1. Given real numbers a, b, c, d, consider the differential equation $E(a, b, c, d)$ given by $y'' + (a \sin x - 3)y' + (be^x + 2)y + c \cos x = d$.

   (a) Find the set S of all $(a, b, c, d) \in \mathbb{R}^4$ such that the solutions to the differential equation $E(a, b, c, d)$ form a real vector space $V(a, b, c, d)$ under addition and scalar multiplication of functions.

   (b) Pick some $(a, b, c, d) \in S$ (your choice), and find a basis for the vector space $V(a, b, c, d)$.

   Explain your assertions.

   Solution. (a) The differential equation $E(a, b, c, d)$ is linear, and is homogeneous if and only if $c = d = 0$. So $S = \{(a, b, c, d) \in \mathbb{R}^4 \mid c = d = 0\}$.

   (b) Take $(0, 0, 0, 0) \in S$. Here $V(0, 0, 0, 0)$ is the set of solutions to the differential equation $y'' - 3y' + 2y = 0$. This is a constant coefficient homogeneous linear differential equation, and a basis for the solutions is given by $\{e^x, e^{2x}\}$, since $r = 1, 2$ are the solutions to the polynomial equation $r^2 - 3r + 2 = 0$.

2. (a) Give an example of a non-abelian group G, generated by two elements g, h, such that the center of G is non-trivial.

   (b) Show that no such example can exist if one additionally requires that g is in the center of G.

   Solution. (a) We can take G to be the dihedral group of order 8, generated by g, h subject to the relations $g^4 = 1$, $h^2 = 1$, $gh = hg^{-1}$. This is non-abelian, but g^2 is in the center.

   (b) If G is generated by g, h, and if g is in the center of G, then every element can be written in the form $g^i h^j$ for i, j integers, by commuting g past h. One then has $(g^i h^j)(g^{i'} h^{j'}) = g^{i+i'} h^{j+j'} = (g^{i'} h^{j'})(g^i h^j)$, again commuting g past h. So the group is abelian.

3. Let $f(x) = 1/x$ for $x \neq 0$. On which of the following intervals is the function f uniformly continuous? Explain your assertions.

   (i) $1 \leq x \leq 2$.

   (ii) $1 < x < 2$.

   (iii) $0 < x < 1$.

Solution. (i) A continuous function is uniformly continuous on any closed interval $[a, b]$, so f is uniformly continuous on $[1, 2]$.

(ii) A function that is uniformly continuous on a set S is also uniformly continuous on each subset of S, and so f is uniformly continuous on $(1, 2)$.

(iii) The function f is not uniformly continuous on $(0, 1)$. To see this, take $\varepsilon = 1$. For any $\delta > 0$, let $\bar{\delta} = \min(1/2, \delta)$, and take $x_1 = \bar{\delta}/2$ and $x_2 = \bar{\delta}$. Then $|x_1 - x_2| = \bar{\delta}/2 < \delta$, but $|f(x_1) - f(x_2)| = 1/\bar{\delta} > 1 = \varepsilon$. This contradicts uniform continuity.

4. For each of the following either give an example of a real square matrix M with the given properties or explain why none exists:

   (a) M is not similar over $\mathbb{R}$ to an upper triangular matrix.

   (b) M is similar over $\mathbb{R}$ to an upper triangular matrix but is not similar over $\mathbb{R}$ to a diagonal matrix.

   (c) M is not similar over $\mathbb{C}$ to an upper triangular matrix.

   (d) M is similar over $\mathbb{C}$ to an upper triangular matrix but is not similar over $\mathbb{C}$ to a diagonal matrix.

Solution. (a) We can take M to be the rotation matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It is not similar to a real triangular matrix since it has no real eigenvalues.

(b) We can take M to be the triangular matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. This is not similar to a diagonal matrix since its eigenvalues are both equal to 1 but M is not similar to the identity matrix.

(c) This does not exist: every square matrix is similar to an upper triangular matrix over an algebraically closed field.

(d) We can take M to be the triangular matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, by the same reasoning as in part (b).

5. Let $f(x) = x^2 + 2y^2 - 2xy + 2x$, let D be the closed disc $x^2 + y^2 \leq 10$, and let D' be the interior of D.

   (a) Does the restriction of f to D achieve a maximum? (I.e., is there a point $(a, b) \in D$ such that $f(a, b) \geq f(x, y)$ for all $(x, y) \in D$?) Similarly, does it achieve a minimum on D? Justify your assertions. (You are not asked to *find* the maximum and minimum if you assert that they exist.)

(b) Does the restriction of f to D' achieve a maximum, and does it achieve a minimum? If it does, find where the maximum (resp. minimum) is achieved. Justify your assertions.

Solution. (a) The function f is continuous (being a polynomial), and the closed disc D is compact (being closed and bounded in $\mathbb{R}^2$); so f achieves both a maximum and a minimum on D.

(b) Any maximum or minimum of f on the open set D' will be a relative maximum or minimum, and so will be at a critical point of f. The critical points of f in $\mathbb{R}^2$ occur where $f_x = f_y = 0$; and here $f_x = 2x - 2y + 2$, $f_y = 4y - 2x$. So the unique critical point is at $(x, y) = (-2, -1) \in D'$. Since $f_{xx} = 2$, $f_{xy} = -2$, and $f_{yy} = 4$, we have $f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} > 0$, so this point is a relative minimum by the second derivative test. In fact, the graph of f is a paraboloid, and its unique critical point on $\mathbb{R}^2$ is its unique extremum. So f has a minimum on D' at $(-2, -1)$ and it has no maximum on D'.

6. Let $f(x, y) = e^{xy^5} + x^{10} + \cos(y^2)$, let $g = \partial f/\partial x$, and let $h = \partial f/\partial y$. Let C be the path in the plane from the origin to the point $(1, 0)$ given by the portion of the graph of $y = \sin^3(\pi x)$ over the interval $0 \le x \le 1$. Evaluate $\int_C g\,dx + h\,dy$. Explain your computations. [Hint: This does not require a brute force calculation of the integral.]

Solution. Since $g\,dx + h\,dy = df = \nabla f \cdot d\mathbf{r}$, the value of the line integral depends only on the endpoints, and is equal to $f(1, 0) - f(0, 0) = (1 + 1 + 1) - (1 + 0 + 1) = 1$. (Alternatively, we may apply Green's Theorem to the region R lying below C and above the x-axis. Since $g_y = f_{xy} = f_{yx} = h_x$, it follows that $\int_{\partial R} g\,dx + h\,dy = 0$. So $\int_C g\,dx + h\,dy$ is equal to the integral along the line segment $[0, 1]$, which is $\int_{x=0}^{1} \frac{d}{dx} f(x, 0)\,dx = f(x, 0)|_{x=0}^{1} = 3 - 2 = 1$.)

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, with graph Γ.

   (a) Show that $\{(x, y) \mid y > f(x)\}$ is an open subset of $\mathbb{R}^2$.

   (b) Show that the complement of Γ in $\mathbb{R}^2$ is disconnected.

   Solution. (a) Suppose that (a, b) lies in the given set S. Thus $c := b - f(a) > 0$. By continuity there exists $\varepsilon > 0$ such that $f(x) < f(a) + \frac{c}{2} = b - \frac{c}{2}$ for $|x - a| < \varepsilon$. So the open rectangle $(a - \varepsilon, a + \varepsilon) \times (b - \frac{c}{2}, b + \frac{c}{2})$ is an open neighborhood of (a, b) in S. Hence S is open.

   (b) By replacing f by $-f$ in part (a), we also have that the set of points S' lying below the graph is open. The complement of the graph is thus the disjoint union of the two open sets S, S', and so it is disconnected.

8. Let V be the span of the four vectors $(1, -1, 0, 1)$, $(2, -1, 1, 6)$, $(-1, 2, 1, 3)$, $(1, 0, 1, 5)$ in $\mathbb{R}^4$. With respect to the usual inner product on $\mathbb{R}^4$, find an orthogonal basis of V, and find the point on V closest to $(1, 1, 1, 1)$.

   Solution. Using row reduction we get

$$\begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 6 \\ -1 & 2 & 1 & 3 \\ 1 & 0 & 1 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 1 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   and so a basis for V is given by $v = (1, -1, 0, 1)$, $w = (0, 1, 1, 4)$. Now apply Gram-Schmidt to get an orthogonal basis of V: $v_1 = v = (1, -1, 0, 1)$, $v_2 = w - \frac{v \cdot w}{v \cdot v} v = (0, 1, 1, 4) - (1, -1, 0, 1) = (-1, 2, 1, 3)$.

   The closest point to $z = (1, 1, 1, 1)$ on V is the orthogonal projection of z onto V. Using the orthogonal basis v_1, v_2 of V, we find that this point is $(1, 1, 1, 1) - \frac{1}{3}(1, -1, 0, 1) - \frac{5}{15}(-1, 2, 1, 3) = (1, \frac{2}{3}, \frac{2}{3}, \frac{-1}{3})$.

9. Let $\{a_n\}$ be a sequence of real numbers. For each of the following, give either a proof or a counter-example:

   (a) If $\sum_{n=1}^{\infty} a_n$ is convergent but not absolutely convergent, then $\sum_{n=1}^{\infty} n a_n$ is divergent.

   (b) If $\sum_{n=1}^{\infty} a_n$ is convergent but not absolutely convergent, then $\sum_{n=1}^{\infty} n^2 a_n$ is divergent.

Solution. (a) This is false. For example let $a_n = (-1)^n \frac{1}{n\log(n+1)}$. Then $\sum_{n=1}^{\infty} a_n$ is convergent (by the alternating series test), and is not absolutely convergent (by the integral test). But $\sum_{n=1}^{\infty} na_n$ is convergent (by the alternating series test).

(b) This is true: If instead $\sum_{n=1}^{\infty} n^2 a_n$ is convergent, then the terms approach 0, and so are bounded; so there is a constant $C > 0$ such that $|n^2 a_n| < C$, or equivalently $|a_n| < C/n^2$, for all n. But then $\sum_{n=1}^{\infty} a_n$ would be absolutely convergent by comparison with $\sum_{n=1}^{\infty} C/n^2$.

10. (a) Show that if $\mathfrak{m}$ is a maximal ideal in $\mathbb{Q}[x]$, then $\mathbb{Q}[x]/\mathfrak{m}$ is a field extension of $\mathbb{Q}$ of finite degree.

    (b) Conversely, show that if K is a field extension of $\mathbb{Q}$ of finite degree, then K is isomorphic to $\mathbb{Q}[x]/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of $\mathbb{Q}[x]$.

Solution. (a) Since $\mathbb{Q}[x]$ is a PID, $\mathfrak{m} = (f)$ for some $f(x) \in \mathbb{Q}[x]$; and f is irreducible since $\mathfrak{m}$ is maximal. Let n be the degree of f. Then $\mathbb{Q}[x]/\mathfrak{m} = \mathbb{Q}[x]/(f)$ is a field because $\mathfrak{m}$ is maximal; and it has degree n over $\mathbb{Q}$, being a $\mathbb{Q}$-vector space with basis $1, x, \ldots, x^{n-1}$.

(b) Since $\mathbb{Q}$ has characteristic zero, the extension $K/\mathbb{Q}$ is separable. By the primitive element theorem, the finite separable field extension K is generated over $\mathbb{Q}$ by a single element α. Let $f(x) \in \mathbb{Q}[x]$ be the minimal polynomial of α over $\mathbb{Q}$. Then f generates the kernel of the surjective homomorphism $\mathbb{Q}[x] \to K$ given by $g(x) \mapsto g(\alpha)$, and so K is isomorphic to $\mathbb{Q}[x]/(f)$. Since K is a field, the ideal (f) is maximal. So K is isomorphic to $\mathbb{Q}[x]/\mathfrak{m}$ with $\mathfrak{m} = (f)$ a maximal ideal.

11. Let $f(x)$ be a differentiable function on the real line such that $f(0) = 0$ and $f'(0) = 1$. Prove directly, from the definition of the derivative, that there exists a positive real number c such that $f(x) > 0$ for all x with $0 < x < c$.

Solution. We are given that $1 = f'(0) = \lim_{h\to 0} f(h)/h$, so there exists $c > 0$ such that all x with $0 < x < c$ satisfy $|\frac{f(x)}{x} - 1| < 1/2$. Hence these x satisfy $f(x)/x > 1/2$; i.e., $f(x) > x/2 > 0$.

12. Let v, w be elements of a finite dimensional real vector space V. Prove that there is a linear transformation $T : V \to \mathbb{R}^2$ such that $T(v) = (1, 0)$ and $T(w) = (0, 1)$ if and only if v, w are linearly independent vectors.

Solution. If there is such a linear transformation T, and if $a, b \in \mathbb{R}$ are not both 0, then $T(av + bw) = aT(v) + bT(w) = (a, b) \neq (0, 0)$. Hence $av + bw \neq 0$. This shows that v, w are linearly independent.

Conversely, if v, w are linearly independent, then there is a basis $v_1, v_2, \ldots, v_n$ of V with $v_1 = v$ and $v_2 = w$. We can define a linear transformation $T : V \to \mathbb{R}^2$ by taking v to $(1, 0)$, taking w to $(0, 1)$, and taking the other basis vectors to 0.