Signature PRINTED NAME

PRELIMINARY EXAMINATION, PART I

Friday, August 28, 2020 9:30am-12:30pm

This examination is based on Penn's code of academic integrity

Instructions:

Sign and print your name above.

This part of the examination consists of six problems, each worth ten points. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete — and justify your assertions. Each problem should be given its own page (or more than one page, if necessary).

If a problem has multiple parts, earlier parts may be useful for later parts. Moreover, if you skip some part, you may still use the result in a later part.

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Score	
1	
2	
3	
4	
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6	

1. For every positive integer n, let $f_n(x)$ be the function on $\mathbb{R}$ given by $f_n(x) := \frac{x^n}{n!}$.

   (a) Determine the subset $S \subseteq \mathbb{R}$ consisting of all of real numbers a such that the sequence $\big(f_n(a)\big)_{n\geq 1}$ converges. For each $a \in S$, find the limit.

   (b) Determine whether the sequence of functions $\big(f_n(x)\big)_{n\geq 1}$ converges uniformly on the subset $S \subseteq \mathbb{R}$ defined in (a).

Justify your assertions.

2. Let $f(x) = x^6 - 1$.

(a) Find all maximal ideals in the polynomial ring $\mathbb{Q}[x]$ that contain $f(x)$.

(b) Find all maximal ideals in the polynomial ring $\mathbb{C}[x]$ that contain $f(x)$.

3. Let f be the function on $[-1,1]$ defined by

$$f(x) = \begin{cases} 3x^2 & \text{if } \; 0 < x \leq 1 \\ -1 + 2x & \text{if } \; -1 \leq x \leq 0 \end{cases}$$

Let F be the function on $[-1,1]$ defined by

$$F(x) = \int_{-1}^{x} f(t)\, dt.$$

(a) Is the function F continuous at $x = 0$?

(b) Is the function F differentiable at $x = 0$?

Justify your assertions.

4. Let $\vec{v}_0$ be the vector $(1, 1, 1)$ in $\mathbb{R}^3$. Let T be the linear operator on $\mathbb{R}^3$ defined by cross product with $\vec{v}_0$:

$$T(\vec{w}) := \vec{v}_0 \times \vec{w}$$

for every element $\vec{w} \in \mathbb{R}^3$. For every real number $x \in \mathbb{R}$, define a linear operator U_x on $\mathbb{R}^3$ by

$$U_x := \exp(xT) = \sum_{n \geq 0} \frac{x^n \, T^n}{n!}.$$

(a) Find the matrix representation A of T with respect to the standard basis of $\mathbb{R}^3$.

(b) Show that for every $x \in \mathbb{R}$, the operator U_x is orthogonal.

(Recall that a real 3×3 matrix B is *orthogonal* if $B \cdot B^t = \mathrm{I}_3 = B^t \cdot B$; and an operator on $\mathbb{R}^3$ is *orthogonal* if it is equal to "multiplication on the left by an orthogonal 3×3 matrix".)

5. Consider the improper integral

$$\iint_{\mathbb{R}^2} \frac{dx\, dy}{(1 + x^2 + y^2)^\alpha}$$

on $\mathbb{R}^2$, with a real parameter $\alpha > 0$. Determine whether this improper integral converges for $\alpha = 1$ and whether it converges for $\alpha = 2$.

(Hint: Use polar coordinates.)

6. Let $E := \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + xy + y^2 \leq 1\}$, with the topology given by the standard metric on $\mathbb{R}^2$. Let E^0 be the interior of the subset $E \subseteq \mathbb{R}^2$.

(Recall that E^0 is the subset consisting of all points $P \in E$ such that there exists $\epsilon > 0$ such that the open disk $D(P; \epsilon)$ in $\mathbb{R}^2$ of radius $\epsilon > 0$ centered at P is contained in E.)

(a) Determine whether E (respectively E^0) is connected, and whether E (respectively E^0) is compact. (Your answer should have 4 parts.)

(b) Prove that $E^0 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + xy + x^2 < 1\}$, and E is equal to the closure in $\mathbb{R}^2$ of E^0.

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Signature Printed Name

Preliminary Examination, Part II

Friday, August 28, 2020 2:00-5:00pm

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Score	
7	
8	
9	
10	
11	
12	

7. Let f be an increasing continuously differentiable function on the real line, and let $g = f'$. Let $a = f(0)$, $b = f(1)$, $c = g(0)$, $d = g(1)$. Let R be the region in the (x, y)-plane lying below the graph of $y = g(x)$, above the x-axis, and between the lines $x = 0$ and $x = 1$. Let C be the boundary of R, oriented counterclockwise. Evaluate

$$\oint_C (xy^2e^{x^2y^2} + 3x^2)dx + (x^2ye^{x^2y^2} + 5x)dy.$$

8. Let $U \in \mathrm{M}_3(\mathbb{Q})$ be a 3×3 matrix with coefficients in $\mathbb{Q}$ such that $U^5 = \mathrm{I}_3$, where I_3 is the 3×3 identity matrix. Prove that $U = \mathrm{I}_3$.

(Hint: You may use the factorization $T^5 - 1 = (T-1)(T^4 + T^3 + T^2 + T + 1)$ in the polynomial ring $\mathbb{Q}[T]$, and the fact that $T^4 + T^3 + T^2 + T + 1$ is irreducible in $\mathbb{Q}[T]$.)

9. Let f be a piecewise continuous function on $\mathbb{R}$ such that $f(x+1) = -f(x)$ for all $x \in \mathbb{R}$. Determine whether the limit

$$\lim_{a\to\infty} \int_0^1 f(ax)\, dx$$

exists. If it does, find it.

10. Let $\mathrm{GL}_2(\mathbb{R})$ be the group of all invertible 2×2 matrices with entries in $\mathbb{R}$. Let H be the subgroup of $\mathrm{GL}_2(\mathbb{R})$ consisting of all diagonal 2×2 matrices $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ with $x, y \neq 0$.

(a) Determine explicitly the *center* $\mathrm{Z}(\mathrm{GL}_2(\mathbb{R}))$ of the group $\mathrm{GL}_2(\mathbb{R})$.

(b) Determine explicitly the *normalizer* subgroup $\mathrm{N}_{\mathrm{GL}_2(\mathbb{R})}(H)$ of $\mathrm{GL}_2(\mathbb{R})$, and the index $(\mathrm{N}_{\mathrm{GL}_2(\mathbb{R})}(H) : H)$.
(Recall that $\mathrm{N}_{\mathrm{GL}_2(\mathbb{R})}(H)$ consists of all elements $g \in \mathrm{GL}_2(\mathbb{R})$ such that $g \cdot H \cdot g^{-1} = H$.)

11. For each of the following, give a proof or a counterexample.

(a) If $\left(a_n\right)_{n\geq 1}$ is a sequence of positive real numbers such that the series $\sum a_n$ converges, then the series $\sum a_n^2$ converges.

(b) If $\left(a_n\right)_{n\geq 1}$ is a sequence of arbitrary real numbers such that the series $\sum a_n$ converges, then the series $\sum a_n^2$ converges.

(c) If f is a continuous function on $\mathbb{R}$, and if $a_n = \frac{1}{n}\sum_{j=1}^{n} f(j/n)$ for all positive integers n, then the sequence $\left(a_n\right)_{n\geq 1}$ converges.

12. Let A be the 4×4 real matrix

$$A = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix}.$$

(a) Show that the minimal polynomial of A is $x^2 - x + 1$.

(b) Let K be the $\mathbb{R}$-linear span in $\mathrm{M}_4(\mathbb{R})$ of A and the identity matrix I_4. Show that K is a subring of $\mathrm{M}_4(\mathbb{R})$, and is isomorphic to $\mathbb{C}$.

(Hint: Use (a).)

(c) Let $V \subseteq \mathrm{M}_4(\mathbb{R})$ be the subset of $\mathrm{M}_4(\mathbb{R})$ consisting of all real 4×4 matrices B such that $AB = BA$. Show that V is stable under left and right matrix multiplication by elements of K (i.e., kv, $vk \in V$ for $k \in K$ and $v \in V$), and that V is a vector space over K.