Signature	PRINTED NAME

PRELIMINARY EXAMINATION, PART I

Friday, August 23rd, 2024 9:30am-12:00pm

This examination is based on Penn's code of academic integrity

Instructions:

Sign and print your name above.

This part of the examination consists of six problems, each is worth ten points. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete — and justify your assertions. Each problem should be given its own page (or more than one page, if necessary).

If a problem has multiple parts, earlier parts may be useful for later parts. Moreover, if you skip some part, you may still use the result in a later part.

Be sure to write your name both on the exam and on any extra sheets you may submit.

Score (for faculty use only)	
1	
2	
3	
4	
5	
6	
GRADER	

1. Suppose A is an $n \times n$ matrix of real numbers for which

$$Ax \cdot x > 0$$

for every $x \in \mathbb{R}^n$. Evaluate the integral

$$\int_{\mathbb{R}^n} e^{-Ax \cdot x} dx.$$

2. Indicate which ones of the rings R_1, R_2, R_3 below are Euclidean, principal ideal domains, unique factorization domains, respectively.

a) $R_1 = \mathbb{Z}[\sqrt{2}] := \{a + b\sqrt{2} \,|\, a, b \in \mathbb{Z}\}$.

b) $R_2 = \mathbb{Z}[\sqrt{-1}][t]$, the polynomial ring in the variable t over the Gaussian integers $\mathbb{Z}[\sqrt{-1}]$.

c) $R_3 = \mathbb{Z}[\sqrt{-3}] := \{a + b\sqrt{-3} \,|\, a, b \in \mathbb{Z}\}$.

3. Suppose M is an $n \times n$ matrix of real numbers and consider the initial value problem: find $x : [0, \infty) \to \mathbb{R}^n$ for which

$$\begin{cases} \dot{x}(t) = Mx(t), & (t \geq 0) \\ x(0) = y. \end{cases}$$

(a) Show that $x(t) = e^{Mt}y$ is a solution of the initial value problem.
(b) Verify that the solution given in (a) is unique.

4. Let A be an $n \times n$ matrix with complex coefficients with entries in $\mathbb{C}$. For each $i = 1, \ldots, n$, let $\mathbf{v}_i$ be the i-th column of A, and let $||\mathbf{v}_i|| = (\bar{\mathbf{v}}_i^t \cdot \mathbf{v})^{\frac{1}{2}}$ be the length of $\mathbf{v}$. Show that

$$|\det(A)| \leq \prod_{i=1}^{n} ||\mathbf{v}_i|| \, .$$

5. Let f be the function on $\mathbb{R}$ defined by: $f(x) = x^2 \sin\{\frac{1}{x}\}$ if $x \neq 0$, and $f(0) = 0$.

(a) Is f differentiable at $x = 0$? If so, find $f'(0)$.
(b) Is f' continuous at $x = 0$?

6. (a) (4 points) Let $\gamma : [0,1] \to \mathbb{R}^2 \smallsetminus \{(0,0)\}$ be a differentiable curve on the punctured plane such that $\gamma(0) = \gamma(1)$. Show that the line integral

$$\frac{1}{2\pi} \int_\gamma \frac{-ydx + xdy}{x^2 + y^2}$$

is an integer.
(b) (6 points) Let $\gamma_1, \gamma_2 : [0,1] \to \mathbb{R}^2 \smallsetminus \{(0,0)\}$ be two differentiable curves on $\mathbb{R}^2 \smallsetminus \{(0,0)\}$ such that $\gamma_i(0) = \gamma_i(1)$ for $i = 1,2$ and the line segment on $\mathbb{R}^2$ connecting $\gamma_1(t)$ with $\gamma_2(t)$ does not pass through the origin for every $t \in [0,1]$. Prove that

$$\frac{1}{2\pi} \int_{\gamma_1} \frac{-ydx + xdy}{x^2 + y^2} = \frac{1}{2\pi} \int_{\gamma_2} \frac{-ydx + xdy}{x^2 + y^2}.$$

[Hint: Consider the family of curves $\gamma_s : [0,1] \to \mathbb{R}^2 \smallsetminus \{(0,0)\}$, $s \in [0,1]$ given by $\gamma_s(t) = s\gamma_2(t) + (1-s)\gamma_1(t)$.]

___________________________ ___________________________

Signature Printed Name

Preliminary Examination, Part II

Friday, August 23rd, 2024 1:30 pm - 4:00 pm

This examination is based on Penn's code of academic integrity

Instructions:

Sign and print your name above.

This part of the examination consists of six problems, each is worth ten points. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete — and justify your assertions. Each problem should be given its own page (or more than one page, if necessary).

If a problem has multiple parts, earlier parts may be useful for later parts. Moreover, if you skip some part, you may still use the result in a later part.

Be sure to write your name both on the exam and on any extra sheets you may submit.

Score (for faculty use only)	
7	
8	
9	
10	
11	
12	
Grader	

7. Suppose that H is a non-trivial closed subgroup of $(\mathbb{R}^2, +)$ which is *not* discrete. Show that H contains a one-dimensional $\mathbb{R}$-vector subspace of $\mathbb{R}^2$. (Hint: Show first that H contains a sequence of non-zero elements which converge to 0. Then use the fact that the unit circle in $\mathbb{R}^2$ is compact.)

8. Let R be the subring of the matrix ring $\mathrm{M}_6(\mathbb{C})$ consisting of all 6×6 matrices with complex entries of block form $\begin{bmatrix} A & 0_2 & 0_2 \\ 0_2 & A & 0_2 \\ 0_2 & 0_2 & A \end{bmatrix}$, with $A \in \mathrm{M}_2(\mathbb{C})$, where 0_2 denotes the 2×2 zero matrix. Show that the centralizer subring $\mathrm{Z}_{\mathrm{M}_2(\mathbb{C})}(R)$ is isomorphic to the matrix ring $\mathrm{M}_3(\mathbb{C})$. (Recall that $\mathrm{Z}_{\mathrm{M}_2(\mathbb{C})}(R)$ is the subring consisting of all elements $x \in \mathrm{M}_6(\mathbb{C})$ such that $xy = yx$ for all $y \in R$.)

9. Let $a < b$ be real numbers. Let f be a continuous function on a closed interval $[a, b]$ such that $f(x) \geq 0$ for all $x \in [a, b]$. Suppose that there is an element $x_0 \in [a, b]$ such that $f(x_0) > 0$. Prove that $\int_a^b f(x)\, dx > 0$.

10. For each prime number p, let $\mathbb{Z}_{(p)}$ be the ring consisting of all rational numbers x such that there exists an integer d with $\gcd(d,p)=1$ and $dx \in \mathbb{Z}$. In other words $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$.

Let R be the product ring $R = \prod_p \mathbb{Z}_{(p)}$, where p runs through all prime numbers. Let 1_R be the unity element of R. Prove that the quotient group $R/\mathbb{Z}\cdot 1_R$ of $(R,+)$ is torsion free.

11. Let w be a complex number. Let R, r be positive real numbers such that $R > r > |w|$. Show that

$$\frac{r-|w|}{R^2 - r|w|} \leq \left|\frac{z-w}{R^2 - z\bar{w}}\right| \leq \frac{r+|w|}{R^2 + r|w|} \qquad \forall\, z \text{ with } |z| = r.$$

[Hint: The image of the circle $\{z \in \mathbb{C} : |z| = r\}$ under the fractional linear transformation $z \mapsto \frac{z-w}{R^2 - z\bar{w}}$ is a circle. What is the line connecting the origin with the center of the latter circle (if $w \neq 0$)?]

12. Let G be the subgroup of $\mathrm{GL}_3(\mathbb{C})$ consisting of the six 3×3 permutation matrices. Consider the left action of G on $\mathbb{C}^3$ defined by matrix multiplication. Find all vector subspaces of $\mathbb{C}^3$ which are stable under the action of G.