QUADRATIC FORMS, LOCAL-GLOBAL PRINCIPLES, AND FIELD INVARIANTS

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ABSTRACT

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The Hasse-Minkowski Theorem states that a quadratic form defined over a global field is isotropic if and only if it is isotropic over all completions of the field, and is one of the first examples of a *local-global principle* for quadratic forms. In this thesis, we investigate local-global principles for quadratic forms over more general fields and their use in answering several questions about quadratic forms. First, we study the validity of the local-global principles for isotropy and isometry of quadratic forms over finitely generated field extensions with respect to various sets of discrete valuations. Next, we use the local-global principle for isotropy to study anisotropic universal quadratic forms, particularly over semi-global fields. Finally, we use the Witt index to ask refined questions about the local-global principle for isotropy and about universal quadratic forms.

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CHAPTER 1

Introduction

1.1. Motivating questions

The main objects studied in this thesis are quadratic forms q over a field k of characteristic $\neq 2$, i.e., homogeneous degree 2 polynomials over k. For example, the polynomials

$$q_1(x,y) = xy, \quad q_2(x,y,z) = x^2 - 7y^2 + 3z^2, \quad q_3(x_1,\ldots,x_n) = x_1^2 + \cdots + x_n^2,$$

are all quadratic forms. Given a quadratic form q over k, there are some basic questions one could ask that are at the heart of quadratic form theory.

1. Does q have a non-trivial zero (i.e., is q *isotropic*)? That is, is there some non-zero x such that $q(x) = 0$?

2. Which non-zero elements of k are represented by q? That is, for which $a \in k^\times$ is there some x such that $q(x) = a$?

3. Does q represent *all* non-zero elements of k (i.e., is q *universal*)?

Despite the algebraic simplicity of quadratic forms, over a general field k, these questions are difficult to answer. However, over certain fields we have tools at our disposal that make answering these questions much easier. One of the most famous examples of such a tool is the Hasse-Minkowski Theorem:

Theorem (Hasse-Minkowski)**.** *Let q be a quadratic form over $\mathbb{Q}$. Then*

q is isotropic over $\mathbb{Q}$ if and only if q is isotropic over $\mathbb{Q}_p$ for all primes p and isotropic over $\mathbb{R}$.

This "local-global principle for isotropy" is one of the first examples of a local-global principle, and is an incredibly useful result because determining if a quadratic form is isotropic over $\mathbb{Q}$ is difficult, but is much easier to determine over the complete fields $\mathbb{Q}_p$ and $\mathbb{R}$. Furthermore, there are results like the First Representation Theorem [Lam05, Corollary I.3.5] that relate isotropy to determining which elements of k are represented by a particular quadratic form. Thus the Hasse-Minkowski Theorem, which is known to hold over any global field, can be used to answer all three of the basic questions above for quadratic forms over global fields.

It is beneficial, then, to see if this local-global principle generalizes to fields other than global fields. A common way to phrase this local-global principle over a more general field k is in terms of discrete valuations on k.

Definition 1.1.1. Let k be a field of characteristic $\neq 2$ and let V be a non-empty set of non-trivial discrete valuations on k. We say that a quadratic form q over k *satisfies the local-global principle for isotropy with respect to* V if q being isotropic over k_v for all $v \in V$ implies that q is isotropic over k. Here k_v denotes the completion of k with respect to the metric on k induced by v. We say that the local-global principle for isotropy holds over k with respect to V if all quadratic forms over k satisfy the local-global principle for isotropy with respect to V.

If the local-global principle for isotropy holds over k with respect to V, then so does the local-global principle for isometry with respect to V. Roughly speaking, two quadratic forms over k are isometric if there is an invertible linear change of variables taking one quadratic form to the other (see Section 2.1 for a more precise definition).

Definition 1.1.2. Let k be a field of characteristic $\neq 2$ and let V be a non-empty set of non-trivial discrete valuations on k. We say that a pair of quadratic forms q_1, q_2 over k *satisfies the local-global principle for isometry with respect to* V if q_1 and q_2 being isometric over k_v for all $v \in V$ implies that q_1 and q_2 are isometric over k. We say that the local-global principle for isometry holds over k with respect to V if all pairs of quadratic forms over k satisfy the local-global principle for isometry.

As we will see, these local-global principles for quadratic forms over a field k can be considered with respect to other types of overfields that are not the completions of k with respect to discrete valuations, and there are instances where these local-global principles fail. In this thesis, we will study when the local-global principles for isotropy and isometry hold, and use the local-global principle for isotropy to explore the three basic questions above, as well as to study various field invariants associated to quadratic forms.

1.2. Main results and structure

The format of this thesis is as follows. In Chapter 2, we will recall the basic notions of quadratic form theory needed in this thesis, as well as recall background information about discrete valuations.

In Chapter 3, we will focus on the local-global principles for isotropy and isometry of quadratic forms over finitely generated field extensions of fields $\ell \in \mathscr{A}_i(2)$ for some $i \geq 0$ (see Definition 2.1.11) with respect to various sets of discrete valuations. The main result of Chapter 3 is the following (see Section 3.2.2 for terminology):

Theorem (3.2.4)**.** *Let k be an algebraically closed field of characteristic $\neq 2$ that is not the algebraic closure of a finite field. Let K be any finitely generated field extension of transcendence degree $r \geq 2$ over k, and let V be any non-empty set of non-trivial divisorial discrete valuations on K that satisfies the finite support property. Then for any integer $n \neq 3$ such that*

$$2^{r-1} < n \leq 2^r,$$

there exists an n-dimensional quadratic form over K that violates the local-global principle for isotropy with respect to V.

Previously, Auel and Suresh [AS22] showed that if K is any finitely generated field extension of transcendence degree $r \geq 2$ over an algebraically closed field k of characteristic $\neq 2$ that is not the algebraic closure of a finite field, then there is a quadratic form over K of dimension 2^r that violates the local-global principle for isotropy with respect to the set of all discrete valuations on K. This field has u-invariant 2^r (see Definition 2.1.9), so this dimension is a natural first place to look for counterexamples to the local-global principle for isotropy, as any quadratic form of dimension $> 2^r$ is isotropic over K, thus automatically satisfies the local-global principle for isotropy. Theorem 3.2.4

partially generalizes [AS22, Theorem 1] by finding counterexamples of dimension $< 2^r$, but not with respect to the set of all discrete valuations on K. An example of a set of discrete valuations on K/k to which Theorem 3.2.4 applies is the set of discrete valuations on K induced by prime divisors on a projective integral regular k-scheme with function field K.

As one might expect, the smaller the set of discrete valuations is, the easier it is to violate these local-global principles. However, over rational function fields, the local-global principle for isometry does in fact hold with respect to a small set of discrete valuations (relative to the set of all discrete valuations on the field; see Proposition 3.1.2). Despite that, there are numerous examples of quadratic forms over these fields that violate the local-global principle for isotropy with respect to this same set of discrete valuations. Indeed, in Section 3.1, we prove the following result (see Section 2.1 for notation and terminology):

Theorem (3.1.7)**.** *Let ℓ be a field of characteristic $\neq 2$. Assume $\ell \in \mathscr{A}_i(2)$ for some $i \geq 0$ and $u(\ell) = 2^i$. For any integer $r \geq 1$ let $L_r = \ell(x_1, \ldots, x_r)$, and for $r \geq 2$ let V_r be the set of discrete valuations on L_r that are trivial on L_{r-1}. Then for $r \geq 2$ and any integer $n \neq 3$ such that*

$$2^{i+r-1} < n \leq 2^{i+r},$$

there exists an n-dimensional quadratic form over L_r that violates the local-global principle for isotropy with respect to V_r.

In Chapter 4, we use the local-global principle for isotropy to study universal quadratic forms over a field k. We are particularly interested in studying the m-invariant of k, defined in [GVG92] to be the minimal dimension of an anisotropic universal quadratic form over k, and in studying the set $\mathrm{AU}(k)$ of all possible dimensions of anisotropic universal quadratic forms over k. Most of our investigation is focused on these problems over a *semi-global field* F, i.e., the function field of a curve over a complete discretely valued field.

Inspired by the strong u-invariant of k, $u_s(k)$, defined in [HHK09], in Section 4.2.1 we define the strong m-invariant of k, $m_s(k)$, (see Definition 4.2.16) and show

Theorem (4.2.24)**.** *Let K be a complete discretely valued field with residue field k of characteristic $\neq 2$ such that $m_s(k) = u_s(k)$. Then*

$$m_s(K) = 2m_s(k).$$

Theorem 4.2.24 is analogous to [HHK09, Theorem 4.10], which states that, if K is a complete discretely valued field with residue field k of characteristic $\neq 2$, then $u_s(K) = 2u_s(k)$.

Another main result of Chapter 4 is about the set $\mathrm{AU}(F)$ of all possible dimensions of anisotropic universal quadratic forms over certain semi-global fields F (see Section 4.2 for terminology).

Proposition (4.2.32)**.** *Let k be a field of characteristic $\neq 2$ with $m_s(k) = u_s(k) < \infty$. For any integer $n \geq 1$ let K be an n-local field over k with valuation ring T. Let $\mathscr{X}$ be a regular projective connected T-curve with closed fiber X. Let $X_1, \ldots, X_s$ be the irreducible components of X, and for $1 \leq i \leq s$, let η_i be the unique generic point of X_i. Let Γ be the reduction graph of $\mathscr{X}$, and let F be the function field of $\mathscr{X}$. Then*

$$\mathrm{AU}(F) \subseteq \begin{cases} \{2\} \cup \bigcup_{i=1}^{s} \{r_1 + r_2 \mid r_1, r_2 \in \mathrm{AU}(\kappa(\eta_i))\} & \text{if } \Gamma \text{ is not a tree,} \\ \bigcup_{i=1}^{s} \{r_1 + r_2 \mid r_1, r_2 \in \mathrm{AU}(\kappa(\eta_i))\} & \text{if } \Gamma \text{ is a tree.} \end{cases}$$

Finally, in Chapter 5 we use the Witt index (see Section 2.2) to explore refined notions of both the local-global principle for isotropy and the m-invariant. We begin by defining a refined notion of the local-global principle for isotropy, $\mathrm{LGP}(r, s)$ (see Section 5.1). In Section 5.1.1, we show that over purely transcendental extensions of fields $\ell \in \mathscr{A}_i(2)$ for some $i \geq 0$, there are numerous counterexamples to this refined local-global principle for isotropy.

Theorem (5.1.3)**.** *Let ℓ be a field of characteristic $\neq 2$ such that $\ell \in \mathscr{A}_i(2)$ for some $i \geq 0$ and $u(\ell) = 2^i$. For any integer $r \geq 1$ let $L_r = \ell(x_1, \ldots, x_r)$, and for $r \geq 2$ let V_r be the set of discrete valuations on L_r that are trivial on L_{r-1}. Then for $r \geq 2$ and any integer n such that $0 \leq n < 2^{i+r-2}$, there is a quadratic form over L_r that violates $\mathrm{LGP}\left(2^{i+r-2} - n, 1\right)$ with respect to V_r.*

Due to the existence of these counterexamples, we are led to ask if there are certain conditions we can impose on a quadratic form to ensure it satisfies LGP(r, s) for some integers $r, s \geq 1$. With this goal in mind, in Section 5.1.2 we define the condition of a quadratic form being an I^n-neighbor for some n (see Definition 5.1.6) and prove the following result (see Section 5.1.2 for terminology):

Proposition (5.1.16)**.** *Let k be a field of characteristic $\neq 2$ equipped with a non-empty set V of non-trivial discrete valuations with respect to which the local-global principle for isometry holds. Let q be an I^n-neighbor of complementary dimension r for some $n \geq 1$. Then*

$$q \textit{ satisfies } \mathrm{LGP}\left(\frac{\dim q + r - 2^n}{2} + 1, \frac{\dim q - r}{2}\right) \textit{ with respect to } V.$$

We conclude Section 5.1 by studying whether or not we can find integers $r, s \geq 1$ such that all quadratic forms over k satisfy LGP(r, s).

In Section 5.2 we use the Witt index to refine the notion of the m-invariant of a field k, and define $m_{i,j}(k)$ for any integers $i, j \geq 1$ (see Definition 5.2.1). One of the main results of this section is the following, which gives natural bounds for $m_{i,j}(k)$ in terms of $m(k), u(k), i$, and j.

Proposition (5.2.9)**.** *Let k be any field of characteristic $\neq 2$, and let $i, j \geq 1$ be any positive integers. Then*

$$\max\{1, m(k) + 2j - 1 - i\} \leq m_{i,j}(k) \leq \max\{1, u(k) + 2j - 1 - i\}.$$

To conclude Chapter 5, we show that there is a connection between the refined local-global principle for isotropy and these refined m-invariants.

Proposition (Corollary 5.3.3)**.** *Let k be a field of characteristic $\neq 2$, let V be a non-empty set of non-trivial discrete valuations on k, and let $i, j \geq 1$ be positive integers. If all quadratic forms over k of dimension $m_{i,j}(k) + i - 1$ satisfy* LGP(r, j) *with respect to V for some integer $r \geq 1$, then so do all quadratic forms over k of dimension $< m_{i,j}(k)$.*

CHAPTER 2

Notation and Preliminaries

2.1. Quadratic forms

In this section, we recall the necessary basic information from quadratic form theory (see, e.g., [Lam05, Chapter I]). A *quadratic form* q over a field k (occasionally referred to just as a *form*) is a homogeneous degree two polynomial over k. The number of variables appearing in q is called the *dimension of* q. To an n-dimensional quadratic form q over k, we can associate a symmetric $n \times n$ matrix M_q over k, and we say that q is *regular* if the matrix M_q is nonsingular. All quadratic forms considered in this thesis will be regular. Given a quadratic form q over k and a field extension K/k, we let q_K denote the quadratic form q considered as a quadratic form over K. While quadratic forms can be considered over fields of characteristic 2, the theory in that setting is slightly more complicated, and therefore, throughout this thesis, we will assume that k has characteristic $\neq 2$.

An n-dimensional quadratic form q over k is *isotropic* if there exists some $x = (x_1, \ldots, x_n) \in k^n \setminus \{0\}$ such that $q(x) = 0$. If no such x exists, then we say that q is *anisotropic* (over k).

Example 2.1.1. Consider the quadratic form $q_1(x, y) = x^2 - y^2$. Then q_1 is isotropic over any field k since $q_1(1, 1) = 0$. Now consider the form $q_2(x, y) = x^2 - 2y^2$. Over $\mathbb{Q}$, q_2 is anisotropic since $\sqrt{2} \notin \mathbb{Q}$. However, if we consider q_2 over a larger field like $\mathbb{R}$, then q_2 is isotropic since $q_2(\sqrt{2}, 1) = 0$.

Two n-dimensional quadratic forms q_1, q_2 over k are *isometric* over k if there exists some $C \in \mathrm{GL}_n(k)$ such that $q_1(x) = q_2(Cx)$ for all $x \in k^n$. If q_1 and q_2 are isometric, we write $q_1 \simeq q_2$.

Example 2.1.2. Over any field k of characteristic $\neq 2$, the quadratic forms $q_1(x_1, x_2) = x_1 x_2$ and $q_2(x_1, x_2) = x_1^2 - x_2^2$ are isometric, i.e., $q_1 \simeq q_2$. Indeed, for $C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \in \mathrm{GL}_2(k)$, if $x = (x_1, x_2)$, we have

$$q_1(Cx) = q_1(x_1 + x_2, x_1 - x_2) = x_1^2 - x_2^2 = q_2(x).$$

Diagonal quadratic forms will be of particular interest to us. Given elements $a_1, \ldots, a_n \in k^\times$, we will let $\langle a_1, \ldots, a_n \rangle$ denote the n-dimensional quadratic form $a_1 x_1^2 + \cdots + a_n x_n^2$. By definition, not every quadratic form over k is a diagonal form. However, if char $k \neq 2$, by [Lam05, Corollary I.2.4], every regular quadratic form over k can be *diagonalized*. That is, given any regular quadratic form q over a field k of characteristic $\neq 2$, we can find elements $a_1, \ldots, a_n \in k^\times$ such that $q \simeq \langle a_1, \ldots, a_n \rangle$. We will call the elements $a_1, \ldots, a_n$ the *entries of* q.

Example 2.1.3 (Important Example). Let k be any field of characteristic $\neq 2$, and let $q \simeq \langle a_1, \ldots, a_n \rangle$ be a quadratic form over k. Then for any $b_1, \ldots, b_n \in k^\times$,

$$q \simeq \left\langle b_1^2 a_1, \ldots, b_n^2 a_n \right\rangle .$$

That is, q is isometric to the quadratic form obtained by multiplying each entry of q by a non-zero square.

To a quadratic form $q \simeq \langle a_1, \ldots, a_n \rangle$ over k, we can associate two elements of $k^\times / k^{\times 2}$ (i.e., two square classes of k): the determinant and discriminant of q. The *determinant of* q is defined by

$$\det(q) := a_1 \cdots a_n \in k^\times / k^{\times 2},$$

and the *discriminant of* q (or *signed determinant of* q) is defined by

$$\operatorname{disc}(q) = d_\pm(q) := (-1)^{\frac{n(n-1)}{2}} \det(q) \in k^\times / k^{\times 2}.$$

Given two quadratic forms $q_1 = \langle a_1, \ldots, a_n \rangle, q_2 = \langle b_1, \ldots, b_m \rangle$ over k, there are two natural ways to combine these forms. First, we can "add" the forms together by taking their *orthogonal sum*:

$$q_1 \perp q_2 := \langle a_1, \ldots, a_n, b_1, \ldots, b_m \rangle.$$

Next, we can "multiply" these forms together by taking their *tensor product*:

$$q_1 \otimes q_2 := \langle \ldots, a_i b_j, \ldots \rangle.$$

We say that two quadratic forms q_1, q_2 over k are *similar* if there exists some $a \in k^\times$ such that $q_1 \simeq a \cdot q_2 := \langle a \rangle \otimes q_2$. For any quadratic form q over k, we let $-q$ denote the form $(-1) \cdot q$. Given quadratic forms q, φ over k, we say that q is a *subform* of φ, denoted $q \subseteq \varphi$, if there exists some quadratic form ψ over k such that $\varphi \simeq q \perp \psi$.

At various points of this thesis, particularly in Chapter 4, we will be interested in the set of non-zero values of a quadratic form q over a field k. The set of elements of $k^\times$ represented by q over k will be denoted by $D_k(q)$, i.e.,

$$D_k(q) = \left\{ a \in k^\times \mid q(x) = a \text{ for some } x \right\}.$$

We note that for any $a, b \in k^\times$, $a \in D_k(q)$ if and only if $ab^2 \in D_k(q)$. The form q is *universal* if $D_k(q) = k^\times$.

Example 2.1.4. Let k be any field of characteristic $\neq 2$, and let $q(x, y) = x^2 - y^2$. Then for any $a \in k^\times$, we have

$$q\left(\frac{a+1}{2}, \frac{a-1}{2}\right) = a.$$

Therefore $D_k(q) = k^\times$, hence q is universal.

Throughout this thesis, we will repeatedly use the following two elementary results that provide necessary and sufficient conditions for an element $a \in k^\times$ to belong to $D_k(q)$ for a regular quadratic form q over k.

Theorem 2.1.5 (Representation Criterion). *Let q be a regular quadratic form over a field k of characteristic $\neq 2$ and let $a \in k^\times$. Then $a \in D_k(q)$ if and only if there is a regular quadratic form q' over k such that $q \simeq \langle a \rangle \perp q'$.*

Proof. See, e.g., [Lam05, Theorem I.2.3]. □

Theorem 2.1.6 (First Representation Theorem)**.** *Let q be a regular quadratic form over a field k of characteristic $\neq 2$. Then for any $a \in k^\times$,*

$$a \in D_k(q) \text{ if and only if } q \perp \langle -a \rangle \text{ is isotropic.}$$

Proof. See, e.g., [Lam05, Corollary I.3.5]. □

The *hyperbolic plane* over k is the isotropic two-dimensional quadratic form $\mathbb{H} := \langle 1, -1 \rangle$. We note here that by [Lam05, Theorem I.3.2], for any $a \in k^\times$, we have $a \cdot \mathbb{H} \simeq \mathbb{H}$. We say that an even-dimensional quadratic form q is *hyperbolic* if $q \simeq n\mathbb{H}$ for some positive integer n, where $n\mathbb{H}$ denotes the orthogonal sum of n copies of $\mathbb{H}$.

The next lemma gives a criterion for determining when two quadratic forms are isometric.

Lemma 2.1.7. *Let q_1, q_2 be regular quadratic forms of positive dimension over a field k of characteristic $\neq 2$. Then $q_1 \simeq q_2$ if and only if $\dim q_1 = \dim q_2$ and $q_1 \perp -q_2 \simeq (\dim q_1)\mathbb{H}$.*

Proof. First assume that $q_1 \simeq q_2$. Then $\dim q_1 = \dim q_2 = n$ for some $n \geq 1$. Let $a_1, \dots, a_n \in k^\times$ be such that $q_1 \simeq \langle a_1, \dots a_n \rangle$. Then $q_2 \simeq \langle a_1, \dots, a_n \rangle$, thus

$$q_1 \perp -q_2 \simeq \langle a_1, -a_1, \dots, a_n, -a_n \rangle \simeq n\mathbb{H}.$$

Conversely, suppose $\dim q_1 = \dim q_2 = n$ and $q_1 \perp -q_2 \simeq n\mathbb{H}$. We prove that $q_1 \simeq q_2$ by induction on $n \geq 1$.

For the base case $n = 1$, since $q_1 \perp -q_2 \simeq \mathbb{H}$, the form $q_1 \perp -q_2$ is isotropic. By [Lam05, Corollary I.3.6], q_1 and q_2 must represent a common element $a \in k^\times$. But since q_1 and q_2 are one-dimensional, by the Representation Criterion (Theorem 2.1.5), we have $q_1 \simeq \langle a \rangle \simeq q_2$, proving the base case.

Now suppose that for some $n \geq 1$ the claim is true for all quadratic forms of dimension n. Let q_1 and q_2 be regular $(n+1)$-dimensional quadratic forms over k such that $q_1 \perp -q_2 \simeq (n+1)\mathbb{H}$. Then, in particular, $q_1 \perp -q_2$ is isotropic, and so by [Lam05, Corollary I.3.6], q_1 and q_2 represent a common element $a \in k^\times$. By the Representation Criterion, we can write $q_1 \simeq \langle a \rangle \perp q_1'$ and $q_2 \simeq \langle a \rangle \perp q_2'$ for some regular n-dimensional quadratic forms q_1', q_2' over k. We have

$$(n+1)\mathbb{H} \simeq q_1 \perp -q_2 \simeq (\langle a \rangle \perp q_1') \perp (\langle -a \rangle \perp -q_2') \simeq \mathbb{H} \perp q_1' \perp -q_2'.$$

By Witt Cancellation [Lam05, Theorem I.4.2], the isometries above imply that $q_1' \perp -q_2' \simeq n\mathbb{H}$. By the induction hypothesis, we conclude that $q_1' \simeq q_2'$. This then implies that $q_1 \simeq q_2$, completing the proof by induction. □

Lemma 2.1.7 allows us to give an equivalent formulation of the local-global principle for isometry.

Corollary 2.1.8. *Let k be a field of characteristic $\neq 2$ equipped with a non-empty set V of non-trivial discrete valuations. Then the local-global principle for isometry holds over k with respect to V if and only if every even-dimensional quadratic form over k that is hyperbolic over k_v for all $v \in V$ is also hyperbolic over k.*

A particularly important type of quadratic form, called a Pfister form, can be built from simple quadratic forms using the tensor product. For any elements $a_1, \ldots, a_n \in k^\times$, the *$n$-fold Pfister form* $\langle\langle a_1, \ldots, a_n \rangle\rangle$ is the 2^n-dimensional quadratic form over k defined by

$$\langle\langle a_1, \ldots, a_n \rangle\rangle := \langle 1, a_1 \rangle \otimes \cdots \otimes \langle 1, a_n \rangle.$$

A Pfister form is isotropic if and only if it is hyperbolic [Lam05, Theorem X.1.7]. If φ is a Pfister form over k, we can write $\varphi \simeq \langle 1 \rangle \perp \varphi_{ps}$, and φ_{ps} is called the *pure subform of* φ. We will use this notation for any quadratic form that represents 1, i.e., if $q \simeq \langle 1, a_2, \ldots, a_n \rangle$, then we let $q_{ps} := \langle a_2, \ldots, a_n \rangle$. The form q_{ps} is well-defined up to isometry by Witt Cancellation [Lam05, Theroem I.4.2].

A certain measure of the complexity of quadratic forms over a field k is the u-invariant of k [Lam05, Definition XI.6.1].

Definition 2.1.9. Let k be a field. The *u-invariant of k*, denoted $u(k)$, is the maximal dimension of an anisotropic quadratic form over k. If no such maximum exists, then $u(k) = \infty$.

Examples 2.1.10. (a) $u(\mathbb{Q}) = u(\mathbb{R}) = \infty$,

(b) $u(\mathbb{C}) = 1$,

(c) $u(\mathbb{F}_p) = 2$.

In general, given a field k, calculating $u(k)$ is a challenging problem, and various techniques have been developed to solve this problem for particular k (see, e.g., [HHK09, Lee13, PS10]). However, for the fields considered in this paper, the u-invariant is known.

First, there are C_i fields (see, e.g, [Pfi79, Definition, §2]). If k is a C_i field for some $i \geq 0$, then $u(k) \leq 2^i$. By [Pfi79, Propositions 2, 3], if k is a C_i field for some $i \geq 0$, and K is a finitely generated field extension of transcendence degree r over k, then K is a C_{i+r} field, thus $u(K) \leq 2^{i+r}$.

More generally, there are non-C_i fields for which the u-invariant is known. For example, for a prime p, $u(\mathbb{Q}_p) = 4$, but $\mathbb{Q}_p$ is not a C_2 field. However, $\mathbb{Q}_p$ does satisfy the property $\mathscr{A}_2(2)$ considered in [Lee13], which we now recall (see [Lee13, Definition 2.1] for a more general version of this definition).

Definition 2.1.11. For any integer $i \geq 0$, a field ℓ of characteristic $\neq 2$ *satisfies property* $\mathscr{A}_i(2)$, written $\ell \in \mathscr{A}_i(2)$, if every system of s quadratic forms defined over ℓ in $n > s \cdot 2^i$ common variables has a nontrivial simultaneous zero in an odd degree extension of ℓ.

By [Pfi79, Proposition 1], if k is a C_i field, then $k \in \mathscr{A}_i(2)$. According to [Lee13, Proposition 2.2], if $\ell \in \mathscr{A}_i(2)$ for some $i \geq 0$, then $u(\ell) \leq 2^i$. Moreover, much like C_i fields, if $\ell \in \mathscr{A}_i(2)$ for some $i \geq 0$, and L is a finitely generated extension of transcendence degree r over ℓ, then by [Lee13, Theorems 2.3, 2.5], $L \in \mathscr{A}_{i+r}(2)$. For such a field L/ℓ, we conclude that $u(L) \leq 2^{i+r}$.

2.2. Witt index and the Witt ring

Given a regular quadratic form q over a field k of characteristic $\neq 2$, by the Witt Decomposition Theorem [Lam05, Theorem I.4.1], there exists a unique non-negative integer called the *Witt index of* q, denoted $i_W(q)$, and an anisotropic quadratic form q' over k such that

$$q \simeq i_W(q)\mathbb{H} \perp q'.$$

The quadratic form q' is unique up to isometry, and is called the *anisotropic part of* q, which we will denote by q_{an}. The form q is isotropic over k if and only if $i_W(q) \geq 1$.

At various points of this thesis we will need to understand the Witt index of an orthogonal sum of two quadratic forms q, φ over k. We start by studying the orthogonal sum of two anisotropic forms.

Lemma 2.2.1. *Let q, φ be regular anisotropic quadratic forms over a field k of characteristic $\neq 2$. Then for any integer $r \geq 1$, $i_W(q \perp \varphi) \geq r$ if and only if there is a regular r-dimensional form σ_r over k such that $\sigma_r \subseteq q$ and $-\sigma_r \subseteq \varphi$.*

Proof. We first prove the "if" statement. Let σ_r be the regular r-dimensional quadratic form such that $q \simeq \sigma_r \perp q'$ for some q' and $\varphi \simeq -\sigma_r \perp \varphi'$ for some φ'. Then

$$q \perp \varphi \simeq (\sigma_r \perp q') \perp (-\sigma_r \perp \varphi') = (\sigma_r \perp -\sigma_r) \perp (q' \perp \varphi') \simeq r\mathbb{H} \perp (q' \perp \varphi').$$

Therefore $i_W(q \perp \varphi) \geq r$, as desired.

We now prove the "only if" statement by induction on $r \geq 1$. For the base case, suppose we have $i_W(q \perp \varphi) \geq 1$, i.e., $q \perp \varphi$ is isotropic. Then by [Lam05, Corollary I.3.6], there is some $a \in k^\times$ such that $a \in D_k(q)$ and $-a \in D_k(\varphi)$. By the Representation Criterion, we can write $q \simeq \langle a \rangle \perp q'$ for some q', and $\varphi \simeq \langle -a \rangle \perp \varphi'$ for some φ'. Letting $\sigma_1 = \langle a \rangle$ then proves the base case.

Now assume the claim is true for some integer $r \geq 1$, and suppose $i_W(q \perp \varphi) \geq r+1$. Then, in particular, $i_W(q \perp \varphi) \geq r$, so by the induction hypothesis, there is some r-dimensional quadratic form σ_r over k such that $q \simeq \sigma_r \perp q'$ and $\varphi \simeq -\sigma_r \perp \varphi'$ for regular anisotropic quadratic forms q', φ' over k. Since $i_W(q \perp \varphi) \geq r+1$, by Witt Cancellation [Lam05, Theorem I.4.2], we conclude $i_W(q' \perp \varphi') \geq 1$. So by the base case, there is a 1-dimensional form σ_1 over k such that $\sigma_1 \subseteq q'$ and $-\sigma_1 \subseteq \varphi'$. Letting $\sigma_{r+1} = \sigma_r \perp \sigma_1$ completes the proof of this direction by induction. □

Corollary 2.2.2. *Let q, φ be regular anisotropic quadratic forms over a field k of characteristic $\neq 2$. Then*

$$i_W(q \perp \varphi) \leq \min\{\dim q, \dim \varphi\}.$$

Proof. The claim is true if either $\dim q = 0$ or $\dim \varphi = 0$ since q and φ are anisotropic. So we may assume that q, φ are both positive-dimensional. Without loss of generality, assume $\dim q \leq \dim \varphi$, and by contradiction, assume $i_W(q \perp \varphi) > \dim q$. By Lemma 2.2.1, we can find a subform of q with dimension larger than $\dim q$, which is impossible. □

We can now study the Witt index of the orthogonal sum of two quadratic forms that may or may not be anisotropic.

Lemma 2.2.3. *Let q, φ be regular quadratic forms over a field k of characteristic $\neq 2$. Then*

$$i_W(q \perp \varphi) \leq i_W(q) + i_W(\varphi) + \min\{\dim q_{an}, \dim \varphi_{an}\}.$$

Proof. Write $q \simeq i_W(q)\mathbb{H} \perp q_{an}$, $\varphi \simeq i_W(\varphi)\mathbb{H} \perp \varphi_{an}$. Then

$$q \perp \varphi \simeq (i_W(q) + i_W(\varphi))\mathbb{H} \perp (q_{an} \perp \varphi_{an}).$$

Therefore

$$i_W(q \perp \varphi) = i_W(q) + i_W(\varphi) + i_W(q_{an} \perp \varphi_{an}) \leq i_W(q) + i_W(\varphi) + \min\{\dim q_{an}, \dim \varphi_{an}\},$$

where this last inequality follows from Corollary 2.2.2. □

As an immediate consequence of Lemma 2.2.3, we arrive at the conclusion of [Lam05, Exercise I.16(2)].

Corollary 2.2.4. *Let q, φ be regular quadratic forms over a field k of characteristic $\neq 2$. Then*

$$i_W(q \perp \varphi) \leq i_W(q) + \dim \varphi.$$

Proof. We have $\dim \varphi = 2i_W(\varphi) + \dim \varphi_{an}$. By Lemma 2.2.3, we have

$$\begin{aligned} i_W(q \perp \varphi) &\leq i_W(q) + i_W(\varphi) + \min\{\dim q_{an}, \dim \varphi_{an}\} \leq i_W(q) + i_W(\varphi) + \dim \varphi_{an} \\ &= i_W(q) + \dim \varphi - i_W(\varphi) \leq i_W(q) + \dim \varphi. \end{aligned}$$

□

From Corollary 2.2.4 we deduce a particular case of [Lam05, Exercise I.14].

Corollary 2.2.5. *Let $n \geq 1$ be any positive integer, and for any $0 < r \leq n$, let q be an $(n+r)$-dimensional regular subform of the hyperbolic form $n\mathbb{H}$ over a field k of characteristic $\neq 2$. Then $i_W(q) \geq r$.*

Proof. Because q is a subform of $n\mathbb{H}$, there is an $(n-r)$-dimensional regular form φ over k such that $q \perp \varphi \simeq n\mathbb{H}$. By Corollary 2.2.4, we have

$$n = i_W(n\mathbb{H}) = i_W(q \perp \varphi) \leq i_W(q) + \dim \varphi = i_W(q) + (n-r).$$

The inequality above implies that $i_W(q) \geq r$. □

To a field k of characteristic $\neq 2$, we can associate a ring, called the *Witt ring of k*, denoted $W(k)$, whose elements are *Witt equivalence classes* of regular quadratic forms over k. As we saw above, we can write any regular quadratic form q over k as $q \simeq i_W(q)\mathbb{H} \perp q_{an}$, where q_{an} is an anisotropic quadratic form over k, unique up to isometry. Two regular quadratic forms q_1, q_2 over k are *Witt* *equivalent* if $q_{1,an} \simeq q_{2,an}$. Equivalently, q_1 and q_2 are Witt equivalent if $q_1 \perp r_1\mathbb{H} \simeq q_2 \perp r_2\mathbb{H}$ for some integers $r_1, r_2 \geq 0$. We note here that, by definition, the zero-dimensional form 0 over k is Witt equivalent to any hyperbolic form $n\mathbb{H}$ for $n \geq 1$. For a regular quadratic form q over k, we let $[q]$ denote its Witt equivalence class, and let

$$W(k) = \{\text{Witt equivalence classes of regular quadratic forms over } k\}.$$

The operations of $\perp$ and $\otimes$ give us an addition and multiplication on $W(k)$. Indeed, both of these operations are well-defined on Witt equivalence classes, with

$$[q_1] \perp [q_2] = [q_1 \perp q_2], \quad [q_1] \otimes [q_2] = [q_1 \otimes q_2].$$

The class $[0]$ is the additive identity on $W(k)$ under $\perp$. For any n-dimensional quadratic form q over k, $[-q]$ is the additive inverse of $[q]$ since $q \perp -q \simeq n\mathbb{H}$. Therefore the set $W(k)$ forms a ring under the operations $\perp, \otimes$. (For a slightly different approach to defining the Witt ring, see [Lam05, Chapter II.1]).

Furthermore, we have a well-defined group homomorphism

$$\begin{aligned} \dim : W(k) &\to \mathbb{Z}/2\mathbb{Z} \\ [q] &\mapsto \dim q \mod 2. \end{aligned}$$

The kernel of this map is denoted by $I(k)$, and is called the *fundamental ideal of* $W(k)$. Therefore, a quadratic form q over k represents an element in $I(k)$ if and only if q has even dimension. Later in this thesis, we will consider powers of the fundamental ideal, $I^n(k)$, for $n \geq 1$ and their relationship to a refined local-global principle for isotropy (see Section 5.1.2). By a slight abuse of notation, we will write $q \in I^n(k)$ if $[q] \in I^n(k)$. It is known that n-fold Pfister forms generate the ideal $I^n(k)$ as an abelian group [Lam05, Proposition X.1.2], and certain invariants associated to a quadratic form q can be used to determine if $q \in I^n(k)$. For example, $q \in I^2(k)$ if and only if $\dim q$ is even and $d_\pm(q) = 1 \in k^\times/k^{\times 2}$ [Lam05, Corollary II.2.2].

These powers of the fundamental ideal have played a key role in several other areas of algebra, perhaps most notably in the Milnor Conjecture. Indeed, by Voevodsky's proof of the Milnor Conjecture (see, e.g., [OVV07, Voe03]), for any integer $n \geq 1$,

$$I^n(k)/I^{n+1}(k) \cong H^n(k, \mu_2),$$

where $H^n(k, \mu_2)$ denotes the n-th Galois cohomology group of k with μ_2 coefficients.

2.3. Discrete valuations

In this section, we recall some basic notions about discrete valuations on a field.

Definition 2.3.1. A *discrete valuation* on a field k is a surjective map $v : k \to \mathbb{Z} \cup \{\infty\}$ satisfying the following three properties for all $x, y \in k$:

1. $v(x) = \infty$ if and only if $x = 0$,
2. $v(xy) = v(x) + v(y)$,
3. $v(x + y) \geq \min\{v(x), v(y)\}$.

A *uniformizer for* v is any element $\pi \in k$ such that $v(\pi) = 1$. If k is a subfield of another field K and v is a discrete valuation on K, we say that v is *trivial on* k if $v(x) = 0$ for all $x \in k^\times$.

Given a discrete valuation v on a field k, we can define several algebraic objects associated to v. First, there is the *valuation ring of* v, $\mathcal{O}_v := \{x \in k \mid v(x) \geq 0\}$. The valuation ring $\mathcal{O}_v$ is a local ring, whose unique maximal ideal is given by $\mathfrak{m}_v := \{x \in k \mid v(x) > 0\}$. The *residue field of* v is the quotient $\kappa_v := \mathcal{O}_v/\mathfrak{m}_v$. Lastly, the discrete valuation v induces a metric on k, and we let k_v denote the completion of k with respect to this v*-adic* metric, and for a quadratic form q over k, we will write q_v for the form q_{k_v}.

Throughout this thesis, we will be considering various discrete valuations on finitely generated field extensions of some positive transcendence degree. Here we consider a simple example.

Example 2.3.2. Let $K = k(t)$. For any $\alpha \in K^\times$, we can write $\alpha = t^n \frac{f}{g}$, where $f, g \in k[t]$ are polynomials not divisible by t. The *t-adic valuation of α* is defined by $v_t(\alpha) = n$. The discrete valuation v_t on K is trivial on k, and we have

$$\mathcal{O}_{v_t} = k[t]_{(t)}, \;\; \mathfrak{m}_{v_t} = (t), \;\; \kappa_{v_t} \cong k, \;\; K_{v_t} \cong k((t)).$$

Given any monic irreducible polynomial $\pi \in k[t]$, in the same way that we defined the t-adic valuation v_t on $K = k(t)$ in Example 2.3.2, we can define the π-adic valuation v_π on K. For such a discrete valuation v_π, we will let K_π denote the completion of K with respect to v_π, and denote the residue field of v_π by $\kappa_\pi \cong k[t]/(\pi)$.

Throughout this thesis, we will repeatedly use a theorem of Springer [Lam05, Proposition VI.1.9] about the behavior of quadratic forms over a complete discretely valued field, which we will refer to as Springer's Theorem. Let K be a complete discretely valued field with valuation ring $\mathcal{O}$, residue field κ, char $\kappa \neq 2$, and uniformizer π. Let $q = \langle a_1, \ldots, a_n \rangle$ be a quadratic form over K. Then by multiplying and dividing the entries of q by even powers of π, we can write

$$q \simeq \langle u_1, \ldots, u_r \rangle \perp \pi \cdot \langle u_{r+1}, \ldots, u_n \rangle,$$

where each $u_i \in \mathcal{O}^\times$. Let $q_1 = \langle u_1, \ldots, u_r \rangle$, $q_2 = \langle u_{r+1}, \ldots, u_n \rangle$, and let $\overline{q}_1 = \langle \overline{u}_1, \ldots, \overline{u}_r \rangle$, $\overline{q}_2 = \langle \overline{u}_{r+1}, \ldots, \overline{u}_n \rangle$, where $\overline{u}_i$ is the non-zero image of u_i in κ. We call $\overline{q}_1, \overline{q}_2$ the *first* and *second residue forms of q*, respectively.

Theorem 2.3.3 (Springer's Theorem)**.** *In the above notation, q is anisotropic over K if and only if both residue forms $\overline{q}_1, \overline{q}_2$ are anisotropic over κ.*

To conclude this chapter, we make a small observation about local-global principles for quadratic forms over a field k with respect to various sets of discrete valuations on k. Let V and W be two non-empty sets of non-trivial discrete valuations on k, and suppose that $V \subseteq W$. If quadratic forms over k satisfy the local-global principle (for isotropy/isometry) with respect to V, then they also satisfy the local-global principle with respect to W. Equivalently, if a quadratic form q over k violates the local-global principle with respect to W, then q also violates the local-global principle with respect to V.

CHAPTER 3

Local-Global Principles for Isotropy and Isometry

In this chapter, we will focus on the local-global principles for isotropy and isometry over finitely generated field extensions with respect to various sets of discrete valuations. We will be particularly interested in finitely generated field extensions of fields $\ell \in \mathscr{A}_i(2)$ for some $i \geq 0$ (see Definition 2.1.11).

3.1. A small set of discrete valuations

Let us first focus on rational function fields in one variable over a field k of characteristic $\neq 2$. Let $K = k(t)$, and let $\mathscr{P}$ be the set of monic irreducible polynomials in $k[t]$. Then if $V_{K/k}$ is the set of all discrete valuations on K that are trivial on k, by [EP05, Theorem 2.1.4], we know

$$V_{K/k} = V_{\mathscr{P}} \cup \{v_\infty\}\,,$$

where $V_{\mathscr{P}} = \{v_\pi : \pi \in \mathscr{P}\}$ and v_∞ is the degree valuation with respect to t. Relative to the set of all discrete valuations on K, the set $V_{K/k}$ is small, but provides enough local data for the local-global principle for isometry to hold (see Proposition 3.1.2), and for certain quadratic forms over K to satisfy the local-global principle for isotropy (see Proposition 3.1.4). However, the main result of this section (Theorem 3.1.7) shows that there are numerous counterexamples over K to the local-global principle for isotropy with respect to $V_{K/k}$ for certain ground fields k. We note that several of the results in this section are well-known to experts (e.g., Proposition 3.1.2, Lemma 3.1.3, Proposition 3.1.4, Corollary 3.1.5), but the proofs do not seem to be written explicitly in the literature, so we include proofs for the sake of completeness.

Lemma 3.1.1. *Let k be a field of characteristic $\neq 2$, and let q be an even-dimensional quadratic form over k. Then q is hyperbolic over k if and only if q is hyperbolic over $k((t))$.*

Proof. The forward implication is trivial since $k \subset k((t))$. It therefore remains to prove the reverse implication. Let $\dim q = 2n$ for some $n \geq 1$. We prove the claim by induction on $n \geq 1$.

For the base case $n = 1$, we have a two-dimensional quadratic form q over k that becomes hyperbolic, hence isotropic, over $k((t))$. By Springer's Theorem (Theorem 2.3.3), one of the residue forms of q must be isotropic over the residue field k of $k((t))$. But, because q is defined over k, the first residue form of q is q itself and q has no second residue form. So q must be isotropic over k, proving the base case.

Now suppose that for some $n \geq 1$, any $2n$-dimensional quadratic form over k that becomes hyperbolic over $k((t))$ must be hyperbolic over k, and let q be a $2(n+1)$-dimensional quadratic form over k that becomes hyperbolic over $k((t))$. In particular, q is isotropic over $k((t))$, which, again by Springer's Theorem, implies that q is isotropic over k. Over k, we can therefore write $q \simeq \mathbb{H} \perp q'$, where q' is a quadratic form over k of dimension $2n$. Because $q \simeq \mathbb{H} \perp q'$ is hyperbolic over $k((t))$, the form q' defined over k must become hyperbolic over $k((t))$ as well. By the induction hypothesis, this implies that q' must be hyperbolic over k, i.e., $q' \simeq n\mathbb{H}$. This then implies that $q \simeq (n+1)\mathbb{H}$, proving the claim by induction. □

Proposition 3.1.2. *Let k be any field of characteristic $\neq 2$, let $K = k(t)$, and let $V_{K/k}$ be the set of discrete valuations on K that are trivial on k. Then the local-global principle for isometry holds over K with respect to $V_{K/k}$.*

Proof. It suffices to consider $V_{\mathscr{P}} \subset V_{K/k}$, and, by Corollary 2.1.8, it suffices to show that any even-dimensional quadratic form over K that is hyperbolic over K_π for all $\pi \in \mathscr{P}$ is hyperbolic over K.

Recall the Milnor exact sequence on Witt groups [Mil69, Theorem 5.3]:

$$0 \to W(k) \to W(K) \xrightarrow{\delta} \bigoplus_{\pi \in \mathscr{P}} W(\kappa_\pi) \to 0.$$

Suppose q is a quadratic form over K that is hyperbolic over K_π for all $\pi \in \mathscr{P}$. The map δ factors through $\bigoplus_{\pi \in \mathscr{P}} W(K_\pi)$, so the class $[q]$ of q in $W(K)$ lies in $\ker \delta$. By exactness, there is a quadratic form q_0 over k such that $[q_0] = [q] \in W(K)$.

By assumption, q is in particular hyperbolic over $K_t = k((t))$, so q_0 must be hyperbolic over $k((t))$ as well. By Lemma 3.1.1, because the k-form q_0 is hyperbolic over $k((t))$, it must also be hyperbolic over k. Therefore $0 = [q_0] = [q]$, hence q is hyperbolic over K. $\square$

A quadratic form q over a field k is a *Pfister neighbor* if q is similar to a subform of a Pfister form φ over k with $\dim \varphi < 2 \dim q$ (see, e.g., [Lam05, Definition X.4.16]). In this situation, φ is unique up to isometry by [Lam05, Proposition X.4.17], and is called the *Pfister form associated to q*.

Lemma 3.1.3. *Let q be a Pfister neighbor over a field k of characteristic $\neq 2$, with associated Pfister form φ. Then q is isotropic if and only if φ is isotropic.*

Proof. By definition, there is some $a \in k^\times$ such that $a \cdot q \subseteq \varphi$, and from this we can see that if q is isotropic, then so is φ.

Conversely, let $2n = \dim \varphi$. By definition, we must have $\dim q = n + r$ for some $r \geq 1$. Now, suppose that φ is isotropic. Then $\varphi \simeq n\mathbb{H}$ is hyperbolic since it is an isotropic Pfister form, hence $q \subseteq n\mathbb{H}$. This then implies, by Corollary 2.2.5, that $i_W(q) \geq r \geq 1$, so q is isotropic. $\square$

Proposition 3.1.4. *Let $K = k(t)$, where k is any field of characteristic $\neq 2$. Then Pfister neighbors over K satisfy the local-global principle for isotropy with respect to $V_{K/k}$.*

Proof. Let q be a Pfister neighbor over K with associated Pfister form φ, and assume that q_v is isotropic over K_v for all $v \in V_{K/k}$. By Lemma 3.1.3, this implies that φ_v is isotropic, and hence hyperbolic, over K_v for all $v \in V_{K/k}$. By Proposition 3.1.2, φ must be hyperbolic over K. By Lemma 3.1.3, we conclude that q is isotropic over K. $\square$

Corollary 3.1.5. *Let $K = k(t)$, where k is any field of characteristic $\neq 2$. Then the following quadratic forms over K satisfy the local-global principle for isotropy with respect to $V_{K/k}$:*

(a) Pfister forms,

(b) regular quadratic forms of dimension 2 or 3,

(c) regular four-dimensional quadratic forms with trivial determinant.

Proof. Pfister forms are Pfister neighbors, and by [Lam05, Examples X.4.18], over any field of characteristic $\neq 2$, regular two- and three-dimensional quadratic forms, as well as four-dimensional quadratic forms with trivial determinant, are Pfister neighbors. So by Proposition 3.1.4, these quadratic forms over K satisfy the local-global principle for isotropy with respect to $V_{K/k}$. □

Remarks 3.1.6.

1. For any field k of characteristic $\neq 2$ equipped with a non-empty set V of non-trivial discrete valuations with respect to which the local-global principle for isometry holds, the same proof as above shows that Pfister neighbors over k satisfy the local-global principle for isotropy with respect to V.

2. For any integer $r \geq 1$, let $K_r = k(x_1, \ldots, x_r)$ be a purely transcendental field extension of transcendence degree r over a field k of characteristic $\neq 2$. Let V_r be the set of discrete valuations on K_r that are trivial on K_{r-1} (here taking $K_0 = k$). Then $K_r \cong K_{r-1}(x_r)$, so with respect to V_r, the local-global principle for isometry is satisfied, and Pfister neighbors over K_r satisfy the local-global principle for isotropy.

The following result shows that, even when the local-global principle for isometry holds over purely transcendental field extensions of fields $\ell \in \mathscr{A}_i(2)$ for some i (see Definition 2.1.11), the local-global principle for isotropy can fail in several dimensions.

Theorem 3.1.7. *Let ℓ be a field of characteristic $\neq 2$. Assume $\ell \in \mathscr{A}_i(2)$ for some $i \geq 0$ and $u(\ell) = 2^i$. For any integer $r \geq 1$ let $L_r = \ell(x_1, \ldots, x_r)$, and for $r \geq 2$ let V_r be the set of discrete valuations on L_r that are trivial on L_{r-1}. Then for $r \geq 2$ and any integer $n \neq 3$ such that*

$$2^{i+r-1} < n \leq 2^{i+r},$$

there exists an n-dimensional quadratic form over L_r that violates the local-global principle for *isotropy with respect to* V_r.

The proof is constructive and uses several lemmas which leverage both Springer's Theorem and the explicit description of $V_{K/k}$ for $K = k(t)$.

Lemma 3.1.8. *Let q be an anisotropic quadratic form over a field k of characteristic $\neq 2$. Then for any integer $r \geq 1$, the quadratic form $\langle\langle x_1, \ldots, x_r\rangle\rangle \otimes q$ is anisotropic over $k(x_1, \ldots, x_r)$. In particular, the Pfister form $\langle\langle x_1, \ldots, x_r\rangle\rangle$ is anisotropic over $k(x_1, \ldots, x_r)$.*

Proof. The second statement of the lemma follows from the first by taking $q = \langle 1\rangle$, so it suffices to prove the first statement, which we do by inducting on $r \geq 1$.

First, suppose $r = 1$. Then by working over $k((x_1))$ and writing

$$\langle\langle x_1\rangle\rangle \otimes q = q \perp x_1 \cdot q,$$

we see that both the first and second residue forms are equal to q, which is anisotropic over the residue field k by assumption. So by Springer's Theorem, $\langle\langle x_1\rangle\rangle \otimes q$ is anisotropic over $k((x_1))$, which contains $k(x_1)$, thus proving the base case.

Now suppose that for some $r \geq 1$, the form $\langle\langle x_1, \ldots, x_r\rangle\rangle \otimes q$ is anisotropic over $k(x_1, \ldots, x_r)$. Over $k(x_1, \ldots, x_r, x_{r+1})$, we can write $\langle\langle x_1, \ldots, x_{r+1}\rangle\rangle \otimes q$ as

$$\begin{aligned} \langle\langle x_1, \ldots, x_r, x_{r+1}\rangle\rangle \otimes q &= (\langle\langle x_{r+1}\rangle\rangle \otimes \langle\langle x_1, \ldots, x_r\rangle\rangle) \otimes q \\ &= \langle\langle x_{r+1}\rangle\rangle \otimes (\langle\langle x_1, \ldots, x_r\rangle\rangle \otimes q)\,. \end{aligned}$$

By the induction hypothesis, $\langle\langle x_1, \ldots, x_r\rangle\rangle \otimes q$ is anisotropic over $k(x_1, \ldots, x_r)$, so by the base case (with $k(x_1, \ldots, x_r)$ replacing k and x_{r+1} replacing x_1), $\langle\langle x_{r+1}\rangle\rangle \otimes (\langle\langle x_1, \ldots, x_r\rangle\rangle \otimes q)$ is anisotropic over $k(x_1, \ldots, x_r)(x_{r+1}) \cong k(x_1, \ldots, x_r, x_{r+1})$, completing the proof of the lemma by induction. □

Now, recall from Section 2.1 that for a quadratic form q over k that represents 1, q_{ps} denotes the quadratic form over k such that $q \simeq \langle 1\rangle \perp q_{ps}$.

Lemma 3.1.9. *Let k be any field of characteristic $\neq 2$, and let q be an anisotropic quadratic form over k that represents 1. Then the quadratic form*

$$\varphi = \langle x_2+1, -x_2-x_1\rangle \perp \langle 1, -x_2\rangle \otimes q_{ps} \perp x_1 \cdot (\langle\langle x_2\rangle\rangle \otimes q)$$

is anisotropic over $k(x_1, x_2)$.

Proof. We actually show that φ is anisotropic over the field $k(x_2)((x_1))$ which contains $k(x_1, x_2)$. By Lemma 3.1.8, the second residue form of φ is anisotropic over the residue field $k(x_2)$. So by Springer's Theorem, the lemma is proven if we show that the first residue form of φ,

$$\varphi_1 = \langle x_2+1, -x_2\rangle \perp \langle 1, -x_2\rangle \otimes q_{ps}$$

is anisotropic over $k(x_2)$. Rewrite φ_1 as

$$\varphi_1 = \langle x_2+1\rangle \perp q_{ps} \perp -x_2 \cdot (\langle 1\rangle \perp q_{ps}) = (\langle x_2+1\rangle \perp q_{ps}) \perp x_2 \cdot (-q)$$

and consider φ_1 over $k((x_2))$. The first residue form of φ_1 over k is q, and the second residue form of φ_1 is $-q$. By our choice of q, both residue forms of φ_1 are anisotropic over k. This implies that φ_1 is anisotropic over $k((x_2))$ which contains $k(x_2)$, and thus completes the proof of the lemma. □

Remark 3.1.10. If q is any diagonal quadratic form over a field k, then scaling q by its first entry results in a quadratic form over k that represents 1. Moreover, if q is anisotropic, then q remains anisotropic after scaling by its first entry, and we can therefore apply Lemma 3.1.9.

Lemma 3.1.11. *Let ℓ be a field of characteristic $\neq 2$. Assume $\ell \in \mathscr{A}_i(2)$ for some $i \geq 0$ and $u(\ell) = 2^i$. Let $L_2 = \ell(x_1, x_2)$, and let V_2 be the set of discrete valuations on L_2 that are trivial on $L_1 = \ell(x_1)$. Let q be an anisotropic 2^i-dimensional quadratic form over ℓ that represents 1, and* *let φ be the 2^{i+2}-dimensional quadratic form over L_2 defined by*

$$\varphi = \langle x_2 + 1, -x_2 - x_1 \rangle \perp \langle 1, -x_2 \rangle \otimes q_{ps} \perp x_1 \cdot (\langle\langle x_2 \rangle\rangle \otimes q) \,.$$

If ψ is any subform of φ such that $\dim \psi > 2^{i+1}$ and

$$\langle x_2 + 1, -x_2 - x_1, x_1, x_1 x_2 \rangle \subseteq \psi,$$

then ψ is isotropic over $L_{2,v}$ for all $v \in V_2$.

Proof. We prove the lemma by considering several cases for $v \in V_2$.

Case 1: $v = v_\infty$ is the degree valuation with uniformizer x_2^{-1}.

The form ψ contains the subform $\langle x_2 + 1, -x_2 - x_1 \rangle = x_2 \cdot \left\langle 1 + x_2^{-1}, -1 - x_1 x_2^{-1} \right\rangle$. Scaling by x_2^{-2}, we have

$$\langle x_2 + 1, -x_2 - x_1 \rangle \simeq x_2^{-1} \cdot \left\langle 1 + x_2^{-1}, -1 - x_1 x_2^{-1} \right\rangle,$$

whose second residue form is $\langle 1, -1 \rangle$, which is isotropic. The second residue form of ψ is therefore isotropic over the residue field L_1, hence ψ is isotropic over $L_{2,v}$ by Springer's Theorem.

Case 2: $v = v_\pi$, where $\pi = x_2$, $x_2 + 1$, or $x_2 + x_1$ is a divisor of at least one entry of ψ.

The form ψ contains the subforms $\langle -x_2 - x_1, x_1 \rangle, \langle x_1, x_1 x_2 \rangle$, and $\langle x_2 + 1, x_1, x_1 x_2 \rangle$, each of which reduces to an isotropic form over the respective residue field κ_π. So the first residue form of ψ is isotropic over the residue field, hence ψ is isotropic over $L_{2,\pi}$.

Case 3: $v = v_\pi$, where $\pi \in L_1[x_2]$ is a monic irreducible polynomial different from x_2, $x_2 + 1$, and $x_2 + x_1$.

Let $n = \dim \psi$. In this case, each entry of ψ is a unit in $\mathcal{O}_{v_\pi}$, so ψ reduces to an n-dimensional quadratic form over the residue field κ_π. Since κ_π is a finite extension of L_1, it satisfies property $\mathscr{A}_{i+1}(2)$, thus $u(\kappa_\pi) \leq 2^{i+1} < n$ (see Section 2.1). So the first residue form of ψ is isotropic over κ_π, which implies that ψ is isotropic over $L_{2,\pi}$.

This covers all cases of $v \in V_2$, so the proof is complete. $\square$

We can now prove Theorem 3.1.7.

Proof of Theorem 3.1.7. We first observe that if $r > 2$, then $L_r = \ell(x_1, \ldots, x_r)$ is isomorphic to $\ell(x_1, \ldots, x_{r-2})(x_{r-1}, x_r)$. If $\ell \in \mathscr{A}_i(2)$, then $\ell(x_1, \ldots, x_{r-2}) \in \mathscr{A}_{i+r-2}(2)$, and by Lemma 3.1.8, if γ is an anisotropic form over ℓ, then the form $\langle\langle x_1, \ldots, x_{r-2}\rangle\rangle \otimes \gamma$ is anisotropic over $\ell(x_1, \ldots, x_{r-2})$. Hence $u\left(\ell(x_1, \ldots, x_{r-2})\right) = 2^{i+r-2}$. It therefore suffices to prove the theorem for $r = 2$.

By assumption, $u(\ell) = 2^i$, so there exists a 2^i-dimensional anisotropic quadratic form q over ℓ, which we can assume represents 1. By Lemmas 3.1.9 and 3.1.11, if φ is the 2^{i+2}-dimensional form over L_2 defined by

$$\varphi = \langle x_2 + 1, -x_2 - x_1\rangle \perp \langle 1, -x_2\rangle \otimes q_{ps} \perp x_1 \cdot \left(\langle\langle x_2\rangle\rangle \otimes q\right),$$

then any subform ψ of φ such that $n = \dim \psi > 2^{i+1}$ and $\langle x_2 + 1, -x_2 - x_1, x_1, x_1x_2\rangle \subseteq \psi$, in particular φ itself, violates the local-global principle for isotropy over L_2 with respect to V_2. $\square$

Examples 3.1.12. The following are special cases of Theorem 3.1.7.

1. For any prime $p \neq 2$, the field $\mathbb{F}_p \in \mathscr{A}_1(2)$ and $u\left(\mathbb{F}_p\right) = 2$. Then for any $\alpha \in \mathbb{F}_p^\times \setminus \mathbb{F}_p^{\times 2}$, the five-dimensional quadratic form over $\mathbb{F}_p(x_1, x_2)$ defined by

$$\langle x_2 + 1, -x_2 - x_1, -\alpha, x_1, x_1x_2\rangle$$

violates the local-global principle for isotropy with respect to V_2.

2. By [Lee13, Corollary 2.7], for any prime p, the field $\mathbb{Q}_p \in \mathscr{A}_2(2)$ and $u\left(\mathbb{Q}_p\right) = 4$. Let u be a lift of a non-square in $\mathbb{F}_p^\times$ to $\mathbb{Q}_p$. Then the nine-dimensional quadratic form over $\mathbb{Q}_p(x_1, x_2)$

defined by

$$\langle x_2+1, -x_2-x_1, -p, -u, x_2u, x_1, x_1x_2, -x_1u, -x_1x_2u\rangle$$

violates the local-global principle for isotropy with respect to V_2.

Remarks 3.1.13. 1. If $i=0$ and $r=2$, the assumption that $n \neq 3$ in Theorem 3.1.7 is necessary. Indeed, by Corollary 3.1.5, three-dimensional quadratic forms over L_2 *satisfy* the local-global principle for isotropy with respect to V_2.

2. In some instances, the assumption in Theorem 3.1.7 that $r \geq 2$ is necessary. For example, if $p \neq 2$ is a prime, then $\ell = \mathbb{F}_p \in \mathscr{A}_1(2)$, and the Hasse-Minkowski Theorem says that the local-global principle for isotropy holds over $\mathbb{F}_p(x)$ with respect to all discrete valuations on $\mathbb{F}_p(x)$. Any discrete valuation on $\mathbb{F}_p$ is trivial, so the conclusion of Theorem 3.1.7 is false if $\ell = \mathbb{F}_p$ and $r=1$.

3.2. Divisorial discrete valuations

Let K/k be a finitely generated field extension of transcendence degree $r \geq 1$. A discrete valuation v on K, trivial on k, is *divisorial* if there exists some normal k-variety $\mathscr{X}$ with function field K and some prime divisor D on $\mathscr{X}$ such that v is equivalent to the discrete valuation on K induced by D. Because K has transcendence degree r over k, if v is a divisorial discrete valuation on K, then its residue field κ_v is a finitely generated field extension of transcendence degree $r-1$ over k.

Given a field k equipped with a non-empty set V of non-trivial discrete valuations, we say that V *satisfies the finite support property* if, given any $a \in k^\times$, the set

$$\{v \in V \mid v(a) \neq 0\}$$

is finite. Sets of discrete valuations that satisfy the finite support property arise naturally, and have also been considered in, e.g., [CRR19, RR22]. If $\mathscr{X}$ is a projective integral regular k-scheme with function field K, then by [Har77, Lemma II.6.1], the set $V_{\mathscr{X}}$ of discrete valuations on K induced by prime divisors on $\mathscr{X}$ satisfies the finite support property. We saw a particular example of this in Section 3.1 for $K = k(t)$: if $\mathscr{X} = \mathbb{P}^1_k$, then $V_{\mathscr{X}} = V_{K/k}$. Much like what we saw in Section 3.1, for certain ground fields k and k-varieties $\mathscr{X}$ with function field K, the local-global principle for isometry over K is satisfied with respect to $V_{\mathscr{X}}$ (Proposition 3.2.1), but numerous counterexamples exist over K to the local-global principle for isotropy with respect to $V_{\mathscr{X}}$ (Theorem 3.2.4).

3.2.1. The local-global principle for isometry

Let k be any field of characteristic $\neq 2$, and for any integer $r \geq 1$ let $K_r = k(x_1, \ldots, x_r)$ be a purely transcendental field extension of transcendence degree r over k. We saw in Section 3.1 that the local-global principle for isometry holds over K_r with respect to the set V_r of discrete valuations on K_r that are trivial on K_{r-1} (here taking $K_0 = k$). Consequently, for any set V of discrete valuations on K_r that contains V_r, the local-global principle for isometry holds with respect to V; in particular, with respect to the set of all discrete valuations on K_r.

By [KMRT98, Example VII.29.28], we know that, given an n-dimensional quadratic form q over a field k, the pointed Galois cohomology set $H^1(k, \mathrm{O}_n(q))$ is in bijection with the set of isometry classes of n-dimensional quadratic forms over k, with (the isometry class of) q being the distinguished element. By [KMRT98, Example VII.29.29], the pointed Galois cohomology set $H^1(k, \mathrm{SO}_n(q))$ is in bijection with the set of isometry classes of n-dimensional quadratic forms over k with the same discriminant as q, again with q being the distinguished element. If W is a non-empty set of non-trivial discrete valuations on k, then a quadratic form φ over k is isometric to q over k_w for all $w \in W$ if and only if (the isometry class of) φ belongs to the kernel of the global-to-local map

$$H^1(k, \mathrm{O}_n(q)) \to \prod_{w \in W} H^1(k_w, \mathrm{O}_n(q)).$$

The kernel of this global-to-local map gives a measure of the failure of the local-global principle for isometry with respect to W.

Let $K_r = k(x_1, \ldots, x_r)$ be as above, let $\mathscr{X}$ be a smooth projective integral k-variety with function field K_r, and let $V_{\mathscr{X}}$ be the set of discrete valuations on K_r induced by prime divisors on $\mathscr{X}$. For an n-dimensional quadratic form q over K_r, let

$$Ш_{\mathscr{X}}(K_r, \mathrm{O}_n(q)) = \ker\left(H^1(K_r, \mathrm{O}_n(q)) \to \prod_{v \in V_{\mathscr{X}}} H^1(K_{r,v}, \mathrm{O}_n(q))\right),$$

$$Ш_{\mathscr{X}}(K_r, \mathrm{SO}_n(q)) = \ker\left(H^1(K_r, \mathrm{SO}_n(q)) \to \prod_{v \in V_{\mathscr{X}}} H^1(K_{r,v}, \mathrm{SO}_n(q))\right),$$

$$Ш^i_{\mathscr{X}}(K_r, \mu_2) = \ker\left(H^i\left(K_r, \mu_2\right) \to \prod_{v \in V_{\mathscr{X}}} H^i(K_{r,v}, \mu_2)\right), \ i \geq 1.$$

Since $\mu_2 = \{\pm 1\}$ is contained in K_r, for any j we can identify the Galois modules μ_2 and $\mu_2^{\otimes j}$, which allows us to identify $H^i\left(K_r, \mu_2^{\otimes j}\right)$ and $H^i\left(K_r, \mu_2\right)$ for all i.

For any discrete valuation v on K_r with residue characteristic $\neq 2$, we have well-defined residue homomorphisms (see [GMS03, II, §7])

$$\partial_v^i : H^i\left(K_r, \mu_2\right) \to H^{i-1}\left(\kappa_v, \mu_2\right).$$

Let $V_{K_r/k}$ be the set of all discrete valuations on K_r that are trivial on k. Then for any $v \in V_{K_r/k}$, since char $k \neq 2$, the residue field κ_v has characteristic $\neq 2$ as well. Moreover, the set $V_{K_r/k}$ equals the set of discrete valuations on K_r with residue characteristic $\neq 2$ whose valuation ring contains k, as this last condition forces invertible elements of k to have valuation 0. For any $i \geq 1$, we consider the following unramified cohomology groups:

$$H^i_{nr}\left(K_r/k, \mu_2\right) = \bigcap_{v \in V_{K_r/k}} \ker \partial_v^i,$$

$$H^i\left(K_r, \mu_2\right)_{\mathscr{X}} = \bigcap_{v \in V_{\mathscr{X}}} \ker \partial_v^i.$$

Once again, the following result is well-known to experts, but does not seem to be written explicitly in the literature. We include a proof using unramified cohomology for the sake of completeness.

Proposition 3.2.1. *Let k be any field of characteristic $\neq 2$, and for any integer $r \geq 1$ let $K_r =$*

$k(x_1, \ldots, x_r)$. Let $\mathscr{X}$ be a smooth projective integral k-variety with function field K_r, and let $V_{\mathscr{X}}$ be the set of discrete valuations on K_r induced by prime divisors on $\mathscr{X}$. Then for any n-dimensional quadratic form q over K_r, the set $Ш_{\mathscr{X}}(K_r, \mathrm{O}_n(q))$ is trivial; i.e., the local-global principle for isometry holds over K_r with respect to $V_{\mathscr{X}}$.

Remark 3.2.2. If $V_r \subseteq V_{\mathscr{X}}$, this follows from Proposition 3.1.2, so there is nothing to prove. However, $V_{\mathscr{X}}$ does not necessarily contain V_r. For example, consider $\mathbb{P}^1_k \times \mathbb{P}^1_k$ with coordinates x_1, x_2. Let P be the point given by $x_1 = x_2 = 0$, and let L be the line $x_2 = 0$. By blowing up $\mathbb{P}^1_k \times \mathbb{P}^1_k$ at P, then blowing down the proper transform of L, we arrive at a smooth projective k-variety $\mathscr{X}$ with function field K_2 such that the x_2-adic valuation, which belongs to V_2, is not contained in $V_{\mathscr{X}}$.

Proof of Proposition 3.2.1. We have the following short exact sequence of groups:

$$1 \to \mathrm{SO}_n(q) \to \mathrm{O}_n(q) \to \mu_2 \to 1.$$

Taking the associated 6-term exact sequence in Galois cohomology, we have

$$\cdots \to \mathrm{O}_n(q)(K_r) \to \mu_2 \to H^1(K_r, \mathrm{SO}_n(q)) \xrightarrow{\alpha} H^1(K_r, \mathrm{O}_n(q)) \xrightarrow{\beta} H^1(K_r, \mu_2).$$

The map $\mathrm{O}_n(q)(K_r) \to \mu_2$ is surjective, so by exactness, α has trivial kernel.

Letting $\alpha_Ш$ and $\beta_Ш$ be the restrictions of α and β, respectively, we arrive at the complex

$$Ш_{\mathscr{X}}(K_r, \mathrm{SO}_n(q)) \xrightarrow{\alpha_Ш} Ш_{\mathscr{X}}(K_r, \mathrm{O}_n(q)) \xrightarrow{\beta_Ш} Ш^1_{\mathscr{X}}(K_r, \mu_2), \qquad (3.2.1)$$

which fits into the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & Ш_{\mathscr{X}}(K_r, \mathrm{SO}_n(q)) & \xrightarrow{\alpha_{Ш}} & Ш_{\mathscr{X}}(K_r, \mathrm{O}_n(q)) & \xrightarrow{\beta_{Ш}} & Ш^1_{\mathscr{X}}(K_r, \mu_2) \\
 & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & H^1(K_r, \mathrm{SO}_n(q)) & \xrightarrow{\alpha} & H^1(K_r, \mathrm{O}_n(q)) & \xrightarrow{\beta} & H^1(K_r, \mu_2) \\
 & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \prod_{v \in V_{\mathscr{X}}} H^1(K_{r,v}, \mathrm{SO}_n(q)) & \xrightarrow{\prod \alpha_v} & \prod_{v \in V_{\mathscr{X}}} H^1(K_{r,v}, \mathrm{O}_n(q)) & \xrightarrow{\prod \beta_v} & \prod_{v \in V_{\mathscr{X}}} H^1(K_{r,v}, \mu_2)
\end{array}$$

A diagram chase shows that the sequence (3.2.1) is exact. Therefore, to prove Proposition 3.2.1, it suffices to show that the first and third terms of (3.2.1) are trivial. For the third term of (3.2.1), we show more: $Ш^i_{\mathscr{X}}(K_r, \mu_2)$ is trivial for all $i \geq 1$.

For any $i \geq 1$, by the definition of $Ш^i_{\mathscr{X}}(K_r, \mu_2)$ and unramified cohomology, we have

$$Ш^i_{\mathscr{X}}(K_r, \mu_2) \subseteq H^i\left(K_r, \mu_2\right)_{\mathscr{X}}.$$

By [CT95, Theorems 4.1.1, 4.1.5], for any $i \geq 1$,

$$H^i\left(K_r, \mu_2\right)_{\mathscr{X}} \xrightarrow{\sim} H^i_{nr}\left(K_r/k, \mu_2\right) \xrightarrow{\sim} H^i(k, \mu_2).$$

So we may view $Ш^i_{\mathscr{X}}(K_r, \mu_2) \subseteq H^i(k, \mu_2)$. Since $\mathscr{X}$ is rational, there is a codimension one point x on $\mathscr{X}$ whose induced discrete valuation v_x has residue field isomorphic to K_{r-1}. The map $\psi_x : H^i(k, \mu_2) \to H^i(K_{r,v_x}, \mu_2)$ factors as

$$H^i(k, \mu_2) \xrightarrow{\sim} H^i_{nr}(K_{r-1}/k, \mu_2) \hookrightarrow H^i(K_{r-1}, \mu_2) \xrightarrow{\sim} H^i_{nr}(K_{r,v_x}, \mu_2) \hookrightarrow H^i(K_{r,v_x}, \mu_2).$$

Here, the first isomorphism follows from [CT95, Theorem 4.1.5]; the second and fourth maps are inclusions; and the third map is an isomorphism by the Gersten conjecture (see, e.g., [CT95, pp. 27]) and [Art62, Theorem III.4.9]. The restriction of the injection ψ_x to $Ш^i_{\mathscr{X}}(K_r, \mu_2) \subseteq \ker \psi_x$ is trivial by the definition of Ш. Hence $Ш^i_{\mathscr{X}}(K_r, \mu_2)$ is trivial.

In particular, we have shown that the third term of (3.2.1) is trivial, and therefore, to prove Proposition 3.2.1, it suffices to show that the first term of (3.2.1), $Ш_{\mathscr{X}}(K_r, \mathrm{SO}_n(q))$, is trivial. By [CRR19, Theorem 3.4], if $s = \lfloor \log_2 n \rfloor + 1$, and if $Ш^i_{\mathscr{X}}(K_r, \mu_2)$ has finite order ω_i for $i = 1, \ldots, s$, then the size of $Ш_{\mathscr{X}}(K_r, \mathrm{SO}_n(q))$ is bounded above by $\omega_1 \cdots \omega_s$. We have shown that $\omega_i = 1$ for any $i \geq 1$, and therefore $Ш_{\mathscr{X}}(K_r, \mathrm{SO}_n(q))$ is trivial. This completes the proof of Proposition 3.2.1. □

Remark 3.2.3. The statement of [CRR19, Theorem 3.4] assumed that $\dim q \geq 5$, but this assumption was not used in the proof.

3.2.2. The local-global principle for isotropy

The goal of this section is to prove the following theorem:

Theorem 3.2.4. *Let k be an algebraically closed field of characteristic $\neq 2$ that is not the algebraic closure of a finite field. Let K be any finitely generated field extension of transcendence degree $r \geq 2$ over k, and let V be any non-empty set of non-trivial divisorial discrete valuations on K that satisfies the finite support property. Then for any integer $n \neq 3$ such that*

$$2^{r-1} + 1 \leq n \leq 2^r,$$

there is an n-dimensional quadratic form over K that violates the local-global principle for isotropy with respect to V.

Example 3.2.5. Let $K = k(x, y, z)$, where k is an algebraically closed field of characteristic $\neq 2$ that is not the algebraic closure of a finite field, and let $\mathscr{X} = \mathbb{P}^3_k$. Since $V_{\mathscr{X}}$ contains the set of discrete valuations on $k(x, y, z)$ that are trivial on $k(x, y)$, by Proposition 3.1.2, the local-global principle for isometry holds over K with respect to $V_{\mathscr{X}}$. Moreover, by Corollary 3.1.5, quadratic forms over K of dimensions two and three satisfy the local-global principle for isotropy with respect to $V_{\mathscr{X}}$, as do four-dimensional quadratic forms over K with trivial determinant. In particular, we see that in some instances the assumption in Theorem 3.2.4 that $n \neq 3$ is necessary. By Theorem 3.2.4, there are counterexamples to the local-global principle for isotropy over K with respect to $V_{\mathscr{X}}$ in dimensions five through eight. Since $u(K) = 8$, quadratic forms of dimension > 8 over K are isotropic, thus automatically satisfy the local-global principle for isotropy. The case of four-dimensional quadratic forms over K with non-trivial determinant remains open.

Before proving Theorem 3.2.4, we prove several results related to the local-global principle for isotropy over finitely generated field extensions of fields $\ell \in \mathscr{A}_i(2)$ for some i (defined in Section 2.1). First we show that under certain assumptions, if quadratic forms of a particular dimension n satisfy the local-global principle for isotropy, then so do quadratic forms of dimension $\geq n$.

Proposition 3.2.6. *Let ℓ be a field of characteristic $\neq 2$, and suppose that $\ell \in \mathscr{A}_i(2)$ for some $i \geq 0$. For any integer $r \geq 1$ such that $i + r > 2$, let L be a finitely generated field extension of transcendence degree r over ℓ. Let V be a non-empty set of non-trivial divisorial discrete valuations on L, trivial on ℓ, that satisfies the finite support property. Then for any integer n such that*

$$2^{i+r-1} + 2 \leq n < 2^{i+r},$$

if regular n-dimensional quadratic forms over L satisfy the local-global principle for isotropy with respect to V, then so do regular $(n+1)$-dimensional quadratic forms over L.

Proof. This proof closely mirrors the proof of the Hasse-Minkowski Theorem for quadratic forms of dimension at least 5 found in [Lam05, pp. 172].

Let n be any integer such that $2^{i+r-1} + 2 \leq n < 2^{i+r}$, and suppose regular n-dimensional quadratic forms over L satisfy the local-global principle for isotropy with respect to V. Let q be a regular $(n+1)$-dimensional quadratic form over L, and suppose that q is isotropic over L_v for all $v \in V$. Write $q = q_1 \perp q_2$, where $q_1 = \langle a_1, a_2 \rangle$, and $q_2 = \langle a_3, \ldots, a_{n+1} \rangle$. Note that $\dim q_2 = n - 1 \geq 2^{i+r-1} + 1$.

Consider the following two disjoint subsets of V, whose union is V:

$$S = \left\{v \in V \;\middle|\; q_{2,v} \text{ is isotropic over } L_v\right\},$$
$$T = \left\{v \in V \;\middle|\; q_{2,v} \text{ is anisotropic over } L_v\right\}.$$

Claim: T is a finite set.

Indeed, let $U \subseteq V$ be the subset defined by

$$U = \left\{v \in V \;\middle|\; v(a_3) = v(a_4) = \cdots = v(a_{n+1}) = 0\right\}.$$

The set V satisfies the finite support property, so $V \setminus U$ is a finite set. For any $v \in U$, each entry of $q_{2,v}$ is a unit in $\mathcal{O}_v$, so $q_{2,v}$ reduces to an $(n-1)$-dimensional quadratic form over the residue field κ_v. Each such v is a divisorial discrete valuation on L that is trivial on ℓ, so κ_v is a finitely generated field extension of transcendence degree $r-1$ over the field $\ell \in \mathscr{A}_i(2)$. Hence $u(\kappa_v) \leq 2^{i+r-1} < n-1$ (see Section 2.1). Therefore, for any $v \in U$, $q_{2,v}$ is isotropic over L_v by Springer's Theorem, hence $U \subseteq S$. This implies that $T = V \setminus S$ is contained in the finite set $V \setminus U$, proving the claim.

For any $v \in T$, because q is isotropic over L_v, there exists some $z_v \in L_v^\times$ such that $z_v \in D_{L_v}(q_{1,v})$ and $-z_v \in D_{L_v}(q_{2,v})$ [Lam05, Corollary I.3.6]. Thus, for any $v \in T$, we can write

$$z_v = a_1 x_v^2 + a_2 y_v^2 = q_{1,v}(x_v, y_v)$$

for some x_v, $y_v \in L_v$. Since T is a finite set, by Weak Approximation we can find x, $y \in L$ sufficiently close to x_v, y_v, respectively, for all $v \in T$, so that the element

$$q_1(x, y) = a_1 x^2 + a_2 y^2 =: z \in L$$

is as close as desired to $z_v \neq 0$ for every $v \in T$. So $z \neq 0$, and x and y can be selected so that z_v/z is close enough to 1 in L_v to guarantee that z_v and z belong to the same square class of L_v for all $v \in T$.

Here $z \in D_L(q_1)$, so we may write $q_1 \simeq \langle z, w\rangle$ for some $w \in L^\times$. Let $q^* = \langle z\rangle \perp q_2$, so that $q \simeq \langle w\rangle \perp q^*$. We next observe that the n-dimensional quadratic form q^* is isotropic over L_v for all $v \in V$. Indeed, if $v \in S$, then $q_{2,v}$ is isotropic over L_v, so q^* must be isotropic over L_v as well. For $v \in T$, since z and z_v belong to the same square class of L_v and $-z_v \in D_{L_v}(q_{2,v})$, we see that $-z \in D_{L_v}(q_{2,v})$ for all $v \in T$. Therefore $q^* = \langle z\rangle \perp q_2$ is isotropic over L_v for all $v \in T$. So q^* is isotropic over L_v for all $v \in S \cup T = V$, as asserted. By assumption, this implies that q^* is isotropic over L. Thus $q \simeq \langle w\rangle \perp q^*$ is isotropic over L as well, completing the proof. □

Corollary 3.2.7. *Let ℓ be a field of characteristic $\neq 2$ such that $\ell \in \mathscr{A}_i(2)$ for some $i \geq 0$, and let L be a finitely generated field extension of transcendence degree $r \geq 1$ over ℓ such that $i + r \geq 2$. Let V be a non-empty set of non-trivial divisorial discrete valuations on L, trivial on ℓ, that satisfies the finite support property, and suppose there exists a 2^{i+r}-dimensional quadratic form over L that violates the local-global principle for isotropy with respect to V. Then for any integer n such that*

$$2^{i+r-1} + 2 \leq n \leq 2^{i+r},$$

there exists an n-dimensional quadratic form over L that violates the local-global principle for isotropy with respect to V.

Proof. If $i+r = 2$, the result is true by assumption since $2^{2-1}+2 = 2^2$. So for $i+r > 2$, suppose by contradiction that the corollary is false. Let n^* be the largest integer between $2^{i+r-1} + 2$ and 2^{i+r} such that n^*-dimensional quadratic forms over L satisfy the local-global principle for isotropy with respect to V. By assumption, $n^* < 2^{i+r}$. Applying Proposition 3.2.6, since n^*-dimensional quadratic forms over L satisfy the local-global principle for isotropy with respect to V, then so do quadratic forms over L of dimension $n^* + 1$. This contradicts the definition of n^*, and so for each n such that $2^{i+r-1} + 2 \leq n < 2^{i+r}$, there must be an n-dimensional counterexample to the local-global principle for isotropy over L with respect to V. □

It remains to investigate the local-global behavior of quadratic forms with dimension $2^s + 1$ for some $s \geq 2$. Following [AS22, Section 3], over a field k of characteristic $\neq 2$, given any elements $a_1, \ldots, a_s, d \in k^\times$, let $\langle\langle a_1, \ldots, a_s; d\rangle\rangle$ denote the 2^s-dimensional quadratic form over k obtained by multiplying the last entry, $a_1 \cdots a_s$, of the Pfister form $\langle\langle a_1, \ldots, a_s\rangle\rangle$ by d. For instance, if $s = 2$ we have

$$\langle\langle a_1, a_2; d\rangle\rangle = \langle 1, a_1, a_2, a_1a_2d\rangle.$$

Such a form $\langle\langle a_1, \ldots, a_s; d\rangle\rangle$ is a *twisted Pfister form* in the sense of Hoffmann [Hof96].

As observed by Auel and Suresh [AS22], by using a "trick" of Bogomolov, these twisted Pfister forms can be used to generate counterexamples to the local-global principle for isotropy over function fields. Namely, let k be an algebraically closed field of characteristic $\neq 2$. Let K be a finitely generated field extension of transcendence degree $r \geq 1$ over k and let W be any non-empty set of non-trivial discrete valuations on K. According to Bogomolov's trick (see [Bog95, Proof of Theorem 1.1], [AS22, Corollary 1.2]), since k is algebraically closed we can present K as an odd degree extension of $k(x_1, \ldots, x_r)$ for some transcendence basis $x_1, \ldots, x_r$ of K/k. As such, any $w \in W$ restricts to a non-trivial discrete valuation on $k(x_1, \ldots, x_r)$, so let

$$V = \left\{ w|_{k(x_1,\ldots,x_r)} \mid w \in W \right\}.$$

Suppose we have found a quadratic form q over $k(x_1, \ldots, x_r)$ that violates the local-global principle for isotropy with respect to V. Since K is an odd degree extension of $k(x_1, \ldots, x_r)$, q_K remains anisotropic over K by Springer's Theorem on odd degree extensions [Lam05, Theorem VII.2.7]. For any $v \in V$, q_v is isotropic over $k(x_1, \ldots, x_r)_v$, and since v is the restriction of some $w \in W$, $k(x_1, \ldots, x_r)_v$ is contained in K_w. Hence q_{K_w} is isotropic over K_w for all $w \in W$, and therefore q_K violates the local-global principle for isotropy over K with respect to W. In particular, these observations of Auel and Suresh [AS22, Corollary 1.2, Proposition 1.3] prove the following:

Lemma 3.2.8 (Auel-Suresh)**.** *Let K be any finitely generated field extension of transcendence degree $r \geq 1$ over an algebraically closed field k of characteristic $\neq 2$. If there is an n-dimensional quadratic form over the rational function field $k(x_1, \ldots, x_r)$ that violates the local-global principle for isotropy* *with respect to the set of all discrete valuations on $k(x_1, \dots, x_r)$, then there is an n-dimensional quadratic form over K that violates the local-global principle for isotropy with respect to the set of all discrete valuations on K.*

Now, for any algebraically closed field k of characteristic $\neq 2$ that is not the algebraic closure of a finite field, let $k_0 \subset k$ be a subfield of k equipped with a discrete valuation v_0 whose residue field has characteristic $\neq 2$. Then for any integer $r \geq 2$ if we let

$$f_r = \prod_{i=1}^{r} x_i(x_i - 1)(x_i - \lambda_i),$$

where each $\lambda_i \in k_0 \backslash \{0, 1\}$ satisfies $v_0(\lambda_i) > 0$, [AS22, Theorem 4.1] states that over $k(x_1, \dots, x_r)$, the 2^r-dimensional twisted Pfister form $\langle\langle x_1, \dots, x_r; f_r \rangle\rangle$ violates the local-global principle for isotropy with respect to all discrete valuations on $k(x_1, \dots, x_r)$. In particular, [AS22, Theorem 4.1], together with Bogomolov's trick, proves the dimension 2^r case of Theorem 3.2.4. We will use variants on the form $\langle\langle x_1, \dots, x_r; f_r \rangle\rangle$ to prove the case of dimension $2^{r-1} + 1$.

Lemma 3.2.9. *Let k be any field of characteristic $\neq 2$. Then for any integer $r \geq 1$ the twisted Pfister form $\langle\langle x_1, \dots, x_r; -1 \rangle\rangle$ is anisotropic over $k(x_1, \dots, x_r)$.*

Proof. Let $F = k(\sqrt{-1})$. Then over $F(x_1, \dots, x_r)$, we have

$$\langle\langle x_1, \dots, x_r; -1 \rangle\rangle_{F(x_1, \dots, x_r)} \simeq \langle\langle x_1, \dots, x_r \rangle\rangle.$$

By Lemma 3.1.8, $\langle\langle x_1, \dots, x_r \rangle\rangle$ is anisotropic over $F(x_1, \dots, x_r)$ which contains $k(x_1, \dots, x_r)$. So the form $\langle\langle x_1, \dots, x_r; -1 \rangle\rangle$ must be anisotropic over $k(x_1, \dots, x_r)$. □

Lemma 3.2.10. *Let k be any field of characteristic $\neq 2$, and for any integer $r \geq 2$ let $K_r = k(x_1, \dots, x_r)$. For $1 \leq i \leq r$, let $g_i(x_i) \in k[x_i]$ be polynomials of positive degree such that $g_i(0) \in k^{\times 2}$, and let $f_r = \prod_{i=1}^{r} x_i g_i(x_i)$. Suppose the 2^r-dimensional quadratic form $q_r = \langle\langle x_1, \dots, x_r; f_r \rangle\rangle$ is* *anisotropic over K_r. Then the (2^r+1)-dimensional quadratic form*

$$\widetilde{q}_r = q_r \perp \left\langle -x_{r+1}^2 - x_1 \cdots x_r \right\rangle$$

is anisotropic over $K_{r+1} = k(x_1, \ldots, x_{r+1})$.

Proof. We first observe that

$$q_r = \langle\langle x_1, \ldots, x_r; f_r \rangle\rangle \simeq \left\langle 1, x_1, \ldots, x_r, \ldots, x_2 \cdots x_r, \prod_{i=1}^{r} g_i(x_i) \right\rangle.$$

Moreover, for each $i = 1, \ldots, r$, since $g_i(0) \neq 0$, $g_i(x_i)$ is a unit in $\mathcal{O}_{v_{x_i}}$, with reduction $g_i(0) \in \kappa_{x_i}$. The form q_r is anisotropic over K_r, and we may write $q_r \simeq \langle 1 \rangle \perp q_r'$, so by [Lam05, Theorem IX.2.1],

$$x_1 \cdots x_r \in D_{K_r}\left(q_r'\right) \Longleftrightarrow x_{r+1}^2 + x_1 \cdots x_r \in D_{K_{r+1}}(q_r).$$

<u>Claim</u>: $x_1 \cdots x_r \notin D_{K_r}(q_r')$.

The claim implies that q_r does not represent $x_{r+1}^2 + x_1 \cdots x_r$ over K_{r+1}; or equivalently, the quadratic form

$$\widetilde{q}_r = q_r \perp \left\langle -x_{r+1}^2 - x_1 \cdots x_r \right\rangle$$

is anisotropic over K_{r+1}. It therefore suffices to prove the claim, which is equivalent to showing that the form $q_r' \perp \langle -x_1 \cdots x_r \rangle$ is anisotropic over K_r.

We prove the stronger claim, that $q_r' \perp \langle -x_1 \cdots x_r \rangle$ is anisotropic over the x_1-adic completion of K_r, which is $k(x_2, \ldots, x_r)((x_1))$, with residue field $k(x_2, \ldots, x_r)$. Since

$$q_r' \perp \langle -x_1 \cdots x_r \rangle \simeq \langle\langle x_2, \ldots, x_r \rangle\rangle_{ps} \perp \left\langle \prod_{i=1}^{r} g_i(x_i) \right\rangle \perp x_1 \cdot \langle\langle x_2, \ldots, x_r; -1 \rangle\rangle,$$

where $\langle\langle x_2, \ldots, x_r \rangle\rangle_{ps}$ is the pure subform of $\langle\langle x_2, \ldots, x_r \rangle\rangle$, we see that the second residue form of $q_r' \perp \langle -x_1 \cdots x_r \rangle$ is the twisted Pfister form $\langle\langle x_2, \ldots, x_r; -1 \rangle\rangle$, which is anisotropic over $k(x_2, \ldots, x_r)$

by Lemma 3.2.9. By Springer's Theorem, to prove the claim it suffices to show that the first residue form of $q_r' \perp \langle -x_1 \cdots x_r \rangle$ is anisotropic over $k(x_2, \ldots, x_r)$. The first residue form is

$$\varphi_r := \langle\langle x_2, \ldots, x_r \rangle\rangle_{ps} \perp \left\langle g_1(0) \prod_{i=2}^{r} g_i(x_i) \right\rangle \simeq \langle\langle x_2, \ldots, x_r \rangle\rangle_{ps} \perp \left\langle \prod_{i=2}^{r} g_i(x_i) \right\rangle,$$

where this last isometry follows because $g_1(0) \in k^\times$ is a square. We now prove, by induction on $r \geq 2$, that φ_r is anisotropic over $k(x_2, \ldots, x_r)$.

First, suppose $r = 2$. Then

$$\varphi_2 \simeq \langle\langle x_2 \rangle\rangle_{ps} \perp \langle g_2(x_2) \rangle = \langle g_2(x_2) \rangle \perp x_2 \cdot \langle 1 \rangle.$$

Now consider φ_2 over $k((x_2))$. The first residue form of φ_2 is $\langle g_2(0) \rangle$, and the second residue form of φ_2 is $\langle 1 \rangle$. Both residue forms are anisotropic over k, so by Springer's Theorem, φ_2 is anisotropic over $k((x_2)) \supset k(x_2)$, proving the base case.

Now suppose that for some $r \geq 2$, φ_r is anisotropic over $k(x_2, \ldots, x_r)$, and consider φ_{r+1} over $k(x_2, \ldots, x_r)((x_{r+1}))$, whose residue field is $k(x_2, \ldots, x_r)$. We have

$$\begin{aligned} \varphi_{r+1} &\simeq \langle\langle x_2, \ldots, x_r, x_{r+1} \rangle\rangle_{ps} \perp \left\langle \prod_{i=2}^{r+1} g_i(x_i) \right\rangle \\ &= \langle\langle x_2, \ldots, x_r \rangle\rangle_{ps} \perp \left\langle \prod_{i=2}^{r+1} g_i(x_i) \right\rangle \perp x_{r+1} \cdot \langle\langle x_2, \ldots, x_r \rangle\rangle. \end{aligned}$$

The second residue form of φ_{r+1} is $\langle\langle x_2, \ldots, x_r \rangle\rangle$, which is anisotropic over $k(x_2, \ldots, x_r)$ by Lemma 3.1.8. Since $g_{r+1}(0) \in k^\times$ is a square, the first residue form of φ_{r+1} is

$$\langle\langle x_2, \ldots, x_r \rangle\rangle_{ps} \perp \left\langle g_{r+1}(0) \prod_{i=2}^{r} g_i(x_i) \right\rangle \simeq \langle\langle x_2, \ldots, x_r \rangle\rangle_{ps} \perp \left\langle \prod_{i=2}^{r} g_i(x_i) \right\rangle \simeq \varphi_r.$$

By the induction hypothesis, φ_r is anisotropic over $k(x_2, \ldots, x_r)$, so the first residue form of φ_{r+1} is anisotropic. Both residue forms of φ_{r+1} are anisotropic over $k(x_2, \ldots, x_r)$, so φ_{r+1} is anisotropic over $k(x_2, \dots, x_r)((x_{r+1}))$, which contains $k(x_2, \dots, x_r, x_{r+1})$, completing the proof of the claim by induction, and the proof of the lemma as a whole. ☐

Lemma 3.2.11. *Let ℓ be a field of characteristic $\neq 2$ such that $\ell \in \mathscr{A}_i(2)$ for some $i \geq 2$. Let $a_1, \dots, a_i, d \in \ell^\times$ be elements such that $-a_1 \cdots a_i \notin \ell^{\times 2}$ and the twisted Pfister form over ℓ defined by $q_i = \langle\langle a_1, \dots, a_i; d\rangle\rangle$ is isotropic over ℓ_v for all discrete valuations v on ℓ. Then the $(2^i + 1)$-dimensional quadratic form $\widetilde{q_i}$ over $\ell(x)$ defined by*

$$\widetilde{q_i} = q_i \perp \left\langle -x^2 - a_1 \cdots a_i \right\rangle$$

is isotropic over $\ell(x)_w$ for all discrete valuations w on $\ell(x)$.

Proof. We prove the lemma by considering several cases for the discrete valuation w on $\ell(x)$.

<u>Case 1</u>: w is non-trivial on ℓ.

In this case, if $v = w|_\ell$, then ℓ_v is contained in $\ell(x)_w$, and q_i is isotropic over ℓ_v by assumption. So $\widetilde{q_i}$ is isotropic over $\ell(x)_w$.

The remaining cases cover the situation when w is trivial on ℓ.

<u>Case 2</u>: $w = w_\infty$ is the degree valuation with respect to x. Thus, $\ell(x)_w = \ell\left(\left(x^{-1}\right)\right)$.

Multiplying the last entry of $\widetilde{q_i}$ by x^{-2}, we have

$$\widetilde{q_i} \simeq q_i \perp \left\langle -1 - a_1 \cdots a_i x^{-2} \right\rangle .$$

Now $-1 - a_1 \cdots a_i x^{-2}$ is an x^{-1}-adic unit with reduction -1. Since $\langle 1\rangle$ is a subform of q_i, the first residue form of $\widetilde{q_i}$ over κ_w contains $\langle 1, -1\rangle$, which is isotropic. Therefore $\widetilde{q_i}$ is isotropic over $\ell(x)_w$ by Springer's Theorem.

<u>Case 3</u>: $w = w_\pi$ is the π-adic valuation for $\pi = x^2 + a_1 \cdots a_i$, which is irreducible since $-a_1 \cdots a_i \notin \ell^{\times 2}$.

In this case, over the residue field $\kappa_\pi \cong \ell\left(\sqrt{-a_1 \cdots a_i}\right)$ we have

$$a_1 = \frac{\left(\sqrt{-a_1 \cdots a_i}\right)^2}{-a_2 \cdots a_i}.$$

The form $\widetilde{q}_i$ contains the subform $\langle a_1, a_2 \cdots a_i \rangle$, whose residue form mod π is

$$\overline{\langle a_1, a_2 \cdots a_i \rangle} = \left\langle \frac{\left(\sqrt{-a_1 \cdots a_i}\right)^2}{-a_2 \cdots a_i}, a_2 \cdots a_i \right\rangle \simeq \langle -a_2 \cdots a_i, a_2 \cdots a_i \rangle,$$

which is isotropic over κ_π. Thus the first residue form of $\widetilde{q}_i$ is isotropic over κ_π, so $\widetilde{q}_i$ must be isotropic over $\ell(x)_\pi$.

Case 4: $w = w_\pi$, where $\pi \in \ell[x]$ is any monic irreducible polynomial different from $x^2 + a_1 \cdots a_i$.

In this case, each entry of $\widetilde{q}_i$ is a unit in $\mathcal{O}_{v_\pi}$, so $\widetilde{q}_i$ reduces to a $(2^i + 1)$-dimensional form over κ_π. The field κ_π is a finite extension of $\ell \in \mathscr{A}_i(2)$, so $\kappa_\pi \in \mathscr{A}_i(2)$. Therefore $u\left(\kappa_\pi\right) \leq 2^i$. So the first residue form of $\widetilde{q}_i$ must be isotropic over κ_π, which implies that $\widetilde{q}_i$ is isotropic over $\ell(x)_\pi$.

These cases cover all possibilities for discrete valuations on $\ell(x)$, so the proof is complete. □

We can now prove Theorem 3.2.4.

Proof of Theorem 3.2.4. By [AS22, Theorem 1], there is a 2^r-dimensional quadratic form over K that violates the local-global principle for isotropy with respect to V. This completes the proof if $r = 2$, so suppose $r \geq 3$. The field k is algebraically closed, so $k \in \mathscr{A}_0(2)$. Moreover, any discrete valuation v on K is trivial on k since any $x \in k^\times$ has m-th roots for all $m \in \mathbb{Z}$, so $v(x) \in \mathbb{Z}$ must be divisible by all $m \in \mathbb{Z}$, hence $v(x) = 0$. So by Corollary 3.2.7, for any n such that $2^{r-1} + 2 \leq n \leq 2^r$, there is an n-dimensional quadratic form over K that violates the local-global principle for isotropy with respect to V. It therefore remains to find a quadratic form of dimension $2^{r-1} + 1$ over K that violates the local-global principle for isotropy with respect to V.

The set V is contained in the set of all discrete valuations on K, so by Lemma 3.2.8, the proof will be complete if we can find a $(2^{r-1}+1)$-dimensional quadratic form over the rational function field $k(x_1, \ldots, x_r)$ that violates the local-global principle for isotropy with respect to all discrete valuations on $k(x_1, \ldots, x_r)$. Let $k_0 \subset k$ be a subfield with a discrete valuation v_0 with residue characteristic $\neq 2$, and for $1 \leq i \leq r-1$, let $\lambda_i \in k_0 \setminus \{0, 1\}$ be elements such that $v_0(\lambda_i) > 0$. If we let

$$f_{r-1} = \prod_{i=1}^{r-1} x_i(x_i - 1)(x_i - \lambda_i),$$

then by [AS22, Theorem 4.1], the quadratic form $q_{r-1} = \langle\langle x_1, \ldots, x_{r-1}; f_{r-1}\rangle\rangle$ over $k(x_1, \ldots, x_{r-1})$ violates the local-global principle for isotropy with respect to the set of all discrete valuations on $k(x_1, \ldots, x_{r-1})$. That is, q_{r-1} is anisotropic over $k(x_1, \ldots, x_{r-1})$, but is isotropic over the completion at each discrete valuation on that field. Because the field k is algebraically closed, each λ_i appearing in f_{r-1} is a square in $k^\times$, so by Lemma 3.2.10, the $(2^{r-1}+1)$-dimensional form over $k(x_1, \ldots, x_r)$ defined by

$$\widetilde{q}_{r-1} = q_{r-1} \perp \left\langle -x_r^2 - x_1 \cdots x_{r-1} \right\rangle$$

is anisotropic over $k(x_1, \ldots, x_r)$. The field $k \in \mathscr{A}_0(2)$, so $k(x_1, \ldots, x_{r-1}) \in \mathscr{A}_{r-1}(2)$ by [Lee13, Theorem 2.3]. Therefore, by Lemma 3.2.11 where $\ell = k(x_1, \ldots, x_{r-1})$, the form $\widetilde{q}_{r-1}$ is isotropic over $k(x_1, \ldots, x_r)_v$ for all discrete valuations v on $k(x_1, \ldots, x_r)$. Thus $\widetilde{q}_{r-1}$ violates the local-global principle for isotropy with respect to all discrete valuations on $k(x_1, \ldots, x_r)$, completing the proof. □

CHAPTER 4

Universal Quadratic Forms

In this chapter, we will study universal quadratic forms over a field k. Recall that a quadratic form q over k is universal if it represents all non-zero elements of k, i.e., if $D_k(q) = k^\times$. Any isotropic quadratic form over k is universal, so we will be particularly interested in studying anisotropic universal quadratic forms over k.

4.1. Preliminaries

The question of determining which quadratic forms are universal goes back hundreds of years. For example, a universality result is given by Lagrange's Four Square Theorem of 1770: all positive integers can be written as the sum of four squares. In other words, the quadratic form $\langle 1, 1, 1, 1 \rangle$ over $\mathbb{Q}$ represents all positive integers. Over the rational numbers, the question of universality is typically restricted to asking if a quadratic form represents all positive integers, and has led to several celebrated results, e.g., the "15-Theorem" of Conway-Schneeberger [Bha00, Con00], the "290-Theorem" of Bhargava-Hanke [BH05], and the "451-Theorem" of Rouse [Rou14]. These questions have also been asked over various totally real numbers fields in, e.g., [BK18, KY21]. What these fields have in common is that they all have infinite u-invariant. In this chapter, we will be interested instead in studying universal quadratic forms over fields with finite u-invariant.

To this point, we have only needed the definition of the u-invariant involving isotropy:

$$u(k) = \max_{\dim q \geq 1} \{q \text{ an anisotropic quadratic form over } k\}.$$

However, using the First Representation Theorem (Theorem 2.1.6), we can give an equivalent definition of $u(k)$ in terms of universal quadratic forms (see, e.g., [Lam05, pp. 399]).

Lemma 4.1.1. *Let k be a field of characteristic $\neq 2$. Then*

$$u(k) = \min_{n \geq 1} \{\textit{all } n\textit{-dimensional quadratic forms over } k \textit{ are universal}\},$$

where the minimum of the empty set is ∞.

Proof. To begin, assume $u(k) < \infty$. We first show that any quadratic form q over k of dimension $u(k)$ must be universal. Let $a \in k^\times$ be arbitrary. Then the form $q \perp \langle -a \rangle$ has dimension $u(k)+1 > u(k)$, and must be isotropic by the definition of $u(k)$. So, by the First Representation Theorem, $a \in D_k(q)$. Since this holds for any $a \in k^\times$, q must be universal. Therefore

$$u(k) \geq \min_{n \geq 1} \{\text{all } n\text{-dimensional quadratic forms over } k \text{ are universal}\}.$$

The opposite inequality follows immediately if $u(k) = 1$, and if $u(k) \geq 2$, to prove the opposite inequality we must find a quadratic form over k of dimension $< u(k)$ that is not universal. Let q be any anisotropic quadratic form over k of dimension $n = u(k) \geq 2$, and write $q \simeq \langle a_1, \ldots, a_n \rangle$. The subform $q' = \langle a_1, \ldots, a_{n-1} \rangle \subseteq q$ is anisotropic since it is a subform of an anisotropic quadratic form. Therefore, by the First Representation Theorem, $-a_n \notin D_k(q')$, hence q' is not universal over k. We have found a quadratic form of dimension $< u(k)$ that is not universal over k, and therefore

$$u(k) \leq \min_{n \geq 1} \{\text{all } n\text{-dimensional quadratic forms over } k \text{ are universal}\}.$$

Now assume $u(k) = \infty$. Then for any positive integer n, there is an $(n+1)$-dimensional anisotropic quadratic form q_{n+1} over k. By the First Representation Theorem, any n-dimensional subform of q_{n+1} is not universal. Therefore, there is no positive integer n such that all n-dimensional quadratic forms over k are universal, completing the proof. □

By Lemma 4.1.1, any anisotropic quadratic form over k of dimension $u(k)$ must be universal, and by the original definition of $u(k)$, any quadratic form over k of dimension $> u(k)$ is isotropic. Therefore, the following is a third equivalent definition of $u(k)$:

$$u(k) = \max_{\dim q \geq 1} \{q \text{ an anisotropic universal quadratic form over } k\}.$$

In words, $u(k)$ is the maximal dimension of an anisotropic universal quadratic form over k. This then naturally leads to the question of determining the minimal dimension of an anisotropic universal quadratic form over k, which is precisely the definition of the m-invariant of k [GVG92].

Definition 4.1.2 (Gesquière-Van Geel)**.** Let k be a field. The *m-invariant of k* is defined by

$$m(k) = \min_{\dim q \geq 1} \{q \text{ an anisotropic universal quadratic form over } k\}.$$

If there are no anisotropic universal quadratic forms over k, then $m(k) = \infty$.

Examples 4.1.3. (a) $m(\mathbb{R}) = \infty$,

(b) $m(\mathbb{C}) = 1$,

(c) $m(\mathbb{F}_p) = 2$.

We now record some preliminary results about the m-invariant. Immediately from its definition, we see that $1 \leq m(k) \leq u(k)$ for any field k. Moreover, there are certain integers that can never be the m-invariant of a field k.

Lemma 4.1.4. *Let k be any field of characteristic $\neq 2$. Then $m(k) \neq 3, 5$.*

Proof. See [GVG92, 1.1a), pp. 194]. □

Remark 4.1.5. Lemma 4.1.4 is similar to [Lam05, Proposition XI.6.8], which states that, for any field k, $u(k) \neq 3, 5$, or 7. For *linked* fields k (see, e.g, [Lam05, pp. 370] for the definition of a linked field), Gesquière and Van Geel showed that $m(k) \neq 7$ [GVG92, 1.1b), pp. 195].

We can make the statement $m(k) \leq u(k)$ more precise. Indeed, $m(k)$ and $u(k)$ are always separated by a power of 2.

Proposition 4.1.6. *Let k be a field of characteristic $\neq 2$, and let $n \geq 0$ be the largest integer such that $2^n \leq u(k)$. Then $m(k) \leq 2^n \leq u(k)$.*

Proof. If $u(k) = 1$, then $n = 0$ and $m(k) = 1 \leq 2^0$ [GVG92, Proposition 1.3]. If $u(k) \geq 2$, this is precisely the statement of [GVG92, Corollary 1.6]. □

We immediately deduce

Corollary 4.1.7. *If k is a field of characteristic $\neq 2$ with $m(k) = u(k) < \infty$, then $m(k) = u(k) = 2^n$ for some integer $n \geq 0$.*

Proof. Let $n \geq 0$ be the largest integer such that $2^n \leq u(k)$. Then by Proposition 4.1.6 we have

$$u(k) = m(k) \leq 2^n \leq u(k).$$

This implies that $m(k) = u(k) = 2^n$. □

Remark 4.1.8. In [GVG92] it was observed that determining when $m(k) = 2$ can be related to studying the *Kaplansky radical* of k, denoted $R(k)$, defined by Kaplansky in [Kap69]. By [Lam05, Proposition XII.6.1], $R(k)$ consists of the elements $a \in k^\times$ such that $\langle 1, -a \rangle$ is universal over k. By definition, $R(k) \subseteq k^\times$, and because isotropic quadratic forms are universal, we have $k^{\times 2} \subseteq R(k)$. By [GVG92, 1.2, pp. 195], $m(k) = 2$ if and only if $k^{\times 2} \subset R(k) \subset k^\times$, where both inclusions are strict. The Kaplansky radical was also studied in [BL14], where Becher and Leep called a field k *radical-free* if $R(k) = k^{\times 2}$, which is equivalent to $m(k) > 2$.

Much like the u-invariant, determining the m-invariant of a given field is a challenging problem. For some simple fields, like $\mathbb{C}$, $\mathbb{F}_p$, and $\mathbb{R}$, we know exact values of the m-invariant, all of which agree with the respective u-invariant. However, there are fields k for which $m(k) < u(k)$. For example, in [GVG92, Example 2.10], for any positive integer $n \geq 2$, Gesquière and Van Geel constructed a field k with $m(k) = 4$ and $u(k) = 2n$, and in [Hof94, Proposition 4.3], Hoffmann constructed a field k with $m(k) = 6$, which by Proposition 4.1.6 must have $m(k) < u(k)$. We will see other examples of such fields as we continue (see Remark 4.2.14).

For fields k with $m(k) = u(k) < \infty$, the only possible dimension of an anisotropic universal quadratic form over k is this common value $m(k) = u(k)$. However, if $m(k) < u(k)$, there may be other such dimensions. With this in mind, we define the set $\mathrm{AU}(k)$ by

$$\mathrm{AU}(k) = \{\dim q \mid q \text{ an anisotropic universal quadratic form over } k\}.$$

From the above observations, for a field k with $u(k) < \infty$, we see that $\mathrm{AU}(k)$ is non-empty, with $m(k), u(k) \in \mathrm{AU}(k)$.

Throughout this chapter, we will use the local-global principle for isotropy with respect to various sets of overfields to compute both lower bounds and exact values of the m-invariant for certain types of fields, as well as to compute the set $\mathrm{AU}(k)$. As a part of this process, it is beneficial to collect several results about universal quadratic forms over complete discretely valued fields.

Lemma 4.1.9. *Let K be a complete discretely valued field with residue field k of characteristic $\neq 2$. Let q be a quadratic form over K, and for a uniformizer π of K, write $q \simeq q_1 \perp \pi \cdot q_2$, where the entries of q_1 and q_2 are units in the valuation ring of K. Then q is anisotropic and universal over K if and only if both residue forms $\overline{q}_1, \overline{q}_2$ have positive dimension and are anisotropic and universal over k.*

Proof. First assume that both residue forms $\overline{q}_1$ and $\overline{q}_2$ are anisotropic and universal over k, with $\dim \overline{q}_1, \dim \overline{q}_2 > 0$. Then by Springer's Theorem (Theorem 2.3.3), the form q is anisotropic over K. Now let $a \in K^\times$ be arbitrary. After multiplying a by an even power of π (which does not affect whether or not a is represented by q), we may assume that $a = u$ or $a = \pi u$ for some unit u in the valuation ring of K. Consider the quadratic form $q \perp \langle -a \rangle$ over K. If $a = u$, then the first residue form of $q \perp \langle -a \rangle$ is $\overline{q}_1 \perp \langle -\overline{u} \rangle$. If $a = \pi u$, then the second residue form of $q \perp \langle -a \rangle$ is $\overline{q}_2 \perp \langle -\overline{u} \rangle$. In either case, since both $\overline{q}_1$ and $\overline{q}_2$ are universal over k, one of the residue forms of $q \perp \langle -a \rangle$ is isotropic over k. Therefore $q \perp \langle -a \rangle$ is isotropic over K by Springer's Theorem, and because $a \in K^\times$ was arbitrary, we conclude that q is universal over K.

Conversely, suppose that $q = q_1 \perp \pi \cdot q_2$ is anisotropic and universal over K. By Springer's Theorem, both residue forms $\overline{q}_1, \overline{q}_2$ must be anisotropic over k, and we now show that $\dim \overline{q}_1, \dim \overline{q}_2 > 0$. By contradiction, suppose not, and without loss of generality, assume $\dim \overline{q}_1 = 0$, i.e., $q \simeq \pi \cdot q_2$. Then for any unit u in the valuation ring of K, the quadratic form

$$q \perp \langle -u \rangle \simeq \langle -u \rangle \perp \pi \cdot q_2$$

is anisotropic over K by Springer's Theorem since both $\langle -\overline{u} \rangle$ and $\overline{q}_2$ are anisotropic over k. This contradicts our assumption that q is universal over K, and therefore $\dim \overline{q}_i > 0$ for $i = 1, 2$. So all that remains to show is that both residue forms are universal over k. By contradiction, suppose at least one of the residue forms is not universal over k. Without loss of generality, we may assume $\overline{q}_1$ is not universal over k. Then there exists some $\overline{u} \in k^\times$ such that $\overline{q}_1 \perp \langle -\overline{u} \rangle$ is anisotropic over k. Letting u be a unit lift of $\overline{u}$ to K, we therefore have that

$$q \perp \langle -u \rangle \simeq (q_1 \perp \langle -u \rangle) \perp \pi \cdot q_2$$

is anisotropic over K. Therefore u is not represented by q, which contradicts our assumption that q is universal. □

As an immediate corollary of Lemma 4.1.9, for a complete discretely valued field K with residue field k, the set $\mathrm{AU}(K)$ is completely determined by $\mathrm{AU}(k)$.

Corollary 4.1.10. *Let K be a complete discretely valued field with residue field k of characteristic $\neq 2$. Then*

$$\mathrm{AU}(K) = \{r_1 + r_2 \mid r_1, r_2 \in \mathrm{AU}(k)\}.$$

Proof. Let q be any anisotropic universal quadratic form over K, and write $q \simeq q_1 \perp \pi \cdot q_2$, where π is a uniformizer, and all entries of q_1, q_2 are units in the valuation ring of K. Then by Lemma 4.1.9, since q is anisotropic and universal over K, both residue forms $\overline{q}_1, \overline{q}_2$ must be anisotropic and universal over k. Therefore $\dim \overline{q}_1, \dim \overline{q}_2 \in \mathrm{AU}(k)$. Since q was arbitrary, this shows

$$\mathrm{AU}(K) \subseteq \{r_1 + r_2 \mid r_1, r_2 \in \mathrm{AU}(k)\}.$$

To show the reverse containment, let $r_1, r_2 \in \mathrm{AU}(k)$ be given, and let $\overline{\varphi}_1, \overline{\varphi}_2$ be anisotropic universal quadratic forms over k of dimensions r_1, r_2, respectively. Then for lifts φ_1, φ_2 of $\overline{\varphi}_1, \overline{\varphi}_2$ to K, the $(r_1 + r_2)$-dimensional quadratic form φ defined over K by

$$\varphi = \varphi_1 \perp \pi \cdot \varphi_2$$

is anisotropic and universal over K by Lemma 4.1.9. This completes the proof. □

Corollary 4.1.10 then implies the following result, which was stated without proof in [GVG92, Lemma 2.1].

Corollary 4.1.11. *Let K be a complete discretely valued field with residue field k of characteristic $\neq 2$. Then*

$$m(K) = 2m(k).$$

Proof. This follows immediately from Corollary 4.1.10 since the m-invariant of a field F is the smallest element of $\mathrm{AU}(F)$. □

Lemma 4.1.12. *Let k be any field of characteristic $\neq 2$, and let v be any non-trivial discrete valuation on k. If a quadratic form q over k is universal over k, then q is universal over k_v.*

Proof. To show that q is universal over k_v, it suffices to show that $q \perp \langle -b_v \rangle$ is isotropic over k_v for all $b_v \in k_v^\times$. Because k is dense in k_v, we can find an element $b \in k^\times$ close enough to b_v in k_v to ensure that b and b_v belong to the same square class of k_v. That is, in k_v, b/b_v is a square.

Because q is universal over k, the form $q \perp \langle -b \rangle$ is isotropic over k. Then over k_v, we have $q \perp \langle -b_v \rangle \simeq q \perp \langle -b \rangle$, and therefore $q \perp \langle -b_v \rangle$ is isotropic over k_v, proving the lemma. □

To conclude this section, we prove a lemma that will be used several times in Section 4.2.

Lemma 4.1.13. *Let k be a field of characteristic $\neq 2$, and let $\{k_i\}_{i\in I}$ be a set of overfields of k. Suppose there exists an integer $n \geq 2$ such that all $(n+1)$-dimensional quadratic forms over k that are isotropic over k_i for all $i \in I$ are isotropic over k. Let q be an n-dimensional quadratic form over k that is either isotropic or universal over k_i for all $i \in I$. Then q is universal over k. In particular, if q is anisotropic over k, then $m(k) \leq n$.*

Proof. The second statement follows immediately from the first by the definition of $m(k)$, so it suffices to prove the first statement. Let $a \in k^\times$ be arbitrary, and consider the $(n+1)$-dimensional quadratic form $q \perp \langle -a \rangle$ over k. Because q is either isotropic or universal over k_i for all $i \in I$, then $q \perp \langle -a \rangle$ is isotropic over k_i for all $i \in I$. This, by assumption, implies that $q \perp \langle -a \rangle$ is isotropic over k. Since $a \in k^\times$ was arbitrary, q must be universal over k. □

4.2. Universal quadratic forms over semi-global fields

A *semi-global field* is a one-variable function field F over a complete discretely valued field K. Such fields have been extensively studied in, e.g., [HH10, HHK13, HHK15a]. For such fields F, the local-global principle for isotropy with respect to particular sets of overfields has been used to calculate $u(F)$ (see, e.g., [CPS12, HHK09, HHK15b]). In this section, we will use the local-global principle for isotropy to study anisotropic universal quadratic forms over semi-global fields.

We begin by giving a lower bound on the m-invariant of any finitely generated field extension of transcendence degree one.

Lemma 4.2.1. *Let k be any field of characteristic $\neq 2$, and let L be any finitely generated field extension of transcendence degree one over k. Then $m(L) \geq 2$.*

Proof. By [GVG92, Proposition 1.3], $m(L) = 1$ if and only if $u(L) = 1$, which holds if and only if L is quadratically closed. Therefore, to prove the lemma, it suffices to show that L is not quadratically closed.

First consider the field $k(t)$. This field is not quadratically closed since t is not a square in $k(t)$. Furthermore, the field $k(t)(\sqrt{-1}) \cong k(\sqrt{-1})(t)$ is not quadratically closed either, since t is once again not a square. Now, since L is a finitely generated field extension of transcendence degree one over k, we can write L as a finite extension of $k(t)$. If L were quadratically closed, then $k(t)(\sqrt{-1}) \subseteq L$. By [Lam05, Corollary VIII.5.11], L being quadratically closed implies that $k(t)(\sqrt{-1})$ is quadratically closed, which we just showed is not the case. Therefore L is not quadratically closed, completing the proof. □

As an immediate consequence of Lemma 4.2.1, we have

Corollary 4.2.2. *Let F be a semi-global field over a complete discretely valued field K with residue characteristic $\neq 2$. Then $m(F) \geq 2$.*

We now recall notation and terminology from the patching framework developed by Harbater and Hartmann (see [HH10, Section 6]).

Notation 4.2.3. Let T be a complete discrete valuation ring with uniformizer t, residue field k, and fraction field K. Let F be a one-variable function field over K, and let $\mathscr{X}$ be a *normal model* of F over T, i.e., a normal connected projective T-curve with function field F. Such a normal model always exists (e.g., write F as a finite extension of $K(x)$, and then take the normalization of $\mathbb{P}^1_T$ in F). Let X denote the closed fiber of $\mathscr{X}$. For each point $P \in X$ (not necessarily closed), let R_P denote the local ring of $\mathscr{X}$ at P, let $\widehat{R}_P$ denote the completion of R_P with respect to its maximal ideal, and let F_P be the fraction field of $\widehat{R}_P$. For each subset U of X that is contained in an irreducible component of X and does not meet other components, let R_U be the subring of F consisting of rational functions that are regular on U, let $\widehat{R}_U$ denote the t-adic completion of R_U, and let F_U be the fraction field of $\widehat{R}_U$. A *branch* of X at a closed point P is a height one prime $\wp$ of $\widehat{R}_P$ that contains t. We let $R_\wp$ denote the localization of $\widehat{R}_P$ at $\wp$, $\widehat{R}_\wp$ the completion of $R_\wp$ with respect to its maximal ideal, and $F_\wp$ the fraction field of $\widehat{R}_\wp$.

Example 4.2.4. Let $T = k[[t]]$ for a field k, and let $F = k((t))(x)$. Then $\mathscr{X} = \mathbb{P}^1_T$ is a normal (in fact, regular) model of F over T, with closed fiber $X = \mathbb{P}^1_k$. Let P be the closed point of X given by $x = t = 0$. Then $\widehat{R}_P = k[[x,t]]$ and $F_P = k((x,t))$. If $U \subset X$ is the complement of the point $P \in X$, then $\widehat{R}_U = k[x^{-1}][[t]]$ and $F_U = \operatorname{frac}\left(\widehat{R}_U\right)$ (see [Har13, Section 6.2]).

In fact, given such a field F, by [Abh69, Lip75], there are *regular models* $\mathscr{X}$ over T of F. In [HHK15a, Section 6], Harbater, Hartmann, and Krashen defined a bipartite graph associated to the closed fiber X of a regular model $\mathscr{X}$ called the *reduction graph*, and showed that the isomorphism class of this graph does not depend on the choice of regular model [HHK15a, Corollary 7.8]. Harbater, Hartmann, and Krashen then showed that, for various algebraic objects over F, the validity of the local-global principle depends on whether or not the reduction graph of a regular model of F is a tree. The local-global principle that we are most interested in is the local-global principle for isotropy of quadratic forms.

Theorem 4.2.5 (Harbater-Hartmann-Krashen)**.** *Let T be a complete discrete valuation ring with residue field k of characteristic $\neq 2$ and fraction field K. Let F be a one-variable function field over K, and let $\mathscr{X}/T$ be a normal model of F with closed fiber X. Let q be a quadratic form of dimension ≥ 3 over F. Then*

$$q \text{ is isotropic over } F \text{ if and only if } q \text{ is isotropic over } F_P \text{ for all } P \in X.$$

Moreover, two-dimensional quadratic forms over F satisfy this local-global principle for isotropy if and only if the reduction graph of a regular model of F is a tree.

Proof. See [HHK15a, Theorem 9.3] for the statement about quadratic forms of dimension ≥ 3, and [HHK15a, Corollary 9.7] for the statement about two-dimensional quadratic forms. □

We will now show that we can use the reduction graph of a regular model of a semi-global field F to determine when $m(F) = 2$.

Lemma 4.2.6. *Let T be a complete discrete valuation ring of residue characteristic $\neq 2$ with fraction field K. Let F be a one-variable function field over K with a regular model whose reduction graph* *is not a tree. Then* $m(F) = 2$.

Proof. By Corollary 4.2.2, we know that $m(F) \geq 2$. Therefore, to prove the result, it suffices to show that $m(F) \leq 2$. Let $\mathscr{X}$ be the regular model of F whose reduction graph is not a tree, and let X be the closed fiber of $\mathscr{X}$. Since the reduction graph of $\mathscr{X}$ is not a tree, there exists a two-dimensional quadratic form q over F that is anisotropic over F but isotropic over F_P for all $P \in X$ [HHK15a, Corollary 9.7]. However, three-dimensional quadratic forms over F satisfy the local-global principle for isotropy with respect to these overfields F_P [HHK15a, Theorem 9.3], and therefore by Lemma 4.1.13, q is universal over F, i.e., $m(F) \leq 2$. □

The next lemma is a rephrasing of Lemma 4.1.12 over a semi-global field F in terms of points P on the closed fiber of a particular regular model of F.

Lemma 4.2.7. *Let T be a complete discrete valuation ring of residue characteristic $\neq 2$ with fraction field K, and let F be a one-variable function field over K. Let $\mathscr{X}/T$ be a regular model of F with closed fiber X such that distinct branches at any closed point of X lie on distinct irreducible components of X. Let q be a universal quadratic form over F. Then for any point $P \in X$, q is universal over F_P.*

Proof. There are two types of points $P \in X$: generic points of irreducible components of X, and closed points. We consider these types of points separately.

First, suppose that $P \in X$ is the generic point η of an irreducible component of X. The codimension one point η on $\mathscr{X}$ induces a discrete valuation $v(\eta)$ on F, and in this case the field $F_P = F_\eta$ is the completion of F with respect to $v(\eta)$. Therefore, by Lemma 4.1.12, since q is universal over F, it must also be universal over F_P.

Next, suppose that $P \in X$ is a closed point. To show that q is universal over F_P, by the First Representation Theorem, it suffices to show that, for any $b_P \in F_P^\times$, the form $q \perp \langle -b_P \rangle$ is isotropic over F_P. To that end, let $b_P \in F_P^\times$ be arbitrary. Because we assumed distinct branches at P lie on distinct irreducible components of X, the conditions of [HHK13, Corollary 3.3(c)] are automatically satisfied. Therefore, there is some $b \in F^\times$ and $c \in F_P^\times$ such that $b_P = bc^2$. The form q is universal over F, and therefore the form $q \perp \langle -b \rangle$ is isotropic over F. Now over F_P, we have

$$q \perp \langle -b_P \rangle = q \perp \langle -bc^2 \rangle \simeq q \perp \langle -b \rangle.$$

Therefore $q \perp \langle -b_P \rangle$ is isotropic, completing the proof. □

As we continue to study anisotropic universal quadratic forms q over a semi-global field F, we will want to consider particular regular models of F determined by the quadratic form q.

Definition 4.2.8. Let T be a complete discrete valuation ring of residue characteristic $\neq 2$ with fraction field K. Let F be a one-variable function field over K, and let $\mathscr{X}/T$ be a regular model of F. Let $q = \langle a_1, \dots, a_n \rangle$ be a regular quadratic form over F, and let D be the union of the supports of the divisors of the a_i considered as rational functions on $\mathscr{X}$. We call $\mathscr{X}$ a *normal crossings q-model of F* if

1. the singularities of D are normal crossings, and
2. distinct branches at any closed point of the closed fiber X of $\mathscr{X}$ lie on distinct irreducible components of X.

Remark 4.2.9. Given a quadratic form q over a semi-global field F, we can always find a normal crossings q-model of F by taking a suitable blow-up of a given regular model. Indeed, if we start with a regular model $\mathscr{X}''$ of F, then by [HHK09, Lemma 4.7], we can find a blow-up $\mathscr{X}'$ of $\mathscr{X}''$ so that condition 1 of Definition 4.2.8 is satisfied. By blowing up $\mathscr{X}'$ at points of intersection of its closed fiber (which does not affect the validity of condition 1), we arrive at a regular model $\mathscr{X}$ satisfying conditions 1 and 2 of Definition 4.2.8. We will frequently see normal crossings q-models arise in this way.

The next lemma will be useful when considering anisotropic universal quadratic forms over the fields F_P for closed points P, and is similar to [HHK15a, Lemma 9.9].

Lemma 4.2.10. *Let R be a regular complete local domain of dimension two, whose residue field k has characteristic $\neq 2$. Let E be the fraction field of R, let $\{x, y\}$ be a generating set of the maximal ideal of R, and let E_y be the completion of E with respect to the y-adic valuation. Let $q = \langle a_1, \ldots, a_n \rangle$ be a regular quadratic form over E such that $a_i = u_i x^{r_i} y^{s_i}$ for some integers $r_i, s_i \geq 0$ and some $u_i \in R^\times$. Then*

(a) $m(E_y) = 4m(k)$,

(b) if q is anisotropic and universal over E, then it is anisotropic and universal over E_y.

Proof. (a) The field E_y is a complete discretely valued field, whose residue field $\kappa(y)$ is the fraction field of the complete discrete valuation ring R/yR. The field $\kappa(y)$ is therefore also a complete discretely valued field, with residue field k. Therefore, applying Corollary 4.1.11 twice, we arrive at

$$m(E_y) = 2m(\kappa(y)) = 2\,(2m(k)) = 4m(k).$$

(b) The field E_y is the completion of the field E with respect to the discrete valuation induced on E by y, and because q is universal over E, it is universal over E_y by Lemma 4.1.12. Furthermore, q is of the form in the hypothesis of [HHK15a, Lemma 9.9], and because q is anisotropic over E, by [HHK15a, Lemma 9.9(a)], it must also be anisotropic over E_y.

□

We will use the following corollary of Lemma 4.2.10 at various points of this section.

Corollary 4.2.11. *Let T be a complete discrete valuation ring of residue characteristic $\neq 2$ with fraction field K. Let F be a one-variable function field over K and let q be a regular quadratic form over F. Let $\mathscr{X}/T$ be a normal crossings q-model of F with closed fiber X, and let $P \in X$ be any closed point. If q is anisotropic and universal over F_P, then $\dim q \geq 4m(\kappa(P))$, where $\kappa(P)$ is the* *residue field of the ring $\widehat{R}_P$.*

Proof. The field F_P is the fraction field of the complete regular local domain $\widehat{R}_P$ of dimension two, whose residue field $\kappa(P)$ has characteristic $\neq 2$. If D is the union of the supports of the divisors of the entries of q considered as rational functions on $\mathscr{X}$, then because $\mathscr{X}$ is a normal crossings q-model of F, we can find a local system of parameters $\{x, y\}$ of $\widehat{R}_P$ so that any component of D that passes through P must belong to the zero locus of xy on $\mathscr{X}$. Therefore, after scaling q by an element of the form $x^r y^s$, we may assume that q is of the form in the hypothesis of Lemma 4.2.10. Scaling q by $x^r y^s$ does not affect q being anisotropic and universal over F_P, and therefore q must be anisotropic and universal over $F_{P,y}$ by Lemma 4.2.10(b). This implies $\dim q \geq m(F_{P,y})$. Applying Lemma 4.2.10(a), we have

$$\dim q \geq m(F_{P,y}) = 4m(\kappa(P)).$$

□

We now have the terminology and results needed to show that the converse of Lemma 4.2.6 is true.

Lemma 4.2.12. *Let T be a complete discrete valuation ring of residue characteristic $\neq 2$ with fraction field K. Let F be a one-variable function field over K. If $m(F) = 2$, then there is a regular model of F whose reduction graph is not a tree.*

Proof. Because $m(F) = 2$, there is an anisotropic universal quadratic form q over F with $\dim q = 2$. Let $\mathscr{X}/T$ be a normal crossings q-model of F with closed fiber X. Then by Lemma 4.2.7, for any point $P \in X$, since q is universal over F, it must also be universal over F_P. We will show that q is isotropic over each F_P, which implies that q is a two-dimensional counterexample to the local-global principle for isotropy with respect to the points $P \in X$, which proves the claim that the reduction graph of a regular model of F is not a tree [HHK15a, Corollary 9.7].

We first consider the case that $P \in X$ is a closed point, and assume, by contradiction, that q is anisotropic over F_P. So q is anisotropic and universal over F_P. Applying Corollary 4.2.11, we have

$\dim q \geq 4m(\kappa(P))$. Since $m(\kappa(P)) \geq 1$, this implies that $\dim q \geq 4$, which is a contradiction since $\dim q = 2$. So the form q must be isotropic over F_P for all closed points P.

Now suppose $P \in X$ is the generic point η of an irreducible component of X. In this case, the field F_η is a complete discretely valued field with residue field $\kappa(\eta)$ a finitely generated transcendence degree one extension of the residue field k of K. By Lemma 4.2.1, we have $m(\kappa(\eta)) \geq 2$, and therefore, by Corollary 4.1.11, $m(F_\eta) = 2m(\kappa(\eta)) \geq 4$. Therefore, the two-dimensional form q that is universal over F_η must be isotropic over F_η.

We have shown that q is isotropic over F_P for any point $P \in X$, which completes the proof. □

Combining Corollary 4.2.2, Lemma 4.2.6, and Lemma 4.2.12 together gives us the following.

Proposition 4.2.13. *Let T be a complete discrete valuation ring of residue characteristic $\neq 2$ with fraction field K. Let F be a one-variable function field over K. Then $m(F) \geq 2$, and $m(F) = 2$ if and only if there is a regular model of F whose reduction graph is not a tree.*

Remark 4.2.14. As a consequence of Proposition 4.2.13, we can find numerous fields F such that $m(F) < u(F)$. Indeed, let k be a finite field of odd characteristic, and for any integer $n \geq 1$ let $K_n = k((t_1)) \cdots ((t_n))$ be the field of iterated Laurent series over k. Let F_n be a semi-global field over K_n with a regular model whose reduction graph is not a tree (such an F_n always exists [HKP21, Lemma 5.1]). Then by Proposition 4.2.13, $m(F_n) = 2$, but $u(F_n) = 2^{n+2}$ [HHK09, Corollary 4.14].

4.2.1. The strong m-invariant

In [HHK09], Harbater, Hartmann, and Krashen defined the strong u-invariant of a field k in order to calculate the u-invariant of certain semi-global fields (most notably to show $u(\mathbb{Q}_p(x)) = 8$). In this section we define the strong m-invariant of a field k in order to calculate the m-invariant of certain semi-global fields, and show that it relates to the strong u-invariant of k in ways that are analogous to how $m(k)$ and $u(k)$ relate to one another. We begin by recalling the definition of the strong u-invariant of k.

Definition 4.2.15 ([HHK09])**.** Let k be a field. The *strong u-invariant* of k, denoted by $u_s(k)$, is the smallest real number n such that

- $u(k') \leq n$ for all finite field extensions k'/k, and
- $u(K') \leq 2n$ for all finitely generated field extensions K'/k of transcendence degree one.

If these u-invariants are arbitrarily large we say that $u_s(k) = \infty$.

It is noted in [HHK09] that, since the u-invariant, if finite, is always an integer, the strong u-invariant always belongs to $\frac{1}{2}\mathbb{N}$. Motivated by this definition, we introduce the following definition.

Definition 4.2.16. Let k be a field. The *strong m-invariant* of k, denoted by $m_s(k)$, is the largest integer n such that

- $m(k') \geq n$ for all finite field extensions k'/k, and
- $m(k'(t)) \geq 2n$ for all finite field extensions k'/k.

If these m-invariants are arbitrarily large we say that $m_s(k) = \infty$.

Example 4.2.17. Let k be a finite field of odd characteristic. Then $m_s(k) = 2$. Indeed, for any finite extension k'/k, $m(k') = 2$ since $u(k') = 2$, and $m(k'(t)) = 4$ by, e.g., [GVG92, Example 2.8].

The main goal of this section is to prove Theorem 4.2.24: for a complete discretely valued field K with residue field k satisfying $m_s(k) = u_s(k)$, we have $m_s(K) = 2m_s(k)$. This is an analog of [HHK09, Theorem 4.10], which states that $u_s(K) = 2u_s(k)$. With this goal in mind, we make two observations about the differences between the definitions of $u_s(k)$ and $m_s(k)$.

In the definition of $m_s(k)$, we are asking for the *largest* integer n that gives *lower* bounds on the m-invariants of certain field extensions of k. This is the opposite of $u_s(k)$, which is asking for the smallest n that gives upper bounds on the u-invariants of certain field extensions of k. This difference is due to the fact that finding upper bounds (respectively lower bounds) for $m(k)$ (respectively $u(k)$) is "easy", while finding lower bounds for $m(k)$ (respectively upper bounds for $u(k)$) is challenging. The next key difference between $m_s(k)$ and $u_s(k)$ is in the second requirement of the definitions.

In the definition of $u_s(k)$, we consider all finitely generated field extensions K'/k of transcendence degree one, whereas in the definition of $m_s(k)$, we consider only those transcendence degree one extensions of k of the form $K' = k'(t)$ for some k'/k finite. This is due to the fact that the m-invariant behaves less predictably than the u-invariant over arbitrary finitely generated field extensions of transcendence degree one, as the following example illustrates.

Example 4.2.18. Let p be an odd prime number, and let $F = \mathbb{Q}_p(x)$. We will see in Corollary 4.2.26 that $m(\mathbb{Q}_p(x)) = 8$. Consider the quadratic extension of F given by $F' = \mathbb{Q}_p(x)\left(\sqrt{x(1-x)(x-p)}\right)$. In [CPS12, Appendix], Colliot-Thélène, Parimala, and Suresh showed that the function $(1-x)$ is not a square in F', but is a square in F'_v for every discrete valuation v on F'. Therefore, there is a two-dimensional counterexample to the local-global principle for isotropy over F' with respect to the set of all discrete valuations on F'. By [HHK15a, Theorem 9.11(a)], this implies that the reduction graph of a regular model of F' is not a tree, and therefore, by Lemma 4.2.6, $m(F') = 2$. If the second condition in the definition of m_s considered *all* finitely generated field extensions of transcendence degree one, then the strong m-invariant of $\mathbb{Q}_p$ would be 1, and therefore Theorem 4.2.24 would be false since $m_s(\mathbb{F}_p) = 2$.

We begin by investigating some basic properties of $m_s(k)$.

Lemma 4.2.19. *Let k be any field. Then $m_s(k) \leq u_s(k)$.*

Proof. Let k'/k be any finite extension. Then

- $m(k') \leq u(k') \leq u_s(k)$,
- $m(k'(t)) \leq u(k'(t)) \leq 2u_s(k)$.

Therefore, by definition, $m_s(k) \leq u_s(k)$. □

Corollary 4.2.20. *For a field k of characteristic $\neq 2$, if $u_s(k) = 1$, then $m_s(k) = 1$.*

Proof. For any field E of characteristic $\neq 2$, $m(E) \geq 1$, and by Lemma 4.2.1, $m(E(t)) \geq 2$, so $m_s(k) \geq 1$. By Lemma 4.2.19, we have $m_s(k) \leq u_s(k) = 1$, hence $m_s(k) = 1$, as claimed. □

The next lemma shows that the analog of Proposition 4.1.6 holds for m_s and u_s.

Lemma 4.2.21. *Let k be any field of characteristic $\neq 2$. If n is the largest integer such that $u_s(k) \geq 2^n$, then $m_s(k) \leq 2^n \leq u_s(k)$.*

Proof. Because n is the largest integer such that $2^n \leq u_s(k)$, we have $u_s(k) < 2^{n+1}$. Therefore, for any finite extension k'/k, we have $u(k') < 2^{n+1}$ and $u(k'(t)) < 2^{n+2}$. This, in turn, implies that $m(k') \leq 2^n$ and $m(k'(t)) \leq 2^{n+1}$. Indeed, by Proposition 4.1.6, for a field E, if ℓ is the largest integer such that $2^\ell \leq u(E)$, then $m(E) \leq 2^\ell$. Therefore, by definition, $m_s(k) \leq 2^n$, as desired. □

Corollary 4.2.22. *Let k be a field of characteristic $\neq 2$ such that $m_s(k) = u_s(k) < \infty$. Then $m_s(k) = u_s(k) = 2^n$ for some integer $n \geq 0$. Moreover, for all finite field extensions k'/k we have*

(a) $m(k') = u(k') = 2^n$,

(b) $m(k'(t)) = u(k'(t)) = 2^{n+1}$.

Proof. Let $n \geq 0$ be the largest integer such that $2^n \leq u_s(k)$. Then by Lemma 4.2.21, we have $m_s(k) \leq 2^n \leq u_s(k)$. So, because $m_s(k) = u_s(k)$, we have $m_s(k) = u_s(k) = 2^n$.

Now let k'/k be any finite field extension. The proof of the remaining claims of the corollary follow from these inequalities:

(a) $2^n = m_s(k) \leq m(k') \leq u(k') \leq u_s(k) = 2^n$.

(b) $2^{n+1} = 2m_s(k) \leq m(k'(t)) \leq u(k'(t)) \leq 2u_s(k) = 2^{n+1}$.

□

By studying how u_s and m_s change under finite extension, we see that if $m_s = u_s$, then these invariants are stable under finite extension.

Lemma 4.2.23. *Let k be any field of characteristic $\neq 2$, and let k'/k be any finite field extension. Then*

$$m_s(k) \leq m_s(k') \leq u_s(k') \leq u_s(k).$$

In particular, if $m_s(k) = u_s(k)$, then $m_s(k) = m_s(k') = u_s(k') = u_s(k)$.

Proof. The second claim follows immediately from the first, so it suffices to prove the first claim. By Lemma 4.2.19, we know that $m_s(k') \leq u_s(k')$. It therefore suffices to prove the first and third inequalities.

Let k''/k' be any finite field extension of k'. Then since k'' is also a finite extension of k, we have $m(k'') \geq m_s(k)$ and $m(k''(t)) \geq 2m_s(k)$, so $m_s(k') \geq m_s(k)$ by definition. Moreover, we have $u(k'') \leq u_s(k)$ by the definition of $u_s(k)$.

Now, let K'/k' be any finitely generated field extension of k' of transcendence degree one. Then K' is also a finitely generated field extension of k of transcendence degree one, so $u(K') \leq 2u_s(k)$. Therefore, by definition, $u_s(k') \leq u_s(k)$, completing the proof of the lemma. □

Let T be a complete discrete valuation ring with residue field k of characteristic $\neq 2$ and fraction field K. Using Springer's Theorem, one can show that $u(K) = 2u(k)$, and [HHK09, Theorem 4.10] states that the same conclusion holds for the strong u-invariant, i.e., $u_s(K) = 2u_s(k)$. We are now ready to prove

Theorem 4.2.24. *Let T be a complete discrete valuation ring with fraction field K and residue field k of characteristic $\neq 2$. If $m_s(k) = u_s(k)$, then*

$$m_s(K) = 2m_s(k).$$

A main ingredient in the proof of Theorem 4.2.24 is the following result.

Proposition 4.2.25. *Let K be a complete discretely valued field with residue field k of characteristic*

$\neq 2$, and suppose $m_s(k) \geq n$ for some integer $n \geq 1$. Then $m_s(K) \geq 2n$.

Proof. By definition, since $m_s(k) \geq n$, then for any finite field extension k'/k we have

$$m(k') \geq n \text{ and } m(k'(t)) \geq 2n.$$

Now let K'/K be any finite field extension of K. Because K' is a finite field extension of the complete discretely valued field K, the field K' is also a complete discretely valued field, with residue field k' a finite field extension of k. By Corollary 4.1.11, we have

$$m(K') = 2m(k') \geq 2n.$$

To complete the proof, we must show that $m(K'(x)) \geq 4n$.

Let S be the valuation ring of K', let $F = K'(x)$, and let q be any anisotropic universal quadratic form over F. We want to show that $\dim q \geq 4n$. By [BL14, Proposition 3.4], we know that $m(F) > 2$, and therefore $\dim q \geq 3$. By taking a suitable blow-up of $\mathbb{P}^1_S$, we arrive at a normal crossings q-model $\mathscr{X}$ of F with closed fiber X. By our choice of model $\mathscr{X}$, for any point $P \in X$, because q is universal over F, it must also be universal over F_P by Lemma 4.2.7. Moreover, because q is anisotropic over F with $\dim q \geq 3$, by [HHK15a, Theorem 9.3], there must be a particular point $P_* \in X$ such that q is anisotropic over F_{P_*}. Thus q is anisotropic and universal over F_{P_*}. This point P_* is either a closed point or the generic point of an irreducible component of X.

If the point P_* is a closed point, then because q is anisotropic and universal over F_{P_*}, by Corollary 4.2.11, we know $\dim q \geq 4m(\kappa(P_*))$. The field $\kappa(P_*)$ is a finite extension of k, thus $m(\kappa(P_*)) \geq n$. This implies that $\dim q \geq 4n$ in this case.

Now suppose that P_* is the generic point ξ of an irreducible component X_ξ of the closed fiber X of $\mathscr{X}$. In this case, the field $F_{P_*} = F_\xi$ is a complete discretely valued field, whose residue field $\kappa(\xi)$ is the function field of X_ξ. Because $\mathscr{X}$ is a blow-up of the regular S-curve $\mathbb{P}^1_S$, the irreducible component X_ξ, being either an exceptional divisor or birational to the closed fiber of $\mathbb{P}^1_S$, has function field $\kappa(\xi) \simeq k''(t)$ for some finite extension k'' of k'. The field k' is a finite extension of k, so k'' is also a finite extension of k. Because $m_s(k) \geq n$ by assumption, applying Corollary 4.1.11, we have

$$m(F_\xi) = 2m(\kappa(\xi)) = 2m(k''(t)) \geq 4n.$$

Since q is anisotropic and universal over F_ξ, we conclude $\dim q \geq m(F_\xi) \geq 4n$.

We have shown that any anisotropic universal quadratic form q over $K'(x)$ must have dimension at least $4n$, and we conclude that $m(K'(x)) \geq 4n$, as desired. □

Proof of Theorem 4.2.24. By Proposition 4.2.25, we have $m_s(K) \geq 2m_s(k)$. By Lemma 4.2.19, we have $m_s(K) \leq u_s(K)$. By [HHK09, Theorem 4.10], we have $u_s(K) = 2u_s(k)$. Finally, since $u_s(k) = m_s(k)$, we have $m_s(K) \leq 2m_s(k)$, which completes the proof. □

Theorem 4.2.24 then allows us to calculate exact values of the m-invariant of rational function fields in one variable over certain complete discretely valued fields, as illustrated by the following corollary.

Corollary 4.2.26. *(a) If p is an odd prime, then $m(\mathbb{Q}_p(x)) = 8$.*

(b) If k is an algebraically closed field of characteristic $\neq 2$, then $m\left(k(x)((y))(z)\right) = 8$.

(c) If p is an odd prime and $r \geq 1$ is any positive integer, then $m\left(\mathbb{F}_p((t_1))\cdots((t_r))(x)\right) = 2^{r+2}$.

Proof. (a) For a prime $p \neq 2$, $m_s(\mathbb{F}_p) = u_s(\mathbb{F}_p) = 2$, and so $m_s(\mathbb{Q}_p) = 2m_s(\mathbb{F}_p) = 4 = u_s(\mathbb{Q}_p)$ by Theorem 4.2.24. Hence $m(\mathbb{Q}_p(x)) = 8$ by Corollary 4.2.22(b).

(b) For any finite extension L of $k(x)$, we have $m(L) \geq 2$ by Lemma 4.2.1, and $u(L) \leq 2$ since L is a C_1 field. Therefore $m(L) = u(L) = 2$. Furthermore, by [BL14, Proposition 3.4], $m(L(t)) \geq 4$, so $m_s(k(x)) = 2$. For any finitely generated transcendence degree one extension E of $k(x)$, E is a C_2 field, so $u(E) \leq 4$. Therefore $u_s(k(x)) = 2$. So $m_s(k(x)) = u_s(k(x)) = 2$, which implies that $m_s(k(x)((y))) = u_s(k(x)((y))) = 4$ by Theorem 4.2.24. Finally, we conclude that $m\left(k(x)((y))(z)\right) = 8$ by Corollary 4.2.22(b).

(c) For a prime $p \neq 2$, we have $m_s(\mathbb{F}_p) = u_s(\mathbb{F}_p) = 2$. So for any integer $r \geq 1$, applying Theorem 4.2.24 inductively, we have

$$m_s\left(\mathbb{F}_p((t_1))\cdots((t_r))\right) = u_s\left(\mathbb{F}_p((t_1))\cdots((t_r))\right) = 2^{r+1}.$$

Therefore $m\left(\mathbb{F}_p((t_1))\cdots((t_r))(x)\right) = 2^{r+2}$ by Corollary 4.2.22(b).

□

We conclude this section by calculating the u-invariant of a one-variable function field F over a field k with $m_s(k) = u_s(k)$.

Proposition 4.2.27. *Let k be a field of characteristic $\neq 2$ such that $m_s(k) = u_s(k) < \infty$. Let F be a one-variable function field over k. Then $u(F) = 2u_s(k)$.*

Proof. The proof of this result closely mirrors that of [HHK09, Corollary 4.13(c)].

By Corollary 4.2.22, we have $m_s(k) = u_s(k) = 2^n$ for some $n \geq 0$. So by definition,

$$u(F) \leq 2u_s(k) = 2^{n+1}.$$

To show the reverse inequality, let $\mathcal{X}$ be a normal (equivalently, regular) k-curve with function field F, and choose a closed point ξ on $\mathcal{X}$. The local ring $\mathcal{O}_{\mathcal{X},\xi}$ of $\mathcal{X}$ at ξ is a discrete valuation ring with fraction field F and residue field $\kappa(\xi)$, which is a finite extension of k. By Corollary 4.2.22(a), we have $u(\kappa(\xi)) = 2^n$. So, applying [HHK09, Lemma 4.9] to $\mathcal{O}_{\mathcal{X},\xi}, F$, we have

$$u(F) \geq 2u(\kappa(\xi)) = 2^{n+1} = 2u_s(k),$$

giving us the desired inequality. □

4.2.2. Semi-global fields over n-local fields

We begin by recalling that a *non-Archimedean local field* is a complete discretely valued field K with finite residue field. If a non-Archimedean local field K has characteristic 0, then K is a finite extension of $\mathbb{Q}_p$ for some prime p, and if K has characteristic p, then K is the field $\mathbb{F}_q((t))$ of Laurent series over $\mathbb{F}_q$, with q a power of p [FK00, Classification Theorem]. The notion of an n-local field generalizes this idea of a local field (see, e.g., [FK00, Section 1.1]).

Definition 4.2.28. Let k be a field, and let $n \geq 1$ be a positive integer. A complete discretely valued field K is called an *n-local field over k* if K fits into a chain of fields

$$K = K_n, K_{n-1}, \ldots, K_1, K_0 = k,$$

where, for $1 \leq i \leq n$, K_i is a complete discretely valued field with residue field K_{i-1}.

We will occasionally refer to finite extensions of a field k as *0-local fields over k*.

Example 4.2.29. Let $k = \mathbb{F}_p$ be a finite field. The field $\mathbb{Q}_p$ is a 1-local field over k, and for any $n \geq 1$ the field $k((t_1))((t_2))\cdots((t_n))$ of iterated Laurent series over k is an n-local field over k.

For any $n \geq 1$ and any n-local field K over a field k, the invariants $u(K), m(K), u_s(K)$, and $m_s(K)$ are completely determined by those of k.

Proposition 4.2.30. *Let k be a field of characteristic $\neq 2$ with $u(k) < \infty$. For any integer $n \geq 1$ let K be an n-local field over k. Then*

(a) $u(K) = 2^n u(k)$,

(b) $m(K) = 2^n m(k)$,

(c) $u_s(K) = 2^n u_s(k)$,

(d) if $m_s(k) = u_s(k)$, *then* $m_s(K) = 2^n m_s(k)$.

Proof. All of these results follow by induction on n together with one other result. For (a), use [Lam05, Corollary VI.1.10] (see also [Lam05, Examples XI.6.2(7)]). For (b), use Corollary 4.1.11. For (c), use [HHK09, Theorem 4.10] (see also the paragraph before [HHK09, Corollary 4.13]). For (d), use Theorem 4.2.24. □

The main goal of this section is to study the set $\mathrm{AU}(F)$ of possible dimensions of anisotropic universal quadratic forms over a semi-global field F over an n-local field. We now introduce notation that will be used throughout this section.

Notation 4.2.31. Let F be a semi-global field, and let $\mathscr{X}$ be a regular model of F with closed fiber X. If η is the generic point of an irreducible component of X, then we let $\kappa(\eta)$ denote the residue field of the local ring $\mathcal{O}_{\mathscr{X},\eta}$ of $\mathscr{X}$ at η, and we let $\widehat{\mathcal{O}}_{\mathscr{X},\eta}$ denote the completion of $\mathcal{O}_{\mathscr{X},\eta}$ with respect to its maximal ideal.

As the next result suggests, we can use information coming from a regular model of a semi-global field F to learn about $\mathrm{AU}(F)$.

Proposition 4.2.32. *Let k be a field of characteristic $\neq 2$ with $m_s(k) = u_s(k) < \infty$. For any integer $n \geq 1$ let K be an n-local field over k with valuation ring T. Let $\mathscr{X}$ be a regular projective connected T-curve with closed fiber X. Let $X_1, \ldots, X_s$ be the irreducible components of X, and for $1 \leq i \leq s$, let η_i be the unique generic point of X_i. Let Γ be the reduction graph of $\mathscr{X}$, and let F be the function field of $\mathscr{X}$. Then*

$$\mathrm{AU}(F) \subseteq \begin{cases} \{2\} \cup \bigcup_{i=1}^{s} \{r_1 + r_2 \mid r_1, r_2 \in \mathrm{AU}(\kappa(\eta_i))\} & \textit{if } \Gamma \textit{ is not a tree,} \\ \bigcup_{i=1}^{s} \{r_1 + r_2 \mid r_1, r_2 \in \mathrm{AU}(\kappa(\eta_i))\} & \textit{if } \Gamma \textit{ is a tree.} \end{cases}$$

Proof. If Γ is not a tree, then by Lemma 4.2.6, $m(F) = 2 \in \mathrm{AU}(F)$. If Γ is a tree, then $m(F) > 2$, so $2 \notin \mathrm{AU}(F)$. It therefore suffices to show that if q is any anisotropic universal quadratic form over F with $\dim q \geq 3$, then $\dim q \in \{r_1 + r_2 \mid r_1, r_2 \in \mathrm{AU}(\kappa(\eta_i))\}$ for some $i = 1, \ldots, s$.

We first make an observation about the fields $\kappa(\eta_i)$. Let κ be the residue field of K. Because K

is an n-local field over k, the field κ is an $(n-1)$-local field over k. Since $m_s(k) = u_s(k) < \infty$, by Corollary 4.2.22, we have $m_s(k) = u_s(k) = 2^r$ for some $r \geq 0$, thus $m_s(\kappa) = u_s(\kappa) = 2^{r+n-1}$. For all $i = 1, \ldots, s$ the field $\kappa(\eta_i)$ is a finitely generated transcendence degree one extension of κ. Hence $u(\kappa(\eta_i)) = 2u_s(\kappa) = 2^{r+n}$ by Proposition 4.2.27, which implies that $2^{r+n} \in \mathrm{AU}(\kappa(\eta_i))$ for all $i = 1, \ldots, s$.

Because char $F \neq 2$, we may assume that q is a diagonal form $\langle a_1, \ldots, a_d \rangle$ with $a_j \in F^\times$. By taking a suitable blow-up $\mathscr{X}'$ of $\mathscr{X}$, we may assume that $\mathscr{X}'$ is a normal crossings q-model of F over T with closed fiber X' (see Definition 4.2.8).

Because q is universal over F and because of our choice of model $\mathscr{X}'$, by Lemma 4.2.7, for all points $P \in X'$, q is universal over F_P. Moreover, because $\dim q \geq 3$ and q is anisotropic over F, by [HHK15a, Theorem 9.3], there must be a point $P_* \in X'$ such that q is anisotropic (and universal) over F_{P_*}.

First suppose that this point P_* is a closed point. Then because q is anisotropic and universal over F_{P_*}, by Corollary 4.2.11, we have $\dim q \geq 4m(\kappa(P_*))$. The field $\kappa(P_*)$ is a finite extension of the residue field κ of K, and is therefore an $(n-1)$-local field over k. Thus

$$m(\kappa(P_*)) = u(\kappa(P_*)) = 2^{r+n-1}.$$

Therefore $\dim q \geq 2^{r+n+1}$. Since $m_s(k) = u_s(k) = 2^r$, then $m_s(K) = u_s(K) = 2^{r+n}$ by Proposition 4.2.30(d). The field F is a one-variable function field over K, so $u(F) = 2^{r+n+1}$ by Proposition 4.2.27, and since q is anisotropic over F with $\dim q \geq u(F) = 2^{r+n+1}$, we have $\dim q = 2^{r+n+1}$. Now, for all $i = 1, \ldots, s$, as we observed above, $2^{r+n} \in \mathrm{AU}(\kappa(\eta_i))$. Therefore

$$\dim q = 2^{r+n} + 2^{r+n} \in \{r_1 + r_2 \mid r_1, r_2 \in \mathrm{AU}(\kappa(\eta_i))\}$$

for all $i = 1, \ldots, s$, proving the claim if $P_* \in X'$ is a closed point.

Now suppose that the point $P_* \in X'$ is the generic point ξ of an irreducible component X'_ξ of X'.

The form q is anisotropic and universal over $F_{P_*} = F_\xi$, so $\dim q \in \mathrm{AU}(F_\xi)$. In this case, F_ξ is a complete discretely valued field with residue field $\kappa(\xi)$, so by Corollary 4.1.10,

$$\mathrm{AU}(F_\xi) = \{r_1 + r_2 \mid r_1, r_2 \in \mathrm{AU}(\kappa(\xi))\}.$$

Because $\mathscr{X}'$ is a blow-up of the regular T-curve $\mathscr{X}$, the irreducible component X'_ξ is either an exceptional divisor, in which case $X'_\xi \simeq \mathbb{P}^1_{\kappa'}$ for some finite extension κ' of κ by, e.g., [Liu02, Theorem 9.3.8], or X'_ξ is birational to an irreducible component X_i of X.

If $X'_\xi \simeq \mathbb{P}^1_{\kappa'}$, then $\kappa(\xi) \simeq \kappa'(t)$. Now, the field κ' is a finite extension of κ, and as we have seen, $m_s(\kappa) = u_s(\kappa) = 2^{r+n-1}$. So $m(\kappa'(t)) = u(\kappa'(t)) = 2^{r+n}$ by Corollary 4.2.22(b), which implies that $m(F_\xi) = u(F_\xi) = 2^{r+n+1}$. As we saw above, $2^{r+n+1} = 2u(\kappa(\eta_i))$ for all $i = 1, \ldots, s$. So in this case,

$$\mathrm{AU}(F_\xi) = \left\{2^{r+n+1}\right\} \subseteq \{r_1 + r_2 \mid r_1, r_2 \in \mathrm{AU}(\kappa(\eta_i))\}$$

for all $i = 1, \ldots, s$.

If X'_ξ is birational to an irreducible component X_i of X, then $\kappa(\xi) \simeq \kappa(\eta_i)$, which implies that $\mathrm{AU}(\kappa(\xi)) = \mathrm{AU}(\kappa(\eta_i))$. Therefore, by Corollary 4.1.10,

$$\mathrm{AU}(F_\xi) = \{r_1 + r_2 \mid r_1, r_2 \in \mathrm{AU}(\kappa(\eta_i))\}.$$

In either case of X'_ξ, we have

$$\dim q \in \mathrm{AU}(F_\xi) \subseteq \bigcup_{i=1}^{s} \{r_1 + r_2 \mid r_1, r_2 \in \mathrm{AU}(\kappa(\eta_i))\},$$

which completes the proof. □

We will now show that in certain situations the containment in Proposition 4.2.32 is an equality. Indeed, if k is a finite field of odd characteristic, then in Proposition 4.2.33 we show that we have equality in Proposition 4.2.32 for any semi-global field over a 1-local field over k, and in Proposition

4.2.36 we show that we have equality in Proposition 4.2.32 for certain semi-global fields over a 2-local field over k. Furthermore, if K is an n-local field over any field k with $m_s(k) = u_s(k)$ (not necessarily a finite field), then in Proposition 4.2.40 we show that the containment in Proposition 4.2.32 is an equality for a semi-global field F over K with a smooth model.

Proposition 4.2.33. *Let T be the valuation ring of a 1-local field K over a finite field k of odd characteristic. Let $\mathscr{X}$ be a regular projective connected T-curve with closed fiber X. Let $X_1, \ldots, X_s$ be the irreducible components of X, and for $1 \leq i \leq s$, let η_i be the unique generic point of X_i. Let Γ be the reduction graph of $\mathscr{X}$, and let F be the function field of $\mathscr{X}$. Then*

$$\mathrm{AU}(F) = \begin{cases} \{2\} \cup \bigcup_{i=1}^{s} \{r_1 + r_2 \mid r_1, r_2 \in \mathrm{AU}(\kappa(\eta_i))\} & \textit{if } \Gamma \textit{ is not a tree,} \\ \bigcup_{i=1}^{s} \{r_1 + r_2 \mid r_1, r_2 \in \mathrm{AU}(\kappa(\eta_i))\} & \textit{if } \Gamma \textit{ is a tree.} \end{cases}$$

Proof. We know that $m_s(k) = u_s(k) = 2$. So by Proposition 4.2.32, we have the containment $\subseteq$. Therefore we need only to show that we have the opposite containment $\supseteq$.

For any irreducible component X_i of X with generic point η_i, the residue field $\kappa(\eta_i)$ is the function field of a curve over k, i.e., $\kappa(\eta_i)$ is a global function field. By [GVG92, Example 2.8], $m(\kappa(\eta_i)) = 4$, and by, e.g., [Lam05, Example XI.6.2(5)], $u(\kappa(\eta_i)) = 4$. Therefore $m(\kappa(\eta_i)) = u(\kappa(\eta_i)) = 4$, which implies that $\mathrm{AU}(\kappa(\eta_i)) = \{4\}$. Therefore, for each $i = 1, \ldots, s$,

$$\{r_1 + r_2 \mid r_1, r_2 \in \mathrm{AU}(\kappa(\eta_i))\} = \{8\}.$$

So to prove the proposition, we need to show that $8 \in \mathrm{AU}(F)$, and if Γ is not a tree, then $2 \in \mathrm{AU}(F)$.

Because $m_s(k) = u_s(k) = 2$, we have $m_s(K) = u_s(K) = 4$ by Theorem 4.2.24. By Proposition 4.2.27, we conclude $u(F) = 8$ (see also [HHK09, Corollary 4.14(b)]), and therefore $8 \in \mathrm{AU}(F)$. Furthermore, if Γ is not a tree, then by Lemma 4.2.6, $m(F) = 2$, hence $2 \in \mathrm{AU}(F)$. □

Corollary 4.2.34. *Let K be a 1-local field over a finite field k of characteristic $\neq 2$, and let F be a one-variable function field over K. Then* *(a)* $\mathrm{AU}(F) = \{8\}$ *or* $\{2, 8\}$, *with* $2 \in \mathrm{AU}(F)$ *if and only if the reduction graph of a regular model of* F *is not a tree.*

(b) $m(F) = 2$ *or* 8, *with* $m(F) = 2$ *if and only if the reduction graph of a regular model of* F *is not a tree.*

Proof. Part (a) follows from Lemma 4.2.12 and Proposition 4.2.33 since the residue field $\kappa(\eta)$ of the generic point η of any irreducible component of the closed fiber of any regular model of F has $\mathrm{AU}(\kappa(\eta)) = \{4\}$. Part (b) follows immediately from part (a) since $m(F)$ is the smallest element of $\mathrm{AU}(F)$. □

We now want to determine the set $\mathrm{AU}(F)$ for certain semi-global fields F over 2-local fields over finite fields. In order to do that, we need to prove the following preliminary result.

Lemma 4.2.35. *Let* T *be the valuation ring of a 2-local field* K *over a finite field* k *of odd characteristic. Let* $\mathscr{X}$ *be a regular connected projective* T*-curve with regular closed fiber* X. *Let* η *be the unique generic point of* X, *and let* F *be the function field of* $\mathscr{X}$. *If there is a two-dimensional anisotropic universal quadratic form* q *over* $\kappa(\eta)$, *then there is a two-dimensional quadratic form* $\widetilde{q}$ *over* F *such that*

(a) the first residue form of the quadratic form $\widetilde{q}_\eta$ *over* F_η *is isometric to* q,

(b) $\widetilde{q}$ *is anisotropic over* F_η,

(c) $\widetilde{q}$ *is isotropic over* F_P *for all closed points* $P \in X$.

Proof. Because the quadratic form q is universal over $\kappa(\eta)$, it represents 1, so we may assume $q \simeq \langle 1, -d \rangle$ for some $d \in \kappa(\eta)$. Furthermore, d is not a square in $\kappa(\eta)$ since q is anisotropic.

For any closed point $P \in X$, because X is regular, the local ring $\mathcal{O}_{X,P}$ is a discrete valuation ring with associated discrete valuation v_P. The residue field $\kappa(P)$ of $\mathcal{O}_{X,P}$ is a finite extension of κ, the residue field of K. Therefore $\kappa(P)$ is a 1-local field over the finite field k, which implies that

$m(\kappa(P)) = 4$. So by Corollary 4.1.11, we have

$$m\left(\kappa(\eta)_{v_P}\right) = 2m\left(\kappa(P)\right) = 8.$$

Since q is universal over $\kappa(\eta)$ by assumption, then it must also be universal over $\kappa(\eta)_{v_P}$ by Lemma 4.1.12. But, because $\dim q = 2 < 8$, the form q must be isotropic over $\kappa(\eta)_{v_P}$. Because $q \simeq \langle 1, -d\rangle$, then q being isotropic is equivalent to d being a square, hence $d \in \kappa(\eta)_{v_P}^{\times 2}$ for all closed points $P \in X$.

Consider the quadratic extension $\kappa(\eta)\left(\sqrt{d}\right)$ of $\kappa(\eta)$ and let Y be the normalization of X in $\kappa(\eta)\left(\sqrt{d}\right)$. Because d is a square in $\kappa(\eta)_{v_P}$ for all closed points P, the connected degree two cover $Y \to X$ is étale. In fact, for all closed points P, the cover $Y \to X$ is split over P, meaning that $Y \times_X P$ consists of two copies of the point P (see [HHK15a, Section 5]).

Next, because $\mathscr{X}$ is proper over the complete local Noetherian ring T, by [Gro71, Theorem X.2.1], the connected étale cover $Y \to X$ lifts uniquely to a connected degree two étale cover $\mathscr{Y} \to \mathscr{X}$ such that $\mathscr{Y} \times_{\mathscr{X}} X = Y$. The field F has characteristic $\neq 2$, so by Kummer Theory the field extension E of F corresponding to the cover $\mathscr{Y} \to \mathscr{X}$ is given by $E = F\left(\sqrt{\widetilde{d}}\right)$ for a non-square $\widetilde{d} \in F^{\times}$. Furthermore, $\widetilde{d}$ is a unit in the complete local ring $\widehat{\mathcal{O}}_{\mathscr{X},\eta}$ of $\mathscr{X}$ at η, and its image d' in $\kappa(\eta)$ differs from d by a non-zero square in $\kappa(\eta)$, i.e., $d' = dc^2$ for some $c \in \kappa(\eta)^{\times}$. Recall that d is not a square in $\kappa(\eta)$, therefore d' is not a square in $\kappa(\eta)$ either.

Consider any closed point $P \in X$. We have

$$\mathscr{Y} \times_{\mathscr{X}} P = \mathscr{Y} \times_{\mathscr{X}} (X \times_X P) = (\mathscr{Y} \times_{\mathscr{X}} X) \times_X P = Y \times_X P.$$

Thus the cover $\mathscr{Y} \to \mathscr{X}$ is split over P as well. By Hensel's Lemma [Mum99, pp. 177], $\mathscr{Y} \to \mathscr{X}$ being split over the closed point P is equivalent to $\mathscr{Y} \to \mathscr{X}$ being split over $\widehat{R}_P$, which is equivalent to $\mathscr{Y} \to \mathscr{X}$ being split over F_P since $\mathscr{Y}$ is normal (see the paragraph before Proposition 5.1 in [HHK15a]). This implies that $\widetilde{d} \in F_P^{\times 2}$.

Finally, consider the two-dimensional quadratic form $\widetilde{q} = \left\langle 1, -\widetilde{d}\right\rangle$ over F. As we observed above,

$\widetilde{d} \in \widehat{\mathcal{O}}^{\times}_{\mathscr{X},\eta}$ and its image d' in $\kappa(\eta)$ can be written as $d' = dc^2$ for some $c \in \kappa(\eta)^{\times}$. Therefore, the first residue form of $\widetilde{q}_{\eta}$, $\langle 1, -d' \rangle = \langle 1, -dc^2 \rangle$, is isometric to $q \simeq \langle 1, -d \rangle$ over $\kappa(\eta)$, proving (a). Part (b) then follows from Springer's Theorem, since the first residue form of $\widetilde{q}_{\eta}$ is isometric to the anisotropic form q over $\kappa(\eta)$ and $\widetilde{q}_{\eta}$ has no second residue form. For all closed points $P \in X$, as we saw in the previous paragraph, $\widetilde{d} \in F_P^{\times 2}$, which is equivalent to $\widetilde{q}$ being isotropic over F_P, which proves (c). □

Proposition 4.2.36. *Let T be the valuation ring of a 2-local field K over a finite field k of odd characteristic. Let $\mathscr{X}$ be a regular connected projective T-curve with regular closed fiber X. Let η be the unique generic point of X, and let F be the function field of $\mathscr{X}$. Then*

$$\mathrm{AU}(F) = \{r_1 + r_2 \mid r_1, r_2 \in \mathrm{AU}(\kappa(\eta))\}.$$

Proof. Because the closed fiber X of $\mathscr{X}$ is regular, the reduction graph of $\mathscr{X}$ is trivial. Therefore $2 \notin \mathrm{AU}(F)$, and since $m_s(k) = u_s(k) = 2$, by Proposition 4.2.32 we have

$$\mathrm{AU}(F) \subseteq \{r_1 + r_2 \mid r_1, r_2 \in \mathrm{AU}(\kappa(\eta))\}.$$

To complete the proof, we must show the reverse containment. That is, for all $r_1, r_2 \in \mathrm{AU}(\kappa(\eta))$ we must find an anisotropic universal quadratic form over F of dimension $r_1 + r_2$.

Since $m_s(k) = u_s(k) = 2$, then $m_s(K) = u_s(K) = 8$ by Proposition 4.2.30(d). Therefore, by Proposition 4.2.27, $u(F) = 16 \in \mathrm{AU}(F)$. Furthermore, the residue field κ of K is a 1-local field over k, and the field $\kappa(\eta)$ is a one-variable function field over κ. So by Corollary 4.2.34(a), we have $\mathrm{AU}(\kappa(\eta)) = \{2, 8\}$ or $\{8\}$. If $\mathrm{AU}(\kappa(\eta)) = \{8\}$, then we are done, since $8 + 8 = 16 = u(F) \in \mathrm{AU}(F)$.

So to complete the proof we must show that if $\mathrm{AU}(\kappa(\eta)) = \{2, 8\}$ then there are anisotropic universal quadratic forms over F of dimension 4 and 10. Because $2 \in \mathrm{AU}(\kappa(\eta))$, there is a two-dimensional anisotropic universal quadratic form q over $\kappa(\eta)$. Then by Lemma 4.2.35(a), there is a two-dimensional quadratic form $\widetilde{q}$ over F that, when considered over F_{η}, has first residue form isometric over $\kappa(\eta)$ to q.

Let $\pi_\eta \in F^\times$ be a uniformizer for the discrete valuation v_η induced on F by η. Let $\widetilde{\varphi}_1$ be the four-dimensional quadratic form over F defined by

$$\widetilde{\varphi}_1 = \widetilde{q} \perp \pi_\eta \cdot \widetilde{q}.$$

Over F_η, the first and second residue forms of $\widetilde{\varphi}_1$ are both isometric to the anisotropic universal form q over $\kappa(\eta)$. Therefore, both the first and second residue forms of $\widetilde{\varphi}_1$ are anisotropic and universal over $\kappa(\eta)$, thus $\widetilde{\varphi}_1$ is anisotropic and universal over F_η by Lemma 4.1.9. For all closed points $P \in X$, since $\widetilde{q}$ is isotropic over F_P by Lemma 4.2.35(c), so is $\widetilde{\varphi}_1$. Therefore the four-dimensional quadratic form $\widetilde{\varphi}_1$ is either universal or isotropic over F_Q for all points $Q \in X$. Moreover, by [HHK15a, Theorem 9.3], quadratic forms of dimension 5 over F satisfy the local-global principle for isotropy with respect to the overfields F_Q for all $Q \in X$. So by Lemma 4.1.13, $\widetilde{\varphi}_1$ is universal over F, and since $\widetilde{\varphi}_1$ is anisotropic over F_η, it must also be anisotropic over F. The form $\widetilde{\varphi}_1$ is therefore a four-dimensional anisotropic universal quadratic form over F.

Now let $\psi = \langle a_1, \ldots, a_8 \rangle$ be any 8-dimensional anisotropic universal quadratic form over $\kappa(\eta)$. Let $\widetilde{a}_1, \ldots, \widetilde{a}_8$ be unit lifts in $\mathcal{O}^\times_{\mathscr{X},\eta} \subseteq F$ of $a_1, \ldots, a_8$, and let $\widetilde{\psi} = \langle \widetilde{a}_1, \ldots, \widetilde{a}_8 \rangle$. Let

$$\widetilde{\varphi}_2 = \widetilde{q} \perp \pi_\eta \cdot \widetilde{\psi}.$$

Then by an argument similar to that of the previous paragraph, the 10-dimensional quadratic form $\widetilde{\varphi}_2$ over F is anisotropic and universal over F. □

Remark 4.2.37. In the context of Proposition 4.2.36, we can use the geometry of $\kappa(\eta)$ to determine if $\mathrm{AU}(F) = \{4, 10, 16\}$ or $\{16\}$. Indeed, the field $\kappa(\eta)$ is itself a semi-global field over a 1-local field over a finite field. So by Corollary 4.2.34(a), $\mathrm{AU}(\kappa(\eta)) = \{2, 8\}$ or $\{8\}$, with $2 \in \mathrm{AU}(\kappa(\eta))$ if and only if the reduction graph of a regular model of $\kappa(\eta)$ is not a tree. So if the reduction graph of a regular model of $\kappa(\eta)$ is not a tree, we have $\mathrm{AU}(F) = \{4, 10, 16\}$ and $m(F) = 4$, and if the reduction graph of a regular model of $\kappa(\eta)$ is a tree, then we have $\mathrm{AU}(F) = \{16\}$ and $m(F) = 16$.

Next, we consider n-local fields K over fields k with $m_s(k) = u_s(k)$ that are not necessarily finite fields, and study $\mathrm{AU}(F)$ for a semi-global field F over K with a smooth model. The overall strategy is similar to the proof of Proposition 4.2.36, but will require slightly different machinery. We begin by proving a lemma analogous to Springer's Theorem.

Lemma 4.2.38. *Let R be a complete local ring with maximal ideal $\mathfrak{m}$, residue field κ of characteristic $\neq 2$, and fraction field K. For $n \geq 2$ let $a_1, \ldots, a_n \in R^\times$ be units in R with images $\overline{a}_1, \ldots, \overline{a}_n \in \kappa^\times$, and consider the quadratic form $q = \langle a_1, \ldots, a_n \rangle$ over K. If the quadratic form $\overline{q} = \langle \overline{a}_1, \ldots, \overline{a}_n \rangle$ is isotropic over κ, then q is isotropic over K.*

Proof. Since $\overline{q}$ is isotropic over κ, there exist elements $\overline{b}_1, \ldots, \overline{b}_n \in \kappa$, not all zero, such that

$$\overline{q}\left(\overline{b}_1, \ldots, \overline{b}_n\right) = \overline{a}_1 \overline{b}_1^2 + \cdots + \overline{a}_n \overline{b}_n^2 = 0.$$

Potentially after renumbering, we may assume that $\overline{b}_1, \ldots, \overline{b}_r \neq 0$ for some $2 \leq r \leq n$ and $\overline{b}_{r+1}, \ldots, \overline{b}_n = 0$. Then over κ we have

$$\overline{a}_1 \overline{b}_1^2 + \cdots + \overline{a}_r \overline{b}_r^2 = 0.$$

Let $b_2, \ldots, b_r \in R^\times$ be unit lifts of $\overline{b}_2, \ldots, \overline{b}_r$, and consider the polynomial $f(x) \in R[x]$ given by

$$f(x) = q(x, b_2, \ldots, b_r, 0, \ldots, 0) = a_1 x^2 + a_2 b_2^2 + \cdots + a_r b_r^2.$$

After reducing modulo $\mathfrak{m}$, we have

$$\overline{f}(\overline{b}_1) = 0 \text{ and } \frac{\partial \overline{f}}{\partial x}(\overline{b}_1) = 2\overline{a}_1 \overline{b}_1 \neq 0,$$

where the last equality holds because $\overline{a}_1, \overline{b}_1 \neq 0$ and κ has characteristic $\neq 2$. By Hensel's Lemma, there is a unit lift $b_1 \in R^\times$ of $\bar{b}_1$ such that

$$f(b_1) = q(b_1, b_2, \ldots, b_r, 0, \ldots, 0) = 0.$$

Therefore q is isotropic over K, as desired. □

We now recall some terminology regarding smooth points of curves over complete discrete valuation rings from [HH10, Section 4]. Let T be a complete discrete valuation ring and let $\mathscr{X}$ be a projective T-curve with closed fiber X. For a closed point $P \in X$ at which $\mathscr{X}$ is smooth, we call an effective divisor $\widehat{P}$ on $\mathscr{X}$ a *lift* of P to $\mathscr{X}$ if the restriction of $\widehat{P}$ to X is P. Such lifts always exist [HH10, pp. 71], and for a general effective divisor $D = \sum_{i=1}^{r} n_i P_i$ on X, if $\widehat{P}_i$ is a lift of P_i to $\mathscr{X}$, then we call $\widehat{D} = \sum_{i=1}^{r} n_i \widehat{P}_i$ a *lift* of D to $\mathscr{X}$.

The next result is similar to Lemma 4.2.35.

Lemma 4.2.39. *Let k be a field of characteristic $\neq 2$ such that $m_s(k) = u_s(k) = 2^r$ for some integer $r \geq 0$. For any integer $n \geq 1$ such that $n + r \geq 2$, let K be an n-local field over k with valuation ring T. Let $\mathscr{X}$ be a smooth connected projective T-curve with closed fiber X, let η be the unique generic point of X, and let F be the function field of $\mathscr{X}$. If there is an anisotropic universal quadratic form q over $\kappa(\eta)$ with $2 \leq \dim q < 2^{r+n}$, then there is a quadratic form $\widetilde{q}$ over F with* $\dim \widetilde{q} = \dim q$ *such that*

(a) the first residue form of the quadratic form $\widetilde{q}_\eta$ over F_η is isometric to q,

(b) $\widetilde{q}$ is anisotropic over F_η,

(c) $\widetilde{q}$ is isotropic over F_P for all closed points $P \in X$.

Proof. The field $\kappa(\eta)$ is the function field of X, which is smooth over the residue field κ of K since $\mathscr{X}$ is smooth over T. Let $P \in X$ be any closed point. Then the local ring $\mathcal{O}_{X,P}$ is a discrete valuation ring with associated discrete valuation v_P. The residue field $\kappa(P)$ of $\mathcal{O}_{X,P}$ is an $(n-1)$-local field over k, and since $m_s(k) = u_s(k) = 2^r$, we have $m(\kappa(P)) = 2^{r+n-1}$. Therefore, by Corollary 4.1.11, we have $m(\kappa(\eta)_{v_P}) = 2^{r+n}$. By assumption, the quadratic form q is universal over $\kappa(\eta)$, so q must be universal over $\kappa(\eta)_{v_P}$ for all closed points $P \in X$ by Lemma 4.1.12. Now, since

$$d := \dim q < 2^{r+n} = m\left(\kappa(\eta)_{v_P}\right),$$

we conclude that the form q must be isotropic over $\kappa(\eta)_{v_P}$ for all closed points $P \in X$.

The form q is universal over $\kappa(\eta)$, so it represents 1 and we may write $q \simeq \langle 1, a_2, \ldots, a_d \rangle$ for some $a_2, \ldots, a_d \in \kappa(\eta)^\times$. Let $\mathcal{P}$ be any non-empty finite set of closed points in X that contains all the closed points $P \in X$ such that $a_i \notin \mathcal{O}_{X,P}^\times$ for some $i = 2, \ldots, d$ (i.e., $v_P(a_i) \neq 0$). For each $P \in \mathcal{P}$, let $n_P > 0$ be a positive integer such that

$$n_P \geq \max_{i=2,\ldots,d} \{-v_P(a_i)\} \text{ and } \sum_{P \in \mathcal{P}} n_P > 2g - 2,$$

where g is the genus of the closed fiber X. Now, if we let $D = \sum_{P \in \mathcal{P}} n_P P$, then D has degree $> 2g - 2$, and by our choice of n_P, for each $i = 2, \ldots, d$ we have $a_i \in \mathscr{L}(X, D)$. Here $\mathscr{L}(X, D)$ is the set of rational functions on X whose pole divisor is at most D.

The closed fiber X is smooth over κ, and therefore for each $P \in \mathcal{P}$ we can find a lift $\widehat{P}$ of P to $\mathscr{X}$ and consider the lift $\widehat{D} = \sum_{P \in \mathcal{P}} n_P \widehat{P}$ of D to $\mathscr{X}$. The map $\mathscr{L}(\mathscr{X}, \widehat{D}) \to \mathscr{L}(X, D)$ is surjective by [HH10, Proposition 4.1(b)], so for each $i = 2, \ldots, d$, there is some $\widetilde{a}_i \in \mathscr{L}(\mathscr{X}, \widehat{D})$ such that $\widetilde{a}_i \in \mathcal{O}_{\mathscr{X},\eta}^\times$ with image a_i in $\kappa(\eta)$. Furthermore, for any $\widetilde{a}_i$ and any closed point $P \notin \mathcal{P}$, we have $\widetilde{a}_i \in \widehat{R}_P^\times$. Indeed, since $\widetilde{a}_i \in \mathscr{L}(\mathscr{X}, \widehat{D})$, then $\widetilde{a}_i$ can have poles only at $\widehat{P}$ for $P \in \mathcal{P}$, so $\widetilde{a}_i \in \widehat{R}_P$ for all closed points $P \notin \mathcal{P}$. If $P \notin \mathcal{P}$ and $\widetilde{a}_i \notin \widehat{R}_P^\times$, then a component of the zeros of $\widetilde{a}_i$ must intersect X at P. Then looking at the image a_i of $\widetilde{a}_i$ in $\kappa(\eta)$, this would imply that a_i has a zero but no poles at P, hence $v_P(a_i) > 0$. This, however, is a contradiction of $P \notin \mathcal{P}$.

Over F, let $\widetilde{q} = \langle 1, \widetilde{a}_2, \ldots, \widetilde{a}_d \rangle$. Then because the image of $\widetilde{a}_i$ in $\kappa(\eta)$ is a_i, the first residue form of $\widetilde{q}_\eta$ over F_η is q, proving (a). Moreover, because q is anisotropic over $\kappa(\eta)$, the form $\widetilde{q}$ must be anisotropic over F_η by Springer's Theorem since the first residue form of $\widetilde{q}_\eta$ is q and $\widetilde{q}_\eta$ has no second residue form. This proves (b). Therefore, to complete the proof, we must show that $\widetilde{q}$ is isotropic over F_P for all closed points $P \in X$.

First, suppose the closed point $P \in X$ does not belong to $\mathcal{P}$. As we saw above, the quadratic form $q = \langle 1, a_2, \ldots, a_d \rangle$ is isotropic over $\kappa(\eta)_{v_P}$, and $a_2, \ldots, a_d \in \mathcal{O}_{X,P}^\times$ since $P \notin \mathcal{P}$. So by Springer's Theorem, the first residue form of q over $\kappa(\eta)_{v_P}$, which is $\overline{q} = \langle 1, \overline{a}_2, \ldots, \overline{a}_d \rangle$, must be isotropic over $\kappa(P)$, where $\overline{a}_i$ is the image of a_i in $\kappa(P)$. Now considering the form $\widetilde{q}$ over F_P, since $P \notin \mathcal{P}$, we have $\widetilde{a}_2, \ldots, \widetilde{a}_d \in \widehat{R}_P^\times$. The ring $\widehat{R}_P$ is a complete local ring with residue field $\kappa(P)$, and for each $i = 2, \ldots, d$, the image of $\widetilde{a}_i$ in $\kappa(P)$ is $\overline{a}_i$. So by Lemma 4.2.38, since $\overline{q}$ is isotropic over $\kappa(P)$, the form $\widetilde{q}$ must be isotropic over F_P, the fraction field of $\widehat{R}_P$.

Now suppose $P \in \mathcal{P}$. The ring $\widehat{R}_P$ is a regular local ring, thus it is a unique factorization domain [AB59, Theorem 5]. So $\widehat{P}$ is given by the locus of an irreducible element $r_P \in \widehat{R}_P \cap \widehat{\mathcal{O}}_{\mathscr{X},\eta}^\times$. Because $\widehat{P}$ is the only component of $\widehat{D}$ that meets X at P, then after multiplying and dividing the entries of $\widetilde{q}$ by even powers of r_P, over F_P we can write

$$\widetilde{q} \simeq \widetilde{q}_P = \widetilde{q}_{1,P} \perp r_P \cdot \widetilde{q}_{2,P},$$

where the entries of $\widetilde{q}_{1,P}$ and $\widetilde{q}_{2,P}$ all belong to $\widehat{R}_P^\times$. Reducing modulo η, over $\kappa(\eta)_{v_P}$ we have

$$q \simeq q_P = q_{1,P} \perp \pi_P \cdot q_{2,P},$$

where π_P is a uniformizer for $\kappa(\eta)_{v_P}$ and the entries of $q_{1,P}$ and $q_{2,P}$ all belong to $\widehat{\mathcal{O}}_{X,P}^\times$. The quadratic form q is isotropic over $\kappa(\eta)_{v_P}$, and therefore so is q_P. So by Springer's Theorem, at least one of the residue forms $\overline{q}_{1,P}$ or $\overline{q}_{2,P}$ of q_P must be isotropic over $\kappa(P)$. This then implies, by Lemma 4.2.38, that one of the quadratic forms $\widetilde{q}_{1,P}$ or $\widetilde{q}_{2,P}$ must be isotropic over F_P, and therefore $\widetilde{q} \simeq \widetilde{q}_P$ is isotropic over F_P as well.

This covers all cases for the closed point $P \in X$, and therefore proves (c). □

Proposition 4.2.40. *Let k be a field of characteristic $\neq 2$ such that $m_s(k) = u_s(k) = 2^r$ for some* *integer $r \geq 0$. For any integer $n \geq 1$ such that $n + r \geq 2$, let K be an n-local field over k with valuation ring T. Let $\mathscr{X}$ be a smooth connected projective T-curve with closed fiber X, let η be the unique generic point of X, and let F be the function field of $\mathscr{X}$. Then*

$$\mathrm{AU}(F) = \{r_1 + r_2 \mid r_1, r_2 \in \mathrm{AU}(\kappa(\eta))\}.$$

Proof. The closed fiber X of $\mathscr{X}$ is smooth, so the reduction graph of $\mathscr{X}$ is trivial, hence $2 \notin \mathrm{AU}(F)$. Moreover, since $m_s(k) = u_s(k) < \infty$, then by Proposition 4.2.32,

$$\mathrm{AU}(F) \subseteq \{r_1 + r_2 \mid r_1, r_2 \in \mathrm{AU}(\kappa(\eta))\}.$$

To complete the proof, we must show the reverse containment holds by finding an anisotropic universal quadratic form over F of dimension $r_1 + r_2$ for all $r_1, r_2 \in \mathrm{AU}(\kappa(\eta))$.

The field $\kappa(\eta)$ is a one-variable function field over the residue field κ of K. The field κ is an $(n-1)$-local field over k, and since $m_s(k) = u_s(k) = 2^r$, we have $m_s(\kappa) = u_s(\kappa) = 2^{r+n-1}$. So by Proposition 4.2.27, we have $u(\kappa(\eta)) = 2^{r+n} \in \mathrm{AU}(\kappa(\eta))$. Furthermore, since $m_s(k) = u_s(k) = 2^r$, then $m_s(K) = u_s(K) = 2^{r+n}$ by Proposition 4.2.30(d). This then implies, by Proposition 4.2.27, that $u(F) = 2^{r+n+1} \in \mathrm{AU}(F)$, which shows that $r_1 + r_2 \in \mathrm{AU}(F)$ if $r_1 = r_2 = 2^{r+n} \in \mathrm{AU}(\kappa(\eta))$.

So to complete the proof, we must consider the case when at least one of $r_1, r_2 \in \mathrm{AU}(\kappa(\eta))$ is less than 2^{r+n}. We note here that by Lemma 4.2.1, we have $m(\kappa(\eta)) \geq 2$, so $r_1, r_2 \geq 2$. Let $r_1, r_2 \in \mathrm{AU}(\kappa(\eta))$ be given, and without loss of generality, assume $2 \leq r_1 < 2^{r+n}$. Let q_1 be an r_1-dimensional anisotropic universal quadratic form over $\kappa(\eta)$. Then by Lemma 4.2.39(a), there is an r_1-dimensional quadratic form $\widetilde{q}_1$ over F such that the first residue form of $\widetilde{q}_{1,\eta}$ over F_η is q_1. Furthermore, for all closed points $P \in X$, the form $\widetilde{q}_1$ is isotropic over F_P by Lemma 4.2.39(c).

Now, let $q_2 = \langle a_1, \ldots, a_{r_2} \rangle$ be any r_2-dimensional anisotropic universal quadratic form over $\kappa(\eta)$, and for unit lifts $\widetilde{a}_1, \ldots, \widetilde{a}_{r_2} \in \mathcal{O}_{\mathscr{X},\eta}^\times$ of $a_1, \ldots, a_{r_2}$ to F, let $\widetilde{q}_2 = \langle \widetilde{a}_1, \ldots, \widetilde{a}_{r_2} \rangle$. Let $\pi_\eta \in F^\times$ be a uniformizer for the discrete valuation v_η on F, and consider the $(r_1 + r_2)$-dimensional quadratic form $\widetilde{\varphi}$ over F defined by

$$\widetilde{\varphi} = \widetilde{q}_1 \perp \pi_\eta \cdot \widetilde{q}_2.$$

Over F_η, the first and second residue forms of $\widetilde{\varphi}_\eta$ are the anisotropic universal forms q_1 and q_2 over $\kappa(\eta)$. Thus $\widetilde{\varphi}_\eta$ is anisotropic and universal over F_η by Lemma 4.1.9. In particular, $\widetilde{\varphi}$ is anisotropic over F. For all closed points $P \in X$, since $\widetilde{q}_1$ is isotropic over F_P by Lemma 4.2.39(c), then so is $\widetilde{\varphi}$. So the anisotropic form $\widetilde{\varphi}$ over F with $\dim \widetilde{\varphi} = r_1 + r_2 \geq 4$ is either universal or isotropic over F_Q for all points $Q \in X$. Therefore $\widetilde{\varphi}$ is universal over F by Lemma 4.1.13, hence $\widetilde{\varphi}$ is an anisotropic universal quadratic form over F with $\dim \widetilde{\varphi} = r_1 + r_2$. This construction can be done for any $r_1, r_2 \in \mathrm{AU}(\kappa(\eta))$ such that $r_1 < 2^{r+n}$, so the proof is complete. □

Propositions 4.2.33, 4.2.36, and 4.2.40 provide evidence that the containment in Proposition 4.2.32 is always an equality.

Conjecture 4.2.41. *With the notation of Proposition 4.2.32,*

$$\mathrm{AU}(F) = \begin{cases} \{2\} \cup \bigcup_{i=1}^{s} \{r_1 + r_2 \mid r_1, r_2 \in \mathrm{AU}(\kappa(\eta_i))\} & \text{if } \Gamma \text{ is not a tree,} \\ \bigcup_{i=1}^{s} \{r_1 + r_2 \mid r_1, r_2 \in \mathrm{AU}(\kappa(\eta_i))\} & \text{if } \Gamma \text{ is a tree.} \end{cases}$$

Example 4.2.42. Let F be a one-variable function field over $K = \mathbb{Q}_p((t))$ for an odd prime p. Assuming Conjecture 4.2.41, we have

$$\mathrm{AU}(F) = \{2, 4, 10, 16\},\ \{2, 16\},\ \{4, 10, 16\},\ \text{or } \{16\}.$$

Furthermore, assuming Conjecture 4.2.41 and using the above information, if E is a one-variable function field over $\mathbb{Q}_p((t))((s))$, then there are ten possibilities for $\mathrm{AU}(E)$.

We finish this section by showing how Conjecture 4.2.41 can be used to find the possible m-invariants of a semi-global field F over an n-local field K over a finite field.

Theorem 4.2.43. *Assume Conjecture 4.2.41. For any integer $n \geq 1$ let K be an n-local field over* *a finite field k of odd characteristic. Let F be a one-variable function field over K. Then*

$$m(F) \in \left\{2^j \mid j = 1, \ldots, n, n+2\right\}.$$

Proof. We prove the claim by induction on $n \geq 1$.

For the base case, let K be a 1-local field over k. Then by Corollary 4.2.34(b), $m(F) = 2$ or 8, which proves the base case.

Now assume for some $n \geq 1$ that if E is any one-variable function field over an n-local field over k, then

$$m(E) \in \left\{2^j \mid j = 1, \ldots, n, n+2\right\}.$$

Let K be an $(n+1)$-local field over k with residue field κ, and let F be a one-variable function field over K. Let T be the valuation ring of K and let $\mathscr{X}/T$ be a regular model of F with closed fiber X. Let $X_1, \ldots, X_s$ be the irreducible components of X, with η_i the generic point of X_i for $i = 1, \ldots, s$. Then by Proposition 4.2.13, $m(F) = 2$ if and only if the reduction graph Γ of $\mathscr{X}$ is not a tree, so to complete the proof, we need to show that if Γ is a tree, then

$$m(F) \in \left\{2^j \mid j = 2, \ldots, n+1, n+3\right\}.$$

So suppose Γ is a tree and let q be any anisotropic universal quadratic form over F. Then since $m_s(k) = u_s(k) = 2$, by Proposition 4.2.32,

$$\dim q \in \bigcup_{i=1}^{s} \{r_1 + r_2 \mid r_1, r_2 \in \mathrm{AU}(\kappa(\eta_i))\}.$$

For each $i = 1, \ldots, s$ and $r_1, r_2 \in \mathrm{AU}(\kappa(\eta_i))$, we have $r_1 + r_2 \geq 2m(\kappa(\eta_i))$ since $r_1, r_2 \geq m(\kappa(\eta_i))$. So if we let $m^* = \min_{1 \leq i \leq s}\{m(\kappa(\eta_i))\}$, we have

$$\dim q \geq 2m^*.$$

Since q was any anisotropic universal quadratic form over F, we conclude $m(F) \geq 2m^*$.

Let i^* be such that $m(\kappa(\eta_{i^*})) = m^*$. By Conjecture 4.2.41, there exists a $2m^*$-dimensional anisotropic universal quadratic form over F. Therefore $m(F) \leq 2m^*$. In the above paragraph, we proved the reverse inequality, so

$$m(F) = 2m^* = 2m(\kappa(\eta_{i^*})).$$

The field $\kappa(\eta_{i^*})$ is a one-variable function field over the residue field κ of K, and κ is an n-local field over k. Therefore, by the induction hypothesis,

$$m^* \in \left\{2^j \mid j = 1, \ldots, n, n+2\right\}.$$

Thus,

$$m(F) = 2m^* \in \left\{2^j \mid j = 2, \ldots, n+1, n+3\right\},$$

completing the proof by induction. □

4.3. Universal quadratic forms over $\mathbb{F}_q((x,y))$

In Section 4.2 we used the local-global principle for isotropy with respect to points on the closed fiber of a regular model of a semi-global field F to determine the possible dimensions of anisotropic universal quadratic forms over F. In this section, we will again use the local-global principle for isotropy, but now with respect to a specified set of discrete valuations, to determine the m-invariant of $\mathbb{F}_q((x,y))$, where $\mathbb{F}_q$ is a finite field of odd characteristic. The local-global principle that we will be using was proven in [Hu12]. We now recall the set of discrete valuations considered in [Hu12].

Let $R = \mathbb{F}_q[[x,y]]$. For any integral regular scheme $\mathcal{M}$ equipped with a proper birational morphism $\mathcal{M} \to \operatorname{Spec} R$, let $\Omega_{\mathcal{M}}$ denote the set of discrete valuations on $\mathbb{F}_q((x,y))$ induced by codimension one points of $\mathcal{M}$. Let Ω_R be the union of all such $\Omega_{\mathcal{M}}$. For any discrete valuation $w \in \Omega_R$, the residue field κ_w is either a henselian discrete valuation ring whose residue field is a finite extension of $\mathbb{F}_q$ or the function field of a curve over $\mathbb{F}_q$ (see, e.g., [Hu13, Corollary 3.16]). In either case,

$m(\kappa_w) = u(\kappa_w) = 4$ (the former case using that Springer's Theorem remains true over henselian discretely valued fields). In particular, by Corollary 4.1.11, $m(\mathbb{F}_q((x,y))_w) = 8$.

Theorem 4.3.1. *Let $\mathbb{F}_q$ be a finite field of odd characteristic. Then*

$$m(\mathbb{F}_q((x,y))) = u(\mathbb{F}_q((x,y))) = 8.$$

Proof. By [HHK15b, Theorem 4.11(b)], $u(\mathbb{F}_q((x,y))) = 4u(\mathbb{F}_q) = 8$. To complete the proof, it suffices to show that $m(\mathbb{F}_q((x,y))) \geq 8$. By [BL14, Theorem 3.7], $m(\mathbb{F}_q((x,y))) > 2$. Therefore any anisotropic universal quadratic form over $\mathbb{F}_q((x,y))$ must have dimension ≥ 3.

Let q be any anisotropic universal quadratic form over $\mathbb{F}_q((x,y))$. Then $\dim q \geq 3$ by the end of the previous paragraph. By [Hu12, Theorems 1.1, 1.2], because q is anisotropic over $\mathbb{F}_q((x,y))$ and $\dim q \geq 3$, there is some $w^* \in \Omega_R$ such that q is anisotropic over $\mathbb{F}_q((x,y))_{w^*}$. Moreover, because q is universal over $\mathbb{F}_q((x,y))$, by Lemma 4.1.12, q must also be universal over $\mathbb{F}_q((x,y))_{w^*}$. Therefore $\dim q \geq m(\mathbb{F}_q((x,y))_{w^*}) = 8$. Since q was any anisotropic universal quadratic form over $\mathbb{F}_q((x,y))$, we conclude that $m(\mathbb{F}_q((x,y))) \geq 8$, which completes the proof. □

4.4. Application of $m(k)$: Round quadratic forms

For a quadratic form q over a field k of characteristic $\neq 2$, let

$$G_k(q) = \left\{a \in k^\times \mid q \simeq a \cdot q\right\}$$

be the *group of similarity factors of* q. We say that q is *round* if $G_k(q) = D_k(q)$ [Lam05, Definition X.1.13], and to show that q is round, it suffices to show that $D_k(q) \subseteq G_k(q)$ [OS18, pp. 392]. For example, any Pfister form over k is a round form [Lam05, pp. 322]. Round forms have been considered in, e.g., [Alp91, OS18, WS77]. For instance, O'Shea completely classified odd-dimensional round quadratic forms over k up to isometry [OS18, Corollary 2.13]. In this section, we will use the invariants $m(k)$ and $u(k)$ to determine the possible dimensions of anisotropic round quadratic forms over k (Proposition 4.4.4 and Corollary 4.4.5).

We begin by recalling the following fact about isotropic round quadratic forms, which was stated without proof in [OS18, pp. 395]. We include a proof for the reader's convenience.

Lemma 4.4.1. *An isotropic quadratic form q over a field k of characteristic $\neq 2$ is round if and only if q is hyperbolic or the anisotropic part q_{an} of q satisfies $D_k(q_{an}) = k^\times = G_k(q_{an})$, i.e., q_{an} is round and universal.*

Proof. For the reverse implication, if q is hyperbolic then it is universal, and for any $a \in k^\times$ and any positive integer n, we have $a \cdot n\mathbb{H} \simeq n(a \cdot \mathbb{H}) \simeq n\mathbb{H}$, so q is round. Now suppose that q is isotropic but not hyperbolic and write

$$q \simeq i_W(q)\mathbb{H} \perp q_{an}$$

with $\dim q_{an} > 0$ and q_{an} round and universal. Because q is isotropic, it is universal, so in order to show that q is round, we must show that $a \cdot q \simeq q$ for all $a \in k^\times$, i.e., $G_k(q) = k^\times$. Since q_{an} is universal and round, we have $a \in D_k(q_{an}) = G_k(q_{an})$, so $a \cdot q_{an} \simeq q_{an}$. We therefore have

$$a \cdot q \simeq a \cdot (i_W(q)\mathbb{H} \perp q_{an}) \simeq a \cdot i_W(q)\mathbb{H} \perp a \cdot q_{an} \simeq i_W(q)\mathbb{H} \perp q_{an} \simeq q.$$

This shows that q is, indeed, round over k.

Conversely, suppose that q is isotropic and round over k. If q is hyperbolic, then there is nothing to show, so suppose that

$$q \simeq i_W(q)\mathbb{H} \perp q_{an}$$

with $\dim q_{an} > 0$. Since q is isotropic, hence universal, and round, we have $a \cdot q \simeq q$ for all $a \in k^\times$. Therefore

$$i_W(q)\mathbb{H} \perp q_{an} \simeq q \simeq a \cdot q \simeq a \cdot i_W(q)\mathbb{H} \perp a \cdot q_{an} \simeq i_W(q)\mathbb{H} \perp a \cdot q_{an}.$$

By Witt Cancellation [Lam05, Theorem I.4.2], we conclude $q_{an} \simeq a \cdot q_{an}$, thus $G_k(q_{an}) = k^\times$. This implies that $D_k(q_{an}) \subseteq G_k(q_{an})$, and therefore q_{an} is round and universal over k. □

Corollary 4.4.2. *Let k be a field of characteristic $\neq 2$ with $m(k) < \infty$. Then round quadratic* *forms over k of dimension $\leq m(k)$ are either anisotropic or hyperbolic.*

Proof. To prove the claim, we must show that any isotropic round quadratic form q over k with $\dim q \leq m(k)$ is hyperbolic, in particular $\dim q$ is even. So, we first show that there are no isotropic odd-dimensional round quadratic forms over k with dimension $\leq m(k)$.

By contradiction, suppose q is an odd-dimensional isotropic round quadratic form over k with $\dim q \leq m(k)$. The existence of an odd-dimensional isotropic round quadratic form over k implies that k is quadratically closed [OS18, Corollary 2.14]. Therefore $m(k) = 1$, hence $\dim q = 1$. This implies that q is anisotropic, which is a contradiction. So no such q can exist.

So any isotropic round quadratic form over k of dimension $\leq m(k)$ must have even dimension, and it remains to show that such a form must be hyperbolic. By contradiction, suppose there exists an even-dimensional round quadratic form q over k with $\dim q \leq m(k)$ that is isotropic but not hyperbolic and write $q \simeq i_W(q)\mathbb{H} \perp q_{an}$ with $0 < \dim q_{an} < m(k)$. By Lemma 4.4.1, q_{an} must be universal over k. However, because $\dim q_{an} < m(k)$, if q_{an} is universal over k, then it must be isotropic, which is a contradiction. Therefore, if q is isotropic, it must be hyperbolic. □

Lemma 4.4.3. *Let k be a field of characteristic $\neq 2$ with $u(k) < \infty$. If a quadratic form q over k is round with $\dim q > \frac{u(k)}{2}$, then q is universal.*

Proof. We need to show that $D_k(q) = k^\times$. Let $a \in k^\times$ be arbitrary. Because q is round over k, by [EKM08, Proposition 9.8(2)], $a \in D_k(q)$ if and only if $q \otimes \langle\langle -a \rangle\rangle$ is isotropic. Since $\dim q > \frac{u(k)}{2}$, we have $\dim(q \otimes \langle\langle -a \rangle\rangle) > u(k)$, and therefore $q \otimes \langle\langle -a \rangle\rangle$ is automatically isotropic, completing the proof. □

We will now use $u(k)$ and $m(k)$ to study the possible dimensions of anisotropic round quadratic forms over k.

Proposition 4.4.4. *Let k be a field of characteristic $\neq 2$ with $u(k) < \infty$. If a quadratic form q*

over k of even dimension is round and there exists an integer $n \geq 0$ such that

$$\frac{u(k)}{2^{n+1}} < \dim q < \frac{m(k)}{2^n},$$

then q is hyperbolic.

Proof. We prove the proposition by induction on $n \geq 0$. First, consider the base case $n = 0$. Let q be an even-dimensional round quadratic form over k with

$$\frac{u(k)}{2} < \dim q < m(k).$$

By Lemma 4.4.3, since $\dim q > \frac{u(k)}{2}$, q must be universal over k. However, because $\dim q < m(k)$, then q being universal over k implies that q is isotropic over k. Applying Corollary 4.4.2, we conclude that q is hyperbolic.

Now assume for some $n \geq 0$ that even-dimensional round quadratic forms φ over k with

$$\frac{u(k)}{2^{n+1}} < \dim \varphi < \frac{m(k)}{2^n}$$

are hyperbolic, and let q be a round quadratic form over k of even dimension with

$$\frac{u(k)}{2^{n+2}} < \dim q < \frac{m(k)}{2^{n+1}}.$$

To prove that q is hyperbolic, it suffices to show that q is universal over k. Indeed, if q is universal, then because $\dim q < m(k)$, it must be isotropic over k, thus hyperbolic by Corollary 4.4.2.

To that end, let $a \in k^\times$ be arbitrary. By [EKM08, Proposition 9.8(2)], $a \in D_k(q)$ if and only $q \otimes \langle\langle -a \rangle\rangle$ is hyperbolic. Because q is round, so is $q \otimes \langle\langle -a \rangle\rangle$ [EKM08, Proposition 9.8(1)]. So, $q \otimes \langle\langle -a \rangle\rangle$ is round with

$$\frac{u(k)}{2^{n+1}} < \dim(q \otimes \langle\langle -a \rangle\rangle) < \frac{m(k)}{2^n}.$$

By the induction hypothesis, $q \otimes \langle\langle -a \rangle\rangle$ is hyperbolic, and therefore $a \in D_k(q)$. Since $a \in k^\times$ was arbitrary, this shows that q is universal over k, which completes the proof of the proposition by induction. □

Corollary 4.4.5. *Let k be a field of characteristic $\neq 2$ with $m(k) = u(k) = 2^n$ for some integer $n \geq 1$. Then*

(a) if q is round over k with $\dim q \geq 2$, then q is even-dimensional,

(b) any anisotropic round quadratic form over k has dimension 2^i for some $i \leq n$,

(c) for each $0 \leq i \leq n$, there exists a 2^i-dimensional round anisotropic quadratic form over k.

Proof. (a) By contradiction, suppose there exists an odd-dimensional quadratic form q over k of dimension ≥ 3. We first observe that $\dim q < 2^n = u(k)$. If $\dim q > 2^n$, then q is isotropic over k, and by [OS18, Corollary 2.14], the existence of an isotropic round odd-dimensional quadratic form over k implies that k is quadratically closed, i.e., $u(k) = 1$. However, $u(k) = 2^n > 1$, so this is a contradiction. This completes the proof if $n = 1$. So we can assume $n \geq 2$ and $3 \leq \dim q < 2^n$. Because q has odd dimension $< 2^n$, we can find an integer r such that $1 < r \leq n$ and

$$2^{r-1} = \frac{u(k)}{2^{n-r+1}} < \dim q < \frac{m(k)}{2^{n-r}} = 2^r.$$

We claim that q is universal. Indeed, if $r = n$, then q is round with $\dim q > \frac{u(k)}{2}$, so q is universal by Lemma 4.4.3. If $r < n$, then let $a \in k^\times$ be arbitrary. The even-dimensional form $q \otimes \langle\langle -a \rangle\rangle$ is round [EKM08, Proposition 9.8(1)], and

$$\frac{u(k)}{2^{n-r}} < \dim(q \otimes \langle\langle -a \rangle\rangle) < \frac{m(k)}{2^{n-r-1}} < m(k).$$

By Proposition 4.4.4, $q \otimes \langle\langle -a \rangle\rangle$ is hyperbolic, which implies $a \in D_k(q)$ [EKM08, Proposition 9.8(2)]. Since a was arbitrary, this implies that q is universal. So, the odd-dimensional round form q is universal with $\dim q < m(k)$, hence q is isotropic. However, as we saw above, this implies that $u(k) = 1$, which is a contraction. So there are no odd-dimensional round quadratic forms over k with $\dim q \geq 3$.

(b) By part (a), if q is odd-dimensional and round over k, then $\dim q = 1 = 2^0$, with $0 < n$. Now consider anisotropic round quadratic forms over k of even dimension. By contradiction, suppose there exists an even-dimensional anisotropic round quadratic form q over k with $\dim q \neq 2^i$ for any $i \leq n$.

Because q is anisotropic over k, we must have $\dim q < u(k) = 2^n$. Because $\dim q \neq 2^i$ for any $i \leq n$, then $n > 2$ and there must be some integer j with $2 < j \leq n$ such that

$$\frac{u(k)}{2^{n-j+1}} = 2^{j-1} < \dim q < 2^j = \frac{m(k)}{2^{n-j}}.$$

By Proposition 4.4.4, the round quadratic form q must be hyperbolic, and therefore isotropic. This is a contradiction, and therefore no such round even-dimensional quadratic form q exists over k.

(c) The $1 = 2^0$-dimensional quadratic form $\langle 1 \rangle$ over k is anisotropic and round. The round quadratic form $\langle 1 \rangle$ is not universal over k since $1 < m(k) = 2^n$. Thus, there is some $a_1 \in k^\times$ such that $a_1 \notin D_k(\langle 1 \rangle)$. Therefore, applying [EKM08, Proposition 9.8], the two-dimensional quadratic form $\langle 1 \rangle \otimes \langle\langle -a_1 \rangle\rangle = \langle\langle -a_1 \rangle\rangle$ is anisotropic and round over k. If $n = 1$, then the proof is complete. If $n > 1$, then $\dim\langle\langle -a_1 \rangle\rangle = 2 < m(k)$ and $\langle\langle -a_1 \rangle\rangle$ is anisotropic over k, so $\langle\langle -a_1 \rangle\rangle$ is not universal over k. As such, there exists some $a_2 \in k^\times$ such that the 4-dimensional quadratic form $\langle\langle -a_1 \rangle\rangle \otimes \langle\langle -a_2 \rangle\rangle \simeq \langle\langle -a_1, -a_2 \rangle\rangle$ is round and anisotropic over k. This completes the proof if $n = 2$. If $n > 2$, then iteratively repeating this procedure, for any i such that $1 \leq i \leq n$, we can find an element $a_i \in k^\times$ such that the 2^i-dimensional form $\langle\langle -a_1, -a_2, \ldots, -a_i \rangle\rangle$ is anisotropic and round over k.

□

CHAPTER 5

Refinements

In this chapter, we use the Witt index to ask refined questions about both the local-global principle for isotropy and the m-invariant. Before we begin, we prove a preliminary lemma that will be used in Sections 5.1.1 and 5.2.

Lemma 5.0.1. *Let k be a field of characteristic $\neq 2$ with $u(k) < \infty$. For any integer $j \geq 1$ and any quadratic form q over k, if $\dim q \geq u(k) + 2j - 1$, then $i_W(q) \geq j$.*

Proof. We induct on $j \geq 1$. First suppose $j = 1$, and let q be a quadratic form over k with $\dim q \geq u(k) + 2 - 1 = u(k) + 1$. Then because $\dim q > u(k)$, q must be isotropic, and therefore $i_W(q) \geq 1$, proving the base case.

Now assume for some $j \geq 1$ that all quadratic forms over k of dimension $\geq u(k) + 2j - 1$ have Witt index at least j, and let q be a quadratic form over k of dimension $\geq u(k)+2(j+1)-1$. Then because $\dim q > u(k)$, q must be isotropic and we can write $q \simeq \mathbb{H} \perp q'$, where $\dim q' \geq u(k)+2j-1$. By the induction hypothesis, $i_W(q') \geq j$, and therefore $i_W(q) \geq j + 1$, proving the claim by induction. □

5.1. Refined local-global principle for isotropy

Recall that a regular quadratic form q over a field k is isotropic if and only if its Witt index $i_W(q)$ is at least 1 (see Section 2.2). Therefore, given a non-empty set V of non-trivial discrete valuations on k, the quadratic form q satisfies the local-global principle for isotropy with respect to V if and only if

$$i_W(q_v) \geq 1 \text{ for all } v \in V \text{ implies } i_W(q) \geq 1.$$

We can therefore use the Witt index to define a refined local-global principle for isotropy as follows.

Definition 5.1.1. Let k be a field of characteristic $\neq 2$, and let $r, s \geq 1$ be any positive integers.

We say that a quadratic form q over k *satisfies* $\mathrm{LGP}(r, s)$ *with respect to* V if

$$i_W(q_v) \geq r \text{ for all } v \in V \text{ implies } i_W(q) \geq s.$$

In particular, q satisfies the local-global principle for isotropy if and only if it satisfies $\mathrm{LGP}(1, 1)$.

Remark 5.1.2. Let k be a field of characteristic $\neq 2$ equipped with a non-empty set V of non-trivial discrete valuations. Let q be a quadratic form over k and let $r, s \geq 1$ be any positive integers. For any integer $r' > r$, if q satisfies $\mathrm{LGP}(r, s)$ with respect to V, then q satisfies $\mathrm{LGP}(r', s)$ with respect to V as well, and for any integer $s' > s$, if q satisfies $\mathrm{LGP}(r, s')$ with respect to V, then it also satisfies $\mathrm{LGP}(r, s)$ with respect to V.

This section is organized as follows. We begin by generalizing Theorem 3.1.7. Indeed, we show that even when the local-global principle for isometry holds over purely transcendental field extensions of fields $\ell \in \mathscr{A}_i(2)$ for some $i \geq 0$ (see Definition 2.1.11), there are numerous counterexamples to $\mathrm{LGP}(r, 1)$ for various integers r (Theorem 5.1.3). Inspired by the existence of these counterexamples, in Section 5.1.2 we find a certain condition that a quadratic form q can satisfy to guarantee that q satisfies $\mathrm{LGP}(r, s)$ for some integers $r, s \geq 1$. This condition is that of being an I^n*-neighbor* (see Definition 5.1.6). To conclude, in Section 5.1.3 we investigate whether or not it is possible to find integers $r, s \geq 1$ such that *all* quadratic forms over a field k satisfy $\mathrm{LGP}(r, s)$ with respect to a non-empty set V of non-trivial discrete valuations on k.

5.1.1. Counterexamples

In this section, we will prove the following generalization of Theorem 3.1.7.

Theorem 5.1.3. *Let ℓ be a field of characteristic $\neq 2$ such that $\ell \in \mathscr{A}_i(2)$ for some $i \geq 0$ and $u(\ell) = 2^i$. For any integer $r \geq 1$ let $L_r = \ell(x_1, \ldots, x_r)$, and for $r \geq 2$ let V_r be the set of discrete valuations on L_r that are trivial on L_{r-1}. Then for $r \geq 2$ and any integer n such that $0 \leq n < 2^{i+r-2}$, there exists a quadratic form over L_r that violates* $\mathrm{LGP}\left(2^{i+r-2} - n, 1\right)$ *with respect to V_r.*

The idea of the proof is as follows. We will construct a 2^{i+r}-dimensional quadratic form over L_r that violates LGP $\left(2^{i+r-2}, 1\right)$ with respect to V_r. Once we have constructed this form, if we take any $\left(2^{i+r} - n\right)$-dimensional subform, then by Corollary 2.2.4, this subform will violate LGP $\left(2^{i+r-2} - n, 1\right)$ with respect to V_r.

Lemma 5.1.4. *Let ℓ be a field of characteristic $\neq 2$ such that $\ell \in \mathscr{A}_i(2)$ for some $i \geq 0$ and $u(\ell) = 2^i$. Let q be an anisotropic quadratic form over ℓ of dimension 2^i. Then over $\ell(x_1, x_2)$, the 2^{i+2}-dimensional form*

$$\varphi := \langle x_2 + 1, -x_1 - x_2, x_1, x_1 x_2 \rangle \otimes q$$

is anisotropic.

Proof. We prove the stronger claim, that φ is anisotropic over $\ell(x_2)((x_1))$. Over $\ell(x_2)((x_1))$, we have

$$\varphi = \left(\langle x_2 + 1, -x_1 - x_2 \rangle \otimes q\right) \perp x_1 \cdot \left(\langle 1, x_2 \rangle \otimes q\right).$$

The second residue form of φ, $\langle 1, x_2 \rangle \otimes q$, is anisotropic over the residue field $\ell(x_2)$ by Lemma 3.1.8. The first residue form is

$$\varphi_1 := \langle x_2 + 1, -x_2 \rangle \otimes q = (x_2 + 1) \cdot q \perp x_2 \cdot (-q).$$

Considering this form φ_1 over $\ell((x_2))$, the first residue form of φ_1 is q, and the second residue form is $-q$, both of which are anisotropic over ℓ by assumption. Both residue forms of φ are anisotropic over $\ell(x_2)$, so by Springer's Theorem (Theorem 2.3.3) φ is anisotropic over $\ell(x_2)((x_1)) \supset \ell(x_1, x_2)$, proving the claim. □

Lemma 5.1.5. *Let ℓ be a field of characteristic $\neq 2$ such that $\ell \in \mathscr{A}_i(2)$ for some $i \geq 0$ and $u(\ell) = 2^i$. Let q be a 2^i-dimensional anisotropic quadratic form over ℓ. Then for any non-trivial discrete valuation v on $\ell(x_1, x_2)$ that is trivial on $\ell(x_1)$, the 2^{i+2}-dimensional quadratic form defined over $\ell(x_1, x_2)$ by*

$$\varphi = \langle x_2 + 1, -x_1 - x_2, x_1, x_1 x_2 \rangle \otimes q$$

satisfies $i_W(\varphi_v) \geq 2^i$.

Proof. We prove the lemma by considering several cases for v.

<u>Case 1</u>: $v = v_\infty$ is the degree valuation with uniformizer x_2^{-1}.

We can write $\langle x_2+1, -x_1-x_2\rangle = x_2 \cdot \left\langle 1+x_2^{-1}, -x_1x_2^{-1}-1\right\rangle$. Scaling this form by x_2^{-2}, we have

$$\langle x_2+1, -x_1-x_2\rangle \simeq x_2^{-1} \cdot \left\langle 1+x_2^{-1}, -x_1x_2^{-1}-1\right\rangle.$$

The second residue form of this quadratic form is $\langle 1, -1\rangle$, which is isotropic over the residue field $\ell(x_1)$, so this quadratic form is isotropic over $\ell(x_1)\left(\left(x_2^{-1}\right)\right)$ by Springer's Theorem. Therefore, since $\dim q = 2^i$, by [Lam05, Corollary I.6.1] we have

$$i_W\left(\left(\langle x_2+1, -x_1-x_2\rangle \otimes q\right)_{v_\infty}\right) \geq 2^i.$$

This implies that $i_W(\varphi_{v_\infty}) \geq 2^i$ since φ contains $\langle x_2+1, -x_1-x_2\rangle \otimes q$ as a subform.

<u>Case 2</u>: $v = v_\pi$, where $\pi = x_2, x_2+1, x_1+x_2$ is a divisor of at least one entry of φ.

In this case, the quadratic forms $\langle -x_1-x_2, x_1\rangle$, $\langle x_1, x_1x_2\rangle$, and $\langle x_2+1, x_1, x_1x_2\rangle$ each reduce to isotropic quadratic forms over the respective residue fields κ_π. By the same argument as in Case 1, since $\dim q = 2^i$ we have

$$i_W(\varphi_{v_\pi}) \geq 2^i.$$

<u>Case 3</u>: $v = v_\pi$ for a monic irreducible polynomial $\pi \in \ell(x_1)[x_2]$ different from x_2, x_2+1, x_1+x_2.

In this case, each entry of φ is a unit in $\mathcal{O}_{v_\pi}$, so φ reduces to a 2^{i+2}-dimensional quadratic form $\overline{\varphi}$ over the residue field κ_π. The field κ_π is a finite extension of $\ell(x_1) \in \mathscr{A}_{i+1}(2)$, therefore $u(\kappa_\pi) \leq 2^{i+1}$ (see Section 2.1). The form $\overline{\varphi}$ has dimension $2^{i+2} = 2^{i+1} + 2\cdot 2^i \geq u(\kappa_\pi) + 2 \cdot 2^i$, so $i_W(\overline{\varphi}) \geq 2^i$ by Lemma 5.0.1. This implies, by Springer's Theorem, that $i_W(\varphi_{v_\pi}) \geq 2^i$.

This covers all cases of v, so the proof is complete. □

We can now prove Theorem 5.1.3.

Proof of Theorem 5.1.3. We first consider the case $r = 2$. Let q be an anisotropic 2^i-dimensional quadratic form over ℓ, and consider the form

$$\varphi_2 := \langle x_2 + 1, -x_1 - x_2, x_1, x_1x_2\rangle \otimes q$$

over $L_2 = \ell(x_1, x_2)$. By Lemma 5.1.4, φ_2 is anisotropic over L_2, and by Lemma 5.1.5, $i_W(\varphi_{2,v}) \geq 2^i$ for all $v \in V_2$. Therefore φ_2 violates LGP $\left(2^i, 1\right)$ with respect to V_2. For any n such that $0 < n < 2^i$, let ψ_n be any $\left(2^{i+2} - n\right)$-dimensional subform of φ_2. Then ψ_n is anisotropic over L_2, and for any $v \in V_2$, by Corollary 2.2.4, we have $i_W(\psi_{n,v}) \geq 2^i - n$. Thus ψ_n violates LGP $\left(2^i - n, 1\right)$ with respect to V_2, completing the proof if $r = 2$.

Now suppose $r \geq 3$. We have $L_r = \ell(x_1, \ldots, x_r) \simeq \ell(x_1, \ldots, x_{r-2})(x_{r-1}, x_r)$, and since $\ell \in \mathscr{A}_i(2)$, we have $\ell(x_1, \ldots, x_{r-2}) \in \mathscr{A}_{i+r-2}(2)$. Moreover, if q is an anisotropic 2^i-dimensional quadratic form over ℓ, then the 2^{i+r-2}-dimensional quadratic form $q \otimes \langle\langle x_1, \ldots, x_{r-2}\rangle\rangle$ is anisotropic over $\ell(x_1, \ldots, x_{r-2})$ by Lemma 3.1.8. Therefore, by applying Lemmas 5.1.4 and 5.1.5 to the 2^{i+r}-dimensional quadratic form

$$\varphi_r := \langle x_r + 1, -x_{r-1} - x_r, x_{r-1}, x_{r-1}x_r\rangle \otimes (q \otimes \langle\langle x_1, \ldots, x_{r-2}\rangle\rangle)$$

over L_r, we conclude that φ_r violates LGP $\left(2^{i+r-2}, 1\right)$ with respect to V_r. For any n such that $0 < n < 2^{i+r-2}$, all $\left(2^{i+r} - n\right)$-dimensional subforms of φ_r violate LGP $\left(2^{i+r-2} - n, 1\right)$ with respect to V_r, completing the proof for $r \geq 3$. □

5.1.2. I^n-neighbors

As we saw in Section 5.1.1, over purely transcendental field extensions L of fields $\ell \in \mathscr{A}_i(2)$ for some $i \geq 0$, there is a set W of discrete valuations on L with respect to which there are numerous counterexamples to LGP$(r,1)$ despite the fact that the local-global principle for isometry is satisfied over L with respect to W. We are then naturally led to ask if there are certain conditions we can impose on a quadratic form q over a field k equipped with a set V of discrete valuations to ensure that q satisfies LGP(r,s) with respect to V for some integers $r, s \geq 1$. In this section, we explore such a condition on quadratic forms over a field k equipped with a set V of discrete valuations with respect to which the local-global principle for isometry holds (Proposition 5.1.16).

We first recall Remark 3.1.6: given a field k equipped with a non-empty set V of non-trivial discrete valuations with respect to which the local-global princple for isometry holds, Pfister neighbors over k satisfy the local-global principle for isotropy with respect to V. Recall that a quadratic form q is a Pfister neighbor if it is similar to a subform of a Pfister form φ over k, where $\dim \varphi < 2 \dim q$. Now, n-fold Pfister forms additively generate $I^n(k)$, the n-th power of the fundamental ideal of the Witt ring $W(k)$ (see Section 2.2), so a Pfister neighbor q is a subform of some 2^n-dimensional quadratic form $\varphi \in I^n(k)$. Here we recall that by $\varphi \in I^n(k)$ we mean that the Witt equivalence class $[\varphi]$ of φ belongs to $I^n(k)$. This motivates the following generalized definition.

Definition 5.1.6. Let $n \geq 1$ be an integer. A quadratic form q over a field k is an *I^n-neighbor of complementary dimension r* if there exists an r-dimensional quadratic form σ_r over k, with $0 \leq r < \dim q$, such that

$$q \perp \sigma_r \in I^n(k).$$

The form σ_r is called a *complementary form of q*.

Remark 5.1.7. For an I^n-neighbor q, we note that the complementary dimension of q is not always unique. Indeed, if $q \perp \sigma_r \in I^n(k)$ and $r < \dim q - 2$, then $q \perp \sigma_r \perp \mathbb{H} \in I^n(k)$. However, the cardinality of the complementary dimension of the I^n-neighbor q is unique. Indeed, $\dim q$ and any complementary dimension r must have the same cardinality, since any form in $I^n(k)$ for $n \geq 1$ must have even dimension.

Examples 5.1.8. Let k be any field of characteristic $\neq 2$.

1. All quadratic forms q over k of dimension ≥ 2 are I^1-neighbors, as a quadratic form belongs to $I^1(k)$ if and only if it has even dimension. So either q or $q \perp \langle 1 \rangle$ belongs to $I^1(k)$.

2. All quadratic forms q over k of dimension ≥ 3 are I^2-neighbors. Indeed, a quadratic form belongs to $I^2(k)$ if and only if it has even dimension and trivial signed determinant. So either $q \perp \langle \pm \det(q) \rangle$ or $q \perp \langle 1, \pm \det(q) \rangle$ belongs to $I^2(k)$, with the sign of $\det(q)$ chosen based on $\dim q$.

3. If q is a Pfister neighbor over k whose associated Pfister form φ has dimension 2^n, then q is an I^n-neighbor.

We now prove some preliminary results about I^n-neighbors. In our study of I^n-neighbors, we will frequently use a result of Arason and Pfister that provides a lower bound for the dimension of anisotropic quadratic forms in I^n.

Theorem 5.1.9. *Let q be an anisotropic positive-dimensional quadratic form over a field k of characteristic $\neq 2$. If $q \in I^n(k)$ for some $n \geq 1$, then $\dim q \geq 2^n$. Equivalently, if a quadratic form φ over k belongs to $I^n(k)$ and $\dim \varphi < 2^n$, then φ is hyperbolic.*

Proof. See, e.g., [Lam05, Hauptsatz X.5.1]. □

As an immediate consequence, we have

Lemma 5.1.10. *If a quadratic form q over a field k of characteristic $\neq 2$ is an I^n-neighbor of complementary dimension r and $n > \log_2(\dim q + r)$, then*

$$i_W(q) \geq \frac{\dim q - r}{2}.$$

In particular, q is isotropic.

Proof. By assumption, there exists an r-dimensional form σ_r over k such that $q \perp \sigma_r \in I^n(k)$. Since $n > \log_2(\dim q + r)$, we have $2^n > \dim q + r$. Therefore, by Theorem 5.1.9, $q \perp \sigma_r$ is hyperbolic, i.e., $q \perp \sigma_r \simeq \left(\frac{\dim q + r}{2}\right)\mathbb{H}$. Then since $\dim q = \frac{\dim q + r}{2} + \frac{\dim q - r}{2}$ and $\dim q - r > 0$, we conclude by Corollary 2.2.5 that $i_W(q) \geq \frac{\dim q - r}{2} \geq 1$, as desired. □

Proposition 5.1.11. *Suppose that q is an I^n-neighbor of complementary dimension r over a field k of characteristic $\neq 2$. If $r < 2^{n-1}$, then all r-dimensional complementary forms of q are isometric.*

Proof. Let σ_r, σ'_r be r-dimensional forms over k such that $q \perp \sigma_r, q \perp \sigma'_r \in I^n(k)$. Then

$$\varphi := (q \perp \sigma_r) \perp -(q \perp \sigma'_r) \in I^n(k).$$

The form φ is Witt equivalent to $\sigma_r \perp -\sigma'_r$, and therefore $\sigma_r \perp -\sigma'_r \in I^n(k)$. By assumption, $r < 2^{n-1}$, so $\dim(\sigma_r \perp -\sigma'_r) < 2^n$. By Theorem 5.1.9, this implies that $\sigma_r \perp -\sigma'_r \simeq r\mathbb{H}$, so σ_r and σ'_r must be isometric by Lemma 2.1.7. □

Remarks 5.1.12.

1. If q is an I^n-neighbor of complementary dimension r over a field k such that $n > \log_2(\dim q + r)$, then Proposition 5.1.11 shows that the form σ_r such that $q \perp \sigma_r \in I^n(k)$ is unique up to isometry. Indeed, since $\dim q > r$, we have

$$n > \log_2(\dim q + r) > \log_2(2r) = \log_2(r) + 1.$$

   In other words, $2^{n-1} > r$, so Proposition 5.1.11 applies.

2. If q is an I^n-neighbor of complementary dimension $r = 2^{n-1}$ over a field k of characteristic $\neq 2$, then any two r-dimensional complementary forms of q that represent a common element of k are isometric. Indeed, if σ_r and σ'_r are r-dimensional complementary forms of q, then as we saw in the proof of Proposition 5.1.11,

$$\sigma_r \perp -\sigma'_r \in I^n(k).$$

   Now, if σ_r and σ'_r represent a common element, then the form $\sigma_r \perp -\sigma'_r$ is isotropic [Lam05, Corollary I.3.6]. So $\sigma_r \perp -\sigma'_r$ is Witt equivalent to a form in $I^n(k)$ with dimension $< 2^n$, and therefore must be hyperbolic by Theorem 5.1.9. By Lemma 2.1.7, we conclude $\sigma_r \simeq \sigma'_r$.

The following example shows that complementary forms of an I^n-neighbor need not be isometric if the conditions of Proposition 5.1.11 are not met.

Example 5.1.13. Let $k = \mathbb{C}(x, y)$, and suppose that q is a four-dimensional quadratic form over k with determinant x. Then $q \perp \langle 1, x\rangle$ and $q \perp \langle y, xy\rangle$ both belong to $I^2(k)$. But by Lemma 2.1.7, $\langle 1, x\rangle \not\simeq \langle y, xy\rangle$ since the Pfister form $\langle\langle x, y\rangle\rangle$ is anisotropic over k.

We will now show that, given an I^n-neighbor q over a field k of characteristic $\neq 2$ equipped with a set V of discrete valuations with respect to which the local-global principle for isometry holds, we can find integers $r, s \geq 1$ such that q satisfies $\mathrm{LGP}(r, s)$ with respect to V (Proposition 5.1.16). Before proving this result, we collect some results about the behavior of quadratic forms in $I^n(k)$.

Theorem 5.1.14. *Let q be a quadratic form over a field k of characteristic $\neq 2$ with $\dim q = 2^n$ for some $n \geq 1$. Then $q \in I^n(k)$ if and only if $q \simeq a \cdot \varphi$ for some $a \in k^\times$ and some n-fold Pfister form φ over k.*

Proof. See [Lam05, Theorem X.5.6]. □

Therefore, if we let $\mathrm{GP}_n(k)$ be the set of quadratic forms q over k such that $q \simeq a \cdot \varphi$ for some $a \in k^\times$ and some n-fold Pfister form φ over k, then Theorem 5.1.14 states

$$\mathrm{GP}_n(k) = \{\text{quadratic forms } q/k \mid \dim q = 2^n \text{ and } q \in I^n(k)\}.$$

Lemma 5.1.15. *Let q be an even-dimensional quadratic form over a field k of characteristic $\neq 2$ with $\dim q \geq 2^n$ for some $n \geq 1$. If q is Witt equivalent to an isotropic form $\pi \in \mathrm{GP}_n(k)$, then q is hyperbolic.*

Proof. Since $\pi \in \mathrm{GP}_n(k)$, we have $\dim \pi = 2^n \leq \dim q$. Therefore, because q is Witt equivalent to π, we must have

$$q \simeq \pi \perp \frac{\dim q - \dim \pi}{2}\mathbb{H}.$$

By definition, because $\pi \in \mathrm{GP}_n(k)$, there must be some $a \in k^\times$ and some n-fold Pfister form φ over k such that $\pi \simeq a\cdot\varphi$. Furthermore, by assumption, π is isotropic, and therefore φ is isotropic as well. Since φ is a Pfister form that is isotropic, then φ must be hyperbolic [Lam05, Theorem X.1.7]. Therefore π is hyperbolic, which implies that q is hyperbolic as well. □

We can now prove

Proposition 5.1.16. *Let k be a field of characteristic $\neq 2$ equipped with a non-empty set V of non-trivial discrete valuations with respect to which the local-global principle for isometry holds. Let q be an I^n-neighbor of complementary dimension r over k for some $n \geq 1$. Then*

$$q \textit{ satisfies } \mathrm{LGP}\left(\frac{\dim q + r - 2^n}{2} + 1, \frac{\dim q - r}{2}\right) \textit{ with respect to } V.$$

Proof. Because q is an I^n-neighbor of complementary dimension r, there exists an r-dimensional form σ_r over k such that $q \perp \sigma_r \in I^n(k)$. We note that, to show that $i_W(q) \geq \frac{\dim q - r}{2}$, it suffices to show that $q \perp \sigma_r$ is hyperbolic. Indeed, suppose

$$q \perp \sigma_r \simeq \frac{\dim q + r}{2}\mathbb{H}.$$

Since $\dim q = \frac{\dim q + r}{2} + \frac{\dim q - r}{2}$ and $\dim q - r > 0$, then by Corollary 2.2.5, $i_W(q) \geq \frac{\dim q - r}{2}$, as desired. So we will show that $i_W(q_v) \geq \frac{\dim q + r - 2^n}{2} + 1$ for all $v \in V$ implies that $q \perp \sigma_r$ is hyperbolic.

We first note that if $\dim q + r < 2^n$, then by Lemma 5.1.10, $i_W(q) \geq \frac{\dim q - r}{2}$ without any assumptions on the Witt index of q over k_v.

So assume that $\dim q + r \geq 2^n$. By our assumption on the Witt index of q over k_v, we have $i_W\left((q \perp \sigma_r)_v\right) \geq \frac{\dim q + r - 2^n}{2} + 1$ for all $v \in V$. Therefore, for each $v \in V$, we can find a 2^n-

dimensional quadratic form q'_v over k_v such that

$$(q \perp \sigma_r)_v \simeq \frac{\dim q + r - 2^n}{2}\mathbb{H} \perp q'_v,$$

where $i_W(q'_v) \geq 1$. Now, since $q \perp \sigma_r \in I^n(k)$, then $(q \perp \sigma_r)_v \in I^n(k_v)$, and therefore $q'_v \in I^n(k_v)$, hence $q'_v \in \mathrm{GP}_n(k_v)$ by Theorem 5.1.14. So $(q \perp \sigma_r)_v$ is Witt equivalent to an isotropic form $q'_v \in \mathrm{GP}_n(k_v)$, so by Lemma 5.1.15,

$$(q \perp \sigma_r)_v \simeq \frac{\dim q + r}{2}\mathbb{H}.$$

This holds for all $v \in V$, and therefore $q \perp \sigma_r \simeq \frac{\dim q + r}{2}\mathbb{H}$ over k since the local-global principle for isometry holds with respect to V. Therefore, $q \perp \sigma_r$ is hyperbolic over k which completes the proof of the proposition. □

Proposition 5.1.16 shows that if a quadratic form q over a field k is an I^n-neighbor for some n, then there are integers r, s such that q satisfies $\mathrm{LGP}(r, s)$. For large n, however, it is challenging to determine when a quadratic form belongs to $I^n(k)$, so it is difficult to verify that q is an I^n-neighbor. Despite this, for $n \leq 3$ this verification can be done. The fundamental ideal $I(k)$ consists of quadratic forms of even dimension, and $I^2(k)$ consists of even-dimensional quadratic forms with trivial discriminant. It is therefore straightforward to check whether or not a quadratic form belongs to $I(k)$ or $I^2(k)$, so it is easy to check when q is an I^n-neighbor for $n = 1, 2$.

For $n = 3$, the Witt and Hasse invariants of q, denoted by $c(q)$ and $s(q)$, respectively, allow us to easily verify whether or not a quadratic form belongs to $I^3(k)$. Indeed, a quadratic form q over k of dimension $2r$ belongs to $I^3(k)$ if and only if $\det q = (-1)^r$ and $c(q) = 1 \in \mathrm{Br}(k)$ [Lam05, pp. 138], where $\mathrm{Br}(k)$ denotes the Brauer group of k. We now briefly recall the definitions of these invariants $c(q)$ and $s(q)$ (see, e.g., [Lam05, Chapter V.3]).

Given an n-dimensional quadratic form q over k, we can define a 2^n-dimensional k-algebra $C(q)$, called the *Clifford algebra of* q (see, e.g., [Lam05, Chapter V.1] for the construction of $C(q)$).

The Clifford algebra $C(q)$ has a $\mathbb{Z}/2\mathbb{Z}$-grading, and we let $C_0(q)$ be the "even part" of $C(q)$. In general, $C(q)$ is not a central simple algebra over k. However, if q is even-dimensional, then $C(q)$ is a central simple k-algebra, and if q is odd-dimensional, then $C_0(q)$ is a central simple k-algebra. The *Witt invariant of* q, $c(q)$, is then defined by

$$c(q) = \begin{cases} [C(q)] & \text{if } \dim q \text{ is even,} \\ [C_0(q)] & \text{if } \dim q \text{ is odd,} \end{cases}$$

where $[C(q)], [C_0(q)]$ denote the classes of these central simple k-algebras in $\mathrm{Br}(k)$.

The Hasse invariant of q has a more straightforward definition using quaternion algebras. For any $a, b \in k^\times$, we let $\left(\frac{a,b}{k}\right)$ denote the generalized quaternion algebra over k, which is the central simple k-algebra generated by i, j such that

$$i^2 = a, \;\; j^2 = b, \;\; ij = -ji.$$

For example, $\left(\frac{-1,-1}{\mathbb{R}}\right)$ is the Hamiltonian quaternions over $\mathbb{R}$.

Now, given a quadratic form $q = \langle a_1, \ldots, a_n \rangle$ over k, we define the *Hasse invariant of* q, $s(q)$, by

$$s(q) = \prod_{1 \leq i < j \leq n} \left(\frac{a_i, a_j}{k}\right) \in \mathrm{Br}(k).$$

If $n = 1$, we take this product to be 1.

The Hasse invariant is rather easy to compute, particularly because, for quadratic forms q_1, q_2 over k, we have (see, e.g., [Lam05, pp. 119])

$$s(q_1 \perp q_2) = s(q_1)s(q_2)\left(\frac{\det(q_1), \det(q_2)}{k}\right).$$

The Hasse invariant is therefore more convenient to work with than the Witt invariant. However, these two invariants are closely related to one another.

Proposition 5.1.17. *Let q be an n-dimensional quadratic form over a field k of characteristic $\neq 2$. Then*

1. *If $n \equiv 1, 2 \mod 8$, then $c(q) = s(q)$.*
2. *If $n \equiv 3, 4 \mod 8$, then $c(q) = s(q) \cdot \left(\frac{-1, -\det(q)}{k}\right)$.*
3. *If $n \equiv 5, 6 \mod 8$, then $c(q) = s(q) \cdot \left(\frac{-1, -1}{k}\right)$.*
4. *If $n \equiv 7, 8 \mod 8$, then $c(q) = s(q) \cdot \left(\frac{-1, \det(q)}{k}\right)$.*

Proof. See, e.g., [Lam05, Proposition V.3.20]. □

We will now use the Witt invariant to find necessary and sufficient conditions for a quadratic form q over k to be an I^3-neighbor of some complementary dimension, repeatedly using Proposition 5.1.17 in the process. Before proceeding, we recall some properties of quaternion algebras in the Brauer group of k that we will use in conjunction with Proposition 5.1.17.

Proposition 5.1.18. *Let k be a field of characteristic $\neq 2$, and let $a, b, c \in k^\times$ be arbitrary. Then*

(a) $\left(\frac{a, a}{k}\right) = \left(\frac{a, -1}{k}\right) \in \mathrm{Br}(k)$,

(b) $\left(\frac{1, a}{k}\right) = \left(\frac{a, -a}{k}\right) = 1 \in \mathrm{Br}(k)$,

(c) $\left(\frac{a, b}{k}\right)\left(\frac{a, c}{k}\right) = \left(\frac{a, bc}{k}\right) \in \mathrm{Br}(k)$,

(d) $\left(\frac{a, b}{k}\right)\left(\frac{a, b}{k}\right) = 1 \in \mathrm{Br}(k)$.

Proof. (a) See, e.g., [Lam05, Corollary III.2.6].

(b) See, e.g., [Lam05, Corollary III.2.8(1)].

(c) See, e.g., [Lam05, Theorem III.2.11].

(d) By part (c), in $\mathrm{Br}(k)$ we have

$$\left(\frac{a,b}{k}\right)\left(\frac{a,b}{k}\right) = \left(\frac{a,b^2}{k}\right) = \left(\frac{a,1}{k}\right) = 1,$$

where the second equality follows from [Lam05, Proposition III.1(1)], and the last equality follows from part (b).

□

Proposition 5.1.19. *Over a field k of characteristic $\neq 2$, an odd-dimensional quadratic form q of dimension ≥ 3 is an I^3-neighbor of complementary dimension 1 if and only if q has trivial Witt invariant, i.e., $c(q) = 1$.*

Proof. Let $d = \det q$, and first assume that q is an I^3-neighbor of complementary dimension 1; i.e., there is a one-dimensional quadratic form σ_1 over k such that $q \perp \sigma_1 \in I^3(k)$. Calculating determinants, we have

$$\sigma_1 \simeq \begin{cases} \langle -d \rangle & \text{if } \dim q \equiv 1 \mod 4, \\ \langle d \rangle & \text{if } \dim q \equiv 3 \mod 4. \end{cases}$$

So $c(q \perp \langle \pm d \rangle) = 1$. Considering cases of $\dim q \mod 8$, a straightforward calculation shows that $c(q) = c(q \perp \langle \pm d \rangle) = 1$, which completes the proof of the forward implication. We show the case of $\dim q \equiv 1 \mod 8$, and the other cases follow in a similar fashion.

Since $\dim q \equiv 1 \mod 8$, we have $\sigma_1 \simeq \langle -d \rangle$, and

$$1 = c(q \perp \langle -d \rangle) = s(q \perp \langle -d \rangle) = s(q)\left(\frac{d,-d}{k}\right) = s(q) = c(q).$$

Conversely, suppose that q has trivial Witt invariant. Then letting

$$q' = \begin{cases} q \perp \langle -d \rangle & \text{if } \dim q \equiv 1 \mod 4, \\ q \perp \langle d \rangle & \text{if } \dim q \equiv 3 \mod 4, \end{cases}$$

we have $q' \in I^2(k)$. Moreover, the same calculations as those above show that $c(q') = c(q) = 1$, and therefore $q' \in I^3(k)$. Hence q is an I^3-neighbor of complementary dimension 1. □

Remark 5.1.20. A seven-dimensional quadratic form q over k is an I^3-neighbor of complementary dimension 1 if and only if q is a Pfister neighbor. So by Proposition 5.1.19, we conclude that q is a Pfister neighbor if and only if $c(q) = 1$. This agrees with an observation of Knebusch [Kne77, pp. 11].

Proposition 5.1.21. *Over a field k of characteristic $\neq 2$, an even-dimensional quadratic form q of dimension ≥ 4 is an I^3-neighbor of complementary dimension 2 if and only if $c(q) = \left(\frac{a, d_{\pm}q}{k}\right)$ for some $a \in k^\times$, where $d_{\pm}q$ is the signed determinant of q.*

Proof. Let $d = \det q$. Then

$$d_{\pm}q = \begin{cases} d & \text{if } \dim q \equiv 0 \mod 4, \\ -d & \text{if } \dim q \equiv 2 \mod 4. \end{cases}$$

We first focus on the reverse implication, so suppose there is some $a \in k^\times$ such that $c(q) = \left(\frac{a, d_{\pm}q}{k}\right)$. Now let

$$q' = \begin{cases} q \perp \langle -a, ad \rangle & \text{if } \dim q \equiv 0 \mod 4, \\ q \perp \langle -a, -ad \rangle & \text{if } \dim q \equiv 2 \mod 4. \end{cases}$$

Then $q' \in I^2(k)$ by construction, and considering cases of $\dim q \mod 8$, calculations show that $c(q') = 1$. So $q' \in I^3(k)$, thus q is an I^3-neighbor of complementary dimension 2. We show this calculation in the case that $\dim q \equiv 2 \mod 8$, and the other cases follow in a similar fashion.

Because $\dim q \equiv 2 \mod 8$, we have $q' = q \perp \langle -a, -ad \rangle$, hence

$$c(q') = c(q \perp \langle -a, -ad \rangle) = s(q \perp \langle -a, -ad \rangle) \left(\frac{-1, -1}{k}\right)$$

$$= s(q)s(\langle -a, -ad\rangle)\left(\frac{d,d}{k}\right)\left(\frac{-1,-1}{k}\right) = s(q)\left(\frac{-a,-ad}{k}\right)\left(\frac{-1,d}{k}\right)\left(\frac{-1,-1}{k}\right)$$
$$= s(q)\left(\frac{-a,-d}{k}\right)\left(\frac{-1,-d}{k}\right) = s(q)\left(\frac{a,-d}{k}\right) = c(q)\left(\frac{a,d_{\pm}q}{k}\right) = 1.$$

Conversely, suppose that q is an I^3-neighbor of complementary dimension 2; i.e., there exists a two-dimensional form σ_2 over k such that $q \perp \sigma_2 \in I^3(k)$. In particular, $q \perp \sigma_2 \in I^2(k)$, so calculating determinants, we must have, for some $a \in k^\times$,

$$\sigma_2 \simeq \begin{cases} \langle -a, ad\rangle & \text{if } \dim q \equiv 0 \mod 4 \\ \langle -a, -ad\rangle & \text{if } \dim q \equiv 2 \mod 4. \end{cases}$$

Since $c(q \perp \langle -a, \pm ad\rangle) = 1$, a similar calculation to the one above shows that $c(q) = \left(\frac{a,d_{\pm}q}{k}\right)$, as desired. □

Remarks 5.1.22. Let q be an even-dimensional quadratic form over a field k of characteristic $\neq 2$.

1. If $c(q) = 1 = \left(\frac{1,d_{\pm}q}{k}\right)$, then by Proposition 5.1.21 q is an I^3-neighbor of complementary dimension 2.

2. If $q \in I^2(k)$, then q is an I^3-neighbor of complementary dimension 2 if and only if $q \in I^3(k)$, since $d_{\pm}q = 1$, and $\left(\frac{a,1}{k}\right) = 1$ for any $a \in k^\times$.

3. If $\dim q = 4$ with $d = \det q$ and $K = k\left(\sqrt{d}\right)$, then q is isotropic if and only if $c(q_K) = 1$ [Lam05, Remark V.3.24]. If q is an I^3-neighbor of complementary dimension 2, then by Proposition 5.1.21 $c(q) = \left(\frac{a,d}{k}\right)$ for some $a \in k^\times$. Therefore $c(q_K) = \left(\frac{a,1}{K}\right) = 1$, which implies that q is isotropic over k. This agrees with Lemma 5.1.10 since $3 > \log_2(6)$.

4. If $\dim q = 6$ with determinant d, then q is an I^3-neighbor of complementary dimension 2 if and only if q is a Pfister neighbor. So by Proposition 5.1.21, q is a Pfister neighbor if and only if $c(q) = \left(\frac{a,-d}{k}\right)$ for some $a \in k^\times$. Thus $c(q_{k(\sqrt{-d})}) = 1$, i.e., $c(q)$ is split by $k\left(\sqrt{-d}\right)$. Using [Lam05, Theorem III.4.1], we recover an observation of Knebusch [Kne77, pp. 10]: q is

a Pfister neighbor if and only if $c(q)$ is Brauer equivalent to a quaternion algebra that is split by $k\left(\sqrt{-d}\right)$.

Proposition 5.1.23. *Over a field k of characteristic $\neq 2$, an odd-dimensional quadratic form q of dimension ≥ 5 is an I^3-neighbor of complementary dimension 3 if and only if its Witt invariant is Brauer equivalent to a quaternion algebra, i.e., $c(q) = \left(\frac{a,b}{k}\right) \in \mathrm{Br}(k)$ for some $a, b \in k^\times$.*

Proof. First suppose that $c(q) = \left(\frac{a,b}{k}\right)$ for some $a, b \in k^\times$. Since q is odd-dimensional, then for any $\alpha \in k^\times$, $c(q) = c(\alpha \cdot q)$ [Lam05, pp. 118]. Moreover, q is an I^3-neighbor if and only if $\alpha \cdot q$ is an I^3-neighbor. So after scaling, we may assume

$$\det q = \begin{cases} -1 & \text{if } \dim q \equiv 1 \mod 4, \\ 1 & \text{if } \dim q \equiv 3 \mod 4. \end{cases}$$

Now let $q' = q \perp \langle a, b, -ab\rangle \in I^2(k)$, where $a, b \in k^\times$ are such that $c(q) = \left(\frac{a,b}{k}\right)$. Again, considering cases depending on $\dim q \mod 8$, a straightforward calculation shows that $c(q') = 1$, which proves that q is an I^3-neighbor of complementary dimension 3. We show the calculation in the case that $\dim q \equiv 3 \mod 8$, and the other cases follow similarly.

Because $\dim q \equiv 3 \mod 8$, we assume in this case that $\det q = 1$. We have

$$\begin{aligned} c(q') &= c(q \perp \langle a, b, -ab\rangle) = s(q \perp \langle a, b, -ab\rangle)\left(\frac{-1,-1}{k}\right) = s(q)s(\langle a, b, -ab\rangle)\left(\frac{1,-1}{k}\right)\left(\frac{-1,-1}{k}\right) \\ &= s(q)s(\langle a, b, -ab\rangle)\left(\frac{-1,-1}{k}\right) = s(q)\left(\frac{a,b}{k}\right)\left(\frac{ab,-ab}{k}\right)\left(\frac{-1,-1}{k}\right) \\ &= s(q)\left(\frac{-1,-1}{k}\right)\left(\frac{a,b}{k}\right) = c(q)\left(\frac{a,b}{k}\right) = 1. \end{aligned}$$

Conversely, suppose that q is an I^3-neighbor of complementary dimension 3; i.e., there is a 3-dimensional quadratic form σ_3 over k such that $q \perp \sigma_3 \in I^3(k)$. Let $d = \det q$. Then we have $(\pm d) \cdot (q \perp \sigma_3) \in I^3(k)$. If $\dim q \equiv 1 \mod 4$, then scale by $-d$, and if $\dim q \equiv 3 \mod 4$, then scale by d. So we have

$$\pm d \cdot q \perp \pm d \cdot \sigma_3 \in I^3(k).$$

By our choice of scaling, this implies that $\det(\pm d \cdot \sigma_3) = -1$, so there exist $a, b \in k^\times$ such that $\pm d \cdot \sigma_3 \simeq \langle a, b, -ab \rangle$. Thus $\pm d \cdot q \perp \langle a, b, -ab \rangle \in I^3(k)$, which implies

$$c\left(\pm d \cdot q \perp \langle a, b, -ab \rangle\right) = 1.$$

Using that q is odd-dimensional, a calculation similar to the one above shows that

$$c(q) = c(\pm d \cdot q) = \left(\frac{a, b}{k}\right),$$

as desired. ☐

Remark 5.1.24. A five-dimensional quadratic form q is an I^3-neighbor of complementary dimension 3 if and only if q is a Pfister neighbor. So Proposition 5.1.23 implies that a five-dimensional quadratic form is a Pfister neighbor if and only if its Witt invariant is Brauer equivalent to a quaternion algebra, recovering an observation of Knebusch [Kne77, pp. 10].

Proposition 5.1.25. *Let q be an even-dimensional quadratic form of dimension ≥ 6 over a field k of characteristic $\neq 2$. Then q is an I^3-neighbor of complementary dimension 4 with a complementary form that represents its determinant if and only the Witt invariant of q is Brauer equivalent to a quaternion algebra, i.e., $c(q) = \left(\frac{a,b}{k}\right) \in \mathrm{Br}(k)$ for some $a, b \in k^\times$.*

Proof. First suppose that $c(q) = \left(\frac{a,b}{k}\right)$ for some $a, b \in k^\times$. Let $d = \det q$, and let

$$q' = \begin{cases} q \perp \langle d \rangle & \text{if } \dim q \equiv 0 \mod 4, \\ q \perp \langle -d \rangle & \text{if } \dim q \equiv 2 \mod 4. \end{cases}$$

A calculation shows that $c(q) = c(q')$. In particular, q' is an odd-dimensional quadratic form whose Witt invariant is Brauer equivalent to a quaternion algebra, therefore q' is an I^3-neighbor of complementary dimension 3 by Proposition 5.1.23. So there is some three-dimensional form σ_3 over k such that $q' \perp \sigma_3 \in I^3(k)$. By calculating determinants, we have $\det \sigma_3 = 1$. From this we see that $q \perp \langle \pm d \rangle \perp \sigma_3 \in I^3(k)$, where $\langle \pm d \rangle \perp \sigma_3$ has determinant $\pm d$. The form $\langle \pm d \rangle \perp \sigma_3$ is a four-dimensional complementary form of q that represents its determinant, proving the reverse implication.

Conversely, suppose that q is an I^3-neighbor of complementary dimension 4 with a complementary form σ_4 that represents its determinant. In particular $q \perp \sigma_4 \in I^3(k)$, and letting $d = \det q$, by calculating determinants we have

$$\det \sigma_4 = \begin{cases} d & \text{if } \dim q \equiv 0 \mod 4, \\ -d & \text{if } \dim q \equiv 2 \mod 4, \end{cases}$$

By assumption, σ_4 represents its determinant, so by the Representation Criterion (Theorem 2.1.5) $\sigma_4 \simeq \langle \pm d \rangle \perp \sigma_3$ for some three-dimensional form σ_3 over k. The form $q \perp \langle \pm d \rangle$ is therefore an odd-dimensional I^3-neighbor of complementary dimension 3, so by Proposition 5.1.23 we have $c(q \perp \langle \pm d \rangle) = \left(\frac{a,b}{k}\right)$ for some $a, b \in k^\times$. Finally, since $c(q) = c(q \perp \langle \pm d \rangle)$ (with $d, -d$ chosen according to $\dim q \mod 4$), we have $c(q) = \left(\frac{a,b}{k}\right)$, as desired. □

5.1.3. All quadratic forms

In Section 5.1.1 we found numerous counterexamples to $\mathrm{LGP}(r, 1)$ for various integers r, and in Section 5.1.2 we saw that by requiring a quadratic form q to be an I^n-neighbor for some n, we can find integers $r, s \geq 1$ such that q satisfies $\mathrm{LGP}(r, s)$. In this section, we will consider all quadratic forms over a field k, and ask whether we can find integers $r, s \geq 1$ such that all quadratic forms over k satisfy $\mathrm{LGP}(r, s)$.

Lemma 5.1.26. *Let k be a field of characteristic $\neq 2$ equipped with a non-empty set V of non-trivial discrete valuations on k. Suppose there exist integers $r, s \geq 1$ such that all quadratic forms over k satisfy $\mathrm{LGP}(r, s)$ with respect to V. Then for any integer $j \geq 1$, all quadratic forms over k satisfy $\mathrm{LGP}(r + j, s + j)$ with respect to V.*

Proof. We prove the lemma by induction on $j \geq 1$. For the base case of $j = 1$, suppose all quadratic forms over k satisfy $\mathrm{LGP}(r, s)$ with respect to V, and let q be a quadratic form over k such that $i_W(q_v) \geq r + 1$ for all $v \in V$. Then in particular, $i_W(q_v) \geq r$ for all $v \in V$, so by assumption, $i_W(q) \geq s \geq 1$ over k. We can therefore write $q \simeq \mathbb{H} \perp q'$ for some q' over k. Now, for all $v \in V$ we have

$$i_W(q_v) = i_W\left(\mathbb{H} \perp q'_v\right) \geq r + 1,$$

which implies $i_W(q'_v) \geq r$ for all $v \in V$. By assumption, the quadratic form q' satisfies $\mathrm{LGP}(r, s)$ with respect to V, and therefore $i_W(q'_v) \geq r$ for all $v \in V$ implies that $i_W(q') \geq s$. This, in turn, implies that $i_W(q) = i_W(\mathbb{H} \perp q') \geq s + 1$, proving the base case.

Now suppose for some $j \geq 1$ that all quadratic forms over k satisfy $\mathrm{LGP}(r+j, s+j)$ with respect to V, and let q be a quadratic form over k such that $i_W(q_v) \geq r + j + 1$ for all $v \in V$. Then in particular, $i_W(q_v) \geq r + j$ for all $v \in V$. By the induction hypothesis, this implies that $i_W(q) \geq s + j \geq 1$, and we may write $q \simeq \mathbb{H} \perp q'$ for some q' over k. The form q' must satisfy $i_W(q'_v) \geq r + j$ for all $v \in V$, and therefore $i_W(q') \geq s+j$ by the induction hypothesis. Therefore $i_W(q) = i_W(\mathbb{H} \perp q') \geq s+j+1$, proving the claim by induction. □

A statement stronger than the converse of Lemma 5.1.26 holds.

Lemma 5.1.27. *Let k be a field of characteristic $\neq 2$ equipped with a non-empty set V of non-trivial discrete valuations, and let $r, s \geq 1$ be positive integers. If there exists some integer $j \geq 1$ such that all quadratic forms over k satisfy* $\mathrm{LGP}(r+j, s+j)$ *with respect to V, then all quadratic forms over k satisfy* $\mathrm{LGP}(r, s)$ *with respect to V.*

Proof. Let q be any quadratic form over k with $i_W(q_v) \geq r$ for all $v \in V$, and let $q' = q \perp j\mathbb{H}$. Then $i_W(q'_v) \geq r + j$ for all $v \in V$, and since all quadratic forms over k satisfy $\mathrm{LGP}(r + j, s + j)$ with respect to V, we conclude that $i_W(q') \geq s + j$. Because $q' = q \perp j\mathbb{H}$, this implies that $i_w(q) \geq s$, and therefore q satisfies $\mathrm{LGP}(r, s)$ with respect to V. □

Lemmas 5.1.26 and 5.1.27 show that, over a field k of characteristic $\neq 2$ equipped with a non-empty set V of non-trivial discrete valuations, if there are positive integers $r, s \geq 1$ such that all quadratic forms over k satisfy $\mathrm{LGP}(r, s)$ with respect to V, then for any integer j (not necessarily positive) such that $r + j, s + j \geq 1$, all quadratic forms over k satisfy $\mathrm{LGP}(r + j, s + j)$ with respect to V. This motivates the definition of the following invariant associated to k and V.

Definition 5.1.28. For a field k of characteristic $\neq 2$ equipped with a non-empty set V of non-trivial discrete valuations, let

$$l(k, V) = \min_{r \geq 1} \{\text{all quadratic forms over } k \text{ satisfy } \mathrm{LGP}(r, 1) \text{ with respect to } V\}.$$

In particular, $l(k, V) = 1$ if and only if all quadratic forms over k satisfy the local-global principle for isotropy with respect to V. The invariant $l(k, V)$ can then be seen as a measure of the failure of the local-global principle for isotropy of quadratic forms over k with respect to V. The remainder of this section will be devoted to studying $l(k, V)$ for various k, V.

One of the first observations we make is that for fields k with finite u-invariant, $u(k)$ can be used to give a natural upper bound on $l(k, V)$ for any non-empty set V of non-trivial discrete valuations on k.

Proposition 5.1.29. *Let k be a field of characteristic $\neq 2$ with $u(k) < \infty$. Then for any non-empty set V of non-trivial discrete valuations on k,*

$$l(k, V) \leq \left\lceil \frac{u(k) + 1}{2} \right\rceil.$$

Proof. To prove the proposition, we must show that, with respect to V, all quadratic forms over k satisfy $\mathrm{LGP}\left(\left\lceil \frac{u(k)+1}{2} \right\rceil, 1\right)$. To that end, let q be a quadratic form over k such that $i_W(q_v) \geq \left\lceil \frac{u(k)+1}{2} \right\rceil$ for all $v \in V$. This implies that

$$\dim q \geq 2 i_W(q_v) \geq 2 \left\lceil \frac{u(k) + 1}{2} \right\rceil \geq u(k) + 1.$$

So $\dim q > u(k)$. By the definition of $u(k)$, this implies that q is isotropic over k, i.e., $i_W(q) \geq 1$. Thus q satisfies LGP $\left(\left\lceil \frac{u(k)+1}{2} \right\rceil, 1\right)$ with respect to V. □

The upper bound for $l(k,V)$ given by Proposition 5.1.29 is not always sharp, as the following example illustrates.

Example 5.1.30. Let k be an algebraically closed field of characteristic $\neq 2$, and for any integer $r \geq 1$ let $K_r = k(x_1, \ldots, x_r)$. Then $u(K_r) = 2^r$. Let V be any set of discrete valuations on K_r with respect to which the local-global principle for isometry holds (e.g., $V = V_r$ is the set of discrete valuations on K_r that are trivial on K_{r-1} (here taking $K_0 = k$)). Then

$$l(K_r, V) \leq 2^{r-1} < \left\lceil \frac{u(K_r)+1}{2} \right\rceil = 2^{r-1} + 1.$$

Indeed, to show that $l(K_r, V) \leq 2^{r-1}$, we must show that all quadratic forms over K_r satisfy LGP$(2^{r-1}, 1)$ with respect to V. To that end, let q be any quadratic form over K_r with $i_W(q_v) \geq 2^{r-1}$ for all $v \in V$. Then $\dim q \geq 2i_W(q_v) \geq 2^r = u(K_r)$. If $\dim q > 2^r$, then q is automatically isotropic over K_r, and therefore q satisfies LGP$(2^{r-1}, 1)$ with respect to V. If $\dim q = 2^r$ and $i_W(q_v) \geq 2^{r-1}$ for all $v \in V$, then $i_W(q_v) = 2^{r-1}$ for all $v \in V$, and we conclude that q_v is hyperbolic over $K_{r,v}$ for all $v \in V$. Because the local-global principle for isometry holds over K_r with respect to V, this implies that q is hyperbolic over K_r, i.e., $i_W(q) = 2^{r-1} \geq 1$. Therefore q satisfies LGP$(2^{r-1}, 1)$ with respect to V.

Knowing that counterexamples to LGP$(r, 1)$ exist over a field k with respect to V also allows us to give lower bounds on $l(k, V)$.

Example 5.1.31. Let ℓ be a field of characteristic $\neq 2$ such that $\ell \in \mathscr{A}_i(2)$ for some $i \geq 0$ and $u(\ell) = 2^i$. For any integer $r \geq 1$ let $L_r = \ell(x_1, \ldots, x_r)$, and for $r \geq 2$ let V_r be the set of discrete valuations L_r that are trivial on L_{r-1}. Then by Theorem 5.1.3, for $r \geq 2$, there exists a 2^{i+r}-dimensional quadratic form over L_r that violates LGP $\left(2^{i+r-2}, 1\right)$ with respect to V_r. This implies

$$l(L_r, V_r) \geq 2^{i+r-2} + 1.$$

We conclude this section by calculating $l(F, V)$ for V the set of all discrete valuations on a semi-global field F.

Proposition 5.1.32. *Let T be a complete discrete valuation ring with fraction field K and residue field k of characteristic $\neq 2$. Let F be a one-variable function field over K, and let V be the set of all discrete valuations on F. Then $l(F, V) \leq 2$, with $l(F, V) = 2$ if and only if the reduction graph of a regular model of F is not a tree.*

Proof. We first show that $l(F, V) \leq 2$. To prove this claim, we must show that all quadratic forms over F satisfy LGP$(2, 1)$ with respect to V. To that end, let q be any quadratic form over F with $i_W(q_v) \geq 2$ for all $v \in V$. This assumption on the Witt index of q_v implies that $\dim q \geq 4$. Moreover, since $i_w(q_v) \geq 2$ for all $v \in V$, then q is isotropic over F_v for all $v \in V$. Because $\dim q \geq 3$, by [CPS12, Theorem 3.1], we conclude that q is isotropic over F, i.e., $i_W(q) \geq 1$. Therefore q satisfies LGP$(2, 1)$ with respect to V.

From the above paragraph, we see that $l(F, V) = 2$ if and only if $l(F, V) \neq 1$, i.e., there is a quadratic form over F that violates LGP$(1, 1)$ with respect to V. By [CPS12, Theorem 3.1], all quadratic forms over F of dimension ≥ 3 satisfy LGP$(1, 1)$ with respect to V. So $l(F, V) = 2$ if and only if there is a two-dimensional quadratic form over F that violates LGP$(1, 1)$ with respect to V. By [HHK15a, Theorem 9.11], such a two-dimensional counterexample exists if and only if the reduction graph of a regular model of F is not a tree. □

5.2. Refined m-invariant

As we saw in Section 5.1, rephrasing the local-global principle for isotropy in terms of the Witt index allows us to ask a more refined question about the local-to-global behavior of quadratic forms over a field k. In a similar fashion, we can use the Witt index to refine the notion of the m-invariant of a field k.

By the First Representation Theorem (Theorem 2.1.6), a quadratic form q is anisotropic and universal over k if and only if $i_W(q) < 1$ and for all one-dimensional quadratic forms σ_1 over k we have $i_W(q \perp \sigma_1) \geq 1$. We can therefore rewrite $m(k)$ as

$$m(k) = \min_{\dim q \geq 1} \{i_W(q) < 1 \text{ and } i_W(q \perp \sigma_1) \geq 1 \text{ for all one-dimensional forms } \sigma_1 \text{ over } k\}.$$

This motivates the following refined version of the m-invariant of k.

Definition 5.2.1. Let k be a field of characteristic $\neq 2$. For positive integers $i, j \geq 1$ let

$$m_{i,j}(k) = \min_{\dim q \geq 1} \{i_W(q) < j \text{ and } i_W(q \perp \sigma_i) \geq j \text{ for all } i\text{-dimensional forms } \sigma_i \text{ over } k\}.$$

If no such quadratic form q exists we say that $m_{i,j}(k) = \infty$.

In particular, $m_{1,1}(k) = m(k)$.

We will begin this section by proving some relatively imprecise results about how the invariants $m_{i,j}(k)$ relate to one another as we increase i or j. We will then show, depsite the fact that finding precise values of $m_{i,j}(k)$ is a challenging problem in general, that there are natural upper and lower bounds for $m_{i,j}(k)$ given in terms of $u(k)$ and $m(k)$ (see, e.g., Proposition 5.2.9). While finding these lower bounds, we will also prove more precise results about how the invariants $m_{i,j}(k)$ relate to one another as we decrease i or j (see, e.g., Proposition 5.2.5 and Lemma 5.2.7).

Lemma 5.2.2. *Let k be any field of characteristic $\neq 2$, and let $i, j \geq 1$ be any positive integers.*

(a) If $i' \geq i$ and $m_{i,j}(k) < \infty$, then $m_{i',j}(k) \leq m_{i,j}(k)$.

(b) If $j' \geq j$ and $m_{i,j'}(k) < \infty$, then $m_{i,j'}(k) \geq m_{i,j}(k)$.

Proof. (a) Let q be a quadratic form over k with $\dim q = m_{i,j}(k)$ such that $i_W(q) < j$ and $i_W(q \perp \sigma_i) \geq j$ for all i-dimensional quadratic forms σ_i over k. Then letting $\sigma_{i'}$ be any

i'-dimensional quadratic form over k and taking any i-dimensional subform $\widetilde{\sigma}_i \subseteq \sigma_{i'}$, we have

$$i_W(q) < j \text{ and } i_W(q \perp \sigma_{i'}) \geq i_W(q \perp \widetilde{\sigma}_i) \geq j.$$

Therefore $m_{i',j}(k) \leq \dim q = m_{i,j}(k)$.

(b) If $j' = j$ there is nothing to prove, so suppose $j' = j + s$ for some $s > 0$. Let q be a quadratic form over k with $\dim q = m_{i,j'}(k)$ such that $i_W(q) < j'$ and $i_W(q \perp \sigma_i) \geq j' > j$ for all i-dimensional forms σ_i over k. If $i_W(q) < j$, then we are done since $m_{i,j'}(k) = \dim q \geq m_{i,j}(k)$ by the definition of $m_{i,j}(k)$. Otherwise, suppose $j \leq i_W(q) < j'$, and let $i_W(q) = j + \ell \geq \ell + 1$ for some $0 \leq \ell \leq s-1$. We can write $q \simeq (\ell+1)\mathbb{H} \perp q'$ for some q' over k with $i_W(q') = j-1 < j$. Moreover, for all i-dimensional forms σ_i over k, we have

$$i_W(q \perp \sigma_i) = i_W\left((\ell+1)\mathbb{H} \perp q' \perp \sigma_i\right) \geq j' = j + s.$$

This implies

$$i_W(q' \perp \sigma_i) \geq j + s - (\ell + 1) \geq j,$$

where this last inequality holds because $\ell \leq s - 1$. Therefore

$$m_{i,j}(k) \leq \dim q' \leq \dim q = m_{i,j'}(k).$$

□

We will now prove that a natural upper bound exists for $m_{i,j}(k)$ in terms of $u(k), i$, and j.

Lemma 5.2.3. *Let k be a field of characteristic $\neq 2$ with $u(k) < \infty$, and let $i, j \geq 1$ be any positive integers. Then*

$$m_{i,j}(k) \leq \max\{1, u(k) + 2j - 1 - i\}.$$

Proof. If $1 \geq u(k) + 2j - 1 - i$, then we need to show that $m_{i,j}(k) \leq 1$. Take any $a \in k^\times$. Then

$i_W(\langle a \rangle) = 0 < 1 \leq j$, and for all i-dimensional forms σ_i over k we have

$$\dim(\langle a \rangle \perp \sigma_i) = 1 + i \geq u(k) + 2j - 1.$$

So by Lemma 5.0.1, we have $i_W(\langle a \rangle \perp \sigma_i) \geq j$. This shows that $m_{i,j}(k) \leq 1$, as desired.

If $1 \leq u(k) + 2j - 1 - i \leq u(k)$, then let q be any anisotropic quadratic form over k of dimension $u(k) + 2j - 1 - i$. Such a form q exists by the definition of $u(k)$. Then $i_W(q) = 0 < j$, and for all i-dimensional forms σ_i over k we have $\dim(q \perp \sigma_i) = u(k) + 2j - 1$. This implies $i_W(q \perp \sigma_i) \geq j$ by Lemma 5.0.1. Thus $m_{i,j}(k) \leq \dim q = u(k) + 2j - 1 - i$.

Finally, suppose $u(k) + 2j - 1 - i > u(k)$, i.e., $2j - 1 > i$. Let q be any anisotropic quadratic form over k of dimension $u(k)$. Define the $(u(k) + 2j - 1 - i)$-dimensional quadratic form $\varphi_{i,j}$ over k as follows:

$$\varphi_{i,j} = \begin{cases} q \perp \frac{2j-1-i}{2}\mathbb{H} & \text{if } i \text{ is odd,} \\ q \perp \langle 1 \rangle \perp \frac{2j-2-i}{2}\mathbb{H} & \text{if } i \text{ is even.} \end{cases}$$

Then

$$i_W(\varphi_{i,j}) = \begin{cases} j - \frac{i+1}{2} & \text{if } i \text{ is odd,} \\ j - \frac{i}{2} & \text{if } i \text{ is even.} \end{cases}$$

In either case, $i_W(\varphi_{i,j}) < j$, and for all i-dimensional forms σ_i over k we have

$$\dim(\varphi_{i,j} \perp \sigma_i) = u(k) + 2j - 1,$$

thus $i_W(\varphi_{i,j} \perp \sigma_i) \geq j$ by Lemma 5.0.1. Hence $m_{i,j}(k) \leq \dim \varphi_{i,j} = u(k) + 2j - 1 - i$.

In all cases for i, j, we have found a quadratic form ψ over k with $\dim \psi = \max\{1, u(k) + 2j - 1 - i\}$ such that $i_W(\psi) < j$ and $i_W(\psi \perp \sigma_i) \geq j$ for all i-dimensional forms σ_i over k, which completes the proof. □

We now shift our focus to finding a similar lower bound for $m_{i,j}(k)$ in terms of $m(k)$, i, and j.

Lemma 5.2.4. *Let k be any field of characteristic $\neq 2$, and let i, j, r be integers such that $i \geq 2$, $j \geq 1$, and $1 \leq r < i$. Let q be a quadratic form over k such that $i_W(q) < j$ and $\dim q \leq m_{i-r,j}(k) - r$. Then there exists an r-dimensional quadratic form σ_r over k such that*

$$i_W(q \perp \sigma_r) < j.$$

Proof. Since $i_W(q) < j$ and $\dim q < m_{i-r,j}(k)$, there exists an $(i-r)$-dimensional form σ_{i-r} over k such that $i_W(q \perp \sigma_{i-r}) < j$. Because $i - r \geq 1$, we can take any entry a_1 of σ_{i-r} and have $i_W(q \perp \langle a_1 \rangle) < j$. If $r = 1$, then we are done, letting $\sigma_1 = \langle a_1 \rangle$.

If $r \geq 2$, then consider the form $q \perp \langle a_1 \rangle$ over k. We have

$$i_W(q \perp \langle a_1 \rangle) < j \text{ and } \dim(q \perp \langle a_1 \rangle) \leq m_{i-r,j}(k) - r + 1 < m_{i-r,j}(k).$$

By the same reasoning as that above, we can find an element $a_2 \in k^\times$ such that $i_W(q \perp \langle a_1, a_2 \rangle) < j$. We can continue repeating this process, ultimately finding r elements $a_1, \ldots, a_r \in k^\times$ such that $i_W(q \perp \langle a_1, \ldots, a_r \rangle) < j$. Letting $\sigma_r = \langle a_1, \ldots, a_r \rangle$ completes the proof. □

Proposition 5.2.5. *Let k be any field of characteristic $\neq 2$, and let $i, j \geq 1$ be any positive integers. For any integer r such that $0 \leq r < i$, if $m_{i,j}(k) < \infty$, then*

$$m_{i,j}(k) \geq m_{i-r,j}(k) - r.$$

Proof. This is trivial for $r = 0$ since both sides equal $m_{i,j}(k)$, so we may assume $r \geq 1$, which implies that $i \geq 2$. By contradiction, assume we can find integers i, j, r with $i \geq 2, j \geq 1$, and $1 \leq r < i$ such that

$$m_{i,j}(k) \leq m_{i-r,j}(k) - r - 1.$$

Let q be an $m_{i,j}(k)$-dimensional quadratic form over k such that $i_W(q) < j$ and $i_W(q \perp \sigma_i) \geq j$ for all i-dimensional forms σ_i over k. Since $i_w(q) < j$ and $\dim q \leq m_{i-r,j}(k) - r$, by Lemma 5.2.4, there is an r-dimensional quadratic form σ_r over k such that $i_W(q \perp \sigma_r) < j$. Now, since

$\dim q = m_{i,j}(k) \leq m_{i-r,j}(k) - r - 1$, we have

$$\dim(q \perp \sigma_r) \leq m_{i-r,j}(k) - 1 < m_{i-r,j}(k).$$

Moreover, since $i_W(q \perp \sigma_r) < j$, this implies that there exists an $(i-r)$-dimensional form σ_{i-r} over k such that $i_W(q \perp \sigma_r \perp \sigma_{i-r}) < j$. But $\dim(\sigma_r \perp \sigma_{i-r}) = i$ so this is a contradiction of our choice of q, which proves the proposition. □

Letting $r = i - 1$ in Proposition 5.2.5, we have

Corollary 5.2.6. *Let k be any field of characteristic $\neq 2$, and let $i, j \geq 1$ be any positive integers. If $m_{i,j}(k) < \infty$, then*

$$m_{i,j}(k) \geq m_{1,j}(k) - i + 1.$$

Proposition 5.2.5 shows how much $m_{i,j}(k)$ can decrease by decreasing i, and the next lemma shows the extent to which $m_{1,j}(k)$ can decrease by decreasing j.

Lemma 5.2.7. *Let k be any field of characteristic $\neq 2$, and let $j \geq 1$ be any positive integer. If $m_{1,j}(k) < \infty$, then*

$$m_{1,j}(k) \geq m_{1,1}(k) + 2j - 2.$$

Proof. Let q be a quadratic form over k with $\dim q = m_{1,j}(k)$ such that

$$i_W(q) < j \text{ and } i_W(q \perp \sigma_1) \geq j$$

for all 1-dimensional forms σ_1 over k. Then, by Corollary 2.2.4,

$$1 + i_W(q) \geq i_W(q \perp \sigma_1) \geq j,$$

which implies that $i_W(q) \geq j - 1$. By assumption, $i_W(q) < j$, so $i_W(q) = j - 1$ and we can write

$q \simeq (j-1)\mathbb{H} \perp q_{an}$, where q_{an} is anisotropic over k. So for all 1-dimensional forms σ_1 over k,

$$i_W(q \perp \sigma_1) = i_W\left((j-1)\mathbb{H} \perp q_{an} \perp \sigma_1\right) \geq j.$$

This implies that $i_W(q_{an} \perp \sigma_1) \geq 1$. Therefore q_{an} is anisotropic and universal over k, hence $\dim q_{an} \geq m(k) = m_{1,1}(k)$. Thus $m_{1,j}(k) = \dim q = 2j - 2 + \dim q_{an} \geq 2j - 2 + m_{1,1}(k)$. □

We can now write a lower bound for $m_{i,j}(k)$ in terms of $m(k)$, i, and j.

Corollary 5.2.8. *Let k be any field of characteristic $\neq 2$, and let $i, j \geq 1$ be any positive integers. If $m_{i,j}(k) < \infty$, then*

$$m_{i,j}(k) \geq \max\{1, m_{1,1}(k) + 2j - 1 - i\}.$$

Proof. If $m_{1,1}(k) + 2j - 1 - i < 1$, then the claim is automatic by definition since $m_{i,j}(k) \geq 1$. So suppose $m_{1,1}(k) + 2j - 1 - i \geq 1$. Then applying Corollary 5.2.6 and Lemma 5.2.7, we have

$$m_{i,j}(k) \geq m_{1,j}(k) - i + 1 \geq m_{1,1}(k) + 2j - 2 - i + 1 = m_{1,1}(k) + 2j - 1 - i.$$

□

Proposition 5.2.9. *Let k be a field of characteristic $\neq 2$ with $u(k) < \infty$, and let $i, j \geq 1$ be any positive integers. Then*

$$\max\{1, m_{1,1}(k) + 2j - 1 - i\} \leq m_{i,j}(k) \leq \max\{1, u(k) + 2j - 1 - i\}.$$

Moreover, if either $u(k) + 2j - 1 - i \leq 1$ or $u(k) = m(k)$, then both inequalities above are equalities.

Proof. The desired inequalities are shown in Lemma 5.2.3 and Corollary 5.2.8.

To prove the second claim, we first observe that $m_{1,1}(k) + 2j - 1 - i \leq u(k) + 2j - 1 - i$ since $m_{1,1}(k) = m(k) \leq u(k)$. So if either $u(k) + 2j - 1 - i \leq 1$ or $u(k) = m(k)$, then the quantities on the left and right sides of the inequalities above are equal, thus the inequalities become equalities. □

Remark 5.2.10. By Proposition 5.2.9, if k is a field of characteristic $\neq 2$ with $m(k) = u(k) < \infty$, then for any integers $i, j \geq 1$ such that $m(k) + 2j - 1 - i \geq 1$, we have

$$m_{i,j}(k) = m(k) + 2j - 1 - i.$$

This can be used to give precise expressions detailing how the invariants $m_{i,j}(k)$ change as we vary i and j. Indeed, if $1 \leq r < i$, then

$$m_{i-r,j}(k) = m(k) + 2j - 1 - (i - r) = m(k) + 2j - 1 - i + r = m_{i,j}(k) + r.$$

Moreover, for any $s \geq 1$,

$$m_{i,j+s}(k) = m(k) + 2(j + s) - 1 - i = m(k) + 2j - 1 - i + 2s = m_{i,j}(k) + 2s.$$

In other words, if $m(k) = u(k) < \infty$, then the invariants $m_{i,j}(k)$ are completely determined by $m(k)$.

Question 5.2.11. How closely do the invariants $m_{i,j}(k)$ follow the behavior seen in Remark 5.2.10 as we vary i and j if $m(k) < u(k)$?

We will first focus on answering Question 5.2.11 as we vary i. By Proposition 5.2.5, for any field k of characteristic $\neq 2$ and integers i, j, r such that $i \geq 2, j \geq 1$, and $1 \leq r < i$, if $m_{i,j}(k) < \infty$, then

$$m_{i,j}(k) \geq m_{i-r,j}(k) - r.$$

However, the only inequality holding in the opposite direction for a general field k is that of Lemma 5.2.2(a), i.e.,

$$m_{i,j}(k) \leq m_{i-r,j}(k).$$

The following example illustrates this point.

Example 5.2.12. Let $p \neq 2$ be a prime, and let F be a semi-global field over $\mathbb{Q}_p$ with a regular model whose reduction graph is not a tree (see, e.g., Example 4.2.18). Then by Lemma 4.2.6, $m_{1,1}(F) = m(F) = 2$, and we claim that

$$m_{2,1}(F) = m_{1,1}(F) = 2.$$

By Lemma 5.2.2(a), $m_{2,1}(F) \leq m_{1,1}(F) = 2$. So to complete the proof of the claim, it suffices to show that $m_{2,1}(F) \neq 1$, which we will do by contradiction.

Suppose there exists some $a \in F^\times$ such that

$$i_W(\langle a \rangle \perp \sigma_2) \geq 1$$

for all two-dimensional forms σ_2 over F. Let V be the set of all discrete valuations on F such that $u(\kappa_v) \geq 2$ (the set V is non-empty since there are discrete valuations v with residue field κ_v a global function field, thus $u(\kappa_v) = 4$). If there exists a $v \in V$ such that $v(a) \equiv 1 \mod 2$, then let $b \in F^\times$ be a unit lift of a non-square $\overline{b} \in \kappa_v^\times$ (such a $\overline{b}$ exists because $u(\kappa_v) \geq 2$). Then the quadratic form over F given by $\langle a \rangle \perp \langle 1, -b \rangle$ is anisotropic over F_v by Springer's Theorem and our choice of b, hence anisotropic over F. But this contradicts our choice of a.

Therefore $v(a) \equiv 0 \mod 2$ for all $v \in V$. Let v^* be any element of V, and let $\pi \in F^\times$ be a uniformizer for v^*. Let $\overline{c} \in \kappa_{v^*}^\times$ be a non-square, and let $c \in F^\times$ be a unit lift of $\overline{c}$. Then by Springer's Theorem, the quadratic form over F given by $\langle a \rangle \perp \langle -ac, \pi \rangle$ is anisotropic over F_{v^*}, hence anisotropic over F. This once again contradicts our choice of a, so no such a can exist. Therefore $m_{2,1}(F) = 2$ as claimed.

Remark 5.2.13. Let F be a field as in Example 5.2.12. For such a field F, $2 = m(F) < u(F) = 8$, and for $i = 2, j = 1$, both inequalities of Proposition 5.2.9 are strict. Indeed,

$$\max\{1, u(F) + 2j - 1 - i\} = u(F) + 2 - 1 - 2 = u(F) - 1 > 2,$$

and since $m_{1,1}(F) = 2$ we have

$$\max\{1, m_{1,1}(F) + 2j - 1 - i\} = 1.$$

We showed in Example 5.2.12 that $m_{2,1}(F) = 2$, so for $i = 2, j = 1$, we have

$$\max\{1, m_{1,1}(F) + 2j - 1 - i\} < m_{i,j}(F) < \max\{1, u(F) + 2j - 1 - i\}.$$

We now focus on answering Question 5.2.11 as we let j vary. As we saw in Remark 5.2.10, if $m(k) = u(k) < \infty$, then for any integers $i, j, s \geq 1$ we have

$$m_{i,j+s}(k) = m_{i,j}(k) + 2s.$$

The next few results will show that, when $m(k) < u(k)$, the invariants $m_{i,j}(k)$ behave more predictably as we vary j than they do as we vary i.

Lemma 5.2.14. *Let k be any field of characteristic $\neq 2$. For any positive integers $i, j \geq 1$, if $m_{i,j}(k) < \infty$, then*

$$m_{i,j+1}(k) \leq m_{i,j}(k) + 2.$$

Proof. Let q be a quadratic form over k of dimension $m_{i,j}(k)$ such that $i_W(q) < j$ and $i_W(q \perp \sigma_i) \geq j$ for all i-dimensional forms σ_i over k. Then the form $q \perp \mathbb{H}$ satisfies $i_W(q \perp \mathbb{H}) = i_W(q) + 1 < j + 1$, and for all i-dimensional quadratic forms σ_i over k, we have

$$i_W((q \perp \mathbb{H}) \perp \sigma_i) = 1 + i_W(q \perp \sigma_i) \geq j + 1.$$

By definition, $m_{i,j+1}(k) \leq \dim(q \perp \mathbb{H}) = m_{i,j}(k) + 2$. □

Corollary 5.2.15. *Let k be any field of characteristic $\neq 2$, and let $i, j, s \geq 1$ be any positive integers.*

If $m_{i,j}(k) < \infty$, then

$$m_{i,j+s}(k) \leq m_{i,j}(k) + 2s.$$

Proof. This follows by induction on $s \geq 1$ and Lemma 5.2.14. □

In the opposite direction, for particular values of i relative to $u(k)$ and j, we can find an inequality that is sharper than the one given in Lemma 5.2.2(b) (Proposition 5.2.17). First, we prove a lemma.

Lemma 5.2.16. *Let k be a field of characteristic $\neq 2$ with $u(k) < \infty$, and let $i \geq 1, j \geq 2$ be positive integers such that $i < u(k) + 2j - 2$. Then $m_{i,j}(k) \geq 2$.*

Proof. By contradiction, suppose there are integers i, j as in the statement of the lemma such that $m_{i,j}(k) = 1$. That is, there is some $a \in k^\times$ such that $i_W(\langle a \rangle \perp \sigma_i) \geq j$ for all i-dimensional forms σ_i over k. From this inequality and Corollary 2.2.4, we conclude $i_W(\sigma_i) \geq j - 1 \geq 1$ for all i-dimensional forms σ_i over k.

If $i \leq u(k)$, then letting σ_i be any i-dimensional anisotropic quadratic form over k, we have reached a contradiction since $i_W(\sigma_i) = 0 < j - 1$.

If $i = u(k) + s$ for some $1 \leq s \leq 2j - 3$, let q be any anisotropic $u(k)$-dimensional quadratic form over k, and let σ_i be the i-dimensional quadratic form over k defined by

$$\sigma_i = \begin{cases} q \perp \frac{s}{2}\mathbb{H} & \text{if } s \text{ is even,} \\ q \perp \langle -a \rangle \perp \frac{s-1}{2}\mathbb{H} & \text{if } s \text{ is odd.} \end{cases}$$

If s is even, then $i_W(\sigma_i) = \frac{s}{2} \leq j - 2 < j - 1$, which is a contradiction. If s is odd, then

$$i_W(\langle a \rangle \perp \sigma_i) = i_W\left(q \perp \frac{s+1}{2}\mathbb{H}\right) = \frac{s+1}{2} \leq j - 1 < j.$$

This is also a contradiction, so no such $a \in k^\times$ can exist. □

Proposition 5.2.17. *Let k be a field of characteristic $\neq 2$ with $u(k) < \infty$. Let $j \geq 1$ be any positive integer, and let i be a positive integer such that $1 \leq i < u(k) + 2j$. Then*

$$m_{i,j+1}(k) \geq m_{i,j}(k) + 1.$$

Proof. By assumption, $1 \leq i < u(k) + 2j = u(k) + 2(j+1) - 2$. Therefore, by Lemma 5.2.16, $m_{i,j+1}(k) \geq 2$. Moreover, since $u(k) < \infty$, then $m_{i,j+1}(k) < \infty$ by Lemma 5.2.3. Let q be a quadratic form over k of dimension $m_{i,j+1}(k)$ such that $i_W(q) < j+1$ and $i_W(q \perp \sigma_i) \geq j+1$ for all i-dimensional forms σ_i over k.

If q is isotropic, then we can write $q \simeq \mathbb{H} \perp q'$ for some form q' over k with $i_W(q') < j$ and $i_W(q' \perp \sigma_i) \geq j$ for all i-dimensional forms σ_i over k. Therefore $\dim q' \geq m_{i,j}(k)$, in which case

$$m_{i,j+1}(k) = \dim q = 2 + \dim q' \geq m_{i,j}(k) + 2 > m_{i,j}(k) + 1.$$

If q is anisotropic, then writing $q \simeq \langle a \rangle \perp q'$ for some $a \in k^\times$, we have $i_W(q') = 0 < j$ and

$$i_W(q \perp \sigma_i) = i_W\left(\langle a \rangle \perp q' \perp \sigma_i\right) \geq j+1.$$

By Corollary 2.2.4, this implies that $i_W(q' \perp \sigma_i) \geq j$, so $\dim q' \geq m_{i,j}(k)$. Thus

$$m_{i,j+1}(k) = \dim q = 1 + \dim q' \geq 1 + m_{i,j}(k).$$

□

If j is sufficiently large compared to i, we can find exact equalities for $m_{i,j}(k)$ as we increase j.

Lemma 5.2.18. *For any field k of characteristic $\neq 2$ and positive integers i and j such that $j \geq \lceil \frac{i}{2} \rceil \geq 1$, if $m_{i,j+1}(k) < \infty$, then*

$$m_{i,j+1}(k) \geq m_{i,j}(k) + 2.$$

Proof. Let q be a quadratic form over k of dimension $m_{i,j+1}(k)$ such that $i_W(q) < j+1$ and $i_W(q \perp \sigma_i) \geq j+1$ for all i-dimensional forms σ_i over k. Consider the i-dimensional quadratic form $\widetilde{\sigma}_i$ defined over k by

$$\widetilde{\sigma}_i = \begin{cases} \frac{i}{2}\mathbb{H} & \text{if } i \text{ even,} \\ \langle 1 \rangle \perp \frac{i-1}{2}\mathbb{H} & \text{if } i \text{ odd.} \end{cases}$$

Then since $i_W(q \perp \widetilde{\sigma}_i) \geq j+1$, we conclude

$$i_W(q) \geq j + 1 - \left\lceil \frac{i}{2} \right\rceil \geq 1.$$

Thus q is isotropic and we may write $q \simeq \mathbb{H} \perp q'$ for some q'. Moreover, $i_W(q') = 1 + i_W(q) < j+1$, so $i_W(q') < j$, and for all i-dimenisonal forms σ_i over k, we have

$$i_W(q \perp \sigma_i) = i_W(\mathbb{H} \perp q' \perp \sigma_i) \geq j+1,$$

which implies that $i_W(q' \perp \sigma_i) \geq j$. Therefore, by definition, $\dim q' \geq m_{i,j}(k)$, and we have

$$m_{i,j+1}(k) = \dim q = \dim q' + 2 \geq m_{i,j}(k) + 2,$$

as claimed. □

Corollary 5.2.19. *Let k be any field of characteristic $\neq 2$, and let i and j be positive integers such that $j \geq \lceil \frac{i}{2} \rceil \geq 1$. For any integer $s \geq 1$, if $m_{i,j}(k) < \infty$, then*

$$m_{i,j+s}(k) = m_{i,j}(k) + 2s.$$

Proof. We first observe that $m_{i,j+s}(k) \leq m_{i,j}(k) + 2s$ by Corollary 5.2.15. The reverse inequality follows immediately by induction on $s \geq 1$ and Lemma 5.2.18. Therefore $m_{i,j+s}(k) = m_{i,j}(k) + 2s$, as desired. □

5.2.1. Refining the u-invariant

Let k be a field of characteristic $\neq 2$. Because any isotropic quadratic form over k is universal, then by Lemma 4.1.1,

$$u(k) = \min_{n\geq 1} \{\text{all } n\text{-dimensional anisotropic quadratic forms over } k \text{ are universal}\}.$$

One way, then, to motivate the definition of $m(k)$ is to start with this definition of $u(k)$ and change the "for all" statement into a "there exists" statement, leaving the rest of the definition unchanged. Once we defined the m-invariant, we saw in Section 5.2 how we could use the Witt index to define refined m-invariants $m_{i,j}(k)$ for integers $i, j \geq 1$. By starting with $m_{i,j}(k)$ and reversing the process laid out in the first sentence, we arrive at a notion of a *refined u-invariant*, $u_{i,j}(k)$. That is, $u_{i,j}(k)$ is the minimal integer $n \geq 1$ such that *all* n-dimensional quadratic forms q over k with $i_W(q) < j$ satisfy $i_W(q \perp \sigma_i) \geq j$ for all i-dimensional forms σ_i over k. Much like $m_{1,1}(k) = m(k)$, we have $u_{1,1}(k) = u(k)$. Moreover, as the next proposition illustrates, for any field k of characteristic $\neq 2$ with finite u-invariant, the invariants $u_{i,j}(k)$ are completely determined by $u(k), i$, and j. This is unlike the behavior of $m_{i,j}(k)$ exhibited by some fields k as we saw in the previous section (see Example 5.2.12).

Proposition 5.2.20. *Let k be a field of characteristic $\neq 2$ with $u(k) < \infty$, and let $i, j \geq 1$ be any positive integers. Then*

$$u_{i,j}(k) = \max\{1, u(k) + 2j - 1 - i\}.$$

Proof. By definition, $u_{i,j}(k) \geq 1$. If $u(k) + 2j - 1 - i \leq 1$, then for all $a \in k^\times$ and all i-dimensional forms σ_i over k, we have

$$\dim(\langle a\rangle \perp \sigma_i) = i + 1 \geq u(k) + 2j - 1.$$

Therefore, by Lemma 5.0.1, $i_W(\langle a\rangle \perp \sigma_i) \geq j$. This implies that $u_{i,j}(k) = 1$.

Now suppose $u(k) + 2j - 1 - i > 1$. For all quadratic forms q over k with $\dim q = u(k) + 2j - 1 - i$ (in particular those with $i_W(q) < j$), and for all i-dimensional quadratic forms σ_i over k, we have $\dim(q \perp \sigma_i) = u(k) + 2j - 1$, so by Lemma 5.0.1, $i_W(q \perp \sigma_i) \geq j$. This implies that $u_{i,j}(k) \leq u(k)+2j-1-i$. To prove the reverse inequality, we find a $(u(k)+2j-2-i)$-dimensional form q over k with $i_W(q) < j$ and an i-dimensional form σ_i over k such that $i_W(q \perp \sigma_i) < j$, as this implies $u_{i,j}(k) > u(k) + 2j - 2 - i$. We do this by analyzing several cases.

Case 1: $i \leq u(k)$. (Note that if $i = u(k)$, then $j \geq 2$ since $u(k) + 2j - 1 - i > 1$).

Let φ be any anisotropic $u(k)$-dimensional quadratic form over k, and let $\sigma_i \subseteq \varphi$ be any i-dimensional subform. Let φ' be the $(u(k)-i)$-dimensional form such that $\varphi \simeq \sigma_i \perp \varphi'$. Now let $q = \varphi' \perp (j-1)\mathbb{H}$. Then $\dim q = u(k) + 2j - 2 - i$, and $i_W(q) = j - 1 < j$ since φ' is anisotropic. Moreover,

$$i_W(q \perp \sigma_i) = i_W(\varphi' \perp (j-1)\mathbb{H} \perp \sigma_i) = i_W(\varphi \perp (j-1)\mathbb{H}) = j - 1 < j.$$

Case 2: $u(k) + 2j - 2 - i \geq 1$ and $2j - 2 - i \geq 0$.

Let $\varphi = \langle 1 \rangle \perp \varphi'$ be a $u(k)$-dimensional anisotropic form over k. If i is even, let $q = \varphi \perp \frac{2j-2-i}{2}\mathbb{H}$. Then $\dim q = u(k) + 2j - 2 - i$, with $i_W(q) = j - 1 - \frac{i}{2} < j$. Now letting $\sigma_i = \frac{i}{2}\mathbb{H}$ we have $i_W(q \perp \sigma_i) = j - 1 < j$.

If i is odd, let $q = \varphi' \perp \frac{2j-1-i}{2}\mathbb{H}$. Then $\dim q = u(k) + 2j - 2 - i$, $i_W(q) = j - \frac{i+1}{2} < j$, and for $\sigma_i = \langle 1 \rangle \perp \frac{i-1}{2}\mathbb{H}$, we have $i_W(q \perp \sigma_i) = i_W(\varphi \perp (j-1)\mathbb{H}) = j - 1 < j$.

Case 3: $2j - 2 - i < 0$, and $u(k) > u(k) + 2j - 2 - i \geq 1$.

Let φ be any $u(k)$-dimensional anisotropic form over k, and let $q \subseteq \varphi$ be any subform of dimension $u(k) + 2j - 2 - i$. Then $\varphi \simeq q \perp \varphi'$ for a form φ' with $\dim \varphi' = i + 2 - 2j$. Let $\sigma_i = \varphi' \perp (j-1)\mathbb{H}$. Then $i_W(q) = 0 < j$, and for the i-dimensional form σ_i we have

$$i_W(q \perp \sigma_i) = i_W(\varphi \perp (j-1)\mathbb{H}) = j - 1 < j.$$

These cover all possibilities for $i, j, u(k)$, and therefore completes the proof. $\square$

5.3. Connecting $m_{i,j}$ and LGP(r, s)

In Section 3.2.2, we saw that a "going-up" result holds in terms of the local-global principle for isotropy (see Proposition 3.2.6). That is, if quadratic forms over certain fields k of a particular dimension n satisfy the local-global principle for isotropy with respect to some set V of discrete valuations on k, then all quadratic forms over k of dimension $> n$ also satisfy the local-global principle for isotropy with respect to V. Moreover, this dimension n is strongly related to the u-invariant of the field k. In this section, we will show that an analogous "going-down" result holds for LGP(r, s) (Corollary 5.3.3).

Lemma 5.3.1. *Let k be a field of characteristic $\neq 2$, let V be a non-empty set of non-trivial discrete valuations on k, let $i, j \geq 1$ be positive integers, and let n be a positive integer such that $n \leq m_{i,j}(k)$. If all n-dimensional quadratic forms over k satisfy* LGP(r, j) *with respect to V for some integer $r \geq 1$, then so do all quadratic forms over k of dimension $< n$.*

Proof. By contradiction, suppose the lemma is false, and let n^* be the largest dimension of a counterexample to LGP(r, j) with respect to V. Let q be an n^*-dimensional counterexample, i.e., $i_W(q_v) \geq r$ for all $v \in V$ but $i_W(q) < j$. Because $n^* < n \leq m_{i,j}(k)$ and $i_W(q) < j$, there must be some i-dimensional form σ_i over k such that $i_W(q \perp \sigma_i) < j$. In particular, taking any 1-dimensional subform $\sigma_1 \subseteq \sigma_i$, we have $i_W(q \perp \sigma_1) < j$. Moreover, for all $v \in V$ we have

$$i_W((q \perp \sigma_1)_v) \geq i_W(q_v) \geq r.$$

Therefore $q \perp \sigma_1$ is an $(n^* + 1)$-dimensional counterexample to LGP(r, j) with respect to V. This is a contradiction of our choice of n^* and therefore no such q can exist. □

Lemma 5.3.2. *Let k be a field of characteristic $\neq 2$ equipped with a non-empty set V of non-trivial discrete valuations. Let $i, j \geq 1$ be positive integers, and let n be a positive integer such that $n < m_{i,j}(k)$. If there is an n-dimensional counterexample to* LGP(r, j) *with respect to V for some integer $r \geq 1$, then there is an $(n + i)$ dimensional counterexample to* LGP(r, j) *with respect to V.*

Proof. Let q be an n-dimensional quadratic form over k with $i_W(q_v) \geq r$ for all $v \in V$ but $i_W(q) < j$. Because $\dim q < m_{i,j}(k)$ and $i_W(q) < j$, there must be an i-dimensional form σ_i over k such that $i_W(q \perp \sigma_i) < j$. Moreover, for all $v \in V$ we have

$$i_W\left((q \perp \sigma_i)_v\right) \geq i_W(q_v) \geq r.$$

So $q \perp \sigma_i$ is an $(n+i)$-dimensional counterexample to $\mathrm{LGP}(r,j)$ with respect to V. □

Corollary 5.3.3. *Let k be a field of characteristic $\neq 2$, let V be a non-empty set of non-trivial discrete valuations on k, and let $i, j \geq 1$ be positive integers. If all quadratic forms over k of dimension $m_{i,j}(k) + i - 1$ satisfy $\mathrm{LGP}(r,j)$ with respect to V for some integer $r \geq 1$, then so do all quadratic forms over k of dimension $< m_{i,j}(k)$.*

Proof. By Lemma 5.3.1, to prove the corollary, it suffices to show that all quadratic forms over k of dimension $m_{i,j}(k) - 1$ satisfy $\mathrm{LGP}(r,j)$ with respect to V. By the contrapositive of Lemma 5.3.2, because all quadratic forms over k of dimension $m_{i,j}(k) + i - 1$ satisfy $\mathrm{LGP}(r,j)$ with respect to V, then so must all quadratic forms over k of dimension $m_{i,j}(k) - 1$, which completes the proof. □

Remark 5.3.4. There are certain situations in which Corollary 5.3.3 could be particularly useful when studying $\mathrm{LGP}(1,1)$, i.e., the local-global principle for isotropy. Let k be a field of characteristic $\neq 2$ with $m(k) = u(k) < \infty$, and let V be a non-empty set of non-trivial discrete valuations on k. By definition, any quadratic form over k of dimension $> u(k)$ is isotropic over k, and therefore satisfies $\mathrm{LGP}(1,1)$ with respect to V. By Corollary 5.3.3, if quadratic forms over k of dimension $m(k) = m_{1,1}(k)$ satisfy $\mathrm{LGP}(1,1)$ with respect to V, then so do all quadratic forms over k of dimension $< m(k)$. Therefore in order to show that all quadratic forms over k satisfy $\mathrm{LGP}(1,1)$ with respect to V, it suffices to show only that all quadratic forms of dimension $m(k) = u(k)$ do.

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