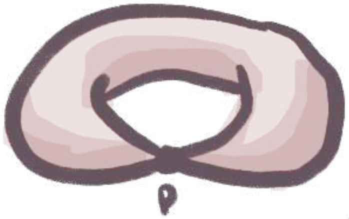
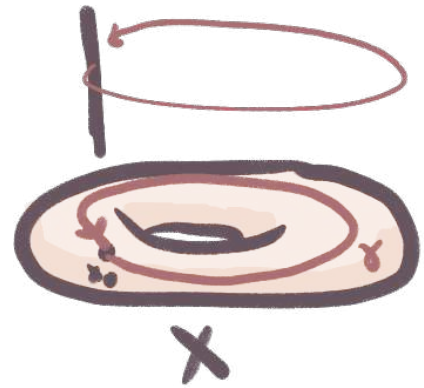
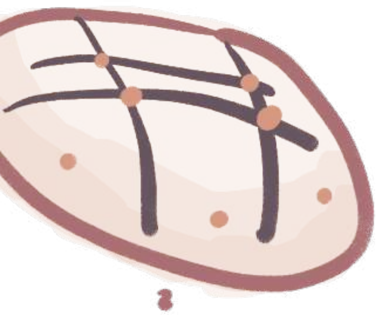
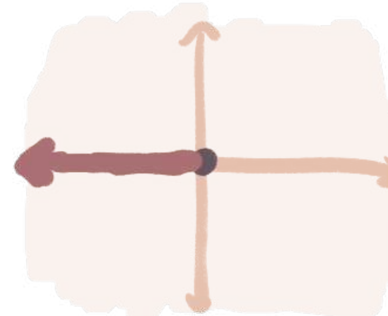




Geometric Langlands over $\mathbb{C}$





X curve

irred rank n local sys E on X $\xrightarrow{?}$ Hecke eigensheaf Aut_E w/ eigenvalue E

Emmett Lennen

02/09/24

Arithmetic Langlands	$G = GL_n$	Geometric Langlands
Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_n(\overline{\mathbb{Q}}_\ell)$		monodromy representation $\mathrm{Rep}(\pi_1(X)) \cong \mathrm{Loc}^{ft}(X)$
$G(F) \backslash G(\mathbb{A}_F) / \prod G(\hat{\mathcal{O}}_{X,x})$		Bun_n
Automorphic representation of $GL_n(\mathbb{A})$ (Hecke eigenfunctions on $G(F) \backslash G(\mathbb{A}_F) / \prod G(\hat{\mathcal{O}}_{X,x})$)		Hecke eigensheaf (type of perverse sheaf or D-module on Bun_n)

Final version of Langlands - Daeboem:

Galois rep $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to {}^L G(\overline{\mathbb{Q}}_\ell)$ $\xleftrightarrow{1:1}$ Cuspidal automorphic rep π of $G(\mathbb{A}_\mathbb{Q})$

Plan:

I. Local Systems, perverse sheaves and D-modules

II. Hecke Correspondences and Hecke Eigensheaves

III. Geometric Satake

PART I

Local Systems

Bundles

Perverse Sheaves

D-modules

Galois Reps to Monodromy:

X curve w/ function field F

Galois representation: $\rho: \mathrm{Gal}(\overline{F}/F) \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$

Think of $\mathrm{Gal}(\overline{F}/F)$ as fundamental group

Monodromy representations $\rho: \pi_1(X, x_0) \longrightarrow GL_n(\mathbb{C})$

$\mathrm{Rep}(\pi_1(X, x_0)) =$ category of monodromy representations

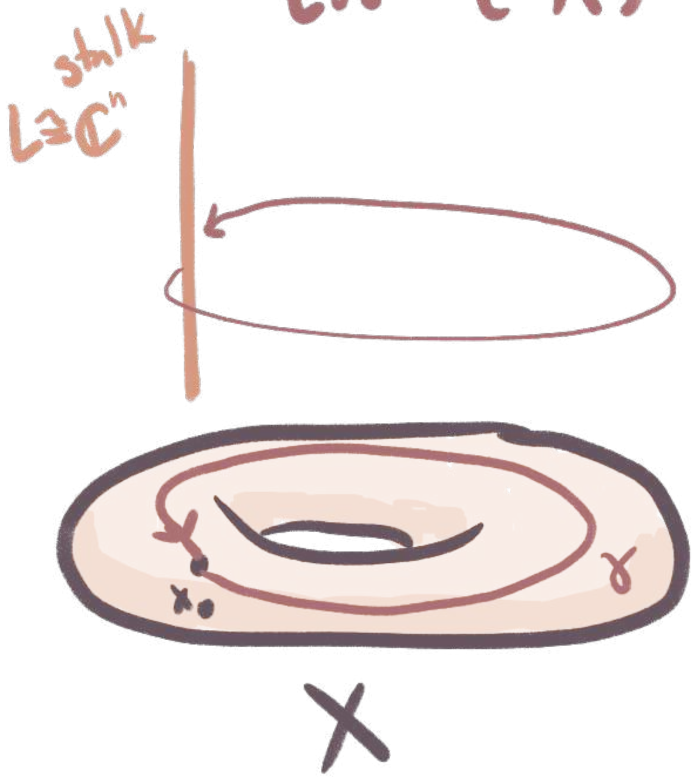
Two different ways of seeing $\mathrm{Rep}(\pi_1(X, x_0))$

A local system of $\mathbb{C}$-vector spaces is a locally constant sheaf $\mathcal{L}$ of $\mathbb{C}$-v.s.
(i.e. $\forall x \in X$, $\exists U \ni x$ s.t. $\mathcal{L}|_U$ is a constant sheaf)

If X path connected, then all stalks of $\mathcal{L}$ are isomorphic and

fin dim v.s.

$$\mathrm{Loc}^{ft}(X) \cong \mathrm{Rep}(\pi_1(X, x_0))$$



$$\longmapsto$$

$$\pi_1(X, x_0) \longrightarrow GL_n(\mathbb{C})$$

$$\gamma \longmapsto (L \cong \mathbb{C}^n \xrightarrow{\gamma} \mathbb{C}^n \cong L)$$

Two different ways of seeing $Rep(\pi_1(X, x_0))$

How can a vector bundle $\mathcal{E}$ give a local system?

Transition functions $U_\alpha \cap U_\beta \longrightarrow GL_n(\mathbb{C})$

should be chosen to be constant . . .

Def: A flat connection ∇ on $\mathcal{E}$ is

$$\nabla : \underset{\text{v.f. on } U}{\underset{\xi}{Vect(U)}} \longrightarrow End(\Gamma(U, \mathcal{E}))$$

$$\xi \longmapsto \nabla_\xi$$

Such that

- $\nabla_\xi (fs) = f \nabla_\xi (s) + (\xi \cdot f)_s \qquad f \in C^\infty(U), \; s \in \Gamma(U, \mathcal{E})$
- $\nabla_{f\xi} = f \nabla_\xi$
- $[\nabla_\xi, \nabla_\eta] = \nabla_{[\xi, \eta]}$ (flatness

Gives preferred local triv by local horizontal sections ($\nabla_\xi s = 0$)

Geometric Langlands over $\mathbb{C}$

X complex curve

$$\mathrm{Loc}^{ft}(X) \xleftrightarrow{1:1} ???$$

all the rest of part 1 and 2 to define Hecke eigensheaves

Previously was: Hecke eigenfunctions on $G(F)\backslash G(\mathbb{A}_F)/\prod G(\widehat{\mathcal{O}}_{x,\cdot})$

$\Updownarrow$ cuspidal automorphic rep. of $G(\mathbb{A}_F)$

Adéles and vector bundles

X curve, F function field

Recall: $\mathbb{A}_F := \mathbb{R} \times \prod_p' F_p$

Claim: We should replace

$$G(F) \backslash G(\mathbb{A}_F) / \prod G(\hat{\mathcal{O}}_{X,x}) \rightsquigarrow \mathrm{Bun}_n$$

Case $n=1$:

$$F^\times \backslash \prod_{x \in X}' F_x^\times / \prod \hat{\mathcal{O}}_{X,x} \cong F^\times \backslash \bigoplus_{x \in X} \mathbb{Z} \cong \mathrm{Pic}\, X.$$

Case triv over $U = X \setminus \{x\}$:

$$\{\text{principal } G\text{-bundles triv on } U\} = G(U) \backslash G(F_x) / G(\mathcal{O}_{X,x})$$

Adéles and vector bundles

X curve, F function field

Recall: $\mathbb{A}_F := \mathbb{R} \times \prod_p' F_p$

Claim: We should replace

$$G(F)\backslash G(\mathbb{A}_F)/\prod G(\hat{\mathcal{O}}_{X,x}) \rightsquigarrow \mathrm{Bun}_n$$

Case triv on U, V s.t. $U \cup V = X$: $E \in \mathrm{Bun}_n$

$$\begin{tikzcd}
 & \mathcal{O}_V^{\oplus n} \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & E|_U \cong \mathcal{O}_V^{\oplus n} \arrow[d] \\
x \in & V \arrow[r, hook] & X & U \arrow[l, hook']
\end{tikzcd}$$

$\Rightarrow G(U)\backslash G(\hat{F}_x)/G(\hat{\mathcal{O}}_{X,x})$

Then take $\lim\limits_{x \in V} G(U)\backslash G(\hat{F}_x)/G(\hat{\mathcal{O}}_{X,x}) = G(F)\backslash G(\hat{F}_{X,x})/G(\hat{\mathcal{O}}_{X,x})$

General: Take further limit over U.

Functions to Sheaves:

Recall: An automorphic representation of $G(\mathbb{A}_F)$ is an irreducible subrep of $L^2(G, \chi)$

$$\left(\text{functions s.t. } F(gz) = F(z)\chi(g), \int_{Z(\mathbb{A}_\mathbb{Q})G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})} |F(g)|^2 dg < \infty \right)$$

What should this translate to?

Answer 1: Use Grothendieck fonctions-faisceaux dictionary

↳ Take X alg var over $\mathbb{F}_q$

$$(\overline{\mathbb{Q}}\text{-valued func}) \xleftarrow{\text{trace of Frobenius}} \text{complex of } \ell\text{-adic sheaves}$$

Functions to Sheaves:

Recall: An automorphic representation of $G(\mathbb{A}_F)$ is an irreducible subrep of $L^2(G,\chi)$

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What should this translate to?

Answer 1: Use Grothendieck fonctions-faisceaux dictionary

↳ Take X alg var over $\mathbb{F}_q$

($\overline{\mathbb{Q}}$-valued func) $\rightsquigarrow$ complex of ℓ-adic sheaves

BUT! Turns out this is too big

Answer 2: Perverse sheaves

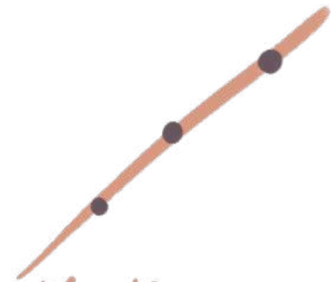
- certain type of complex of constructible sheaves

Constructible Sheaves:

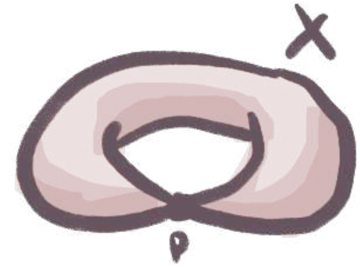
Take an algebraic stratification

$$X = \coprod_{s \in S} X_s$$

loc closed subvar.



$X = U \cup p_1 \cup p_2 \cup p_3$



$X = (X \setminus p) \cup p$

s.t. $\overline{X}_s \cap X_t = (\emptyset \text{ or } X_t)$

Def: A constructible sheaf w.r.t. the stratification is a sheaf $\mathcal{F}$ s.t. $\mathcal{F}|_{X_s}$ is a local system for each $s \in S$.

A complex of sheaves $\dots \to \mathcal{F}^0 \to \mathcal{F}^1 \to \dots$ is if $\mathcal{H}^i(\mathcal{F})$ is for all i.

Examples of Constructible Sheaves:

- Any local system $\mathcal{L} \in \mathrm{Loc}^{ft}(X)$ is constructible w.r.t. the trivial strat. $X = X$

- Let L_x be a skyscraper sheaf at $x \in X$ w/ stalk $L \cong \mathbb{C}^n$. If $i : x \hookrightarrow X$ then

  $i_* L_x$ constructible w.r.t. strat $X = (X \setminus \{x\}) \cup \{x\}$

  should be derived

- Let $\mathcal{L}_{X_s} \in \mathrm{Loc}^{ft}(X_s)$ where $\underset{\text{closed}}{X_s} \xhookrightarrow{i} X$, then

  $i_* \mathcal{L}_{X_s}$ constructible w.r.t strat $X = (X \setminus X_s) \cup X_s$.

  (for $\underset{\text{not closed}}{X_s} \hookrightarrow X$, $i_! \mathcal{L}_{X_s}$ is constructible)

Perverse Sheaves:

Def: A perverse sheaf is a complex of sheaves

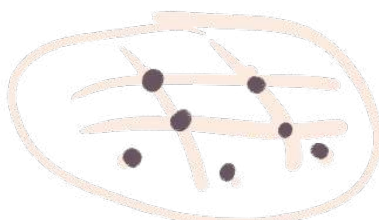
$$\cdots \to \mathcal{F}^{-1} \to \mathcal{F}^{0} \to \mathcal{F}^{1} \to \mathcal{F}^{2} \to \cdots$$

such that
- $\mathcal{H}^i(\mathcal{F})$ is constructible
- $\dim \operatorname{supp} \mathcal{H}^i(\mathcal{F}) \leq -i$
- $\dim \operatorname{supp} \mathcal{H}^i(\mathbb{D}\mathcal{F}) \leq -i$ $\leftarrow$ For the sake of this talk, we ignore this

Picturing a perverse sheaf:

$X = \coprod X_s$, $\quad X^d := \bigcup_{\dim X_s \leq d} X_s$

$i=0$

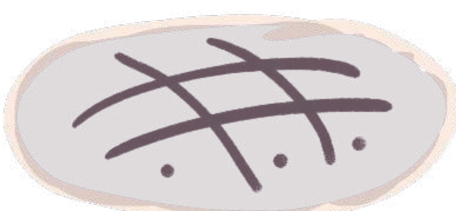


$\operatorname{supp} \mathcal{H}^0(\mathcal{F}) \subseteq X_0$

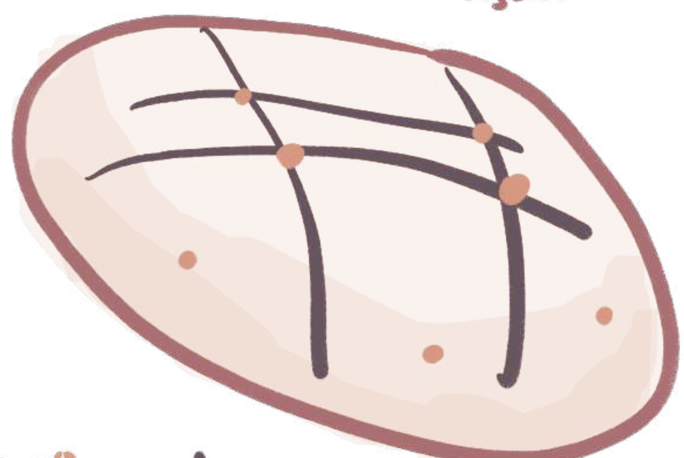
$i=-1$

$\operatorname{supp} \mathcal{H}^{-1}(\mathcal{F}) \subseteq X_1$

$i=-2$



$\operatorname{supp} \mathcal{H}^{-2}(\mathcal{F}) \subseteq X^2$



X

$X^0,\ X^1,\ X^2 = X$

Perverse Sheaves Examples:

- Let $\mathcal{L} \in Loc^{ft}(X)$

$$\begin{array}{ccccccccc} \dots & \to & 0 & \to & \mathcal{L} & \to & 0 & \to & \dots \\ \text{degree} & & \dim X - 1 & & \dim X & & \dim X + 1 & & \end{array}$$

Then $\mathcal{L}[\dim X] \in Perv(X)$ (shifted local systems)

- $\underset{\text{closed}}{X_s} \xhookrightarrow{f} X$ and $\mathcal{L}_{X_s} \in Loc^{ft}(X_s)$ then

$$f_* \mathcal{L}_{X_s}[\dim X_s] \in Perv(X)$$

D-modules: X sm complex alg var

$\mathcal{D}_X :=$ sheaf of differential operators

If $\underset{\mathbb{C}^n}{\overset{\cong}{U}} \subset X$, $\mathcal{D}_X(U) = \langle \underbrace{x_1, \dots, x_n}_{\text{coordinates}}, \underbrace{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}}_{\text{vector fields}} \rangle$.

Def: A D-module is a sheaf of $\mathcal{D}_X$-modules.

Example: Let $\mathcal{O}_X$ = sheaf of analytic function

$\mathcal{D}_X(U) \curvearrowright \mathcal{O}_X(U)$

$$\left(f(z)\frac{\partial}{\partial^{\alpha_1}x_1}\cdots\frac{\partial}{\partial^{\alpha_n}x_n}\right)\cdot g(z) = f(z)\left(\frac{\partial}{\partial^{\alpha_1}x_1}\cdots\frac{\partial}{\partial^{\alpha_n}x_n}g(z)\right)$$

Hence $\mathcal{O}_X(U)$ is a $\mathcal{D}_X$-module.

Relation to solutions of PDEs: $X \subseteq \mathbb{C}^n$

Consider the diff eq $Pu = 0$, $P \in \mathcal{D}_X$, $u \in \mathcal{O}_X$

$M = \mathcal{D}_X / \mathcal{D}_X P$ is a $\mathcal{D}_X$-module

Take $\mathrm{Hom}_{\mathcal{D}_X}(\cdot, \mathcal{O}) : \begin{pmatrix} \text{category of} \\ \text{coh } \mathcal{D}_X\text{-mod} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{category of} \\ \mathbb{C}\text{-mod} \end{pmatrix}$

$$\mathrm{Hom}_{\mathcal{D}_X}(M, \mathcal{O}) = \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{D}_X P, \mathcal{O})$$

$$= \{\varphi \in \underbrace{\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{D}, \mathcal{O})}_{\cong\, \mathcal{O}} : \varphi(P) = 0\}$$

$$\text{Solutions of } P = \{u \in \mathcal{O} : P(u) = 0\}$$

Riemann-Hilbert Correspondence:

$$\text{holonomic } D_X\text{-modules} \xleftrightarrow{1:1} \mathrm{Perv}(X)$$

$$\mathcal{F} \longmapsto R\,\mathrm{Sol}(\mathcal{F}) = R\mathcal{H}om_{D}(\mathcal{F}, \mathcal{O})$$

__Example:__ $X = \mathbb{C}$, $P = 2z\frac{\partial}{\partial z} - 1$, $M = \mathcal{D}/\mathcal{D}\cdot P$

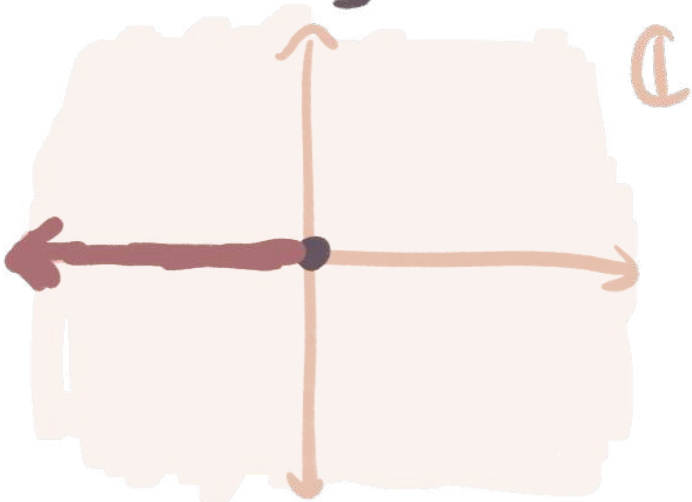
$$M(U) = \{f : U \to \mathbb{C} : 2z\tfrac{\partial}{\partial z} f - f = 0\}$$

Solutions should be $f_0(z) = z^{1/2}$ up to scaling $\alpha \in \mathbb{C}$.

If $\underset{\not\ni 0}{U}$ open, conn, s.c $\xrightarrow[\text{define branch}]{\text{can}}$ $M(U) = \{\alpha e^{\frac{1}{2}\log z} : \alpha \in \mathbb{C}\}$
$\cong \mathbb{C}$

__BUT__ $M(\mathbb{C}) = \emptyset$ since there are no global solutions

Can stratify $\mathbb{C}$



$$\mathcal{M}|_{\mathbb{C}\setminus(\mathbb{R}^- \cup \{0\})} \cong \underline{\mathbb{C}}$$

$$\mathcal{M}|_{\mathbb{R}^-} \cong \underline{\mathbb{C}}$$

$$\mathcal{M}|_0 = \{0\}$$

PART 2

Hecke Correspondences

Hecke Eigensheaves

Hecke Correspondence: X curve

i-th Hecke correspondence is the moduli of quadruples

$$(\underset{\in \mathrm{Bun}_n}{\mathcal{M}}, \underset{\in \mathrm{Bun}_n}{\mathcal{M}'}, x, \beta : \mathcal{M}' \hookrightarrow \mathcal{M}) \in \mathcal{H}ecke_i$$

s.t. $\mathcal{M}/\mathcal{M}' \cong \mathcal{O}_x^{\oplus i}$

$$\begin{tikzcd}
 & \mathcal{H}ecke_i \arrow[dl, "h^{\leftarrow}"'] \arrow[dr, "\mathrm{Supp} \times h^{\rightarrow}"] & \\
\mathrm{Bun}_n & & X \times \mathrm{Bun}_n
\end{tikzcd}$$

$$\mathcal{M} \leftarrow\!\shortmid (\mathcal{M}, \mathcal{M}', x, \beta) \mapsto (x, \mathcal{M}')$$

Hecke Correspondence: What does this look like? Fix x.

$n=2, i=1$

$$\begin{tikzcd} & \mathcal{H}ecke_{1,x} \arrow[dl, "h^{\leftarrow}"'] \arrow[dr, "h^{\rightarrow}"] & \\ Bun_2 & & Bun_2 \end{tikzcd}$$

$\mathcal{M}$

fiber over $\mathcal{M} \in Bun_2$ is $\mathcal{M}' \in Bun_2$ s.t. $\mathcal{M}/\mathcal{M}' \cong \mathcal{O}_x$

This is the same as choosing $\mathcal{L}_x$ line in $\mathcal{M}_x^*$

where $\mathcal{M}'(U) = \{ s \in \mathcal{M}(U) : \langle v, s(x) \rangle = 0 \;\; \forall v \in \mathcal{L}_x \setminus \{0\} \}$.

Thus fiber is $\mathbb{P}^1$ and $\mathcal{H}ecke_{1,x}$ is a $\mathbb{P}^1$-bundle over Bun_2 (on the left, it is also $\mathbb{P}^1$-bund on right)

Hecke Correspondence: What does this look like? Fix x.

Any n, i

$$\begin{tikzcd} & \mathcal{H}ecke_{i,x} \arrow[dl, "h^{\leftarrow}"'] \arrow[dr, "h^{\rightarrow}"] & \\ Bun_n & & Bun_n \end{tikzcd}$$

Similar arguement:

In general, $\mathcal{H}ecke_{i,x}$ is a $Gr(i,n)$ - fibration over the left and right Bun_n.

Hecke Eigensheaves:

$$\begin{tikzcd}
 & \mathcal{H}ecke \arrow[dl, "h^{\leftarrow}"'] \arrow[dr, "\mathrm{supp} \times h^{\rightarrow}"] & h^{\leftarrow *}\mathcal{K} \\
\mathcal{K} \quad Bun_n & & X \times Bun_n \\
 & & (\mathrm{supp} \times h^{\rightarrow})_* h^{\leftarrow *}(\mathcal{K})
\end{tikzcd}$$

Let $H_i(\mathcal{K}) = (\mathrm{supp} \times h^{\rightarrow})_* h^{\leftarrow *}(\mathcal{K})$ ith Hecke function

<u>Def:</u> Let E be a local system of rank n on X. A perverse sheaf $\mathcal{K}$ on Bun_n is a $\mathcal{H}ecke$ eigensheaf w/ eigenvalue E if $\mathcal{K} \neq 0$ and have isomorphisms for $i = 1, \dots, n$

$$\nu_i : H_n^i(\mathcal{K}) \xrightarrow{\sim} \Lambda^i E \boxtimes \mathcal{K}[-i(n-i)]$$

shift by dim fiber of $h^{\rightarrow}$

Hecke Eigensheaves Examples:

- If $G = GL(1)$, can construct a unique Hecke eigensheaf w/ eigenvalue E (rank 1 loc sys)

  (Ron will talk about next week?
  Two proofs in Frenkel's notes)

- Consider $\mathcal{K} = \underline{\mathbb{C}}$

$$H_i(\underline{\mathbb{C}}) = (\mathrm{supp} \times h^{\rightarrow})_* h^{\leftarrow *}(\underline{\mathbb{C}}) = (\mathrm{supp} \times h^{\rightarrow})_* (\underline{\mathbb{C}})$$

Since fibers of $\mathrm{supp} \times h^{\rightarrow}$ are $Gr(i,n)$

$$H_i(\mathbb{C}) = \underline{H^*(G(i,n), \mathbb{C})}$$

So $$H_i(\mathbb{C}) = \Lambda^i(\mathbb{C}[0] \oplus \mathbb{C}[-2] \oplus \dots \oplus \mathbb{C}[-2(n-1)])$$

$$\Rightarrow H_i(\mathbb{C}) \cong \Lambda^i E_0^i \boxtimes \underline{\mathbb{C}}[-i(n-i)]$$ where

$$E_0' = \underline{\mathbb{C}}[-(n-1)] \oplus \underline{\mathbb{C}}[-(n-3)] \oplus \dots \oplus \underline{\mathbb{C}}[n-1]$$

Generally, examples are hard . . .

Conformal field theory can be used to generate the eigensheaves . . .

Geometric Langlands over $\mathbb{C}$

X complex curve

$$\mathrm{Loc}^{ft}(X) \xleftrightarrow{1:1} \text{Hecke Eigensheaf}$$

$$E \xmapsto{?} \text{perverse sheaf } \mathrm{Aut}_E$$

that is a Hecke Eigensheaf w/ Hecke Eigenvalue E.

Geometric Langlands Statements:

- Deligne proved $G = GL(1)$ for all curves
- $\mathbb{P}^1$ ramified at 4 pts w/ $G = SL_2$
  Arinkin and Lysenko (10 pages)
- $\mathbb{P}^1$ ramified at 5 pts w/ $G = GL_2$
  Ron Donagi and Tony Pantev (243 pgs)
- (in progress) genus 2 curves w/ $G = GL_2$
  Ron and Tony (>300 pages)

PART 3

Geometric Satake

Recall Hecke Algebra:

$$\mathcal{H}(G(K), G(\mathcal{O})) = \{ f : GL_n(K) \to K \text{ cmptly supp and } f(x_1 g x_2) = f(g) \ \forall g \in G(K),\ x_1, x_2 \in G(\mathcal{O}) \}$$

Algebra w/ $f_1 * f_2 (g_2) = \int_{G(K)} f_1(g_1 \cdot g_2) f_2(g_1^{-1}) \, d\mu_{g_1}$

Has a basis given by double cosets $G(\mathcal{O}) \backslash G(K) / G(\mathcal{O})$

$$\chi_{G(\mathcal{O}) g G(\mathcal{O})} = \begin{cases} 1 & g \in G(\mathcal{O}) g G(\mathcal{O}) \\ 0 & g \notin G(\mathcal{O}) g G(\mathcal{O}) \end{cases}$$

We used the natural action of $\mathcal{H}$ on L^2 to define automorphic rep.

$T \overset{\text{max}}{\subset} G$ $\qquad$ $W = N(T)/T$

torus

$$\mathcal{H}(G(F), G(\mathcal{O})) \longleftrightarrow \mathcal{H}(T(F), T(\mathcal{O})) \cong \mathbb{C}[X_*(T)_+]$$

where $X_*(T)_+$ are the cochar of T, i.e. $\mathrm{Hom}(\mathbb{C}^*, T)$.
(that are dominant)

<u>Satake Isomorphism:</u> $\mathcal{H}(G(F), G(\mathcal{O})) \cong \mathbb{C}[X_*(T)_+]^W$

$$\cong \mathbb{C}[X^*(T^L)_+]^W$$

<u>Example:</u> $G = GL_n$, $W = S_n$, $X_*(T) = \{\lambda_1, \dots, \lambda_n\}$

$\lambda_i \mapsto \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & \lambda_i & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} \in T$

$$\mathcal{H}(G(F), G(\mathcal{O})) = \mathbb{C}[\lambda_1, \dots, \lambda_n]^{S_n}$$

$$= \mathbb{C}[T_1, \dots, T_{n-1}, T_n^{-1}]$$

T_i = ith symm poly in $\lambda_1, \dots, \lambda_n$

Affine Grassmannian: $F = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$

Let G linear reductive group $/\mathbb{C}$.

Def: The affine Grassmannian is the functor

$$\begin{aligned} Gr_G : (\mathbb{C}\text{-alg}) &\longrightarrow Set \\ R &\longmapsto G(R) = G(R((t)))/G(R[[t]]) \end{aligned}$$

Rmk: Not representable by schemes, but representable by inductive limit of schemes (ind scheme)

The $\mathbb{C}$-points are $G(\mathbb{C}((t)))/G(\mathbb{C}[[t]]) = G(F)/G(\mathcal{O})$

Affine Grassmannian: $\mathcal{O} = \mathbb{C}[[t]]$, $F = \mathbb{C}((t))$

$$G(\mathcal{O})/G(F) \underset{\text{natural stratification}}{=} \coprod_{\lambda \in X_*(T)_+} Gr^\lambda$$

where $Gr^\lambda = (G(\mathcal{O})\text{-orbit of } t^\lambda) = G(\mathcal{O}) \cdot t^\lambda G(\mathcal{O})$

(Think of $t^\lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ if $\lambda = (\lambda_1, \dots, \lambda_n)$)

Each Schubert variety $\overline{Gr^\lambda}$ is f.d. variety.

Example: $G = GL_1(\mathbb{C})$

$$Gr_{GL_1} = GL_1(\mathbb{C}((t)))/GL_1(\mathbb{C}[[t]]) = \mathbb{C}((t))^\times/\mathbb{C}[[t]]^\times$$

$$= (\mathbb{C}((t)) \setminus \{0\}) / \{a = \textstyle\sum a_i t^i \in \mathbb{C}[[t]] : a_0 \neq 0\}$$

$$= \{[t^n] : n \in \mathbb{Z}\} \cong \mathbb{Z}$$

Satake Iso $\mathcal{H}(G(F), G(\mathcal{O})) \cong \mathbb{C}[X_\bullet(T)_+]^W$

basis on double cosets $G(\mathcal{O}) \backslash G(F) / G(\mathcal{O})$

Geometric Satake:

$$\mathrm{Perv}^{G_{\mathcal{O}}}(\mathrm{Gr}_G) \cong \mathrm{Rep}({}^L G)$$

perverse sheaves over stratification $\mathrm{Gr}_G = \coprod_{\lambda \in X_\bullet(T)_+} \mathrm{Gr}^\lambda$

Can think of as over $G(\mathcal{O}) \backslash G(F) / G(\mathcal{O})$

Rmk: This is an isomorphism of <u>monoidal</u> categories

$$(\mathrm{Perv}^{G_{\mathcal{O}}}(\mathrm{Gr}_G), *) \xrightarrow{\sim} (\mathrm{Rep}({}^L G), \otimes).$$

some notion of convolution product

References:

- Lectures on the Langlands Program and Conformal Field Theory by Frenkel
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