})}{\alpha^{2\ell+1}} d\alpha \\ &= i \int \frac{(e^{-i\pi k_j \alpha} - e^{\pi i k_j \alpha})^{2\ell}(e^{\pi i(2\ell k_j)\alpha} - e^{-\pi i(2\ell k_j)\alpha})}{\alpha^{2\ell+1}} d\alpha \\ &= i(2i)^{2k+1} k_j^{2k} \pi^{2k} \int \frac{\sin(\beta)^{2\ell}\sin(2\ell\beta)}{\beta^{2\ell+1}} d\beta \\ &= (-1)^{k+1}(2\pi)^{2k+1} k_j^{2k} \end{aligned} \tag{4.185}$$

And for the second term

$$\begin{aligned} I_2 &= i \int \frac{(1 - e^{2\pi i k_j \alpha})^{2\ell}(e^{-2\pi i(2\ell k_j)\alpha} - e^{-2\pi i(2\ell k_j+M)\alpha})}{\alpha^{2\ell+1}} d\alpha \\ &= i \int \frac{(e^{-2\pi i k_j \alpha} - 1)^{2\ell}(1 - e^{-2\pi i M\alpha})}{\alpha^{2\ell+1}} d\alpha \\ &= \Gamma_{2\ell+1}(k_j, \cdots, k_j, M) \\ &= k_j^{2\ell-1} |k_j| \Gamma_{2\ell+1}(1, \cdots, 1, M/k_j) \end{aligned} \tag{4.186}$$

And by Lemma 4.3.6 we conclude

$$|I_2| \leq |k_j|^{2\ell}(2\pi)^{2\ell}\frac{M}{|k_j|} \leq |k_j|^{2\ell-1}2(2\pi)^{2\ell}M \tag{4.187}$$

Therefore we conclude

$$\Gamma_{2\ell+1}(-k_j, \cdots, -k_j, 2\ell k_j + M) = (-1)^{\ell+1}(2\pi)^{2k+1}k_j^{2\ell} + O(k_j^{2\ell-1}). \tag{4.188}$$

Using this we can estimate $R_{c_1^j \cdots c_{2\ell+1}^j}$

$$(-1)^{\ell+1}R_{c_1^j \cdots c_{2\ell+1}^j} \geq C|\xi|((2\pi)^{2k+1}k_j^{2\ell} + O(k_j^{2\ell-1}))e^{-2\pi t(4\ell k_j+M+2\ell+1)}h(\xi) \tag{4.189}$$

Now we define $\Omega_j = \{(c_1, \cdots, c_j) : c_i \in \{\pm k_j, \pm(2k_j + M)\}, c_1 + \cdots c_{2\ell+1} = \pm M\}$. By Lemma 4.3.9 the only possibilities for Ω_j are the tuples such that one of the elements is equal to $\pm(2\ell k_j + M)$ and the rest $\mp k_j$. Now by summing over all elements of Ω_j we get

$$\begin{aligned} &\left\| \sum_j \sum_{\Omega_j} \gamma_j^{2\ell+1} E(R_{c_1,\cdots,c_{2\ell+1}}) \right\|_{\mathcal{F}_q^{m,p}} \\ &\qquad \geq C(M-2\ell)^{1+m}\sum_j (4\ell+2)\gamma_j^{2\ell+1}k_j^{2\ell}e^{-\pi t(M+2\ell)}\frac{(1-e^{-2\pi t(4\ell k_j+1)})}{2\pi(4\ell k_j+1)} \\ &\qquad \geq C\sum_j \gamma_j^{2\ell+1}k_j^{2\ell-1}e^{-2\pi t(M+2\ell)} \end{aligned} \tag{4.190}$$

For $M > 2\ell$, $tM \leq 1$, $tk_j \gg 1$. This concludes the proof of Lemma 4.3.11.

□

Proof of Lemma 4.3.12. For the upper bound of the high frequency we use the same estimates as in the proof of Lemma 4.3.8,

$$HF_1 = \beta_N^{2\ell+1} \sum_j \sum_{\Lambda_j \setminus \Omega_j} \gamma_j^{2\ell+1} \int_0^t e^{-2\pi(t-\tau)|\xi|} R_{c_1 \cdots c_{2\ell+1}} d\tau \tag{4.191}$$

$\Omega_j = \{(c_1, \cdots, c_{2\ell+1}) : c_i \in \{\pm k_j, \pm(2\ell k_j + M)\}$ and $c_1 + \cdots + c_{2\ell+1} = \pm M\}$,

$$HF_2 = \beta_N^{2\ell+1} \sum_{s_1,\cdots,s_{2\ell+1}} \sum_{\Omega(s_1,\cdots,s_{2\ell+1})} \gamma_{s_1} \cdots \gamma_{s_{2\ell+1}} \int_0^t e^{-2\pi(t-\tau)|\xi|} R_{c_1 \cdots c_{2\ell+1}} d\tau \tag{4.192}$$

From the proof of Lemma 4.3.8, we can apply the estimate 4.169 because Lemma 4.3.10 still apply in this context, then we get

$$\|HF_1\|_{\dot{\mathcal{F}}_q^{m,p}} \leq \beta_N^{2\ell+1} \frac{C}{t^2 k_N} \sum_j \gamma_j^{2k+1} k_j^{2k-1} \tag{4.193}$$

and for HF_2

$$\|HF_2\|_{\dot{\mathcal{F}}_q^{m,p}} \leq \beta_N^{2\ell+1} \frac{C}{t^2 k_N} \left(\sum_j \gamma_j k_j^{\frac{2k-1}{2k+1}} \right)^{2k+1}, \tag{4.194}$$

where the constant depend on m, p, q, ℓ. This complete the proof of Lemma 4.3.12 □

4.4 Norm inflation for the truncated problem in the periodic domain

The goal of this section is to extend the results that we prove for the real lie for a periodic domain, the key to extend the result is an estimate of the convolution of characteristic functions as the one obtained in Lemma 4.2.3, in order to do this we use a series representation of the tangent to identify the most singular part and compare it with the case of the real line.

4.4.1 Convolution of characteristic functions in the periodic domain

The next lemma make a precise error estimate on in a periodic domain instead of the real line, the main difference between this two situation is that estimates for the function $\Gamma(x, y, z)$ from Lemma 4.2.6 do not apply directly to the periodic case. The goal of this section is to extend a version of the estimates to a for the corresponding integral in the periodic case. The rest of the estimates follow directly from using the corresponding notion of Fourier transform in the periodic domain, i.e. the map that takes a periodic function $f : \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}) \to \mathbb{C}$ to the function $\mathcal{F}(f) = \hat{f} : \mathbb{Z} \to \mathbb{C}$ that give the Fourier coefficients of the representation of f as a Fourier series, and for f regular enough we have:

$$f = \sum_{k\in\mathbb{Z}} \hat{f}(k) e^{2\pi i k x}. \tag{4.195}$$

Lemma 4.4.1 (Integral estimate in the periodic case)**.** *Let* $A_1, A_2, A_3 \in \mathbb{R}$, $|A_i > 3|$ *then*

$$\begin{aligned} \Gamma_P(A_1, A_2, A_3) &:= i \int_{\mathbb{T}} \frac{(1-e^{-i\alpha A_1})(1-e^{-i\alpha A_2})(1-e^{-i\alpha A_3})}{\tan^3(\alpha/2)} d\alpha \\ &= \frac{2^3}{(2\pi)^2} \Gamma(A_1, A_2, A_3) + O(|A_1| + |A_2| + |A_3|), \end{aligned} \tag{4.196}$$

where $\Gamma(A_1, A_2, A_3)$ *is defined by* (4.30).

Proof of Lemma 4.4.1. Some estimates first

$$\begin{aligned} J &= \int_{-\pi}^{\pi} \cos^2(x/2)\frac{\sin(Bx)}{x}dx \\ &= \int_{-\pi}^{\pi} \frac{1}{2}(\cos(x)+1)\frac{\sin(Bx)}{x}dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \frac{\cos(x)\sin(Bx)+\sin(Bx)}{x}dx \\ &= \frac{1}{4}\int_{-\pi}^{\pi} \left(\frac{\sin((B+1)x)+\sin((B-1)x)+2\sin(Bx)}{x}\right)dx, \end{aligned} \tag{4.197}$$

now we need the following Lemma

Lemma 4.4.2. *Let $B \in \mathbb{R}$, $|B| > 2$, then*

$$\int_{-\pi}^{\pi} \frac{\sin(Bx)}{x} = \operatorname{sgn}(B)\pi\left(1 + O\left(\frac{1}{|B|}\right)\right) \tag{4.198}$$

Proof of Lemma 4.4.2.

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{\sin(Bx)}{x} &= \operatorname{sgn}(B)\int_{-\pi}^{\pi} \frac{\sin(|B|x)}{x}dx \\ &= \operatorname{sgn}(B)\int_{-\pi}^{\pi} \frac{\sin(|B|x)}{|B|x}|B|dx \\ &= \operatorname{sgn}(B)\int_{-|B|\pi}^{|B|\pi} \frac{\sin(y)}{y}dy \\ &= \operatorname{sgn}(B)\left(\int_{\mathbb{R}} \frac{\sin(y)}{y}dy - 2\int_{|B|\pi}^{\infty} \frac{\sin(y)}{y}dy\right) \end{aligned} \tag{4.199}$$

To estimate the term $\int_{|B|\pi}^{\infty} \frac{\sin(y)}{y}dy$, we use that it behaves like an alternating series

$$\begin{aligned} I &= \int_{|B|\pi}^{\infty} \frac{\sin(y)}{y}dy \\ &= -\int_{\lfloor |B| \rfloor}^{|B|} \frac{\sin(y)}{y}dy + \int_{\lfloor |B| \rfloor}^{\infty} \frac{\sin(y)}{y}dy \\ &= -\int_{\lfloor |B| \rfloor}^{|B|} \frac{\sin(y)}{y}dy + \sum_{k=\lfloor |B| \rfloor}^{\infty} \int_{k\pi}^{(k+1)\pi} \frac{\sin(y)}{y}dy \end{aligned} \tag{4.200}$$

then the sum is a alternating series, and therefore it can be bounded by

$$\begin{aligned}
|I| &\leq \left|\int_{\lfloor |B|\rfloor}^{|B|} \frac{\sin(y)}{y} dy\right| + \left|\int_{\lfloor |B|\rfloor \pi}^{(\lfloor |B|\rfloor+1)\pi} \frac{\sin(y)}{y} dy\right| \\
&\leq 2\left|\int_{\lfloor |B|\rfloor \pi}^{(\lfloor |B|\rfloor+1)\pi} \frac{\sin(y)}{y} dy\right| \\
&\leq \frac{2}{|B|-1}
\end{aligned} \tag{4.201}$$

And so by using that $\int_{\mathbb{R}} \frac{\sin(x)}{x} dx = \pi$ we can finally conclude that if $B > 2$

$$\int_{-\pi}^{\pi} \frac{\sin(Bx)}{x} = \operatorname{sgn}(B)\pi \left(1 + O\left(\frac{1}{|B|}\right)\right). \tag{4.202}$$

□

We will use previous Lemma 4.4.2 to estimate the integral (4.196). For this purpose we need the following expansion of the tangent

$$\tan^3(\alpha/2) = \cos^2(\alpha/2) \sum_{k\in\mathbb{Z}} \frac{1}{(\alpha/2 - k\pi)^3}, \tag{4.203}$$

using this formula we can write $\Gamma(A_1, A_2, A_3) = \sum_{k\in\mathbb{Z}} I_k$ where

$$I_k = i\int_{-\pi}^{\pi} \cos^2(\alpha/2) \frac{(1-e^{-i\alpha A_1})(1-e^{-i\alpha A_2})(1-e^{-i\alpha A_3})}{(\alpha/2 - k\pi)^3} d\alpha. \tag{4.204}$$

Because we expect that the largest contribution comes form the singular term with $k = 0$, we first estimate

$$I_0 = \frac{2^3 i}{2!} \int_{-\pi}^{\pi} \frac{d^2}{d\alpha^2}\left(\frac{1}{\alpha}\right) \cos^2(\alpha/2) F(\alpha) d\alpha, \tag{4.205}$$

where $F(\alpha) = (1-e^{-i\alpha A_1})(1-e^{-i\alpha A_2})(1-e^{-i\alpha A_3})$. Now we proceed to integrate by parts, and noticing that $\cos^2(\alpha/2)$ vanishes up to order 2 at $\pm\pi$, so we do not have boundary terms

$$
\begin{aligned}
I_0 &= 4i\int_{-\pi}^{\pi}\frac{d^2}{d\alpha^2}\left(\frac{1}{\alpha}\right)\cos^2(\alpha/2)F(\alpha)d\alpha \\
&= -4i\int_{-\pi}^{\pi}\frac{d}{d\alpha}\left(\frac{1}{\alpha}\right)\left(\cos(\alpha/2)\sin(\alpha/2)F+\cos^2(\alpha/2)F'\right)d\alpha \\
&= 2i\int_{-\pi}^{\pi}\frac{1}{\alpha}\cos\alpha F d\alpha + 4i\int\frac{1}{\alpha}\sin\alpha F' d\alpha + 4i\int\frac{1}{\alpha}\cos^2(\alpha/2)F'' d\alpha \\
&= I_{0,1}+I_{0,2}+I_{0,3},
\end{aligned}
\tag{4.206}
$$

again we focus on the most singular term, for this purpose we compute

$$
\begin{aligned}
F'' = & -A_1^2e^{-i\alpha A_1}-A_2^2e^{-i\alpha A_2}-A_3^2e^{-i\alpha A_3}+(A_1+A_2)^2e^{-i\alpha(A_1+A_2)} \\
& +(A_1+A_3)^2e^{-i\alpha(A_1+A_3)}+(A_2+A_3)^2e^{-i\alpha(A_2+A_3)}e^{-i\alpha(A_2+A_3)} \\
& -(A_1+A_2+A_3)^2e^{-i\alpha(A_1+A_2+A_3)}.
\end{aligned}
\tag{4.207}
$$

Using the Lemma 4.4.2 we get that

$$I_{0,3}=\frac{2^3}{(2\pi)^2}\Gamma(A_1,A_2,A_3)+O(|A_1|+|A_2|+|A_3|). \tag{4.208}$$

For the other two terms, we use that the quotient $\left|(1-e^{-i\alpha A_i})/\alpha\right| \leq \sqrt{2}|A|$ is bounded and therefore we can bound

$$|I_{0,1}|\leq 2\pi\sqrt{2}|A_i|, \quad |I_{0,2}|\leq C(|A_1|+|A_2|+|A_3|). \tag{4.209}$$

Therefore we conclude that

$$I_0=\Gamma(A_1,A_2,A_3)+O(|A_1|+|A_2|+|A_3|). \tag{4.210}$$

To conclude we need to estimate the nonsingular terms, to do so we use that the numerator is bounded and that $|\alpha/2-k\pi|\geq(|k|-1/2)\pi$ therefore

$$
\begin{aligned}
|I_k| &= \left|\int_{-\pi}^{\pi}\cos^2(\alpha/2)\frac{(1-e^{-i\alpha A_1})(1-e^{-i\alpha A_2})(1-e^{-i\alpha A_3})}{(\alpha/2-k\pi)^3}d\alpha\right| \\
&= \left|\int_{-\pi}^{\pi}\cos^2(\alpha/2)\alpha\frac{(1-e^{-i\alpha A_1})}{\alpha}\frac{(1-e^{-i\alpha A_2})(1-e^{-i\alpha A_3})}{(\alpha/2-k\pi)^3}d\alpha\right| \\
&\leq \frac{C|A_1|\pi}{(|k|-1/2)^3},
\end{aligned}
\tag{4.211}
$$

and therefore summing in k we obtain

$$
\left|\sum_{k\neq 0} I_k\right| \leq C|A_1|\sum_k \frac{1}{(|k|-1/2)^3}\leq C_2|A_1|, \tag{4.212}
$$

which means that $\sum_{k\neq 0} I_k = O(|A_1|)$, which concludes the proof of Lemma 4.4.1.

□

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4.4.2 Norm inflation in the periodic domain for $\ell = 1$

The initial condition is essentially the same as for the case of the real line with two important remarks, first this time instead of using characteristic functions of intervals we can use Kronecker's delta δ_0 and because we are working in a periodic domain all the frequencies must be integers.

Given $N\in\mathbb{N}$ and $\ell\in\mathbb{N}$, we consider $\varphi:\mathbb{Z}\to\mathbb{R}$ of the form

$$
\hat{\varphi} = \beta_N\sum_{j\in S(N)}\gamma_j\Big(P_{k_s}(\xi)+P_{2\ell k_s+M}(\xi)\Big), \tag{4.213}
$$

where $P_A(k)=\delta_0(k-A)+\delta_0(k+A)$, $\{k_s\}_{s\geq 0}$ is a sequence of positive integers that grow very fast, $M>2\ell$ is fixed and $\{\gamma_j\}_j$ a sequence of positive numbers to be chosen later. N is a parameter that will be large in general, $S(N) = \{j : N \leq j \leq (1+\delta)N\}$, and β_N is a scaling factor that also depend on the parameter N.

Lemma 4.4.3 (Size of the Initial data)**.** *Consider* φ *defined by* (4.213) *then*

$$\|\varphi\|_{\dot{\mathcal{F}}_q^{m,p}} \leq C\beta_N \left(\sum_{j\in S(N)} \gamma_j^q k_j^{qm} \right)^{1/q}. \tag{4.214}$$

Proof of Lemma 4.2.1. Because the sequence $\{k_s\}$ is growing fast, at most one of them belong to each C_k annulus. Also, because the C_k are dyadic we can ensure that k_j and $2k_j + M$ belong to different annulus. With this observation in mind we get that if $k_{\bar{j}} \in C_k$ then

$$\sum_{n\in C_k} |n|^{mp} |\hat{\varphi}(n)|^p \leq \left(\beta_N \gamma_{\bar{j}}\right)^p 2^{mp+1} |k_{\bar{j}}|^{mp}. \tag{4.215}$$

Similarly if $2k_{\bar{j}} + M \in C_k$

$$\sum_{n\in C_k} |n|^{mp} |\hat{\varphi}(n)|^p \leq \left(\beta_N \gamma_{\bar{j}}\right)^p 2^{2mp+1} \left| k_{\bar{j}} + \frac{M}{2} \right|^{mp} \leq \left(\beta_N \gamma_{\bar{j}}\right)^p 2^{3mp+1} |k_{\bar{j}}|^{mp}, \tag{4.216}$$

taking the q/p power and summing over k we get

$$\begin{aligned} \sum_k \left(\sum_{n\in C_k} |n|^{mp} |\hat{\varphi}(n)|^p \right)^{q/p} &\leq (\beta_N)^q \sum_{j\in S(N)} \gamma_j^q \left(2^{\frac{q(mp+1)}{p}} |k_j|^{mq} + 2^{\frac{q(3mp+1)}{p}} |k_j|^{mq} \right) \\ &\leq 2 \left(\beta_N 2^{\frac{(3mp+1)}{p}} \right)^q \sum_{j\in S(N)} \gamma_j^q k_j^{mq}, \end{aligned} \tag{4.217}$$

taking the q-th root we obtain

$$\|\varphi\|_{\dot{\mathcal{F}}_q^{m,p}} \leq C\beta_N \left(\sum_{j\in S(N)} \gamma_j^q k_j^{qm} \right)^{1/q}. \tag{4.218}$$

□

Because we can to replicate the result for the real line, our first goal would be to extend the results from Theorem 4.2.8 to the periodic case, more precisely we will prove the following.

Theorem 4.4.4. *Consider the truncation of the Muskat problem given by*

$$\begin{cases} \partial_t f + \Lambda f = T_1 e^{-t\Lambda}\varphi \quad , \quad (x,t) \in [0,T] \times \mathbb{T} \\ f(0) = 0 \qquad\qquad\qquad , \quad x \in \mathbb{T} \end{cases} \tag{4.219}$$

where φ is given by (4.213)*, and T_1 is defined by* (4.7)*. Let $t > 0$ a time such that $t(M+1) < 1$ and $tk_0 \gg 1$. Then the solution f of* (4.219) *satisfy*

$$\|f(t)\|_{\dot{\mathcal{F}}_q^{m,p}} \geq C_1 \beta_N^3 \sum_{j\in S(N)} \gamma_j^3 k_j - \frac{C_2}{t^2}\beta_N^3 \left(\sum_{j\in S(N)} \left(\gamma_j^3 k_j^m\right)^q \right)^{1/q} - \beta_N^3 \frac{C_3}{t^4 k_{\min j}} \left(\sum_{j\in S(N)} \gamma_j \right)^3 , \tag{4.220}$$

where the constants C_1, C_2 and C_3 only depend on M,m,q,p.

As in the case of the real line this result also imply the inflation result in the periodic domain

Corollary 4.4.5. *Let $T > 0$, $R > 0$ and consider the problem* (4.219) *with initial data φ. Then there exists some $0 < \tilde{T} < T$ and some initial φ_R such that for $p \geq 1$*

$$\|\varphi_R\|_{\dot{\mathcal{F}}_\infty^{1/3,p}} < \frac{1}{R} \tag{4.221}$$

and

$$\|f(\tilde{R})\|_{\dot{\mathcal{F}}_\infty^{1/3,p}} \geq R. \tag{4.222}$$

Proof of Corollary 4.4.5. The proof is analogous to Corollary 4.2.9. □

Proof of Theorem 4.4.4. As in the proof of Theorem 4.2.8 we need to look at

$$\begin{aligned} I(n) &= \mathcal{F}\left(T_1 e^{-\tau\Lambda}\varphi\right)(n) \\ &= \frac{1}{3}\int in\left(m_\alpha e^{-\tau|\cdot|}\hat{\varphi}\right) * \left(m_\alpha e^{-\tau|\cdot|}\hat{\varphi}\right) * \left(m_\alpha e^{-\tau|\cdot|}\hat{\varphi}\right) d\alpha, \end{aligned} \tag{4.223}$$

where $m_\alpha = \frac{1-e^{-i\alpha}}{\tan(\alpha/2)}$.

To evaluate $I(n)$ we will expand (4.223) by substituting the initial condition (4.213) and use Lemma 4.4.1. We focus on what happen at the frequency $n = M$, because we exoect the low frequency terms decay slower

$$I(\xi) \;=\; \beta_N^3 \sum_{j\in S(N)} \gamma_j^3 (J_1^j + J_2^j + J_3^j + J_4^j) + HF, \tag{4.224}$$

where

$$\begin{aligned} J_1^j &= \frac{(-1)}{3}\sum_{n\in\mathbb{Z}} in(m_\alpha e^{-\tau|\cdot|}P_{k_j}) * (m_\alpha e^{-\tau|\cdot|}P_{k_j}) * (m_\alpha e^{-\tau|\cdot|}P_{k_j}), \\ J_2^j &= -\sum_{n\in\mathbb{Z}} in(m_\alpha e^{-\tau|\cdot|}P_{k_j}) * (m_\alpha e^{-\tau|\cdot|}P_{k_j}) * (m_\alpha e^{-\tau|\cdot|}P_{2k_j+M}), \\ J_3^j &= -\sum_{n\in\mathbb{Z}} in(m_\alpha e^{-\tau|\cdot|}P_{k_j}) * (m_\alpha e^{-2\pi\tau|\cdot|}P_{2k_j+M}) * (m_\alpha e^{-\tau|\cdot|}P_{2k_j+M}), \\ J_4^j &= \frac{(-1)}{3}\sum_{n\in\mathbb{Z}} in(m_\alpha e^{-\tau|\cdot|}P_{2k_j+M}) * (m_\alpha e^{-\tau|\cdot|}P_{2k_j+M}) \\ &\qquad * (m_\alpha e^{-\tau|\cdot|}P_{2k_j+M}), \end{aligned} \tag{4.225}$$

and HF correspond to the off-diagonal terms

$$\begin{aligned} HF = -\frac{1}{3}\beta_N^3 &\sum_{(s_1,s_2,s_3)\in S}\ \sum_{(a,b,c)\in\Lambda(s_1,s_2,s_3)} \gamma_{s_1}\gamma_{s_2}\gamma_{s_3} \\ &\times \sum_{n\in\mathbb{Z}} in(m_\alpha e^{-\tau|\cdot|}P_a) * (m_\alpha e^{-\tau|\cdot|}P_b) * (m_\alpha e^{-\tau|\cdot|}P_c), \end{aligned} \tag{4.226}$$

where

$$\begin{aligned} S &= \{(s_1, s_2, s_3) \in S(N)^3 : s_1, s_2, s_3 \text{ not all equal } \}, \\ \Lambda(s_1, s_2, s_3) &= \{(a_1, a_2, a_3) : a_i \in \{\pm k_{s_i}, \pm(2k_{s_i} + M)\}, i = 1, 2, 3\}\,. \end{aligned} \tag{4.227}$$

For the estimates of the term J_1, J_3, J_4 and HF the same proofs still holds, the only ingredient that we need is the analogous of Lemma 4.2.3 which we proceed to prove now

Lemma 4.4.6. *Let $A, B, C \in \mathbb{R}$, $M > 4$, $|A|, |B|, |C| \gg M$, $t \leq 1$, $|A + B + C| \geq 2M$ then*

$$\begin{aligned} S &= \sum_{n\in\mathbb{Z}} in(m_\alpha e^{-t|\cdot|}\delta_A) * (m_\alpha e^{-t|\cdot|}\delta_B) * (m_\alpha e^{-t|\cdot|}\delta_C) \\ &\sim (A + B + C)e^{-t(|A|+|B|+|C|)}(\Gamma(A, B, C) + O(|A| + |B| + |C|))\delta_{A+B+C}(n), \end{aligned} \tag{4.228}$$

where $m_\alpha(\xi) = \frac{1-e^{-i\alpha\xi}}{\tan(\alpha/2)}$, $\Gamma(x, y, z)$ is defined by (4.196).

Proof of Lemma 4.2.3. Consider

$$\begin{aligned} S &= \int_{\mathbb{T}} in(m_\alpha e^{-t|\cdot|}\chi_A) * (m_\alpha e^{-t|\cdot|}\chi_B) * (m_\alpha e^{-t|\cdot|}\chi_C) d\alpha \\ &= \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \int_{\mathbb{T}} ne^{-t|n-j|}e^{-t|j-k|}e^{-t|k|} \\ &\qquad \times \frac{i}{\tan^3(\alpha/2)}(1 - e^{-i\alpha(n-j)})(1 - e^{-i\alpha(j-k)})(1 - e^{-i\alpha k})d\alpha \\ &\qquad \times \delta_A(n - j)\delta_B(j - k)\delta_C(k) \\ &= (A + B + C)e^{-t|A|}e^{-t|B|}e^{-t|C|}\Gamma_P(A, B, C)\delta_{A+B+C}(n). \end{aligned} \tag{4.229}$$

Finally by applying Lemma 4.4.1 to Γ_P we obtain the conclusion of the Lemma.

□

Using this lemma we can just follow

Lemma 4.4.7 (Lower bound for J_2). *Let $t > 0$ such that $tk_1 \gg 1$, $t(M+1) < 1$. Then term J_2 satisfies*

$$|\widehat{E(J_2^s)}(n)| = \left| \beta_N^3 \sum_{j \in S(N)} \int_0^t e^{-(t-\tau)|n|} J_2^j d\tau \right| \geq$$

$$c_1 \beta_N^3 \sum_{j \in S(N)} \gamma_j^3 k_j P_M - \frac{c_2}{t^2} \beta_N^3 \sum_{j \in S(N)} \gamma_j^3 \left(P_{2k_j+M} + P_{4k_j+M} \right) \quad (4.230)$$

and consequently

$$\left\| \beta_N^3 \sum_j \gamma_j^3 E(J_2^j)(t) \right\|_{\dot{\mathcal{F}}_q^{m,p}} \geq c_3 \beta_N^3 \sum_j \gamma_j^3 k_j - \frac{c_4}{t^2} \beta_N^3 \left(\sum_j \left(\gamma_j^3 k_j^m \right)^q \right)^{1/q} \quad (4.231)$$

Proof. The proof is completely analogous to the proof of Lemma 4.2.10 by replacing Lemma 4.2.3 by Lemma 4.4.6 □

A similar analysis we can be used to estimate J_1, J_3 and J_4 more precisely

Lemma 4.4.8 (Estimate for J_1, J_3 and J_4). *Under the same conditions of Lemma 4.4.7*

$$|\widehat{E(J_i)}| = \left| \beta_N^3 \sum_{j \in S(N)} \gamma_j^3 \int_0^t e^{-\tau|\xi|} J_i^j d\tau \right| \leq \frac{C}{t^2} \beta_N^3 \sum_{j \in S(N)} \gamma_j^3 \tilde{h}_j(\xi), \qquad i = 1, 3, 4 \quad (4.232)$$

where supp $\tilde{h}_j(\xi) \subset [k_j/2, 7k_j]$ and $\|\tilde{h}_j\|_{\dot{\mathcal{F}}_q^{m,p}} \leq C k_j^m$ where C is independent of j. And by taking the $\dot{\mathcal{F}}_q^{m,p}$ norm, we get

$$\|E(J_i)(t)\|_{\dot{\mathcal{F}}_q^{m,p}} \leq \frac{C}{t^2} \beta_N^3 \left(\sum_j \left(\gamma_j^3 k_j^m \right)^q \right)^{1/q} \quad (4.233)$$

Proof. The proof is completely analogous to the proof of Lemma 4.2.13 □

□

4.4.3 Norm Inflation in the periodic case for $\ell \geq 2$

For the higher order case, it is easy to see that most of the proof can be adapted s in the previous subsection for the case $\ell = 1$, the only thing that we really need to prove is the analogous of Lemma 4.4.1, which give us the ability to use the estimates for the non-periodic case in the periodic case.

Lemma 4.4.9 (Integral estimate in the periodic case)**.** *Let* $A_1, \cdots, A_n \in \mathbb{R}$, $|A_i| > n-1$ *then*

$$\begin{aligned} \Gamma_P(A_1, \cdots, A_n) &:= i \int_{\mathbb{T}} \frac{(1-e^{-i\alpha A_1})\cdots(1-e^{-i\alpha A_n})}{\tan^n(\alpha/2)} d\alpha \\ &= \frac{2^n}{(2\pi)^{n-1}} \Gamma(A_1, \cdots, A_n) + O(|A_1|^{n-2} + \cdots + |A_n|^{n-2}), \end{aligned} \tag{4.234}$$

where $\Gamma(A_1, \cdots, A_n)$ *is defined by* (4.127)*.*

Proof of Lemma 4.4.9. To extend the result of Lemma 4.4.1 to the case of n terms the idea is to follow the same proof with minor adjustments. First we need a expansion for $\frac{1}{\tan^n(\alpha/2)}$ like the one given by Lemma 4.4.10

$$\frac{1}{2^n} \frac{1}{\tan^n(\alpha/2)} = \sum_{k\in\mathbb{Z}} \frac{\cos^{2n-4}(\alpha/2)}{(\alpha + 2\pi k)^n} + \tilde{LS}_n(\alpha), \tag{4.235}$$

where $\tilde{LS}_n(\alpha)$ is less singular than the first term near $2\pi k$ for $k \in \mathbb{Z}$. This formula allow us to integrate by parts as in (4.206) in such a way that the remainder terms are less singular. Note that we have enough vanishing at $\pm\pi$ that we do not get boundary terms from the integration by parts. More specifically we can write

$$\Gamma_P(A_1, \cdots, A_n) = \sum_{k\in\mathbb{Z}} I_k + R, \tag{4.236}$$

where

$$I_k = i\int_{-\pi}^{\pi} \cos^{2n-4}(\alpha/2)\frac{F(\alpha)}{(\alpha/2-k\pi)^n}d\alpha, \tag{4.237}$$

and

$$R = i\int_{-\pi}^{\pi} 2^n\tilde{LS}_n(\alpha)F(\alpha)d\alpha, \tag{4.238}$$

for $F(\alpha) = (1-e^{-i\alpha A_1})\cdots(1-e^{-i\alpha A_n})$. The most singular term in the expansion is I_0 and so we expect the largest contribution to come from it. By integrating by parts we get

$$I_0 = \frac{2^n i}{(n-1)!}\int_{-\pi}^{\pi}\frac{1}{\alpha}\cos^{2n-4}(\alpha/2)\frac{d^{n-1}}{d\alpha^{n-1}}F(\alpha)d\alpha + R_0, \tag{4.239}$$

where R_0 represent the less singular terms coming from the integration by parts. To estimate the main term we need to estimate integrals of the form

$$\begin{aligned}
J &= \int_{-\pi}^{\pi}\cos^{2m}(x/2)\frac{\sin(Bx)}{x}dx \\
&= \int_{-\pi}^{\pi}\frac{1}{2^n}(e^{ix/2}+e^{-ix/2})^{2m}\frac{\sin(Bx)}{x}dx \\
&= \frac{1}{2^n}\int_{-\pi}^{\pi}\sum_{k=0}^{2m}\binom{2m}{k}e^{ix(k-m)}\frac{\sin(Bx)}{x}dx \\
&= \frac{1}{2^n}\int_{-\pi}^{\pi}\left(\binom{2m}{m}+2\sum_{k=1}^{m}\binom{2m}{m-k}\cos(kx)\right)\frac{\sin(Bx)}{x}dx \\
&= \frac{1}{2^n}\int_{-\pi}^{\pi}\frac{1}{x}\sum_{k=-m}^{m}\binom{2m}{m-k}\sin((B+k)x)dx,
\end{aligned} \tag{4.240}$$

and then by applying Lemma 4.4.2 we conclude that for $|B| > m+1$

$$\begin{aligned}\int_{-\pi}^{\pi} \cos^{2m}(x/2)\frac{\sin(Bx)}{x}dx &= \frac{\pi}{2^n}\sum_{k=-m}^{m}\binom{2m}{m-k}\operatorname{sgn}(B+k) \\ &\qquad \times\left(1+O\left(\frac{1}{|B+k|}\right)\right) \\ &= \frac{\pi}{2^n}\sum_{k=-m}^{m}\binom{2m}{m-k}\operatorname{sgn}(B)\left(1+O\left(\frac{1}{|B|}\right)\right) \\ &= \pi\operatorname{sgn}(B)\left(1+O\left(\frac{1}{|B|}\right)\right).\end{aligned} \tag{4.241}$$

we conclude that

$$I_0 = \frac{2^n}{(2\pi)^{n-1}}\Gamma(A_1,\cdots,A_n) + O(|A_1|^{n-2}+\cdots+|A_n|^{n-2}) + R_0 \tag{4.242}$$

Next we can estimate R_0 using (4.209), I_k for $k \neq 0$ using (4.211), and both together to estimate R by separating the cases of singularities near zero and away from zero. This allow us to conclude

$$\begin{aligned} i\int_{\mathbb{T}} &\frac{(1-e^{-i\alpha A_1})\cdots(1-e^{-i\alpha A_n})}{\tan^n(\alpha/2)}d\alpha \\ &= \frac{2^n}{(2\pi)^{n-1}}\Gamma(A_1,\cdots,A_n) + O(|A_1|^{n-2}+\cdots+|A_n|^{n-2}).\end{aligned} \tag{4.243}$$

This concludes the proof of Lemma 4.4.9. □

Now we proceed to prove the formula for that tangent that we used on the proof of Lemma 4.4.9.

Lemma 4.4.10. *Let $n \geq 3$ and $\alpha \in \mathbb{R}\setminus 2\pi k, k\in\mathbb{Z}$, then*

$$\frac{1}{2^n}\frac{1}{\tan^n(\alpha/2)} = \sum_{k\in\mathbb{Z}}\frac{\cos^{2n-4}(\alpha/2)}{(\alpha+2\pi k)^n} + \tilde{LS}_n(\alpha), \tag{4.244}$$

where $|\partial_\alpha^\beta \tilde{LS}_n(\alpha)| \leq C(n,\beta)\sum_{k\in\mathbb{Z}}\frac{1}{|\alpha+2\pi k|^{n-1+\beta}}$.

Proof of Lemma 4.4.10. First we consider the following formula for $\frac{1}{\tan(\alpha/2)}$

$$\frac{1}{2}\frac{1}{\tan(\alpha/2)} = \frac{1}{z} + \sum_{k\geq 0} \frac{2\alpha}{\alpha^2 - (2\pi k)^2}, \tag{4.245}$$

taking derivative in α we get

$$\frac{1}{4}\frac{\sec^2(\alpha/2)}{\tan^2(\alpha/2)} = \frac{1}{4}\frac{1}{\tan^2(\alpha/2)} + \frac{1}{4} = p.v. \sum_{k\in\mathbb{Z}} \frac{1}{(\alpha + 2\pi k)^2}, \tag{4.246}$$

then

$$\frac{1}{4}\frac{1}{\tan^2(\alpha/2)} = \sum_{k\in\mathbb{Z}} \frac{1}{(\alpha + 2\pi k)^2} - \frac{1}{4}, \tag{4.247}$$

taking derivative in α again we get

$$\frac{1}{8}\frac{\sec^2(\alpha/2)}{\tan^3(\alpha/2)} = \sum_{k\in\mathbb{Z}} \frac{1}{(\alpha + 2\pi k)^3}, \tag{4.248}$$

now we make two observations about this formula that will help us to establish our induction, first by writing

$$\frac{1}{\tan^3(\alpha/2)} = \cos^2(\alpha/2) \sum_{k\in\mathbb{Z}} \frac{1}{(\alpha/2 + \pi k)^3}, \tag{4.249}$$

we see that the $\cos^2(\alpha/2)$ term give enough vanishing at the boundary so we can integrate by parts in the proof of Lemma 4.4.9 without getting boundary terms. By taking another derivative we see that

$$\frac{1}{2^4}\frac{\sec^4(\alpha/2)}{\tan^4(\alpha/2)} + \frac{1}{2^4}\frac{2\sec^2(\alpha/2)}{\tan^2(\alpha/2)} = \sum_{k\in\mathbb{Z}} \frac{1}{(\alpha + 2\pi k)^4}, \tag{4.250}$$

which mean that we can write

$$\frac{1}{\tan^4(\alpha/2)} = \cos^4(\alpha/2) \left(\sum_{k\in\mathbb{Z}} \frac{1}{(\alpha/2 + \pi k)^4} + LS_4(\alpha) \right), \tag{4.251}$$

where $\cos^4(\alpha/2)LS_4(\alpha)$ indicates a term that is less singular than the first one at each $2\pi k$, and consequently also all its derivatives are also less singular than the ones of the first term. Using this observation we can formulate our induction in the following way: for any $n \geq 3$

$$\frac{1}{2^n}\frac{\sec^{2n-4}(\alpha/2)}{\tan^n(\alpha/2)} = \sum_{k\in\mathbb{Z}}\frac{1}{(\alpha+2\pi k)^n} + LS_n(\alpha) \tag{4.252}$$

where $LS_n(\alpha)$ satisfy $|\partial_\alpha^\beta \cos^{2n-4}(\alpha/2)LS_n(\alpha)| \leq C(n,\beta)\sum_{k\in\mathbb{Z}}\frac{1}{|\alpha+2\pi k|^{n-1+\beta}}$. We already proved the case $n=3$ with $LS_3(\alpha)=0$, now we assume that our proposition is true for some n, we want to show that is also true for $n+1$. Taking the derivative in α from (4.252) we get

$$\frac{1}{2^{n+1}}\frac{\sec^{2(n+1)-4}(\alpha/2)}{\tan^{n+1}(\alpha/2)} - \frac{1}{n2^{n+1}}\frac{(2n-4)\sec^{2n-4}(\alpha/2)}{\tan^{n-1}(\alpha/2)} = \sum_{k\in\mathbb{Z}}\frac{1}{(\alpha+2\pi k)^{n+1}} - \frac{1}{n}\partial_\alpha LS_n(\alpha) \tag{4.253}$$

and so we get that the identity has the correct structure for $n+1$

$$\begin{aligned}\frac{1}{2^{n+1}}\frac{\sec^{2(n+1)-4}(\alpha/2)}{\tan^{n+1}(\alpha/2)} &= \sum_{k\in\mathbb{Z}}\frac{1}{(\alpha+2\pi k)^{n+1}} - \frac{1}{n}\partial_\alpha LS_n(\alpha) \\ &\quad + \frac{1}{n2^{n+1}}\frac{(2n-4)\sec^{2n-4}(\alpha/2)}{\tan^{n-1}(\alpha/2)} \\ &= \sum_{k\in\mathbb{Z}}\frac{1}{(\alpha+2\pi k)^{n+1}} + LS_{n+1}(s).\end{aligned} \tag{4.254}$$

Now it only remains to prove the estimate for LS_{n+1}

$$\begin{aligned}
\partial_\alpha^\beta \cos^{2(n+1)-4}(\alpha/2) LS_{n+1}(\alpha) &= -\frac{1}{n}\partial_\alpha^\beta \cos^{2(n+1)-4}(\alpha/2)\partial_\alpha LS_n(\alpha) \\
&\quad + \frac{(2n-4)}{n2^{n+1}}\partial_\alpha^\beta \frac{\cos^{2(n+1)-4}(\alpha/2)\sec^{2n-4}(\alpha/2)}{\tan^{n-1}(\alpha/2)} \\
&= I_1 + I_2.
\end{aligned} \tag{4.255}$$

To bound the first term we write

$$\begin{aligned}
-nI_1 &= \partial_\alpha^\beta \left(\cos^{2(n+1)-4}(\alpha/2)\partial_\alpha LS_n(\alpha)\right) \\
&= \partial_\alpha^\beta \left(\cos^2(\alpha/2)\cos^{2n-4}(\alpha/2)\partial_\alpha LS_n\right) \\
&= \partial_\alpha^\beta \left(\cos^2(\alpha/2)\left[\partial_\alpha(\cos^{2n-4}(\alpha/2)LS_n) - LS_n\partial\cos^{2n-4}(\alpha/2)\right]\right) \\
&= \partial_\alpha^\beta \left(\cos^2(\alpha/2)\partial_\alpha(\cos^{2n-4}(\alpha/2)LS_n)\right) \\
&\quad + \frac{1}{2}(2n-4)\partial^\beta \left(\sin(\alpha/2)\cos(\alpha/2)\cos^{2n-4}(\alpha/2)LS_n(\alpha)\right).
\end{aligned} \tag{4.256}$$

Then we get that I_1 can be written as a sum of terms that are the derivatives of $\cos^{2n-4}(\alpha/2)LS_n$, which we can control by the induction hypothesis and terms that are bounded and therefore only affect the constants.

$$\begin{aligned}
|I_1| &\leq C_1(n,\beta)\sum_{p=0}^{\beta+1} |\partial_\alpha^p \cos^{2n-4}(\alpha/2)LS_n| \\
&\leq C_1(n,\beta)\sum_{p=0}^{\beta+1} C(n,p)\sum_{k\in\mathbb{Z}} \frac{1}{|\alpha+2\pi k|^{n-1+p}} \\
&\leq C(n,\beta)\sum_{k\in\mathbb{Z}} \frac{1}{|\alpha+2\pi k|^{n\beta}}.
\end{aligned} \tag{4.257}$$

To estimate I_2 we use the following

$$\begin{aligned}
\frac{n2^{n+1}}{2n-4}I_2 &= \partial_\alpha^\beta \frac{\cos^{2(n+1)-4}(\alpha/2)\sec^{2n-4}(\alpha/2)}{\tan^{n-1}(\alpha/2)} \\
&= \partial_\alpha^\beta \frac{\cos^2(\alpha/2)}{\tan^{n-1}(\alpha/2)},
\end{aligned} \tag{4.258}$$

because near $2\pi k$ we have $\frac{1}{\tan(\alpha/2)} \sim \frac{C}{\alpha - 2\pi k}$ and because $\cos(\alpha/2)$ is bounded we get that inside the derivative we have term that is only singular at $2\pi k$ an the order of the singularity is $n-1$, and consequently the those derivatives have the correct order near $2\pi k$ for $k \in \mathbb{Z}$, so we conclude that

$$|I_2| \leq C(n,\beta) \sum_{k\in\mathbb{Z}} \frac{1}{|\alpha + 2\pi k|^{n-1+\beta}}, \tag{4.259}$$

finally putting all together we conclude that

$$\partial_\alpha^\beta \cos^{2(n+1)-4}(\alpha/2) LS_{n+1}(\alpha) \leq C(n,\beta) \sum_{k\in\mathbb{Z}} \frac{1}{|\alpha + 2\pi k|^{(n+1)-1+\beta}}. \tag{4.260}$$

Therefore we conclude by induction that for all $n \geq 3$

$$\frac{1}{2^n} \frac{\sec^{2n-4}(\alpha/2)}{\tan^n(\alpha/2)} = \sum_{k\in\mathbb{Z}} \frac{1}{(\alpha + 2\pi k)^n} + LS_n(\alpha) \tag{4.261}$$

where $LS_n(\alpha)$ satisfy $|\partial_\alpha^\beta \cos^{2n-4}(\alpha/2) LS_n(\alpha)| \leq C(n,\beta) \sum_{k\in\mathbb{Z}} \frac{1}{|\alpha+2\pi k|^{n-1+\beta}}$ and by defining $\tilde{LS}(\alpha) \cos^{2n-4}(\alpha/2) LS_n(\alpha)$ we concludes the proof of Lemma 4.4.9. $\square$

Chapter 5

Norm inflation for a PDE describing epitaxial growth

Abstract:

The goal of this Chapter is to present another application of the techniques presented in Chapter 4 to study the Ill-posedness for the Muskat problem. The problem in consideration comes from material sciences and is known as the epitaxial growth equation. It describes a process for the formation of thin layers of crystal and is described by a fourth order nonlinear parabolic PDE. To study the Ill-posedness of the epitaxial growth equation, we consider a sequence of approximate problems, and then their corresponding Picard's iterations. We obtain the discontinuity of map that takes the initial condition and return the second Picard's iteration of the approximate problem on some appropriate

supercritical space. For each approximate problem a different supercritical space is used and the sequence of such spaces approach the a critical space on the limit. More precisely we prove the existence of a sequence of initial data with arbitrarily small supercritical norm such that the second Picard's iteration of the approximate problem becomes arbitrarily large in the supercritical norm in a arbitrarily short time.

5.1 Introduction

5.1.1 Description of the model

In material sciences the Epitaxial Growth equation is a model that describe a method used to create high quality crystal growth for semiconductors and some other single layer films. When the surface of the crystal can be described as a graph, one of the equation that can be use to describe its evolution is the following (See [25])

$$\begin{cases} \partial_t v = -v^2 \Delta^2(v^3) \quad , \quad \text{in } [0,T] \times \Omega, \\ v(0,x) = v_0(x) \qquad , \quad \text{in } \Omega, \end{cases} \tag{5.1}$$

where $\Omega = \mathbb{R}^d$ or $\mathbb{T}^d$. In what follows we focus in the periodic case. By the maximum principle, if $v(x,t)$ is a solution of (5.1) and $v_0(x) > 0$ then $v(x,t) > 0$ for all $(x,t) \in \Omega \times [0,T]$. Then by writing

$$\partial_t \left(\frac{1}{v} \right) = \Delta^2(v^3), \tag{5.2}$$

we obtain that the quantity $\int_\Omega \frac{1}{v} dx$ is conserved, and therefore by rescaling we can assume that $\int_\Omega \frac{1}{v} dx = 1$ for all $t > 0$. Consider the change of variables $\frac{1}{v} = 1 + w$ and we get

$$\begin{cases} \partial_t w & = & \Delta^2 \dfrac{1}{(1+w)^3} & , \quad \text{in } [0,T] \times \Omega \\ w(0,x) & = & w_0 & , \quad \text{in } \Omega, \end{cases} \tag{5.3}$$

where $\int_\Omega w_0 dx = \int_\Omega w dx = 0$. Finally because of the zero average condition, we can take $w(t,x) = \Delta u(3t,x)$ to obtain the equation

$$\begin{cases} \partial_t u = \dfrac{1}{3} \Delta \left(\dfrac{1}{(1+\Delta u)^3} \right) & , \quad \text{in } [0,T] \times \Omega \\ u(0) = u_0 & , \quad \text{in } \Omega. \end{cases} \tag{5.4}$$

Another model that is sometimes used to study the Epitaxial growth is given by

$$\begin{cases} \partial_t f = \Delta e^{-\Delta f} & , \quad \text{in } [0,T] \times \Omega, \\ f(0,x) = f_0(x) & , \quad \text{in } \Omega, \end{cases} , \tag{5.5}$$

both models have very similar properties and particular our analysis also applies to (5.5) with minor changes, because we only use finite truncations of the Taylor expansion of the nonlinear part, up to changing the coefficients in that expansion, both models behave in the same way for our purposes.

5.1.2 An approximation of the Epitaxial Growth Equation

To study the Epitaxial growth equation we want to consider a family of approximations of the equation and study the continuity of the solution map at the origin for such systems. To construct such approximated systems we consider the following

Taylor expansion

$$\frac{2}{(1-x)^3} = \frac{d^2}{dx^2}\frac{1}{1-x} = \frac{d^2}{dx^2}\sum_k x^k = \sum_{k\geq 2} k(k-1)x^{k-2}. \tag{5.6}$$

Then for $|\Delta u| < 1$ we can write (5.4) as

$$\partial_t u = \sum_{k=1}^{\infty} \frac{(k+2)(k+1)}{6}\Delta(-\Delta u)^k. \tag{5.7}$$

Next we consider the family of equations obtained by considering only finitely many terms in this expansion (5.7). More precisely given $\ell \geq 2$ we consider the truncated expansion of the epitaxial growth equation of order ℓ to be

$$\partial_t u + \Delta^2 u = \sum_{k=2}^{\ell} \frac{(k+2)(k+1)}{6}\Delta(-\Delta u)^k, u(0) = u_0. \tag{5.8}$$

Next consider the Picard's itertion of the problem, we set $u^{(0)} = 0$ and consider the sequence $\{u^{(k)}\}_{k\geq 0}$ given by

$$\begin{cases} \partial_t u^{(k)} + \Delta^2 u^{(k)} = \displaystyle\sum_{k=2}^{\ell} \frac{(k+2)(k+1)}{6}\Delta(-\Delta u)^k \quad , \quad k \geq 1 \\ u^{(k)}(0) = u_0. \end{cases} \tag{5.9}$$

Under appropriate regularity assumptions a fixed point of the Picard's iteration is a solution of (5.8) and by using the Duhamel's formula

$$\begin{aligned} u^{(k)} &= e^{-t\Delta^2}u_0 + \int_0^t e^{-(t-\tau)\Delta^2}\sum_{j=2}^{\ell}\frac{(j+2)(j+1)}{6}\Delta(-\Delta u^{(k-1)})^j d\tau \\ &= e^{-t\Delta^2}u_0 + Tu^{(k-1)}. \end{aligned} \tag{5.10}$$

we see that the convergence properties of the Picard's iteration, depend on the mapping properties of the operator T. For regular enough spaces the existence of solutions of the problem can be studied by applying the Banach fixed point theorem to (5.10). In our case we want to study the equation (5.10) in a supercritical space, and therefore the convergence of the Picard's iteration is expected to be a difficult problem, but even without know that, valuable information can be obtained by studying the mapping properties of T. More specifically we will show that for fixed $\ell \in \mathbb{N}$ then the map T is not continuous at the origin in the space $L^\infty([0,T]; \mathcal{F}_q^{\frac{2\ell-2}{\ell},p})$ for $p \geq 1$, $q > \ell$.

Another way of looking at this mapping property, is to look at the second Picard's iteration of (5.9) given by

$$\partial_t u^{(1)} + \Delta^2 u^{(1)} = 0 \Rightarrow u^{(1)} = e^{-t\Delta^2}\varphi, \tag{5.11}$$

$$\begin{cases} \partial_t u^{(2)} + \Delta^2 u^{(2)} = \displaystyle\sum_{k=2}^{\ell} \frac{(k+2)(k+1)}{6} \Delta(-\Delta u^{(1)})^k & , \quad (0,T)\times\Omega \\ u^{(2)}(0) = \varphi & , \quad x \in \Omega \end{cases} \tag{5.12}$$

then by the Duhamel's formula

$$\begin{aligned} u^{(2)} &= e^{-t\Delta^2}u_0 + \int_0^t e^{-(t-\tau)\Delta^2} \sum_{k=2}^{\ell} \frac{(k+2)(k+1)}{6} \Delta(-\Delta e^{-\tau\Delta^2}u_0)^k d\tau. \\ &= e^{-t\Delta^2}u_0 + \sum_{k=2}^{\ell} g_k, \end{aligned} \tag{5.13}$$

where

$$\hat{g}_k = -\frac{(k+2)(k+1)}{6} \int_0^t e^{-(t-\tau)|\xi|^4} |\xi|^2 (|\cdot|^2 e^{-t|\cdot|^4\tau} u_0)^{*k} d\tau, \text{ for } k \geq 2, \tag{5.14}$$

where $f^{*k} = f * \cdots * f$ k times and $f^{*0} = 1$. Then for the second Picard's iteration we can look at the continuity of the map T in a special case. Consider the map

$\tilde{T} : X \to L^\infty([0,T];X)$ that takes a function $u_0 \in X$ and return its second Picard's iteration $u^{(2)}$ then

$$\tilde{T}u_0 = e^{-\Delta^2 t}u_0 + Te^{-\Delta^2}u_0, \tag{5.15}$$

and therefore the continuity of T implies the continuity of $\tilde{T}$, in this work we will show that the map $\tilde{T}$ is discontinuous at the origin in a supercritical space, which implies the discontinuity of T at the origin.

5.2 Known results

In physics, the study of crystal surfaces has a long history, here we focus in the developments for the equation in terms of the well posedness, see [29] and references therein for details.

In [29] the existence of global weak solutions for (5.5) in bounded domains of $\mathbb{R}^N$ with $W^{2,\infty}(\Omega) \cap W^{4,2}(\Omega)$ initial data is obtained. In [30] the existence of weak solutions for (5.1) in bounded domains of $\mathbb{R}^N$ for initial data $v_0 \in W^{2,2}(\Omega)$ with $(\Delta u_0)^{-3} \in W^{2,2}(\Omega)$. In [25] the existence global weak solution for (5.5) in a N dimensional periodic domain for $L^2(\mathbb{T}^N)$ initial data with small $\mathcal{F}^{2,1}(\mathbb{T}^N)$ norm. In [28] the well posedness is established $\mathbb{R}^N$ for solutions of (5.5) with $L^2(\mathbb{R}^N)$ initial data with small $\mathcal{F}^{2,1}(\mathbb{R}^N)$ norm. In [22] an iterative strategy and the existence of strong solutions is established for (5.5) in bounded domains of $\mathbb{R}^N$, for $v_0 \in L^2$ initial data with zero mean and finite energy $\phi(v_0) = \int e^{-\Delta v_0} < \infty$.

For references related to Ill-posedness results for fluid equations on which out strategy is based see Section 1.7. Up to our knowledge there are no other works dealing with the question of norm inflation for the epitaxial growth equation.

5.3 Main Results

The goal of this section is to prove the following theorem

Theorem 5.3.1 (Norm inflation for the truncated problem)**.** *Let* $\Omega = \mathbb{T}$. *Given* $T > 0$, $R > 1$, $q > \ell$, $p \geq 1$, *there exists some* $\tilde{t} < T$ *and some initial condition* $\varphi \in \mathcal{F}_q^{\frac{2\ell-4}{\ell},p}(\Omega)$ *such that the solution* u *of the initial value problem* (5.12) *satisfy*

$$\|u(\tilde{t})\|_{\mathcal{F}_q^{\frac{2\ell-4}{\ell},p}} \geq R, \tag{5.16}$$

$$\|\varphi\|_{\mathcal{F}_q^{\frac{2\ell-4}{\ell},p}} \leq 1/R. \tag{5.17}$$

Remark 5.3.2. Note that the initial data given by Theorem 5.3.1 depend on the choice of time, and consequently we cannot claim blow up for a specific solution after a short time, but we can say that there is always a solution with small initial data that becomes big after a short time.

The strategy for the proof is similar to the one used for the Muskat problem in Chapter 4. We consider an initial condition with several high frequency terms that can interact to produce a low frequency component as a result of the nonlinear interaction. Then we analyze separately what happens at low and high frequencies, and then use that the high frequency part decay much faster than the low frequency part, and use this to estimate the size of the solution.

The proof of Theorem 5.3.1 will be split in two lemmas. For the remainder of the section we fix $\ell \geq 2$.

Lemma 5.3.3 (Size estimate for the lower order terms)**.** *Let $2 \leq k < \ell$, $0 < T < 1$, $q > \ell$, $p \geq 1$ and suppose that we take $M > 0$ and $N_0 \in \mathbb{N}$ such that $TM^4 < 1$ and $Tk_{N_0}^4 \gg 1$ and consider g_k as defined by* (5.14) *with $u_0 = \varphi \in \dot{\mathcal{F}}_q^{\frac{2\ell-4}{\ell},p}$ given by* (5.21) *then for any $N \geq N_0$, there are constants and constants C_k, $k = 1, \cdots, \ell-1$ such that*

$$\|g_k(T)\|_{\mathcal{F}_q^{m,p}} \leq \frac{C}{T^3 k_N^4} \frac{(k+2)(k+1)}{6} 4^k \ell^{2k-4} \left(\sum_{j=N}^{(1+\delta)N} \gamma_j k_j^{\frac{2k-4}{k}} \right)^k, \quad k < \ell. \tag{5.18}$$

Lemma 5.3.4 (Size estimate for the main term)**.** *Consider g_ℓ as defined by* (5.14) *then under the same assumptions as Lemma 5.3.3 and using the same u_0 we have*

$$\|g_\ell(T)\|_{\mathcal{F}_q^{m,p}} \geq \frac{C_1}{\ell^2} \sum_{j=N}^{(1+\delta)N} \left(\gamma_j k_j^{\frac{2\ell-4}{\ell}} \right)^\ell - \frac{C_2}{T^3 k_N^4} \left(\sum_{j=N}^{(1+\delta)N} \gamma_j k_j^{\frac{2\ell-4}{\ell}} \right)^\ell. \tag{5.19}$$

5.3.1 The choice of the initial condition

Let $\delta > 0$, $\ell \in \mathbb{N}$, $N \in \mathbb{N}$, and $\{k_j\}_{j=1}^\infty$ a sequence of positive integers that grow very fast. More specifically given $\ell > 0$, $0 < \alpha < 1$, and $\delta > 0$, the sequence $\{k_j\}_j$ must satisfy: $\ell^2 k_j < k_{j+1}$ and

$$\left(\sum_{j=N}^{(1+\delta)N} \frac{1}{j^{\frac{1-\alpha}{\ell}}} \right)^\ell < \frac{1}{N} k_N^4. \tag{5.20}$$

We consider initial data similar to the one used in [26], more specifically we consider φ of the form

$$\hat{\varphi} = \sum_{j=N}^{(1+\delta)N} \gamma_j (P_{k_j} + P_{(\ell-1)k_j+1}), \tag{5.21}$$

where $P_A(k) = \delta_A(k) + \delta_{-A}(k)$ and $\delta_A(k)$ is the Kronecker's delta at the point k and $\{\gamma_j\}_j$ is a sequence of positive numbers that depend on $\{k_j\}_j$.

Remark 5.3.5. Note that this φ as defined by (5.21) is real valued in physical space because $\bar{\varphi}(\xi) = \varphi(-\xi) = \varphi(\xi)$ and consequently

$$\Re\varphi = \frac{\varphi + \bar{\varphi}}{2} = \varphi. \tag{5.22}$$

Lemma 5.3.6 (Size of Initial data)**.** *Consider* φ *as defined by* (5.21)*, then*

$$\|\varphi\|_{\mathcal{F}_q^{s,p}} \leq C(\ell, m, q) \left(\sum_{j=N}^{(1+\delta)N} |\gamma_j|^q |k_j|^{sq} \right)^{1/q}. \tag{5.23}$$

Proof of Lemma 5.3.6.

$$\begin{aligned} \|\varphi\|_{\mathcal{F}_q^{m,p}} &= \left(\sum_{j=N}^{(1+\delta)N} |\gamma_j|^q \left(|k_j|^{mq} + |(\ell-1)k_j + M|^{mq} \right) \right)^{1/q} \\ &= \left(\sum_{j=N}^{(1+\delta)N} |\gamma_j|^q |k_j|^{mq} \left(1 + \left| (\ell - 1) + \frac{M}{k_j} \right|^{mq} \right) \right)^{1/q} \\ &\leq C(\ell, m, q) \left(\sum_{j=N}^{(1+\delta)N} \gamma_j^q k_j^{mq} \right)^{1/q}, \end{aligned} \tag{5.24}$$

note that every k_j and $(\ell - 1)k_j$ belong to a different annulus C_k and therefore the p norm do not appear in the computation. This concludes the proof of Lemma 5.3.6. □

5.3.2 Estimate Lower order terms: Proof of Lemma 5.3.3

The idea of Lemma 5.3.3 is that when we substitute φ in $\hat{g}_k$ we get an expansion that is a sum of of terms that are supported far away from the origin because of our choice of φ. For the lower order terms the choice of the γ_i is enough to establish the smallness, but for g_ℓ a more delicate analysis is required.

Proof of Lemma 5.3.3. First we substitute (5.21) in (5.14) to obtain that for $n \in \mathbb{Z}$

$$\begin{aligned} \hat{g}_k(n) &= -\frac{(k+2)(k+1)}{6}\int_0^t e^{-(t-\tau)|n|^4}|n|^2(|\cdot|^2\hat{u}_1)^{*k}d\tau \\ &= -\frac{(k+2)(k+1)}{6}\sum_{j_1=N}^{(1+\delta)N}\sum_{a_1\in\Lambda_{j_1}}\cdots\sum_{j_k=N}^{(1+\delta)N}\sum_{a_k\in\Lambda_{j_k}}\gamma_{j_1}\cdots\gamma_{j_k} \\ &\qquad \times\hat{J}(a_1,\cdots,a_k)(n), \end{aligned} \tag{5.25}$$

where $\Lambda_j = \{\pm k_j, \pm((\ell-1)k_j+1)\}$

$$\hat{J}(a_1,\cdots,a_k)(n) = \int_0^t e^{-(t-\tau)|n|^4}|n|^2(e^{-\tau|\cdot|^4}\delta_{a_1}) * \cdots * (e^{-\tau|\cdot|^4}\delta_{a_k})d\tau. \tag{5.26}$$

Now because $|\cdot|^2e^{-|\cdot|^4\tau}\delta_a = a^2e^{-a^4\tau}\delta_a$ and

$$(\delta_a * \delta_b)(\xi) = \sum_{k\in\mathbb{Z}}\delta_a(\xi-k)\delta_b(k) = \delta_a(\xi-b) = \delta_{a+b}(\xi), \tag{5.27}$$

we obtain that $\hat{J}(a_1,\cdots,a_k)$ can be written as

$$\begin{aligned} \hat{J}(a_1,\cdots,a_k)(n) &= \int_0^t e^{-(t-\tau)|n|^4}|n|^2(|\cdot|^2e^{-|\cdot|^4\tau}\delta_{a_1}) * \cdots * (|\cdot|^2e^{-|\cdot|^4\tau}\delta_{a_k})d\tau \\ &= a_1^2\cdots a_k^2(a_1+\cdots+a_n)^2\delta_{a_1+\cdots+a_k}(n) \\ &\qquad \times\int_0^t e^{-(t-\tau)(a_1+\cdots+a_k)^4}e^{-\tau(a_1^4+\cdots+a_k^4)}d\tau. \end{aligned} \tag{5.28}$$

Note that if $a_1+\cdots+a_k = 0$ then $\hat{J}(a_1,\cdots,a_k) = 0$, therefore to estimate $\hat{J}$ we can assume that $a_1+\cdots+a_k \neq 0$. The following lemma is key for the estimates of smallness of high frequency terms.

Lemma 5.3.7. *Let $k \leq \ell$ and consider $a_i \in \{\pm k_{j_i}, \pm((\ell-1)k_{j_i}+M)\}$ such that $a_1 + \cdots + a_n \neq 0$, $M > \ell$, $\ell^2 k_j < k_{j+1}/2$, $\ell M < k_N/2$, and suppose we have one of the following*

a) $k < \ell$, and take , $\ell^2 k_j < k_{j+1}/2$, $\ell M < k_N/2$,

b) $k = \ell$, $a_i \in \Lambda_{j_i}$ with not all j_i equal,

c) $k = \ell$, $a_1 + \cdots + a_n \neq \pm M$, $a_i \in \Lambda_j$ with the same j for all i.

then

i) $|a_1 + \cdots + a_k|^4 > \frac{|a_i|^4}{2}$ some i,

ii) at least one among $|a_1 + \cdots + a_k|^4$ and $|a_1^4 + \cdots + a_k^4 - |a_1 + \cdots + a_k|^4|$, can be bounded below by $\max_j |a_j|^4/2$.

Proof of Lemma 5.3.7. To prove part $i)$ the key is to understand the implications of the hypothesis $a_1 + \cdots + a_n \neq 0$. To do this lets first assume that for all $i \in \{1, \cdots, n\}$ $a_i \in \Lambda_j = \{\pm k_j, \pm((\ell-1)k_j + M)\}$, then we write $a_i = b_i k_j + \varepsilon_i$ with $b_i \in \{\pm 1, \pm(\ell-1)\}$ and $\varepsilon_i \in \{-M, 0, M\}$. By grouping all the $b_i \in \{\pm 1\}$ together and all the $b_i \in \{\pm(\ell-1)\}$ together, we can write

$$b_1 + \cdots + b_n = p + q(\ell-1), \tag{5.29}$$

where $|p| < \ell$ and $|q| < \ell$, this means that p is only divisible by $(\ell-1)$ if $p = 0$, and therefore the only solution to $p + q(\ell-1) = 0$ for $|p| < \ell$, $|q| < \ell$ is $p = q = 0$. Now $q = 0$ means that we have the same number of $b_i = -(\ell-1)$ than

$b_i = \ell - 1$ and consequently their corresponding ε_i cancel exactly, which implies that $\varepsilon_1 + \cdots + \varepsilon_n = 0$. By contrapositive $a_1 + \cdots + a_n \neq 0$ imply that $b_1 + \cdots + b_n \neq 0$. By out assumption in k_N we also know that

$$|\varepsilon_1 + \cdots + \varepsilon_n| \leq nM < \ell M \leq k_N/2, \tag{5.30}$$

and therefore we can conclude that

$$|a_1 + \cdots + a_n| \geq |b_1 + \cdots + b_n| k_j - |\varepsilon_1 + \cdots + \varepsilon_n| \geq k_j - \frac{k_N}{2} \geq \frac{k_j}{2}. \tag{5.31}$$

This proves part i) in the case that for all i, $a_i \in \Lambda_j$. For the general case in which $a_i \in \Lambda_{j_i}$ where the j_i could be different, in this case again we can write $a_i = b_i k_{j_i} + \varepsilon_i$ where b_i and ε_i as before, we group the terms b_i whose corresponding a_i belong to the same Λ_j, by doing this we get

$$\begin{aligned} a_1 + \cdots + a_n &= (p_N + (\ell - 1) q_N) k_N + (p_{N+1} + (\ell - 1) q_{N+1}) k_{N+1} \\ &\quad + \cdots + (p_{\lfloor (1+\delta) N \rfloor} + (\ell - 1) q_{\lfloor (1+\delta) N \rfloor}) k_{\lfloor (1+\delta) N \rfloor} + (\varepsilon_1 + \cdots \varepsilon_n) \end{aligned} \tag{5.32}$$

where $|p_j| < \ell$, $|q_j| < \ell$. This is obtained from the number of terms in case a) or because not all a_i belong to the same Λ_j in the case b). Now because $\ell^2 k_j < k_{j+1}/2$

we obtain that

$$
\begin{aligned}
|I_{N_2}| &= |(p_N + (\ell-1)q_N)k_N + (p_{N+1} + (\ell-1)q_{N+1})k_{N+1} \\
&\qquad + \cdots + (p_{N_2} + (\ell-1)q_{N_2})k_{N_2}| \\
&\leq |p_N + (\ell-1)q_N|k_N + |p_{N+1} + (\ell-1)q_{N+1}|k_{N+1} \\
&\qquad + \cdots + |p_{N_2} + (\ell-1)q_{N_2}|k_{N_2} \qquad (5.33) \\
&\leq |\ell-1 + (\ell-1)(\ell-1)|k_N + |\ell-1 + (\ell-1)(\ell-1)|k_{N+1} \\
&\qquad + \cdots + |\ell-1 + (\ell-1)(\ell-1)|k_{N_2} \\
&\leq \sum_{j=N}^{N_2} \ell(\ell-1)k_j.
\end{aligned}
$$

By induction we now prove that $\sum_{j=N}^{r} \ell(\ell-1)k_j \leq k_{r+1}$. For $r = N$ this is direct from out assumption in k_j

$$\sum_{j=N}^{N} \ell(\ell-1)k_j = \ell(\ell-1)k_N < \ell^2 k_N \leq k_{N+1}. \qquad (5.34)$$

Now assume that $\sum_{j=N}^{r} \ell(\ell-1)k_j < k_{r+1}$ we want to show that $\sum_{j=N}^{r+1} \ell(\ell-1)k_j < k_{r+2}/2$

$$
\begin{aligned}
\sum_{j=N}^{r+1} \ell(\ell-1)k_j &= \sum_{j=N}^{r} \ell(\ell-1)k_j + \ell(\ell-1)k_r \\
&\leq k_{r+1} + \ell(\ell-1)k_{r+1} \qquad (5.35) \\
&= \ell^2 k_{r+1} \\
&< k_{r+2}/2.
\end{aligned}
$$

Now using this we get that

$$\begin{aligned} |a_1 + \cdots + a_n| &\geq |(p_{\lfloor (1+\delta)N \rfloor} + (\ell-1) q_{\lfloor (1+\delta)N \rfloor}) k_{\lfloor (1+\delta)N \rfloor}| \\ &\qquad - \sum_{j=N}^{\lfloor (1+\delta)N \rfloor - 1} |p_j + (\ell-1) q_j| k_j - |\varepsilon_1 + \cdots + \varepsilon_n| \\ &> |(p_{\lfloor (1+\delta)N \rfloor} + (\ell-1) q_{\lfloor (1+\delta)N \rfloor}) k_{\lfloor (1+\delta)N \rfloor}| \\ &\qquad - \sum_{j=N}^{\lfloor (1+\delta)N \rfloor - 1} |p_j + (\ell-1) q_j| k_j - k_N \\ &> |(p_{\lfloor (1+\delta)N \rfloor} + (\ell-1) q_{\lfloor (1+\delta)N \rfloor})| k_{\lfloor (1+\delta)N \rfloor} - k_{\lfloor (1+\delta)N \rfloor} \\ &= \left(|(p_{\lfloor (1+\delta)N \rfloor} + (\ell-1) q_{\lfloor (1+\delta)N \rfloor})| - 1 \right) k_{\lfloor (1+\delta)N \rfloor}. \end{aligned} \tag{5.36}$$

Now because $p_{\lfloor (1+\delta)N \rfloor} + (\ell-1) q_{\lfloor (1+\delta)N \rfloor}$ is an integer we conclude that $a_1 + \cdots + a_n$ can only be zero if

$$p_{\lfloor (1+\delta)N \rfloor} + (\ell-1) q_{\lfloor (1+\delta)N \rfloor} = 0. \tag{5.37}$$

By adding this condition we can run the argument again to conclude that $a_1+\cdots+a_n$ can only be zero if for every $j \in [N, (1+\delta)N]$

$$p_j + (\ell-1) q_j = 0. \tag{5.38}$$

Substituting this in (5.32) we obtain that $\varepsilon_1 + \cdots + \varepsilon_n = 0$ also must be zero. Now by the argument of the first part we obtain that $p_j = 0$ and $q_j = 0$ for every $j \in [N, (1+\delta)N]$, and also $\varepsilon_1 + \cdots + \varepsilon_n = 0$. By the contrapositive we obtain that if $a_1 + \cdots + a_n \neq 0$ for at least one $j \in [N, (1+\delta)N]$ we have that $p_j + (\ell-1) q_j \neq 0$.

Let $\hat{j}$ the largest of such j. then from (5.32) we can write

$$\begin{aligned} |a_1 + \cdots + a_n| &\geq |p_j + (\ell-1)q_j|k_j - \sum_{i=N}^{j-1} |p_i + (\ell-1)q_i|k_i - |\varepsilon_1 + \cdots + \varepsilon_n| \\ &\geq |p_j + (\ell-1)q_j|k_j - \frac{1}{2}k_N \\ &\geq \frac{1}{2}k_j. \end{aligned} \tag{5.39}$$

Case c) is a little bit more delicate, in this case we have

$$a_1 + \cdots + a_\ell = (p + q(\ell-1))k_j + \varepsilon_1 + \cdots + \varepsilon_\ell, \tag{5.40}$$

then we need to show that $|p+q(\ell-1)| \geq 1$, because p and q are integers we only need to shot that the quantity is nonzero. Suppose not, then because $|p| \leq \ell$, $|q| \leq \ell$ there are only 3 possibilities $(p,q) = (0,0)$, $(p,q) = (\ell-1,-1)$ and $(p,q) = (-\ell+1,1)$. If $(p,q) = (0,0)$ then there are the same number of b_i equal to $+(\ell-1)$ and $-(\ell-1)$, and therefore their corresponding ε_i cancel exactly to give $\varepsilon_1 + \cdots + \varepsilon_\ell = 0$, which imply that $a_1 + \cdots + a_\ell = 0$, which is a contradiction with the assumptions. The other case is that $(p,q) = (\ell-1,-1)$ (the case $(p,q) = (-\ell+1,1)$ is analogous) then $\ell-1$ of the b_i are equal to 1 and one of them is equal to $-(\ell-1)$. Consequently the corresponding ε_i satisfy $\varepsilon_1+\cdots+\varepsilon_\ell = -M$ and therefore $a_1+\cdots+a_M = -M$, which is also a contradiction with our assumptions. We conclude that $|p + (\ell-1)q| \geq 1$

and therefore

$$\begin{aligned} |a_1+\cdots+a_n| &= |(p+(\ell-1)q)k_j+\varepsilon_1+\cdots+\varepsilon_\ell| \\ &\geq |(p+(\ell-1)q)|k_j-|\varepsilon_1+\cdots+\varepsilon_\ell| \\ &\geq k_j-\frac{1}{2}k_N \\ &\geq \frac{1}{2}k_j. \end{aligned} \tag{5.41}$$

For part *ii*) we use that

$$\left(a_1^4+\cdots+a_k^4-|a_1+\cdots+a_k|^4\right)+|a_1+\cdots+a_k|^4=a_1^4+\cdots+a_k^4\geq \max_j |a_j|^4, \tag{5.42}$$

then we have that the sum of two terms is larger than a positive number, that imply that at least one of them is at least half that amount in modulus.

□

Continuation of proof of Lemma 5.3.3. Integrating in time in (5.28),

$$\hat{J}(a_1,\cdots,a_k)(n)=\frac{a_1^2\cdots a_k^2\,(a_1+\cdots+a_k)^2}{a_1^4+\cdots+a_k^4-|a_1+\cdots+a_k|^4}e^{-t|a_1+\cdots+a_k|^4}$$

$$\times\,(1-e^{-t\left(|a_1|^4+\cdots+|a_k|^4-|a_1+\cdots+a_k|^4\right)})\delta_{a_1+\cdots+a_k}(n) \tag{5.43}$$

$$|\hat{J}(n)|\leq \frac{C}{t^3}\frac{a_1^2\cdots a_n^2\,(a_1+\cdots+a_n)^2}{\left|a_1^4+\cdots+a_n^4-|a_1+\cdots+a_n|^4\right|}\frac{1}{|a_1+\cdots+a_n|^{12}}\delta_{a_1+\cdots+a_k}(n), \tag{5.44}$$

and taking the norm $\mathcal{F}_q^{m,p}$ we get

$$\begin{aligned} \|J\|_{\mathcal{F}_q^{m,p}} &\leq \frac{C}{t^3}\frac{a_1^2\cdots a_k^2|a_1+\cdots+a_k|^{2+m}}{\left|a_1^4+\cdots+a_k^4-|a_1+\cdots+a_k|^4\right|}\frac{1}{|a_1+\cdots+a_k|^{12}} \\ &\leq \frac{C}{t^3}\frac{a_1^2\cdots a_k^2}{\left|a_1^4+\cdots+a_k^4-|a_1+\cdots+a_k|^4\right|}\frac{1}{|a_1+\cdots+a_k|^{8}}, \end{aligned} \tag{5.45}$$

Note that because $\hat{J}(n)$ is supported at a single frequency p and q do not affect the computation of the norm. Now by Lemma 5.3.7 part *ii*) we know that we can bound below at least one among $|a_1+\cdots+a_k|^4$ and $\left|a_1^4+\cdots+a_k^4-|a_1+\cdots+a_k|^4\right|$ by $C\max_j |a_j|^4$ and therefore

$$\|J\|_{\mathcal{F}_q^{m,p}} \leq \frac{C}{t^3 k_N^4}\frac{a_1^2\cdots a_k^2}{\max_j |a_j|^4} \leq \frac{C}{t^3 k_N^4}\left(a_1\cdots a_k\right)^{\frac{2k-4}{k}}. \tag{5.46}$$

Next, summing over all tuples $(a_1,\cdots,a_k)$

$$\begin{aligned}
\|g_k\|_{\mathcal{F}_q^{m,p}} &= \Bigg\| -\frac{(k+2)(k+1)}{6}\sum_{j_1=N}^{(1+\delta)N}\sum_{a_1\in\Lambda_{j_1}}\cdots\sum_{j_k=N}^{(1+\delta)N}\sum_{a_n\in\Lambda_{j_k}}\gamma_{j_1}\cdots\gamma_{j_k} \\
&\qquad \times J(a_1,\cdots,a_k)\Bigg\|_{\mathcal{F}_q^{m,p}} \\
&\leq \frac{(k+2)(k+1)}{6}\sum_{j_1=N}^{(1+\delta)N}\sum_{a_1\in\Lambda_{j_1}}\cdots\sum_{j_k=N}^{(1+\delta)N}\gamma_{j_1}\cdots\gamma_{j_k}\|J(a_1,\cdots,a_k)\|_{\mathcal{F}_q^{m,p}} \\
&\leq \frac{C}{t^3k_N^4}\frac{(k+2)(k+1)}{6}4^k\sum_{j_1=N}^{(1+\delta)N}\cdots\sum_{j_k=N}^{(1+\delta)N}\gamma_{j_1}\cdots\gamma_{j_k}|a_1\cdots a_k|^{\frac{2k-4}{k}} \\
&\leq \frac{C}{t^3k_N^4}\frac{(k+2)(k+1)}{6}4^k\ell^{2k-4}\left(\sum_{j=N}^{(1+\delta)N}\gamma_j k_j^{\frac{2k-4}{k}}\right)^k.
\end{aligned} \tag{5.47}$$

This concludes the proof of Lemma 5.3.3. □

5.3.3 Lower bound for the main term: Proof of Lemma 5.3.4

In this section we prove the main estimate of the norm inflation result. After substituting (5.21) in g_ℓ we split the terms with the objective of isolate the ones that can generate the inflation. Then we establish a lower bound for the low frequency terms that do not decay with N. For the upper bound of the high frequency terms, the idea is similar to the proof of Lemma 5.3.3 but with the additional difficulty that this time we expect that $\gamma_j \sim k_j^{-\frac{2\ell-4}{\ell}}$ and therefore this time we are forced to use the exponential decay to obtain that as N become large, the norm of the high frequency part is small when compared with the low frequency part.

Proof of Lemma 5.3.4. By substituting (5.21) in (5.14) we get that $\hat{g}_\ell$ for $n \in \mathbb{Z}$ can be written as

$$\hat{g}_\ell(n) = -\frac{(\ell+2)(\ell+1)}{6} \sum_{j=N}^{(1+\delta)N} \sum_{a_1 \in \Lambda_j} \cdots \sum_{a_\ell \in \Lambda_j} \gamma_j^\ell \hat{J}(a_1, \cdots, a_\ell)(n) + \widehat{CT}(n), \quad (5.48)$$

where $\Lambda_j = \{\pm k_j, \pm(k_j+1)\}$ and

$$\hat{J}(a_1, \cdots, a_\ell)(n) = \int_0^t e^{-(t-\tau)|n|^4} |n|^2 (|\cdot|^2 e^{-\tau|\cdot|^4} \delta_{a_1}) * \cdots * (|\cdot|^2 e^{-\tau|\cdot|^4} \delta_{a_\ell}) d\tau, \quad (5.49)$$

$$\widehat{CT}(n) = -\frac{(\ell+2)(\ell+1)}{6} \sum_{\substack{j_1, \cdots, j_\ell \in \\ \{N, \cdots, (1+\delta)N\} \\ \text{not all equal}}} \sum_{a_1 \in \Lambda_{j_1}} \cdots \sum_{a_\ell \in \Lambda_{j_\ell}} \gamma_{j_1} \cdots \gamma_{j_\ell} \hat{J}(a_1, \cdots, a_\ell)(n). \quad (5.50)$$

Here CT is the term that involves all the cross terms, i.e. the terms for which not all the factors have the same j in the convolution. For this estimate we focus on the terms where $a_1 + \cdots + a_\ell$ is small compared with other quantities in our problem. In our case, the smallest this sum can be is M. We can split our sum as

$$\hat{g}_\ell = \hat{L}_M + \hat{L}_{-M} + \widehat{HF} + \widehat{CT}, \quad (5.51)$$

where

$$\hat{L}_M = -\frac{(\ell+2)(\ell+1)}{6} \sum_{j=N}^{(1+\delta)N} \sum_{\substack{(a_1, \cdots, a_\ell) \\ \in H_M^{(j)}}} \gamma_j^\ell \hat{J}(a_1, \cdots, a_\ell), \quad (5.52)$$

$$\hat{L}_{-M} = -\frac{(\ell+2)(\ell+1)}{6} \sum_{j=N}^{(1+\delta)N} \sum_{\substack{(a_1,\cdots,a_\ell)\\ \in H_{-M}^{(j)}}} \gamma_j^\ell \hat{J}(a_1,\cdots,a_\ell), \tag{5.53}$$

where $H_B^{(j)}$ is the set of tuples $(a_1,\cdots,a_\ell) \in (\Lambda_j)^\ell$ such that $a_1+\cdots+a_\ell = B$, the term HF represent the high frequency terms.

$$\widehat{HF}(n) = -\frac{(m+2)(m+1)}{6} \sum_{j=N}^{(1+\delta)N} \sum_{a_1\in\Lambda_j} \cdots \sum_{a_m\in\Lambda_j} J(a_1,\cdots,a_\ell)\chi_{|n|>M}. \tag{5.54}$$

Lemma 5.3.8. *Consider L_M and L_{-M} as defined by* (5.52) *and* (5.53) *and let $0<t<1$ such that M and N_0 satisfy $tM^4<1$ and $tK_N^4 \gg 1$ for $N \geq N_0$ then*

$$\|L_M(t)+L_{-M}(t)\|_{\mathcal{F}_q^{m,p}} \geq \frac{C}{\ell^2} \sum_{j=N}^{(1+\delta)N} \left(\gamma_j k_j^{\frac{2\ell-4}{\ell}}\right)^\ell. \tag{5.55}$$

Lemma 5.3.9. *Let CT, HF as defined by* (5.50) *and* (5.54)*. Under the same assumptions as in Lemma 5.3.8 we have that*

$$\|CT(t)\|_{\mathcal{F}_q^{m,p}} \leq \frac{C}{t^2k_N^4} \left(\sum_j \gamma_j k_j^{\frac{2\ell-4}{\ell}}\right)^\ell, \tag{5.56}$$

and

$$\|HF(t)\|_{\mathcal{F}_q^{m,p}} \leq \frac{C}{t^2k_N^4} \left(\sum_j \gamma_j k_j^{\frac{2\ell-4}{\ell}}\right)^\ell. \tag{5.57}$$

Proof of Lemma 5.3.9. This follows from the proof of Lemma 5.3.3, because under the assumptions of the Lemma, the hypothesis Lemma 5.3.7 still applies and therefore the same proof holds. □

Continuation of proof of Lemma 5.3.4. From Lemmas 5.3.8, 5.3.9 we get that

$$\begin{aligned} \|g_\ell\|_{\mathcal{F}_q^{m,p}} &\geq \|L_1+L_2\|_{\mathcal{F}_q^{m,p}} - \|HF\|_{\mathcal{F}_q^{m,p}} - \|CT\|_{\mathcal{F}_q^{m,p}} - \|MF\|_{\mathcal{F}_q^{m,p}} \\ &\geq \frac{C}{\ell^4}\sum_j \left(\gamma_j k_j^{\frac{2\ell-4}{\ell}}\right)^\ell - \frac{C}{t^2k_N^4}\left(\sum_j \gamma_j k_j^{\frac{2\ell-4}{\ell}}\right)^\ell, \end{aligned} \tag{5.58}$$

which conclude the proof of the Lemma 5.3.4. □

Now we proceed to prove the lower bound for the low frequency part.

Proof of Lemma 5.3.8. We need to estimate the term $\hat{J}(a_1, \cdots, a_\ell)$ for $(a_1, \cdots, a_\ell) \in H_M^{(j)}$ i.e. when $a_1 + \cdots + a_\ell = M$.

$$\begin{aligned} \hat{J}(a_1, \cdots, a_\ell) &= \int_0^t e^{-(t-\tau)|\xi|^4} |\xi|^2 (|\cdot|^2 e^{-\tau|\cdot|^4} \delta_{a_1}) * \cdots * (|\cdot|^2 e^{-\tau|\cdot|^4} \delta_{a_\ell}) d\tau \\ &= \frac{a_1^2 \cdots a_\ell^2}{a_1^4 + \cdots + a_\ell^4 - M^4} e^{-Mt} (1 - e^{-t(a_1^4 + \cdots + a_\ell^4 - M^4)}) \delta_M, \end{aligned} \tag{5.59}$$

because $0 < t < 1$ and $tk_N^4 \gg 1$ we can ensure that

$$e^{-M^4 t} \left(1 - e^{-t(a_1^4 + \cdots + a_k^4 - M^4)}\right) > \frac{1}{2} e^{-M^4}, \tag{5.60}$$

then we get

$$\hat{J}(a_1, \cdots, a_\ell) \geq C \frac{a_1^2 \cdots a_\ell^2}{a_1^4 + \cdots + a_\ell^4 - M^4} \delta_M. \tag{5.61}$$

Now using the bound $a_1^4 + \cdots + a_\ell^4 \leq \ell((\ell - 1)k_j + M)^4 \leq C\ell^5 k_j^4$ we get

$$\begin{aligned} \hat{J}(a_1, \cdots, a_\ell) &\geq \frac{C}{\ell^5} e^{-M^4} \frac{a_1^2 \cdots a_\ell^2}{k_j^4} \delta_M \\ &\geq \frac{C}{\ell^5} \frac{a_1^2 \cdots a_\ell^2}{|a_1 \cdots a_\ell|^{4/\ell}} \delta_M \\ &\geq \frac{C}{\ell^5} |a_1 \cdots a_\ell|^{\frac{2\ell - 4}{\ell}} \delta_M. \end{aligned} \tag{5.62}$$

Summing over $(a_1, \cdots, a_\ell) \in H_M^{(j)}$ and over j we get

$$\begin{aligned} \sum_{j=N}^{(1+\delta)N} \sum_{a_i^{(j)} \in \Lambda_j} \gamma_j^\ell \hat{J}(a_1, \cdots, a_\ell) &\geq \sum_{j=N}^{(1+\delta)N} \sum_{a_i \in \Lambda_j} \frac{C}{\ell^5} \gamma_j^\ell |a_1 \cdots a_\ell|^{\frac{2\ell-4}{\ell}} \delta_M \\ &\geq \frac{C}{\ell^4} \sum_{j=N}^{(1+\delta)N} \left(\gamma_j k_j^{\frac{2\ell-4}{\ell}}\right)^\ell \delta_M. \end{aligned} \tag{5.63}$$

The power on ℓ comes from the symmetry of the sum over $a_i \in \Lambda_j$. We obtain

$$\begin{aligned} -\hat{L}_M &= \frac{(\ell+2)(\ell+1)}{6} \sum_{j=N}^{(1+\delta)N} \sum_{H_M^{(j)}} \gamma_j^\ell \hat{J}(a_1, \cdots, a_\ell) \\ &\geq \frac{C}{\ell^2} \sum_{j=N}^{(1+\delta)N} \left(\gamma_j k_j^{\frac{2\ell-4}{\ell}}\right)^\ell \delta_M. \end{aligned} \tag{5.64}$$

Analogously for L_{-M}

$$\begin{aligned} -\hat{L}_{-M} &= \frac{(\ell+2)(\ell+1)}{6} \sum_{j=N}^{(1+\delta)N} \sum_{H_M^{(j)}} \gamma_j^\ell J(a_1, \cdots, a_\ell)) \\ &\geq \frac{C}{\ell^2} \sum_{j=N}^{(1+\delta)N} \left(\gamma_j k_j^{\frac{2\ell-4}{\ell}}\right)^\ell \delta_{-M}, \end{aligned} \tag{5.65}$$

then we conclude

$$\|L_M + L_{-M}\|_{\mathcal{F}_q^{m,p}} \geq \frac{C}{\ell^2} \sum_{j=N}^{(1+\delta)N} \left(\gamma_j k_j^{\frac{2\ell-4}{\ell}}\right)^\ell. \tag{5.66}$$

This concludes the proof of Lemma 5.3.8. □

5.3.4 Norm inflation: Proof of Theorem 5.3.1

In this section we put together our previous estimate to prove the discontinuity of the solution map at the origin as described on the introduction.

Proof of Theorem 5.3.1. First we choose some $M \in \mathbb{N}$, $M > \ell$ so that $TM^4 < 1$. Take $N_0 \in \mathbb{N}$ such that $e^{-T(k_N/2)^4} < \frac{1}{2}$ for $N \geq N_0$. Consider φ, g_k, $k = 1, \cdots, \ell$ as given by Lemmas 5.3.3 and 5.3.4, then $u = \sum_{k=1}^{\ell} g_k$ is a solution of (5.13) with initial condition φ given by (5.21) where the parameters M and N are as stated before. By taking the $\mathcal{F}_q^{m,p}$ norm of u we get

$$\begin{aligned}\|u(T)\|_{\mathcal{F}_q^{m,p}} &\geq \|u_\ell\|_{\mathcal{F}_q^{m,p}} - \sum_k \|u_k\|_{\mathcal{F}_q^{m,p}} - \|e^{-T\Delta^2}\varphi\|_{\mathcal{F}_q^{m,p}} \\ &\geq C_\ell \sum \gamma_j^\ell k_j^{2\ell-4} - \frac{C_{\ell+1}}{T^3 k_N^4}\left(\sum_j \gamma_j k_j^{\frac{2\ell-4}{\ell}}\right)^\ell \\ &\qquad - \sum_{k=2}^{\ell-1} \frac{C_k}{T^3 k_N^4}\left(\sum_j \gamma_j k_j^{\frac{2k-4}{k}}\right)^k - \|e^{-T\Delta^2}\varphi\|_{\mathcal{F}_q^{m,p}}.\end{aligned} \tag{5.67}$$

Now we take $\gamma_j = \frac{1}{k_j^{\frac{2\ell-4}{\ell}}} \frac{1}{j^{\frac{1-\eta}{\ell}}}$, with this choice we get

$$\sum_{j=N}^{(1+\delta)N} \gamma_j k_j^{\frac{2k-4}{k}} = \sum_{j=N}^{(1+\delta)N} \frac{1}{k_j^{\frac{2\ell-4}{\ell}-\frac{2k-4}{k}}} \frac{1}{j^{\frac{1-\eta}{\ell}}} < 1, \tag{5.68}$$

because $\frac{2k-4}{k} < \frac{2\ell-4}{\ell}$ for $k < \ell$. Now we can choose $N_1 > N_0$ so that $\sum_{k=2}^{\ell-1} \frac{C_k}{T^3 k_N^4} < 1$.

For the second term the situation is more delicate, because in this case we get

$$\sum_{j=N}^{(1+\delta)N} \gamma_j k_j^{\frac{2k-4}{k}} = \sum_{j=N}^{(1+\delta)N} \frac{1}{k_j^{\frac{2\ell-4}{\ell}-\frac{2\ell-4}{\ell}}} \frac{1}{j^{\frac{1-\eta}{\ell}}} = \sum_{j=N}^{(1+\delta)N} \frac{1}{j^{\frac{1-\eta}{\ell}}}, \tag{5.69}$$

and because $\frac{1-\eta}{\ell} < 1$ we get that this expression growth with N. Now because by assumption k_N grow very fast, more precisely from our assumption in (5.20) we have that

$$\left(\sum_{j=N}^{(1+\delta)N} \frac{1}{j^{\frac{1-\eta}{\ell}}}\right)^\ell \leq \frac{1}{N} k_N, \tag{5.70}$$

Therefore we can take $N_2 \geq N_1$ such that for all $N \geq N_2$

$$\frac{C_{\ell+1}}{T^3 k_N^4}\left(\sum_{j=N}^{(1+\delta)N} \gamma_j k_j^{\frac{2\ell-4}{\ell}}\right)^\ell < 1. \tag{5.71}$$

Lastly we take a look at the first term, because of our choice of γ_j, this time we have

$$\sum_{j=N}^{(1+\delta)N} \gamma_j^\ell k_j^{2\ell-4} = \sum_{j=N}^{(1+\delta)N} \frac{1}{j^{1-\eta}}, \tag{5.72}$$

We can bound this integral by comparison with the integral

$$\begin{aligned}\sum_{j=N}^{(1+\delta)N} \frac{1}{j^{1-\eta}} &\geq \int_{N+1}^{\lfloor(1+\delta)N\rfloor} \frac{1}{x^{1-\eta}} dx \\ &= \frac{1}{\eta} x^{\eta}\Big|_{N+1}^{\lfloor(1+\delta)N\rfloor} \\ &= \frac{1}{\eta} N^{\eta} \left(\frac{\lfloor(1+\delta)^{\eta} N^{\eta}\rfloor}{N^{\eta}} - 1 \right) \\ &\sim N^{\eta},\end{aligned} \tag{5.73}$$

Therefore this sum grow as N^{η} as $N \to \infty$. Therefore given $R > 0$, we can take $N_3 > N_2$ such that for all $N \geq N_3$

$$C_{\ell} \sum_{j=N}^{(1+\delta)N} \gamma_j^{\ell} k_j^{2\ell-4} \geq R + 4. \tag{5.74}$$

For the term involving the initial condition we first notice that

$$|\mathcal{F}(e^{-t\Delta^2}\varphi)(n)| = |e^{-tn^4}\hat{\varphi}(n)| \leq |\hat{\varphi}(n)|, \tag{5.75}$$

then we get that from Lemma 5.3.6

$$\|e^{-T\Delta^2}\varphi\|_{\mathcal{F}_q^{m,p}} \leq \|\varphi\|_{\mathcal{F}_q^{m,p}} \leq C_0 \left(\sum_{j=N}^{(1+\delta)N} \gamma_j^q k_j^{mq} \right)^{1/q}, \tag{5.76}$$

then we get for $m = \frac{2\ell-4}{\ell}$

$$\left(\sum_{j=N}^{(1+\delta)N} \gamma_j^q k_j^{mq} \right)^{1/q} = \left(\sum_{j=N}^{(1+\delta)N} \frac{1}{j^{\frac{1-\eta}{\ell}q}} \right)^{1/q}. \tag{5.77}$$

Because we want this term to be small, we take q such that $\frac{1-\eta}{\ell}q > 1$, and therefore because $q > \ell$, we can always chose $\eta > 0$ such that this is satisfied, and if that is the case, then the sum go to 0 as $N \to \infty$, therefore we can take $N_4 \geq N_3$ such that for all $N \geq N_4$

$$C_0 \left(\sum_{j=N}^{(1+\delta)N} \gamma_j^q k_j^{mq} \right)^{1/q} < 1/R. \tag{5.78}$$

Finally we can put all together, to obtain that for all $N \geq N_4$ and $q > \frac{\ell}{1-\eta}$ we have

$$\|u(\tilde{T})\|_{\mathcal{F}_q^{\frac{2\ell-4}{\ell},p}} \geq R. \tag{5.79}$$

which concludes the proof of Theorem 5.3.1. □

Chapter 6

Appendix

6.1 Asymptotic Estimate for the convolution integral in the Muskat problem

The goal of this section is to provide asymptotic estimate for n large to the integral

$$I(A_1, \cdots, A_n) = \int_{\mathbb{R}} \left(\frac{1 - e^{-i\alpha A_1}}{\alpha} \right) \cdots \left(\frac{1 - e^{-i\alpha A_n}}{\alpha} \right) d\alpha. \tag{6.1}$$

The estimate obtained in this section is not used in any chapter but it is interesting on its own right so it is included on this appendix. From the Chapter 4 we know an explicit formula for I and a size estimate of the form

$$|I(A_1, \cdots, A_n)| \leq C \frac{|A_1 \cdots A_n|}{\max_i |A_i|}, \tag{6.2}$$

in this section we want to provide an estimate that takes in consideration the signs of the A_i and if possible a lower bound for its magnitude.

Lemma 6.1.1. *Let $A_1, \cdots, A_n, M \in \mathbb{R}$ such that $1 \leq |A_i| \leq M$. Then there exists $N_0 \in N$ such that for $n \geq N_0$*

$$I(A_1, \cdots, A_n) = \sqrt{12} i^n \sqrt{2\pi} \frac{A_1 \cdots A_n}{\sqrt{A_1^2 + \cdots + A_n^2}} \left(e^{-\frac{6|A_1 + \cdots + A_n|^2}{A_1^2 + \cdots + A_n^2}} + O\left(\frac{M^4}{n^{1/3}}\right)\right), \quad (6.3)$$

Remark 6.1.2. The hypothesis of the lemma can also be read as all the $|A_i|$ have the same order of magnitude and therefore after a change of variables we may assume that $1 \leq |A_i| \leq M$ for M not too large.

Proof of Lemma 6.1.1. The first observation is to notice that

$$\begin{aligned} \frac{(1 - e^{-i\alpha A_1})}{\alpha} &= 2i\ e^{-\frac{\alpha}{2} A_1} \frac{(e^{i\frac{\alpha}{2} A_1} - e^{-i\frac{\alpha}{2} A_1})}{2i\alpha} \\ &= 2i\ \mathrm{sgn}(A_1) e^{-i\frac{\alpha}{2} A_1} \frac{\sin(|A_1|\alpha)}{\alpha}. \end{aligned} \quad (6.4)$$

Applying this to I we get

$$\begin{aligned} I(A_1, \cdots, A_n) &= (2i)^n \mathrm{sgn}(A_1 \cdot \cdots \cdot A_n) \\ &\times \int_{\mathbb{R}} e^{-i\frac{\alpha}{2}(A_1 + \cdots + A_n)} \frac{\sin \frac{\alpha}{2}|A_1|}{\alpha} \cdot \cdots \cdot \frac{\sin \frac{\alpha}{2}|A_n|}{\alpha} d\alpha, \end{aligned} \quad (6.5)$$

and because of parity of the integrand

$$\begin{aligned} I(A_1, \cdots, A_n) &= (2i)^n \mathrm{sgn}(A_1 \cdot \ \cdots \ \cdot A_n) \\ &\times \int_{\mathbb{R}} \cos \frac{\alpha}{2} |A_1 + \cdots + A_n| \frac{\sin \frac{\alpha}{2}|A_1|}{\alpha} \cdot \cdots \cdot \frac{\sin \frac{\alpha}{2}|A_n|}{\alpha} d\alpha. \end{aligned} \quad (6.6)$$

To compute this integral we consider the independent random variables X_i for $i = 1, \cdots, n$ and Y defined by

$$P\left(Y = \pm\frac{1}{2}|A_1 + \cdots + A_n|\right) = \frac{1}{2}, \quad X_i \sim U\left(\left[-\frac{A_1}{2}, \frac{A_i}{2}\right]\right), \quad (6.7)$$

and its corresponding Fourier transform

$$E(e^{itY}) = \cos\frac{t}{2}|A_1 + \cdots + A_n|, \quad E(e^{itX_i}) = 2\frac{\sin\frac{t}{2}|A_i|}{|A_i|t}. \tag{6.8}$$

Now we consider the random variable $Z = Y + X_1 + \cdots + X_n$. We know that the pdf_Z is given by the convolution of the densities of Y and X_i, we get

$$E(e^{itZ}) = 2^n \cos\frac{t}{2}|A_1 + \cdots + A_n|\frac{\sin\frac{t}{2}|A_1|}{|A_1|t} \cdot \cdots \cdot \frac{\sin\frac{t}{2}|A_n|}{|A_n|t}, \tag{6.9}$$

and therefore

$$\begin{aligned} \text{pdf}_Z(x) &= \frac{1}{2\pi}\int e^{-ixt}E(e^{itZ})dt, \\ \text{pdf}_Z(x=0) &= \frac{1}{2\pi}\int 2^n \cos\frac{t}{2}|A_1 + \cdots + A_n|\frac{\sin\frac{t}{2}|A_1|}{|A_1|t} \cdot \cdots \cdot \frac{\sin\frac{t}{2}|A_n|}{|A_n|t}dt, \end{aligned} \tag{6.10}$$

and therefore, up to a constant, $I(A_1, \cdots, A_n)$ can be seen as the density of Z at 0

$$I(A_1, \cdots, A_n) = i^n(2\pi)A_1 \cdot \cdots \cdot A_n\text{pdf}_Z(0). \tag{6.11}$$

Note that because pdf(Z) is a Lipschitz continuous function and therefore makes sense to consider its pointwise value, also because the integral $\text{pdf}_Z(0)$ integral is the value of a density function at a point we get that

$$\int e^{-ixt}E(e^{itZ})dt\Big|_{x=0} = \text{pdf}_Z(0) \geq 0. \tag{6.12}$$

Notice that because of this new interpretation we can get more information about the integral, in particular for large n we can apply a version of the central limit theorem to provide a better estimate of the size of the integral $\text{pdf}_Z(0)$. Because the variable Y can only take two values, we can write

$$\begin{aligned}
\mathrm{pdf}_Z(0) &= \mathrm{pdf}_{Y+X_1+\cdots+X_n}(0) \\
&= P(Y=-\frac{1}{2}|A_1+\cdots+A_n|)\mathrm{pdf}_{X_1+\cdots+X_n}\left(\frac{1}{2}|A_1+\cdots+A_n|\right) \\
&\qquad +P(Y=\frac{1}{2}|A_1+\cdots+A_n|)\mathrm{pdf}_{X_1+\cdots+X_n}\left(-\frac{1}{2}|A_1+\cdots+A_n|\right) \\
\mathrm{pdf}_Z(0) &= \frac{1}{2}\mathrm{pdf}_{X_1+\cdots+X_n}\left(\frac{1}{2}|A_1+\cdots+A_n|\right) \\
&\qquad +\frac{1}{2}\mathrm{pdf}_{X_1+\cdots+X_n}\left(-\frac{1}{2}|A_1+\cdots+A_n|\right) \\
&= \mathrm{pdf}_{X_1+\cdots+X_n}\left(\frac{1}{2}|A_1+\cdots+A_n|\right),
\end{aligned} \tag{6.13}$$

in the last step we used the symmetry of $X_1+\cdots+X_n$. To estimate

$$\mathrm{pdf}_{X_1+\cdots+X_n}\left(\frac{1}{2}|A_1+\cdots+A_n|\right), \tag{6.14}$$

we want to use some version of the central limit theorem. Because of variables are not identically distributed we need to use the Lindenberg-Feller theorem with the error estimate given by the Berry-Esseen theorem. First some simple observations that will be useful in our computations.

Lemma 6.1.3. *(Moments of uniform distribution)*

$$E(X_i)=0,\ E(X_i^2)=\frac{1}{12}|A_i|^2, \quad E(|X_i|^3)=\frac{1}{32}|A_i|^3. \tag{6.15}$$

Proof. This can be obtained by direct integration. □

Now we check the hypothesis of the Lindenberg-Feller theorem. Let $\varepsilon>0$, and let $s_n^2=\sum_{i=1}^n VarX_i$, then we need to check that

$$\sum_{i=1}^n E(X_i^2/s_n^2 1_{|X_i|/s_n>\varepsilon}) \to 0 \text{ as } n\to\infty. \tag{6.16}$$

In our particular case we know that $s_n \geq \frac{n}{12}$ and $|X_i| \leq M$ for all i and therefore for $n > (12M)/\varepsilon = N_1$ the sum is identically equal to zero, and so we can apply the theorem, then we get that because pdf_Z is Lipschitz continuous,

$$\begin{aligned}
\mathrm{pdf}_Z(0) &= \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} P\left(\left| X_1 + \cdots + X_n - \frac{|A_1 + \cdots + A_n|}{2} \right| < \varepsilon \right) \\
&= \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} P\left(\left| \frac{X_1 + \cdots + X_n}{s_n} - \frac{|A_1 + \cdots + A_n|}{2s_n} \right| < \frac{\varepsilon}{s_n} \right) \\
&= \lim_{\varepsilon \to 0} \frac{1}{2s_n\varepsilon} P\left(\left| \frac{X_1 + \cdots + X_n}{s_n} - \frac{|A_1 + \cdots + A_n|}{2s_n} \right| < \varepsilon \right) \\
&\overset{LF}{\approx} \lim_{\varepsilon \to 0} \frac{1}{2s_n\varepsilon} P\left(\left| G - \frac{|A_1 + \cdots + A_n|}{2s_n} \right| < \varepsilon \right) \\
&= \frac{1}{s_n\sqrt{2\pi}} e^{-\beta_n^2/2},
\end{aligned} \tag{6.17}$$

where G is a standard Gaussian random variable and $\beta_n = \frac{|A_1 + \ldots + A_n|}{2s_n}$. To obtain a estimate of the approximation error we use the Barry-Esseen Theorem, in our case it tell us that

$$\sup_t \left| P\left(\left| \frac{X_1 + \cdots + X_n}{s_n} \right| \leq t \right) - P(G \leq t) \right| \leq C_{BE}\ s_n^{-3} \sum_{i=1}^{n} \rho_i, \tag{6.18}$$

for a universal constant $C_{BE} > 0$ and $\rho_i = E|X_i|^3$. Under our assumptions in the size of the A_i we get that

$$s_n^2 \geq \frac{n}{12}, \quad \sum_{i=1}^{n} \rho_i \leq \frac{M^3 n}{32}. \tag{6.19}$$

To apply the Barry-Essen theorem we write out probability in the following way

$$\begin{aligned}
\int_{-\varepsilon}^{\varepsilon} \mathrm{pdf}(x)dx &= P\left(\left| \frac{X_1 + \cdots + X_n}{s_n} - \frac{|A_1 + \cdots + A_n|}{2s_n} \right| < \varepsilon \right) \\
&= P\left(\frac{X_1 + \cdots + X_n}{s_n} < \frac{|A_1 + \cdots + A_n|}{2s_n} + \varepsilon \right) \\
&\qquad - P\left(\frac{X_1 + \cdots + X_n}{s_n} \leq \frac{|A_1 + \cdots + A_n|}{2s_n} - \varepsilon \right) \\
&= P(G < \beta_n + \varepsilon) - P(G < \beta_n - \varepsilon) + O\left(\frac{M^3}{\sqrt{n}} \right).
\end{aligned} \tag{6.20}$$

To get an error estimate we have to do the transition between pointwise estimates and averages of integrals over a small balls. For this purpose we consider the Taylor expansion

$$f(x+t) = f(x) + tf'(x) + O(\|f''\|_{L^\infty} t^2), \tag{6.21}$$

integrating we get

$$\frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} f(t)dt = f(x) + O(\|f''\|_{L^\infty}\varepsilon^2). \tag{6.22}$$

Applying (6.22) to the definition of pdf_Z we can write

$$\begin{aligned} \mathrm{pdf}_Z(0) &= \frac{1}{2\varepsilon}P\left(\left|X_1+\cdots+X_n - \frac{|A_1+\cdots+A_n|}{2}\right| < \varepsilon\right) + O(K\varepsilon^2) \\ &= \frac{1}{2\varepsilon}P\left(\left|\frac{X_1+\cdots+X_n}{s_n} - \beta_n\right| < \frac{\varepsilon}{s_n}\right) + O(K\varepsilon^2), \end{aligned} \tag{6.23}$$

here we can apply (6.20) to obtain

$$\mathrm{pdf}_Z(0) = \frac{1}{2\varepsilon}P\left(|G-\beta_n| < \frac{\varepsilon}{s_n}\right) + O\left(K\varepsilon^2 + \frac{M^3}{\sqrt{n}}\right), \tag{6.24}$$

where $K = \left\|\mathrm{pdf}''_{X_1+\ldots+X_n}\right\|_{L^\infty}$. Note that because $\mathrm{pdf}_{X_1+\cdots+X_n}$ is defined as a convolution it gets more regular as $n\to\infty$, in this case it is enough to have $n \geq 3$ to ensure that we can take second derivative. Using (6.22) again we can write the first term as

$$\frac{1}{2\varepsilon}P\left(|G-\beta_n| < \frac{\varepsilon}{s_n}\right) = \frac{1}{s_n}\phi(\beta_n) + O\left(\frac{\varepsilon^2}{s_n^3}\|\phi''\|_{L^\infty}\right), \tag{6.25}$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, and $\|\phi''\|_{L^\infty} < 1/2$. Then we get

$$\mathrm{pdf}_Z(0) = \frac{1}{s_n}\phi(\beta_n) + O\left(\frac{\varepsilon^2}{s_n^3}\|\phi''\|_{L^\infty} + \frac{M^3}{\varepsilon\sqrt{n}} + K\varepsilon^2\right), \tag{6.26}$$

now we choose ε such that $\frac{M^3}{\varepsilon\sqrt{n}} = \varepsilon^2$ to get

$$\text{pdf}_Z(0) = \frac{1}{s_n}\phi(\beta_n) + O\left(\frac{M^2}{s_n^3 n^{1/3}} + \frac{M^2(1+K)}{n^{1/3}}\right), \tag{6.27}$$

the last thing that we need is to estimate the size of $K = \left\|\text{pdf}''_{X_1+\ldots+X_n}\right\|_{L^\infty}$ for this purpose we write

$$\left\|\frac{d^2}{dx^2}\text{pdf}_{X_1+\cdots+X_n}(x)\right\|_{L^\infty} = \left\|\frac{d^2}{dx^2}\frac{1}{|A_1\cdots A_n|}\chi_{[-\frac{A_1}{2},\frac{A_1}{2}]} * \cdots * \chi_{[-\frac{A_n}{2},\frac{A_n}{2}]}\right\|_{L^\infty}. \tag{6.28}$$

Now we need a way of estimating of estimating the derivatives of a convolution of characteristic functions, for this we use the following Lemma

Lemma 6.1.4. *Let A, $B \in \mathbb{R}$ such that $A < B$ then*

$$\frac{d}{dx}\chi_{[A,B]} * g = g(x-A) + g(x-B) \tag{6.29}$$

Using this Lemma we can estimate the derivative in the following way

$$\begin{aligned}
\frac{d^2}{dx^2}\text{pdf}_{X_1+\cdots+X_n}(x) &= \frac{d^2}{dx^2}\chi_{[-\frac{A_1}{2},\frac{A_1}{2}]} * \cdots * \chi_{[-\frac{A_n}{2},\frac{A_n}{2}]} \\
&= \frac{d}{dx}\chi_{[-\frac{A_2}{2},\frac{A_2}{2}]} * \cdots * \chi_{[-\frac{A_n}{2},\frac{A_n}{2}]}(x+\frac{A_1}{2}) \\
&\quad + \frac{d}{dx}\chi_{[-\frac{A_2}{2},\frac{A_2}{2}]} * \cdots * \chi_{[-\frac{A_n}{2},\frac{A_n}{2}]}(x-\frac{A_1}{2}) \\
&= \chi_{[-\frac{A_3}{2},\frac{A_3}{2}]} * \cdots * \chi_{[-\frac{A_n}{2},\frac{A_n}{2}]}(x+\frac{A_1}{2}+\frac{A_2}{2}) \\
&\quad + \chi_{[-\frac{A_3}{2},\frac{A_3}{2}]} * \cdots * \chi_{[-\frac{A_n}{2},\frac{A_n}{2}]}(x+\frac{A_1}{2}-\frac{A_2}{2}) \\
&\quad + \chi_{[-\frac{A_3}{2},\frac{A_3}{2}]} * \cdots * \chi_{[-\frac{A_n}{2},\frac{A_n}{2}]}(x-\frac{A_1}{2}+\frac{A_2}{2}) \\
&\quad + \chi_{[-\frac{A_3}{2},\frac{A_3}{2}]} * \cdots * \chi_{[-\frac{A_n}{2},\frac{A_n}{2}]}(x-\frac{A_1}{2}-\frac{A_2}{2}).
\end{aligned} \tag{6.30}$$

Now by the Young's Inequality for convolution we get that

$$\begin{aligned}
\left|\chi_{[-\frac{A_3}{2},\frac{A_3}{2}]} * \cdots * \chi_{[-\frac{A_n}{2},\frac{A_n}{2}]}\right|_{L^\infty} &\leq \left\|\chi_{[-\frac{A_3}{2},\frac{A_3}{2}]}\|_{L^\infty}\|\chi_{[-\frac{A_4}{2},\frac{A_4}{2}]}\right\|_{L^1}\cdots\|\chi_{[-\frac{A_n}{2},\frac{A_n}{2}]}\|_{L^1} \\
&\leq |A_4|\cdots|A_n|.
\end{aligned} \tag{6.31}$$

Therefore we conclude that

$$\begin{aligned} \left\| \frac{d^2}{dx^2} \mathrm{pdf}_{X_1+\cdots+X_n}(x) \right\|_{L^\infty} &\leq \frac{4}{|A_1 \cdots A_n|} |A_4 \cdots A_n| \\ &\leq \frac{4}{|A_1 A_2 A_3|}, \end{aligned} \tag{6.32}$$

and by symmetry we can conclude that

$$\left\| \frac{d^2}{dx^2} \mathrm{pdf}_{X_1+\cdots+X_n}(x) \right\|_{L^\infty} \leq \frac{4}{|A_1 \cdots A_n|^{3/n}}. \tag{6.33}$$

Applying this estimate we get

$$\frac{M^2}{s_n^6 n^{1/3}} + \frac{M^2(1+K)}{n^{1/3}} \leq C \frac{M^2}{n^{1/3}} \left(\frac{1}{s_n^6} + \frac{4}{|A_1 \cdots A_n|^{3/n}} + 1 \right) \leq C \frac{M^2}{n^{1/3}}. \tag{6.34}$$

Then form (6.27) we get

$$\mathrm{pdf}_Z(0) = \frac{1}{s_n} \phi(\beta_n) + O\left(\frac{M^2}{n^{1/3}} \right), \tag{6.35}$$

and consequently

$$\begin{aligned} I(A_1, \cdots, A_n) &= i^n (2\pi) A_1 \cdot \cdots \cdot A_n \mathrm{pdf}_Z(0) \\ &= i^n (2\pi) A_1 \cdot \cdots \cdot A_n \left(\frac{1}{s_n} \phi(\beta_n) + O\left(\frac{M^2}{n^{1/3}} \right) \right) \\ &= 12 i^n \sqrt{2\pi} \frac{A_1 \cdots A_n}{A_1^2 + \cdots + A_n^2} \left(e^{-\frac{6|A_1 \cdots A_n|^2}{A_1^2 + \cdots + A_n^2}} + O(\frac{M^4}{n^{1/3}}) \right). \end{aligned} \tag{6.36}$$

This concludes the proof of Lemma 6.1.1.

□

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