FERMIONIC DIAGONAL COINVARIANTS

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ABSTRACT

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Let W be a complex reflection group of rank n acting on its reflection representation $V \cong \mathbb{C}^n$. The doubly graded action of W on the exterior algebra $\wedge(V \oplus V^*)$ induces an action on the quotient by the ideal generated by W-invariants with vanishing constant term $\mathrm{FDR}_W = \wedge(V \oplus V^*)/\langle \wedge(V \oplus V^*)^W_+ \rangle$. We describe the bi-graded W-module structure of FDR_W and introduce a variant of Motzkin paths that descends to the standard monomial basis of FDR_W with respect to a certain term order. The top degree of FDR_W exhibits the Narayana refinement of Catalan numbers. When $W = S_n$, the symmetric group, $\mathrm{FDR}_{S_n} \cong R_{n,0,2}$, where $R_{n,0,2}$ is the special case of the Boson-Fermionic diagonal coinvariants with two sets of Fermionic variables. In this case, the (i, j)-th degree component is a difference of Kronecker product of two hook Schur functions.

In addition we consider a module $M_{n,m}$ spanned by m-ary strings of length n. When $m = 2$, as a vector space, $M_{n,2} \cong \mathbb{C}[X_n]/\langle x_1^2, \ldots, x_n^2 \rangle$. The trivial component of $\mathrm{DR}_n \otimes M_{n,2}$ is a weighted sum of q,t-Narayana numbers which is a different q,t-Catalan number than the alternant of DR_n. At $t = 1$, the trivial component equals the inversion generating function for 321-avoiding permutations.

Contents

Chapter 1

Introduction

The classical coinvariant algebra R_n appears in many fields of mathematics. The Macdonald positivity conjecture led to study of the diagonal coinvariants DR_n, from which numerous problems and ideas in combinatorics, representation theory and geometry emerged ever since. Recently the idea of introducing Fermionic (anti-commuting) variables came up to consider Boson-Fermionic diagonal coinvariants with arbitrarily many sets of commuting and anti-commuting variables. The Chapter 2 will cover background materials in symmetric function theory to overview well-known problems and results about the classical coinvariant algebra and the diagonal coinvariants. In Chapter 3, we discuss the Delta conjecture and Boson-Fermionic generalization of the diagonal coinvariants which motivated materials in Chapter 4.

We then define the Fermionic diagonal coinvariants FDR_W and prove main results concerning FDR_W. The key to proofs of main results is the bi-graded version of Lefschetz element in the exterior algebra $\wedge\{\Theta_n, \Xi_n\}$. Some important themes appearing are Kronecker product of representations, the Narayana refinement of Catalan numbers and Motzkin paths. In the Chapter 5, we consider another S_n-module $M_{n,m}$ of m-ary strings with length n and its Kronecker product with the diagonal coinvariants, $\mathrm{DR}_n \otimes M_{n,m}$. We will observe how the common themes of the Narayana refinement and Motzkin paths appear in this Kronecker product as well.

Chapter 2

Diagonal Coinvariants

We cover some background materials to state some classical results for the coinvariant algebra and the diagonal coinvariants. These materials can be found in [4], [21], [31], [37], [40], [45], [46] and [47]. We follow [31] closely.

2.1 Symmetric polynomials

Let S_n denote the symmetric group of n letters. A polynomial $f(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$ is a *symmetric polynomial* if $w.f = f(x_{w_1}, \ldots, x_{w_n})$ for all $w = w_1 \cdots w_n \in S_n$. We denote by Λ_n the ring of symmetric polynomials in n independent variables. We will use X_n to denote the set of variables $\{x_1, \ldots, x_n\}$, X to denote $\{x_1, x_2, \ldots\}$ and write $f(X_n) = f(x_1, \ldots, x_n)$. Often we consider *symmetric functions* $f(X) = f(x_1, x_2, \ldots)$ which are power series in infinitely many variables invariant under any permutation of variables. Formally the ring of symmetric functions Λ is the inverse limit of the symmetric polynomial rings $\Lambda := \varprojlim \Lambda_n$. This ring Λ is naturally a graded algebra $\Lambda = \bigoplus_{d\geq 0} \Lambda^d$ where Λ^d consists of homogeneous elements of degree d. For symmetric functions, it is common to omit the set of variables as it's understood from the context.

An integer *partition* $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of n is a weakly decreasing sequence of positive integers summing up to n. Each λ_i s called a *part* of λ. The *size* of λ is denoted $|\lambda| = \sum_{i=0}^{\ell} \lambda_i$ and the number of parts ℓ is the *length* of λ. For a *weak composition* $\alpha = (\alpha_1, \ldots) \in \mathbb{Z}_{\geq 0}$, we define its size in the same way. We write $\alpha \sim \lambda$ if the nonzero parts of α rearrange to λ.

The ring Λ has many bases indexed by integer partitions. These bases include the *monomial* $\{m_\lambda\}$, *complete homogeneous* $\{h_\lambda\}$, *elementary* $\{e_\lambda\}$ and *power-sum* $\{p_\lambda\}$ basis of symmetric polynomials defined the following way:

$$m_\lambda = \sum_{\alpha\sim\lambda} x^\alpha, \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots, \tag{2.1.1}$$

$$h_k = \sum_{1\leq i_1\leq\cdots\leq i_k} x_{i_1}\cdots x_{i_k}, \quad h_\lambda = h_{\lambda_1}\cdots h_{\lambda_\ell}, \tag{2.1.2}$$

$$e_k = \sum_{1\leq i_1<\cdots<i_k} x_{i_1}\cdots x_{i_k}, \quad e_\lambda = e_{\lambda_1}\cdots e_{\lambda_\ell}, \tag{2.1.3}$$

$$p_k = \sum_{i\geq 1} x_i^k, \quad p_\lambda = p_{\lambda_1}\cdots p_{\lambda_\ell}. \tag{2.1.4}$$

For a partition λ, define $z_\lambda = \prod_i i^{m_i} m_i!$ where m_i is the multiplicity of i in λ. The quantity z_λ is the index of a centralizer of an element in S_n with cycle type λ. In addition to the ordinary multiplication, there is another operation called

Kronecker product defined on the power-sum basis as follows

$$p_\lambda \otimes p_\mu = z_\lambda \delta_{\lambda,\mu} p_\lambda, \tag{2.1.5}$$

where $\delta_{\lambda,\mu}$ is the Kronecker delta evaluating to 1 if $\lambda = \mu$ and else to 0. We also define the *Hall inner product* on Λ on the basis of power-sums

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda,\mu}. \tag{2.1.6}$$

The monomial basis and the homogeneous basis are dual with respect to the Hall inner product

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu}. \tag{2.1.7}$$

Now we define *Schur functions* which form an orthonormal basis of Λ with respect to the Hall inner product. Given a partition λ, we visualize λ by its *diagram* $dg(\lambda)$ which is a left justified alignment of rows of boxes. Rows are counted from bottom to up and the i-th row consists of λ_i many boxes. A *semi-standard Young tableau*, abbreviated by SSYT, of shape λ is a filling of the boxes in $dg(\lambda)$ by nonnegative integers such that filling are strictly increasing in columns and weakly increasing in rows. If we assert that rows are also strictly increasing, we get a *standard Young tableau*. We will write $\mathrm{sh}(T) = \lambda$ to denote the *shape* of T. In another way, a filling T of λ is a map $T : dg(\lambda) \to \mathbb{Z}_{>0}$. Given a semi-standard Young tableaux T of λ, define $x^T = \prod_{c \in dg(\lambda)} x_{T(c)}$. The Schur function indexed by

λ is a generating function for SSYT of shape λ

$$s_\lambda = \sum_{T:\text{shape}(T)=\lambda} x^T. \tag{2.1.8}$$

Schur functions form a self-dual orthonormal basis, $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$. Hence, for any symmetric function f, we may write $f = \sum_\lambda \langle f, s_\lambda \rangle s_\lambda$. More important, Schur functions correspond to the irreducible representations of the symmetric group which we will introduce in the following section.

To express identities in results in diagonal coinvariants, we define *plethystic notation*. Let $E(t_1, t_2, \ldots)$ be a formal series of rational functions in variables t_i. We define the plethystic substitution on the power-sum basis as an algebra homomorphism $p_k[E] = E(t_1^k, t_2^k, \ldots)$. The following examples show how the plethystic substitution is different from the ordinary substitution of variables. We have

$$p_k[X(1-q)] = p_k[X] - p_k[qX] = p_k[X] - q^k p_k[X], \tag{2.1.9}$$

$$p_k[\frac{X}{1-q}] = \sum_i \frac{x_i^k}{1-q^k}. \tag{2.1.10}$$

2.2 Representation theory of S_n

Simply speaking, a linear representation of a finite group is a realization of the group action via matrix representation. Let G be a finite group and V be a $\mathbb{C}$-vector space. A *linear representation* of G is a group homomorphism $\rho : G \to GL(V) = Aut(V)$. An action of G on V induces a linear representation of G and a linear representation $\rho : G \to GL(V)$ endows V a $\mathbb{C}[G]$-module structure. Hence we will use these terms interchangeably without confusion.

Let V be a G-module and W be a subspace of V. We say that W is *G-invariant* if $gW \subseteq W$ for all $g \in G$. If W is G-invariant, it's a G-module itself with the corresponding sub-representation to $GL(W)$. A G-module V is *irreducible* if the 0-dimensional subspace and V itself are the only G-invariable subspaces. As a corollary to Maschke's Theorem, we have the complete reducibility

Theorem 2.2.1 (Maschke)**.** Let G be a finite group. Any G-module V over $\mathbb{C}$ is a direct sum of irreducible submodules.

Let $H \leq G$ be a subgroup and V a G-module. Then H inherits the action of G and thus V is naturally a H-module as well. As a representation, we denote this restriction by $\mathrm{Res}_H^G(V)$. On the other hand, let V be a H-module where $H \leq G$. Then this representation induces a G-module $\mathrm{Ind}_H^G(V)$. Note that this induction is actually an adjoint to the restriction.

For a finite group G, irreducible representations correspond bijectively to conjugacy classes of G. In the case of $G = S_n$, the conjugacy classes of S_n are indexed by integer partitions of n and so are the irreducible representations of S_n. Denote by V^λ the irreducible representation of S_n corresponding to $\lambda \vdash n$. The *Grothendieck group* $\mathcal{R}_n$ is a free $\mathbb{Z}$-algebra generated by the set of isomorphism classes of irreducible representations of S_n where addition is the direct sum of modules. Let $\mathcal{R} = \bigoplus_{n\geq 0} \mathcal{R}_n$ with $\mathcal{R}_0$ being $\mathbb{Z}$. Then $\mathcal{R}$ is the *Grothendieck ring* of S_n with multiplication defined by the induction: let $[V] \in \mathcal{R}_n$ and $[W] \in \mathcal{R}_m$, then

$$[V] * [W] = [\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} V \otimes W]. \tag{2.2.1}$$

Analogous to the Hall inner product, an inner product on $\mathcal{R}$ is defined $\langle V^\lambda, V^\mu \rangle = \delta_{\lambda\mu}$. The multiplicity of an irreducible component V^λ in V is therefore $\langle V, V^\lambda \rangle$. So we have an irreducible decomposition $V = \bigoplus_\lambda c_\lambda V^\lambda = \bigoplus_\lambda \langle V, V^\lambda \rangle V^\lambda$. For such V, its *Frobenius image* is the symmetric function $\mathrm{Frob}(V) = \sum_\lambda c_\lambda s_\lambda$. In fact, this map gives an equivalence of category

Theorem 2.2.2. The ring of symmetric functions Λ and the Grothendieck ring $\mathcal{R}$ are equivalent via $s_\lambda \leftrightarrow V^\lambda$.

Let $V = \bigoplus_i V^i$ be a graded S_n-module where V^i consists of all homogeneous elements of degree i. We define the *Hilbert series* $\mathrm{Hilb}(V; q)$ and the *Frobenius series* $\mathrm{Frob}(V; q)$ by

$$\mathrm{Hilb}(V; q) = \sum_i q^i \dim(V^i), \tag{2.2.2}$$

$$\mathrm{Frob}(V; q) = \sum_i q^i \mathrm{Frob}(V^i) = \sum_i q^i \left(\sum_\lambda \langle V^i, V^\lambda \rangle s_\lambda \right). \tag{2.2.3}$$

For multi-graded modules, above definitions extend naturally. The main theme of the problems in this thesis will be understanding the S_n-structure and the graded dimension of various S_n-modules.

2.3 Classical coinvariant algebra R_n

Let $\mathbb{C}[X_n]$ denote the polynomial ring in n variables. The symmetric group S_n acts naturally by permuting indices of variables. Denote by $\mathbb{C}[X_n]^{S_n}$ be subspace of symmetric polynomials and $\mathbb{C}[X_n]_+^{S_n}$ the subspace of symmetric polynomials with vanishing constant term. Then the classical *coinvariant algebra* R_n is defined by

$$R_n = \mathbb{C}[X_n]/\langle \mathbb{C}[X_n]_+^{S_n} \rangle = \mathbb{C}[X_n]/\langle p_1, \ldots, p_n \rangle. \tag{2.3.1}$$

The space R_n is a graded S_n-algebra with the following properties.

1. (**Borel**) $R_n \cong H^*(Fl_n)$ where Fl_n is the complete flag variety [11].

2. (**Chevalley**) As an ungraded S_n-module, $R_n \cong \mathbb{C}[S_n]$ and $\dim(R_n) = n!$ [16].

3. (**Artin**) R_n has a basis $\left\{\prod_{1 \le i \le n} x_i^{a_i} \mid 0 \le a_i \le i-1\right\}$ called the *Artin basis* of monomials bounded by a staircase [1].

4. (**Stanley-Lusztig**) Let $\mathrm{maj}(-)$ denote the *major index* statistic defined on standard Young tableaux. The S_n-structure of R_n is given by

$$\mathrm{Frob}(R_n; q) = \sum_{T \in SYT(n)} q^{\mathrm{maj}(T)} s_{\mathrm{sh}(T)} \tag{2.3.2}$$

   where $SYT(n)$ is the set of all standard Young tableaux of size n [48].

Besides the Artin basis, R_n has another basis whose elements are indexed by permutations in S_n itself. For $w \in S_n$, define its *descent set* $\mathrm{Des}(w) = \{i \mid w_i > w_{i+1}\}$.

The *Garsia-Stanton basis* of R_n is defined

$$\left\{ \prod_{i \in \mathrm{Des}(w)} x_{w_1} \cdots x_{w_i} \mid w \in S_n \right\}, \tag{2.3.3}$$

as a set of monomials indexed by partitions [25].

In the remaining part of this thesis, we will study generalizations and variants of the algebra R_n and discuss similar results to understand their algebraic structures.

2.4 Diagonal Coinvariants

In this section we give a brief overview of how the theory of diagonal coinvariants developed. Some early works can be found in [23], [24], [27].

The *diagonal coinvariants* DR_n is an extension of R_n by introducing an extra set of variables Y_n. Let $\mathbb{C}[X_n, Y_n]$ denote the polynomial ring in $2n$ variables. Then S_n acts on $\mathbb{C}[X_n, Y_n]$ *diagonally* as a copy of $\Delta S_n \leq S_n \times S_n$, permuting the indices of X variables and Y variables simultaneously. Similarly, $\mathbb{C}[X_n, Y_n]^{S_n}$ denotes the subspace of S_n-invariants and $\mathbb{C}[X_n, Y_n]_+^{S_n}$ denotes the subspace of S_n-invariants with vanishing constant term. The *diagonal coinvariant* algebra is defined similarly by

$$\mathrm{DR}_n = \mathbb{C}[X_n, Y_n] / \langle \mathbb{C}[X_n, Y_n]_+^{S_n} \rangle. \tag{2.4.1}$$

To understand the development of studies of DR_n, we will discuss the *Macdonald polynomials* and related conjectures, which are now mostly theorems. Until now we assumed that our base field is $\mathbb{C}$ for Λ. We consider the scalar extension $\Lambda(q, t) =$

$\Lambda \otimes \mathbb{C}(q,t)$, ring of symmetric functions with coefficients in the field of rational functions in q and t. Macdonald introduced an orthogonal basis of this extension $\{P_\lambda(X;q,t)\}$ [41]. Macdonald polynomials specialize to various bases. For example, $P_\lambda(X;q,q) = s_\lambda(X)$. Macdonald polynomials $P_\lambda(X;q,t)$ have many interesting properties and transformations. We define some quantities on partitions first. For a cell $c \in \mathrm{dg}(\lambda)$, its *arm length* $a(c)$, *coarm length* $a'(c)$, *leg length* $l(c)$ and *coleg length* $l'(c)$ are the number of cells between c and the border of $\mathrm{dg}(\lambda)$. exclusive of c. The quantity $n(\lambda) = \sum_i (i-1)\lambda_i$ is also used often. We have

1. **(Integral form)** $J_\lambda(X;q,t) = \prod_{c\in \mathrm{dg}(\lambda)}(1-q^{a(c)}t^{l(c)+1})P_\lambda(X;q,t)$,
2. **(Modified Macdonald)** $H_\lambda(X;q,t) = J_\lambda[X(1-t)^{-1};q,t]$,
3. **(Modified Macdonald)** $\tilde{H}_\lambda(X;q,t) = t^{n(\lambda)}H_\lambda[X;q,t^{-1}] = \sum_\mu \tilde{K}_{\mu,\lambda}(q,t)s_\mu$.

A symmetric function $f(X;q,t) = \sum_\lambda c_\lambda(q,t)s_\lambda(X) \in \Lambda(q,t)$ is *Schur-positive* if $c_\lambda \in \mathbb{N}[q,t]$ for all λ. In terms of the modified Macdonald, the *Macdonald positivity conjecture* states that $\tilde{H}_\lambda$ is Schur-positive. One way show that a symmetric function is Schur-positive is to show that it is a Frobenius image of an S_n-module. In an attempt to solve the positivity conjecture, Garsia and Haiman introduced modules $V(\lambda)$ affording the left regular representation of S_n such that the Frobenius image is conjecturally $\mathrm{Frob}(V(\lambda);q,t) = \tilde{H}_\lambda$. This conjecture is known to be the $n!$-*conjecture* and implies the Macdonald positivity. The modules (also rings) $V(\lambda)$ are graded quotients of $\mathbb{C}[X,Y]$ such that the only appearance of the trivial representation is the constants $\mathbb{C}$. Hence, each $V(\lambda)$ is in fact a quotient of DR_n which led Garsia and Haiman to study DR_n in their early investigation. Similar to the $n!$-conjecture, they formulated the $(n+1)^{n-1}$-*conjecture* which states that the dimension of DR_n is $(n+1)^{n-1}$.

We define some quantities first. Let $M = (1-q)(1-t)$. For a partition λ, define

$$T_\lambda = t^{n(\lambda)} q^{n(\lambda')}, \quad \omega_\lambda = \prod_{c \in \mathrm{dg}(\lambda)} (q^{a(c)} - t^{l(c)+1})(t^{l(c)} - q^{a(c)+1}), \tag{2.4.2}$$

$$B_\lambda = \sum_{c \in \mathrm{dg}(\lambda)} q^{a'(c)} t^{l'(c)}, \quad \Pi_\lambda = \prod_{c \in \mathrm{dg}(\lambda)}{}' (1 - q^{a'(c)} t^{l'(c)}), \tag{2.4.3}$$

where the prime over the product means that we exclude the cell with zero coarm and zero coleg length. In a series of works [28], [29], [30], Haiman proved the $n!$-conjecture and the $(n+1)^{n-1}$-conjecture. Furthermore, he proved the following Frobenius image expansion

Theorem 2.4.1 (Haiman 2002)**.** For $n \geq 1$,

$$\mathrm{Frob}(\mathrm{DR}_n; q, t) = \sum_{\lambda \vdash n} \frac{T_\lambda M \Pi_\lambda B_\lambda \tilde{H}_\lambda}{\omega_\lambda}. \tag{2.4.4}$$

providing the Macdonald expansion of the symmetric function $\mathrm{Frob}(\mathrm{DR}_n; q, t)$. Haiman's work involved heavy geometry of *Hilbert Schemes*. This expansion is written in terms of the *nabla operator*, ∇, which is an eigenoperator on the modified Macdonald basis $\{\tilde{H}_\lambda\}$ [8]:

$$\nabla \tilde{H}_\lambda = T_\lambda \tilde{H}_\lambda. \tag{2.4.5}$$

From the *Cauchy kernel*, it is shown that

$$e_n = \sum_{\lambda \vdash n} \frac{M \Pi_\lambda B_\lambda \tilde{H}_\lambda}{\omega_\lambda}. \tag{2.4.6}$$

Hence $\nabla e_n = \mathrm{Frob}(\mathrm{DR}_n; q, t)$.

So far we discussed how the diagonal coinvariants were introduced to symmetric function theory and we saw the link between the algebraic (representation theoretic) side and the symmetric function side of the DR_n. In fact, there is another side of the DR_n, the combinatorial side. The dimension $(n+1)^{n-1}$ is an interesting sequence because it counts the number of *parking functions* on n cars.

On the combinatorial side, two main objects are *Dyck paths* and *parking functions*, realized as certain labellings on Dyck paths. A Dyck path of size n is a lattice path from $(0,0)$ to (n,n) consisting of east steps $(1,0)$ and north steps $(0,1)$ that stays weakly above the diagonal $y=x$ line. It is well known that the number of Dyck paths from $(0,0)$ to (n,n) is the *Catalan number* $\frac{1}{n+1}\binom{2n}{n}$. In fact, dimension of the alternating component of DR_n is the Catalan number and is now known as the q,t-Catalan number [22]. A parking function is a labeled Dyck path which is a labeling of the north steps of a Dyck path by distinct integers in $\{1,\ldots,n\}$ such that the labels in each column is increasing (bottom up). We denote by $D(n)$ the set of Dyck paths of size n, $P(\pi)$ the set of parking functions of underlying path π and $P(n) = \bigcup_\pi P(\pi)$.

Given a Dyck path, labeled or not, its rows are numbered from bottom to top. We define the *area sequence* $(a_1,\ldots,a_n)$ where a_i is the number of complete cells between π and $y = x$ in row i. An *area* statistic of a Dyck path or a parking function P is simply the sum of the area sequence, $\text{area}(P) = \sum_i a_i$ which is the total number of complete cells between the underlying path and the main diagonal. Next we define the *dinv*, diagonal inversion, statistic. Given $P \in P(n)$, let l_i denote the label on the north step in row i. Then

$$\begin{aligned} \text{dinv}(P) = & |\{(i,j) \mid 1 \leq i < j \leq n, a_i = a_j \text{ and } l_i < l_j\}| && (2.4.7) \\ & + |\{(i,j) \mid 1 \leq i < j \leq n, a_i = a_j + 1 \text{ and } l_i > l_j\}|. && (2.4.8) \end{aligned}$$

On Dyck paths, dinv is defined the same way without the condition on labels. The pairs counted in the first term are called the primary dinvs and in the second term are called the secondary dinvs. If we define a reading word on parking functions by reading diagonals from top to bottom and each diagonal in SW direction, then the dinv statistic in fact counts the number of pairs of close labels which are inversions.

A *word parking function* is an extension of parking functions allowing repeated labels. The definition of area and dinv statistics extend naturally to word parking functions. Let $WP(n)$ denote the set of word parking functions. Define the *weight* of P to be $\text{wt}(P) = \prod_i x_{l_i}$, the product indicating the number of occurrences of label k in P. Now we can state the *Shuffle Theorem* which was conjectured in [32] and proved after 10 years by [13].

Theorem 2.4.2 (Shuffle Theorem)**.** For $n \geq 1$,

$$\nabla e_n = \sum_{P \in WP(n)} q^{\text{dinv}(P)} t^{\text{area}(P)} \text{wt}(P). \tag{2.4.9}$$

Given a partition (or composition) μ of n, we say that $w = w_1 w_2 \cdots w_n \in S_n$ is a μ-shuffle if it is a shuffle of each of the increasing sequences $[1, 2, \ldots, \mu_1]$, $[\mu_1 + 1, \ldots, \mu_1 + \mu_2], \ldots$ of lengths $\mu_1, \mu_2, \ldots$. Given a word parking function $P \in WP(n)$ with μ_1 many 1's and so on, we can *standardize* P into a parking function by replacing 1's by $1, 2, \ldots, \mu_1$, all the 2's by $\mu_1 + 1, \ldots, \mu_1 + \mu_2$ and so on such that the resulting parking function has its reading word a shuffle of $\mu = (\mu_1, \mu_2, \ldots)$. In this case, we say that P is a μ-shuffle. We state an equivalent statement of the Shuffle Theorem justifying its name

Theorem 2.4.3 (Shuffle Theorem)**.** For $n \geq 1$ and $\mu \vdash n$,

$$\langle \nabla e_n, h_\mu \rangle = \sum_{\substack{P \in WP(n) \\ P \text{a } \mu-\text{shuffle}}} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)}. \tag{2.4.10}$$

Notice how we tried to discuss the results that are parallel to the results on the classical coinvariants. Before we finish this overview, we briefly mention a basis of the DR_n due to Carlsson-Oblomkov [14]. Given a permutation $w \in S_n$, break w into *runs*, increasing subsequences. Define the schedule numbers $w_i(w)$ by the number of $w_j > w_i$ in the same run as w_i greater plus the the number of $w_j < w_i$ in the run after w_i's run. By convention, we attach a run of 0 to w. For example, for $w = 24|3|167|5|0$, runs indicated by bars, the associated schedule numbers in a sequence are

$$(w_i(w)) = (1, 1, 1, 2, 2, 1, 1). \tag{2.4.11}$$

Define a q-number $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$. Then a monomial basis of DR_n is given

Theorem 2.4.4 (Carlsson-Oblomkov)**.** The monomials in

$$\prod_{i \in \mathrm{Des}(w)} \left(\prod_{1 \le j \le i} y_{w_j} \right) \prod_{i=1}^{n} [w_i(w)]_{x_{w_i}}$$

for w ranging over S_n form a monomial basis of DR_n.

Example 2.4.5. For $n = 3$, six permutations produce

$$w = 123 : [3]_{x_1}[2]_{x_2} = 1 + x_1 + x_2 + x_1^2 + x_1x_2 + x_1^2x_2, \quad w = 132 : y_1y_3, \tag{2.4.12}$$

$$w = 213 : y_2[2]_{x_1} = y_2 + y_2x_1, \quad w = 231 : y_2y_3[2]_{x_2} = y_2y_3 + y_2y_3x_2, \tag{2.4.13}$$

$$w = 312 : y_3[2]_{x_3}[2]_{x_1} = y_3 + y_3x_1 + y_3x_3 + y_3x_1x_3, \quad w = 321 : y_3^2y_2, \tag{2.4.14}$$

$$B = \{1, x_1, x_2, x_1^2, x_1x_2, x_1^2x_2, y_2, y_3, y_1y_3, y_2y_3, y_2y_3^2, \tag{2.4.15}$$

$$y_2x_1, y_2y_3x_2, y_3x_1, y_3x_3, y_3x_1x_3\},$$

so the set B is a basis of DR_3. Note that setting all x_i's to 0 yields the Garsia-Stanton basis and setting all y_i's to 0 yields the reversed Artin basis.

Chapter 3

Boson-Fermionic generalization

3.1 The Delta Conjecture

In an attempt to prove the shuffle conjecture, various refinements were introduced. In 2015, Haglund, Remmel and Wilson formulated the *Delta conjecture*, now a theorem, which introduced an extra parameter of z into the combinatorial side of the theorem [33]. On the symmetric function side, the nabla operator is replaced by the more general *Delta operator*. Given $f \in \Lambda$, define eigenoperators Δ_f and Δ'_f as

$$\Delta_f \tilde{H}_\lambda = f[B_\lambda]\tilde{H}_\lambda, \tag{3.1.1}$$

$$\Delta'_f \tilde{H}_\lambda = f[B_\lambda - 1]\tilde{H}_\lambda. \tag{3.1.2}$$

When $f = e_n$, then $\nabla = \Delta_{e_n}$. On the combinatorial side, there are two versions: rise and valley. Here we will discuss the rise version which is recently proved by two independent groups [17], [9], [10]. A *rise* of a Dyck path is a N-step preceded by another N-step. A *rise-decorated (word) parking function* is a (word) parking function with some of its rise steps decorated. In terms of area sequences, the set of rises is $\mathrm{Rise}(P) = \{2 \leq i < n \mid a_i > a_{i-1}\}$. A dinv of a rise-decorated (word) parking function is defined the same way, but the area is defined as $\mathrm{area}(P) = \sum_{i \notin \mathrm{Rise}(P)} a_i$ where we do not include the area of decorated rows. Let $WP(n)^{*k}$ denote the set of decorated word parking functions with k decorations. Then the rise-version of the Delta conjecture says

Theorem 3.1.1 ([17], [9], [10])**.** For $n \geq 1$ and $0 \leq k \leq n-1$,

$$\Delta'_{e_{n-k-1}} e_n = \sum_{P \in WP(n)^{*k}} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} \mathrm{wt}(P). \tag{3.1.3}$$

Parallel to results on the classical and diagonal coinvariants, Haglund and Sergel conjectured a monomial basis associated to the valley version [34]. This conjectured basis builds up on the Carlsson-Oblomkov basis extending the idea behind the schedule formula. On the representation theoretic side, in 2019 Zabrocki introduced the idea of *Fermionic variables* [51]. Let $P_{n,2,1} = \mathbb{C}[X_n, Y_n] \otimes \wedge\{\Theta_n\}$ where $\wedge\{\Theta_n\}$ is the exterior algebra generated by $\{\theta_1, \ldots, \theta_n\}$. In $P_{n,2,1}$, the X_n and Y_n variables commute among themselves and commute with Θ_n variables. However the Θ_n variables anti-commute

$$\theta_i \theta_j = -\theta_j \theta_i \text{ and } \theta_i^2 = 0 \text{ for all } i, j. \tag{3.1.4}$$

As for the diagonal coinvariants, S_n acts on $P_{n,2,1}$ diagonally, permuting the indices simultaneously. Let $I_{n,2,1} = \langle (P_{n,2,1})_+^{S_n} \rangle$ be the ideal generated by the subspace of the invariants with vanishing constant term. Then the *super-diagonal coinvariants* is defined as the quotient $R_{n,2,1} = P_{n,2,1}/I_{n,2,1}$. The ideal $I_{n,2,1}$ is generated by the power-sum like elements [42],

$$p_{r,s} = \sum_{i=1}^{n} x_i^r y_i^s \text{ for } 0 < r+s \leq n, \tag{3.1.5}$$

$$\tilde{p}_{p,q} = \sum_{i=1}^{n} x_i^p y_i^q \theta_i \text{ for } 0 \leq p+q < n. \tag{3.1.6}$$

The commuting variables are called *Bosonic* and the anticommuting variables are called *Fermionic* following the terminology used in physics. The module $R_{n,2,1}$ is tri-graded in a natural way by degrees in each set of variables. Zabrocki conjectured that the character of this module corresponds to the sum of symmetric functions from the Delta conjecture:

$$\mathrm{Frob}(R_{n,2,1}; q, t, z) = \sum_{k=0}^{n-1} z^k \Delta'_{e_{n-k-1}} e_n. \tag{3.1.7}$$

At $z = 0$, this conjecture reduces to Haiman's theorem on the diagonal coinvariants and coefficients of z^k are symmetric functions of the Delta conjecture.

3.2 Boson-Fermionic generalization

A natural extension of the super-diagonal coinvariants is to introduce more sets of Bosonic or Fermionic variables. Indeed, F. Bergeron described many conjectures regarding this extension [5], [6], [7]. To explain these conjectures, we need some more notations and introduce the idea of $GL_k \times S_n$ action. Let X and Θ denote matrices of variables. Let X have dimension $k \times n$ where we view each row as a series of n many Bosonic variables. Similarly, let Θ have dimension $j \times n$ and each row is a series of n many Fermionic variables. Let $P_{n,j,k} = \mathbb{C}[X] \otimes \wedge\{\Theta\}$ denote the ring of polynomials as for $P_{n,2,1}$. Then $GL_j \times GL_k \times S_n$ acts on $P_{n,j,k}$: a triple $(P, Q, w) \in GL_j \times GL_k \times S_n$ acts on $f \in P_{n,j,k}$ by

$$(P, Q, w).f(X; \Theta) = f(PXw; Q\Theta w). \tag{3.2.1}$$

Taking P and Q to be the identity, this action reduces to the usual action of S_n. Denote by $(P_{n,j,k}^{S_n})_+$ the subspace of invariants with vanishing constant terms under the S_n action and $I_{n,j,k} = \langle (P_{n,j,k}^{S_n})_+ \rangle$. Then the general Boson-Fermionic diagonal coinvariants is defined $R_{n,j,k} = P_{n,j,k}/I_{n,j,k}$. So far we used the Frobenius character to encode the S_n-structure of the coinvariants. The purpose of introducing the $GL_j \times GL_k$ action is to view the grading coefficients as plethystic evaluations of products of Schur functions, which are characters of $GL_j \times GL_k$ as well. The Frobenius character of a $GL_j \times GL_k \times S_n$ module, say V, is then

$$\mathrm{Frob}(V) = \sum_{\lambda \vdash n} \sum_{\substack{\ell(\mu) \leq j, \\ \ell(\rho) \leq k}} c_{\mu\rho\lambda} s_\mu(Q) s_\rho(U) s_\lambda(X). \tag{3.2.2}$$

For modules like this, if the coefficients $c_{\mu\rho\lambda}$ do not depend on j and k, such modules are called *coefficient stable*. The dependence on j and k are shown in $\ell(\mu) \leq j$ and $\ell(\rho) \leq k$: if the number of variables is too small, then the plethystic evaluation of Schur function "loses" some terms. For coefficient stable V, then we may write the

Frobenius character as follows

$$\mathrm{Frob}(V) = \sum_{\lambda\vdash n} \sum_{\substack{\ell(\mu)\leq j,\\ \ell(\rho)\leq k}} s_\mu \otimes s_\rho \otimes s_\lambda. \tag{3.2.3}$$

F. Bergeron in fact showed that the purely Bosonic coinvariants are coefficient stable whose character is denoted by

$$\mathcal{E}_n = \sum_{\lambda\vdash n} \sum_{\mu} c_{\mu\lambda} s_\mu \otimes s_\lambda. \tag{3.2.4}$$

It means that by plethystically substituting $Q_j = q_1 + \cdots + q_j$, we have

$$\mathrm{Frob}(R_{n,j,0}; Q_j) = \sum_{\lambda\vdash n} \sum_{\mu} c_{\mu\lambda} s_\mu[Q_j] s_\lambda[X]. \tag{3.2.5}$$

Let ϵ be a formal variable defined by $p_i[\epsilon] = (-1)^i$. Then the supersymmetry conjecture states

Conjecture 3.2.1. The stable expression for the purely Bosonic coinvariants implies the character of the Boson-Fermionic coinvariants:

$$\mathrm{Frob}(R_{n,j,k}; Q, U) = \mathcal{E}_n[Q - \epsilon U; x] \tag{3.2.6}$$

$$= \sum_{\lambda\vdash n} \sum_{\mu} c_{\mu\lambda} s_\mu[Q - \epsilon U] s_\lambda[X]. \tag{3.2.7}$$

Hence, understanding the purely Bosonic diagonal coinvariants should be sufficient to understand the general case. The conjecture also implies that knowledge of purely Fermionic diagonal coinvariants would determine the general case as well which is a reason why it is called *combinatorial supersymmetry conjecture*, see [43].

3.3 Theta operator

So far in this Chapter we focused on the representation theoretic side of the diagonal coinvariants. In this section we briefly mention the *Theta operators* which are studied to explain or conjecture the symmetric function side [19]. Let Π denote an eigenoperator defined by $\Pi\tilde{H}_\lambda = \Pi_\lambda \tilde{H}_\lambda$. Let $f \in \Lambda$. Then for $g[X] \in \Lambda$, the Theta operator Θ_f is defined by

$$\Theta_f\, g[X] = \Pi f\left[\frac{X}{(1-q)(1-t)}\right]\Pi^{-1} g[X]. \tag{3.3.1}$$

The symbol Θ implies that this operator has a close connection to the Fermionic variables. In fact, D'Adderio et. al. prove

Theorem 3.3.1. For $n \geq 1$ and $0 \leq k \leq n-1$,

$$\Theta_{e_k}\nabla e_n = \Delta'_{e_{n-k-1}} e_n. \tag{3.3.2}$$

D'Adderio et. al. conjecture the representation theoretic and the symmetric function side of the super diagonal coinvariants, claiming that the graded components by the Fermionic variables are computed by the Theta operators.

Conjecture 3.3.1. For $n \geq 1$,

$$\mathrm{Frob}(R_{n,2,2}; q, t, u, v) = \sum_{\substack{i,j\geq 0\\ 0\leq i+j\leq n-1}} u^i v^j \Theta_{e_i}\Theta_{e_j}\nabla e_{n-i-j}. \tag{3.3.3}$$

In our discussion of the Delta conjecture, we used the Rise-version. Given a parking function P, label l_i in the i-th row is a *contractible valley* if either $a_i < a_{i-1}$

or $a_i = a_{i-1}$ and $l_i > l_{i-1}$. Consider $WP(n)^{*k,\odot r}$ the set of decorated word parking functions with k decorated rises and r decorated contractible valleys. Then the Theta conjecture combinatorially interprets at $q = 1$ the (r, k)-Fermionic component of the super diagonal coinvariants by interpolating between the rise version and the valley version of the Delta conjecture.

Conjecture 3.3.2. For $n \geq 1$ and $0 \leq r + k \leq n - 1$,

$$\Theta_{e_r}\Theta_{e_k}\nabla e_{n-r-k}\,|_{q=1} = \sum_{P \in WP(n)^{*k,\odot r}} t^{\mathrm{area}(P)}\mathrm{wt}(P). \tag{3.3.4}$$

3.4 Extension to other finite groups

The Boson-Fermionic generalization extends the coinvariants by generalizing the space that the group S_n is acting on. Another possibility of extension is to replace S_n by other finite groups. In his CDM paper [30], Haiman suggests two directions: first to consider complex reflection groups acting on their defining representations and second to look at wreath products of S_n. Let G be a Coxeter group with Coxeter number h and rank n. Let V be its defining (permutation) representation with V^* its dual. Then G acts on $V \oplus V^*$ and $\mathbb{C}[V \oplus V^*]$ which is viewed as polynomial ring in two sets of variables. Define the coinvariant ring $C^G = \mathbb{C}[V \oplus V^*]/(\mathbb{C}[V \oplus V^*]^G_+)$. I. Gordon proved some results on this extension and computed the dimension using the representation theory of *double affine Hecke algebras* [26].

Theorem 3.4.1. There is a canonically defined doubly graded quotient ring R^G of the coinvariants C^G with $\dim R^G = (h+1)^n$ and $\mathrm{Hilb}(R^G; t^{-1}, t) = t^{-hn/2}[h+1]_t^n$.

When $G = S_n$, the Coxeter number is n and rank is $n-1$, so the theorem implies that $R^G = C^G$ with dimension $(n+1)^{n-1}$ which is the result of Haiman.

Chapter 4

Fermionic Diagonal Coinvariants

We had an overview of the development of various generalizations of the coinvariant rings. In this Chapter, we finally discuss the *Fermionic diagonal coinvariants* [38].

Let's first discuss the simplest case of $R_{n,0,1}$ with a single set of Fermions.

Proposition 4.0.1. For $n \geq 1$,

$$\mathrm{Frob}(\wedge\{\Theta_n\}; q) = \sum_{k=0}^{n} q^k h_{n-k} e_k = \sum_{k=0}^{n-1} (q^k + q^{k+1}) s_{(n-k,1^k)}. \tag{4.0.1}$$

Proof. Note that the degree k component is $\mathrm{Ind}_{S_k \times S_{n-k}} V^{(1^k)} \otimes V^{(n-k)}$, an induction product of the sign and trivial representations. The claim follows by simply taking the Frobenius character linearly.

Another way to check is to use the basis of elements of the form $\theta_i - \theta_{i+1}$ and $p_1[\Theta_n]$ in the degree 1 component $\wedge\{\Theta_n\}_1$. Then the S_n-action on degree 1 component decomposes into $V^{(n-1,1)} \oplus V^{(n)}$. Hence the proposition follows by inductively taking wedge products. $\square$

Similar to the second idea of the proof of Proposition 4.0.1, it is known that $I_{n,0,1} = \langle p_1[\Theta_n] \rangle$. Writing $\alpha_i = \theta_i - \theta_{i+1}$, the degree k component has a basis $\{\prod_{j=1}^{k} \alpha_{i_j}\} \cup \{p_1[\Theta_n] \prod_{j=1}^{k-1} \alpha_{i_j}\}$. By multiplying $p_1[\Theta_n]$, elements in the second set become 0 and the first set precisely becomes the second set for the degree $k+1$ component. So in terms of S_n-character, we obtain

$$\mathrm{Frob}(R_{n,0,1}) = \sum_{k=0}^{n-1} q^k s_{(n-k,1^k)}. \tag{4.0.2}$$

Multiplication by the principal generator of $I_{n,0,1}$ being well-controlled is important here. We'll have some parallel arguments for the $R_{n,0,2}$.

The original problem of interest is to understand the algebra $R_{n,0,2}$, which is the quotient of $\wedge\{\Theta_n, \Xi_n\}$ permitting the diagonal S_n-action. The ideal $I_{n,0,2}$ is generated by three elements $p_1[\Theta_n], p_1[\Xi_n]$ and $p_1[\Theta\Xi_n]$, see [42] for the more general statement. Let's look at some Hilbert series data for $R_{n,0,2}$ and explain what motivated myself to study the space $R_{n,0,2}$:

- $\mathrm{Hilb}(R_{1,0,2}; q, t) = 1$,
- $\mathrm{Hilb}(R_{2,0,2}; q, t) = 1 + (q + t)$,
- $\mathrm{Hilb}(R_{3,0,2}; q, t) = 1 + (2q + 2t) + (q^2 + 3qt + t^2)$,
- $\mathrm{Hilb}(R_{4,0,2}; q, t) = 1 + (3q + 3t) + (3q^2 + 8qt + 3t^2) + (q^3 + 6q^2t + 6qt^2 + t^3)$.

The very first observation made was that $\mathrm{Hilb}(R_{n,0,2})$ is approximately

$$\mathrm{Hilb}(R_{n-1,0,2}) \cdot (1 + q + t + qt),$$

where we can interpret $(1+q+t+qt)$ as gradings of generators of $I_{n,0,2}$ (plus the unit 1). In fact it was empirically checked for the small n's that

$$\mathrm{Hilb}(R_{n+1,0,2};q,t) = (1+q+t+qt)\mathrm{Hilb}(R_{n,0,2};q,t) - qt \sum_{i+j=n-1} \dim(R_{n,0,2})_{(i,j)} q^i t^j. \tag{4.0.3}$$

The equation seemed to imply that given a basis B of $R_{n,0,2} \hookrightarrow R_{n+1,0,2}$, the union of four sets

$$B, \quad p_1[\Theta_{n+1}] \cdot B, \quad p_1[\Xi_{n+1}] \cdot B, \quad p_1[\Theta\Xi_{n+1}] \cdot B$$

may be a basis of $R_{n+1,0,2}$. Note that the top degree of $R_{n+1,0,2}$ was conjectured to be n, so the subtracted term in the above equation simply accounts for the elements in $p_1[\Theta\Xi_{n+1}] \cdot B$ of degree $2+(n-1) = n+1$ which are supposed to vanish.

4.1 W-Fermionic Diagonal Coinvariants

In fact, results in this chapter are uniform over any irreducible complex reflection group W. To be more precise, let $V = \mathbb{C}^n$ be an n-dimensional complex vector space. A linear transformation $T \in \mathrm{GL}(V)$ is a *reflection* if it fixes a hyperplane, that is,

$$\dim\{v \in V \mid T(v) = v\} = n-1.$$

A *complex reflection group* W is a subgroup of $\mathrm{GL}(V)$ generated by reflections. We say W is irreducible if there is no nontrivial proper subspace of V that is W-invariant. In this case, the *rank* of W is $\dim V = n$ and V called the *reflection* *representation.*

Let W be an irreducible complex reflection group of rank n and $V \cong \mathbb{C}^n$ its reflection representation. This action induces an action on V^* and so $V \oplus V^*$ and $\wedge(V \oplus V^*)$. The exterior algebra $\wedge(V \oplus V^*)$ is bi-graded by degrees in V and V^*. By choosing a basis $\Theta_n = \{\theta_1, \ldots, \theta_n\}$ of V and its dual $\Xi_n = \{\xi_1, \ldots, \xi_n\}$ of V^*, we may identify

$$\wedge(V \oplus V^*) = \wedge\{\Theta_n, \Xi_n\}. \tag{4.1.1}$$

Definition 4.1.1. The W-fermionic diagonal coinvariant ring is the quotient

$$\mathrm{FDR}_W = \wedge(V \oplus V^*)/\langle \wedge(V \oplus V^*)_+^W \rangle \tag{4.1.2}$$

of $\wedge(V \oplus V^*)$ by the ideal generated by the subspace of W-invariants with vanishing constant term.

There is something different from $R_{n,0,2}$ here. When $W = S_n$, the group S_n has rank $n-1$ and $R_{n,0,2}$ is not the same as FDR_{S_n}. To be more precise, the S_n-action on $R_{n,0,2}$ is induced by the permutation action on $\mathbb{C}^n$, which is not irreducible but the S_n-action on FDR_{S_n} is induced by the irreducible reflection action on $\mathbb{C}^{n-1} \cong \mathbb{C}^n/\mathrm{span}_{\mathbb{C}}(e_1 + \cdots + e_n)$ where e_i's are standard basis vectors. We will explain this in more detail later when we specialize our results to the S_n-case.

4.2 Exterior Groebner Theory

To obtain a combinatorial/monomial basis of FDR_W, we will use a Groebner basis theory specialized to the exterior algebras [12]. For an ordered subset

$$S = (i_1 < \ldots < i_k) \subseteq \{1, 2, \ldots, n\},$$

define an associated monomial

$$\theta_S = \theta_{i_1} \cdots \theta_{i_k}. \tag{4.2.1}$$

We refer to the θ_S as *monomials*. We have the monomial basis $\{\theta_S \mid S \subseteq \{1, 2, \ldots, n\}\}$ of $\wedge\{\Theta_n\}$. Given two monomials θ_S and θ_T, we write $\theta_S \mid \theta_T$ when $S \subseteq T$.

A total order $\prec$ on the basis $\{\theta_S \mid S \subseteq \{1, 2, \ldots, n\}\}$ is a *term order* if

1. $1 = \theta_\emptyset \leq \theta_S$ for all S and
2. for all S, T, U with $S \cap U = T \cap U = \emptyset$, we have $\theta_S < \theta_T$ implies $\theta_{S\cup U} < \theta_{T\cup U}$.

Given a term order $\prec$, for any nonzero element $f \in \wedge\{\Theta_n\}$, define $LM(f)$ to be the leading monomial θ_S with respect to $\prec$ such that θ_S appears in f with nonzero coefficient. Let $I \subseteq \wedge\{\Theta_n\}$ be a two-sided ideal. Define the set of leading monomials of nonzero elements in I

$$LM(I) = \{LM(f) \mid f \in I - \{0\}\}. \tag{4.2.2}$$

The set of *normal forms* for I is the complement

$$N(I) = \{\theta_S \mid S \subseteq \{1, \ldots, n\} \text{ and } \theta_S \notin LM(I)\}. \tag{4.2.3}$$

The set $N(I)$ descends to a basis of the quotient $\wedge\{\Theta_n\}/I$, which is called the *standard monomial basis* with respect to $\prec$.

4.3 Lefschetz Theory for Exterior Algebras

We first define some notions for a single graded $\mathbb{C}$-algebras which are well-known and arise more naturally in geometry as the cohomology rings of smooth complex projective varieties.

Let $A = \oplus_{i=0}^{n}$ be a commutative graded $\mathbb{C}$-algebra. The algebra A is said to satisfy the *Poincare Duality* if $A_n \cong \mathbb{C}$ and the multiplication of complementary degree $A_i \otimes A_{n-i} \to A_n \cong \mathbb{C}$ is a perfect pairing for all i. If A satisfies the Poincare Duality, an element $\ell \in A_1$ is called a *Lefschetz element* if for every $0 \leq i \leq n/2$, the multiplication map

$$\ell^{n-2i} \cdot (-) : A_i \to A_{n-i} \tag{4.3.1}$$

is bijective. If A has a Lefschetz element, we say that A satisfies the *Hard Lefschetz Propety*. In fact, Lefschetz property and its application in combinatorics have been studied in various topics [49], [50].

An important result which we will use in our proof is the result of Hara and Watanabe on the n-fold product $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$, see [35]. The cohomology ring of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ is

$$H^\bullet(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1; \mathbb{C}) = \mathbb{C}[X_n]/\langle x_1^2, \ldots, x_n^2 \rangle, \tag{4.3.2}$$

which is very similar to the $\wedge\{\Theta_n\}$ with a basis of monomials corresponding to subsets of $\{1, \ldots, n\}$. Here the generators x_i's have geometric meaning as the Chern classes. Hara and Watanabe use the following result on the Boolean poset to prove that the element $\ell = x_1 + \cdots + x_n$ is a Lefschetz element for the cohomology ring $H^\bullet(\mathbb{P}^1 \times \cdots \times \mathbb{P}^1; \mathbb{C})$.

Let $B(n)$ denote the Boolean poset of the partial order given by subset relation on all subsets $S \subseteq \{1, \ldots, n\}$. The poset $B(n)$ is graded with the i-th rank given by the family $B(n)_i$ of all subsets with size i. Hara and Watanabe prove that the incidence matrix between complementary ranks is invertible. This result is also due to R. Stanley.

Theorem 4.3.1 (Hara-Watanabe, Stanley)**.** Let $r \leq s \leq n$. Define a $\binom{n}{s} \times \binom{n}{r}$ matrix $M_n(r, s)$ with rows indexed by $B(n)_s$ and columns by $B(n)_r$ where

$$M_n(r, s)_{T,S} = \begin{cases} 1 & \text{if } S \subseteq T \\ \\ 0 & \text{otherwise} \end{cases}. \tag{4.3.3}$$

For all $0 \leq i \leq n$, the square matrix $M_n(i, n - i)$ is invertible.

Here is an example of the matrix $M_4(1,3)$ invertible by the Theorem 4.3.1:

$$\begin{array}{c c} & \begin{array}{cccc} \{1\} & \{2\} & \{3\} & \{4\} \end{array} \\ \begin{array}{c} \{1,2,3\} \\ \{1,2,4\} \\ \{1,3,4\} \\ \{2,3,4\} \end{array} & \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}. \end{array}$$

Hara and Watanabe actually compute the determinant of $M_n(i, n-i)$ in terms of i and n inductively. The induction works, simply speaking, by partitioning $B(n)_i$ into subsets containing 1 and not, following the Pascal's identity $\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}$. This allows to write $M_n(i, n-i)$ into a block matrix to which induction is applied.

We want to study the bigraded version of the Hard Lefschetz Property of the exterior algebra $\wedge\{\Theta_n, \Xi_n\}$. Similar to the Lefschetz element of $\mathbb{C}[X_n]/\langle x_1^2, \ldots, x_n^2\rangle$, we introduce the element

$$\delta_n = p_1[\Theta\Xi_n] = \theta_1\xi_1 + \theta_2\xi_2 + \cdots + \theta_n\xi_i \in \wedge\{\Theta_n, \Xi_n\}_{1,1}. \tag{4.3.4}$$

The multiplication by δ_n^r to the complementary bidegree is a bijection, allowing us to view δ_n as a bigraded version of a Lefschetz element.

Theorem 4.3.2. Let $i + j \leq n$ and $r = n - i - j$. The multiplication map

$$\delta_n^r \cdot (-) : \wedge\{\Theta_n, \Xi_n\}_{i,j} \to \wedge\{\Theta_n, \Xi_n\}_{n-j,n-i} \tag{4.3.5}$$

is a bijection.

Proof. The idea is similar to Theorem 4.3.1: choose bases strategically to decompose the matrix representing $\delta_n^r \cdot (-)$ into simpler blocks.

For two subsets $A, B \subset \{1, 2, \ldots, n\}$, write their difference and intersection

$$A - B = \{a_1 < a_2 < \cdots < a_p\}, \tag{4.3.6}$$

$$B - A = \{b_1 < b_2 < \cdots < b_q\}, \tag{4.3.7}$$

$$A \cap B = \{c_1 < c_2 < \cdots < c_s\}. \tag{4.3.8}$$

Define a monomial for A and B

$$\nu(A, B) = \xi_{c_1}\theta_{c_1}\xi_{c_2}\theta_{c_2} \cdots \xi_{c_s}\theta_{c_s}\theta_{a_1}\theta_{a_2} \cdots \theta_{a_p}\xi_{b_1}\xi_{b_2} \cdots \xi_{b_q}. \tag{4.3.9}$$

The set $\{\nu(A, B) \mid A, B \subset \{1, 2, \ldots, n\}\}$ is a basis of $\wedge\{\Theta_n, \Xi_n\}$. What $\nu(A, B)$ does is setting a convention for the order of multiplication in monomials. Because of this order, we have

$$\delta_n \cdot \nu(A, B) = \sum_{c \notin A \cup B} \nu(A \cup \{c\}, B \cup \{c\}). \tag{4.3.10}$$

Whenever $\xi_c\theta_c$ for $c \notin A \cup B$ finds its place in $\xi_c\theta_c \cdot \nu(A, B)$, we have even number of swaps, making the sign unchanged.

The monomials appearing in the expansion of δ_n^r are of the form

$$\xi_{d_1}\theta_{d_1}\xi_{d_2}\theta_{d_2} \cdots \xi_{d_r}\theta_{d_r}$$

where (d_i) is an increasing sequence. Furthermore, this monomial has coefficient $r!$ corresponding to ordering δ_n's by the order d_i's are chosen. So setting $S =$

$\{1, 2, \ldots, n\} - (A \cup B)$, we have a relation

$$\delta_n^r \cdot \nu(A, B) = \sum_{\substack{T \subseteq S \\ |T| = r}} r! \cdot \nu(A \cup T, B \cup T) \tag{4.3.11}$$

since $\nu(A \cup T, B \cup T) = 0$ for T such that $T \cap (A \cup B) \neq \emptyset$.

Using the $\nu(A, B)$ basis, the matrix representing $\delta_n^r \cdot (-)$ has dimension $\binom{n}{i} \cdot \binom{n}{j} \times \binom{n}{n-j} \cdot \binom{n}{n-i}$. The goal is to show that this matrix is invertible. Notice that all pairs of subsets $C = A \cup T$ and $D = B \cup T$ satisfy have the same difference as A and B, i.e.

$$I = A - B = C - D, \quad J = B - A = D - C. \tag{4.3.12}$$

Hence, the matrix representing $\delta_n^r \cdot (-)$ breaks up into smaller blocks corresponding to this pair of differences (I, J). This matrix is block diagonal, so it suffices to show that each block is invertible.

Fix a pair (I, J). The above relation represents a linear map

$$\operatorname{span}\{\nu(A, B) \mid |A| = i, |B| = j, A - B = I, B - A = J\} \tag{4.3.13}$$

$$\to \operatorname{span}\{\nu(C, D) \mid |C| = n - j, |B| = n - i, C - D = I, D - C = J\}. \tag{4.3.14}$$

This system represents a square matrix of size $\binom{n-|I|-|J|}{k}$ where $k = |A \cap B| = i - |I| = j - |J|$. So the $\nu(A, B)$ really is identified by $A \cap B$. Let $S' = \{1, 2, \ldots, n\} - (I \cup J)$. Thinking R as $A \cap B$, the invertibility of the above map/system is equivalent to the invertibility of the following

$$\delta_n^r \cdot \nu(R,R) = \sum_{\substack{R \subseteq S' \\ |R|=r}} r! \cdot \nu(R \cup T, R \cup T) \tag{4.3.15}$$

representing

$$\operatorname{span}\{\nu(R,R) \mid R \subseteq S', |R| = k\} \to \operatorname{span}\{\nu(R',R') \mid R' \subseteq S', |R'| = k+r\}. \tag{4.3.16}$$

Note that $k + (k+r) = i - |I| + j + |J| + n - i - j = n - |I| - |J| = |S'|$. So the matrix representing this system is the same as $M_{|S'|}(k, |S'| - k)$ up to a factor of $r!$. By the Theorem 4.3.1, we conclude the invertibility.

□

Here is one example illustrating the idea of the proof. Consider $\delta_6^3 \cdot (-)$: $\wedge\{\Theta_6, \Xi_6\}_{2,1} \to \{\Theta_6, \Xi_6\}_{5,4}$. Fixing the difference $A - B = \{5\}, B - A = \{6\}$ and $|A \cap B| = 1$, we have following basis elements in $(2,1)$ degree

$$(\{1,5\},\{1,6\}), \quad (\{2,5\},\{2,6\}), \quad (\{3,5\},\{3,6\}), \quad (\{4,5\},\{4,6\}).$$

Terms appearing in the image involve C, D such that $C - D = \{5\}$, $D - C = \{6\}$ and $|C \cap D| = 4$

$$(\{1,2,3,5\},\{1,2,3,6\}), \quad (\{1,2,4,5\},\{1,2,4,6\}),$$

$$(\{1,3,4,5\},\{1,3,4,6\}), \quad (\{2,3,4,5\},\{2,3,4,6\}).$$

Identifying each element in terms of the intersections, the corresponding block is

$$\begin{array}{c|cccc} & \{1\} & \{2\} & \{3\} & \{4\} \\ \hline \{1,2,3\} & 6 & 6 & 6 & 0 \\ \{1,2,4\} & 6 & 6 & 0 & 6 \\ \{1,3,4\} & 6 & 0 & 6 & 6 \\ \{2,3,4\} & 0 & 6 & 6 & 6 \end{array}$$

which is a scalar multiple of $M_4(1,3)$.

4.4 Casimir Element

We consider the W-Fermionic diagonal coinvariants. Let W be an irreducible complex reflection group of rank n acting on its reflection representation $V = \mathbb{C}^n$. Let $\{\theta_1, \ldots, \theta_n\}$ be a basis of V and $\{\xi_1, \ldots, \xi_n\}$ be the dual basis of V^* such that

$$\xi_i(\theta_j) = \delta_{i,j}. \tag{4.4.1}$$

The Lefschetz element δ_n from the previous section is renamed in terms of W:

$$\delta_W = \delta_n = \theta_1\xi_1 + \theta_2\xi_2 + \cdots + \theta_n\xi_n \in V \otimes V^* \subseteq \wedge(V \oplus V^*). \tag{4.4.2}$$

The element δ_W is called the Casimir element of W. The full general group $GL(V)$ acts on $\wedge(V \oplus V^*)$ and the Casimir element is invariant under this action. This can be checked easily upon applying elementary matrices. Equivalently, the Casimir element is independent of the choice of basis. Since $W \subset GL(V)$, we know that $\delta_W \in \wedge(V \oplus V^*)^W$. In fact, δ_W generates this W-invariant subring.

Theorem 4.4.1. The Casimir element δ_W generates $\wedge(V \oplus V^*)^W \subseteq \wedge(V \oplus V^*)$.

Proof. Let G be a finite group and U, U' be irreducible complex G-modules. We quickly review the tensor product of G-modules which we used in various definitions already. The tensor product $U \otimes U'$ is a G-module by the rule $g.(u \otimes u') = (g.u) \otimes (g.u')$ for $u \in U$, $u' \in U'$ and $g \in G$. Tensoring of two G-modules is equivalent to taking the Kronecker product on the symmetric function side and is also called the Kronecker product of the G-modules. Then the orthogonality of characters of irreducible representations implies

$$\begin{aligned} &\text{multiplicity of the trivial } G\text{-module in } U \otimes U' && (4.4.3)\\ &= \dim(U \otimes U')^G = \langle U \otimes U', 1_G \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{U \otimes U'}(g) && (4.4.4)\\ &= \frac{1}{|G|} \sum_{g \in G} \chi_U(g) \chi_{U'}(g) && (4.4.5)\\ &= \delta_{U' \cong U^*}, && (4.4.6) \end{aligned}$$

where the last equality follows by the assumption that U and U' are irreducible.

By the result of Steinberg ([37], Theorem A, p.250), the exterior powers

$$\wedge^0 V, \wedge^1 V, \ldots, \wedge^n V$$

are all pairwise nonisomorphic irreducible representations of W and the same is true for $\wedge^0 V^*, \wedge^1 V^*, \ldots, \wedge^n V^*$. The bigrading decomposes

$$\wedge(V \oplus V^*) = \oplus_{i,j} \wedge (V \oplus V^*)_{i,j} = \oplus_{i,j} (\wedge^i V \otimes \wedge^j V^*). \tag{4.4.7}$$

By the above paragraph, taking $G = W$, $U = \wedge^i V$ and $U' = \wedge^j V^*$,

$$\dim \wedge(V \oplus V^*)^W_{i,j} = \delta_{i,j}. \tag{4.4.8}$$

On the other hand, $\delta_W \in \wedge(V \oplus V^*)^W_{1,1}$ and its powers $\delta^i_W \in (V \oplus V^*)^W_{i,i}$. Each δ^i_W is nonzero as seen in the proof of Theorem 4.3.2. $\square$

The Casimir element will play a key role in describing the bigraded W-module structure of FDR_W. We state our answer in terms of the Grothendieck ring of W which differs from the Grothendieck ring discussed in section 2. We start with a free $\mathbb{Z}$-algebra generated by the set of isomorphism classes $[U]$ of irreducible representation of W. In terms of short exact sequences, we have a relation $[U] = [U'] + [U'']$ for any short exact sequence

$$0 \to [U'] \to [U] \to [U''] \to 0. \tag{4.4.9}$$

Multiplication in this Grothendieck ring coresponds to Kronecker product instead:

$$[U] \cdot [U'] = [U \otimes U']. \tag{4.4.10}$$

Theorem 4.4.2. Let W be an irreducible complex reflection group acting on its reflection representation $V = \mathbb{C}^n$. Let $0 \leq i, j \leq n$. If $i+j > n$, then $(\mathrm{FDR}_W)_{i,j} = 0$. If $i + j \leq n$, then inside the Grothendieck ring of W,

$$(\mathrm{FDR}_W)_{i,j} = [\wedge^i V] \cdot [\wedge^j V^*] - [\wedge^{i-1} V] \cdot [\wedge^{j-1} V^*], \tag{4.4.11}$$

where we interpret $\wedge^{-1} V = \wedge^{-1} V^* = 0$.

Proof. Choosing a basis and its dual, Theorem 4.4.1 allows us to write

$$\mathrm{FDR}_W = \wedge(V \oplus V^*)/\langle \delta_W \rangle = \wedge\{\Theta_n, \Xi_n\}/\langle \delta_n \rangle. \tag{4.4.12}$$

If $i = 0$ or $j = 0$, then the claim follows trivially as δ_W has bidegree $(1,1)$, so assume $i, j > 0$.

Assume $i + j \leq n$ and let $r = n - i - j + 1$. Theorem 4.3.2 states that the multiplication map

$$\delta_W^r \cdot (-) : \wedge(V \oplus V^*)_{i-1,j-1} \to \wedge(V \oplus V^*)_{n-j+1,n-i+1} \tag{4.4.13}$$

is a linear isomorphism. Whenever a composition $f \circ g$ is a bijection, the map g is an injection and f is a surjection. So the map

$$\delta_W \cdot (-) : \wedge(V \oplus V^*)_{i-1,j-1} \to \wedge(V \oplus V^*)_{i,j} \tag{4.4.14}$$

is an injection which is W-equivariant as δ_W is W-invariant. Since we showed that δ_W is a principal generator of the ideal $\langle \delta_W \rangle$, the theorem follows.

Assume $i + j > n$. Then by the same reasoning,

$$\delta_W \cdot (-) : \wedge(V \oplus V^*)_{i-1,j-1} \to \wedge(V \oplus V^*)_{i,j} \tag{4.4.15}$$

is a W-equivariant surjection proving that $(\mathrm{FDR}_W)_{(i,j)} = 0$. □

From this description, we can deduce the vector space dimension of FDR_W.

Corollary 4.4.3. If W has rank n, then $\dim \mathrm{FDR}_W = \binom{2n+1}{n}$.

Proof. For each k, $\dim \wedge^k V = \wedge^k V^* = \binom{n}{k}$, the theorem says that for $i + j \leq n$,

$$\dim(\mathrm{FDR}_W)_{i,j} = \binom{n}{i}\binom{n}{j} - \binom{n}{i-1}\binom{n}{j-1}. \tag{4.4.16}$$

The product of W-modules being subtracted has degree exactly 2 less. So summing over bidegrees (i, j), the dimensions in the top two degrees survive

$$\begin{aligned} \dim(\mathrm{FDR}_W) &= \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} + \sum_{j=0}^{n-1} \binom{n}{j}\binom{n}{n-1-j} && (4.4.17) \\ &= \binom{2n}{n} + \binom{2n}{n-1} && (4.4.18) \\ &= \binom{2n+1}{n}, && (4.4.19) \end{aligned}$$

where the last equality is the Pascal recursion for binomials. □

The top degree of FDR_W exhibits an interesting combinatorial refinement. The *Narayana* numbers refine the Catalan numbers by

$$\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}, \quad \mathrm{Nar}(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}, \tag{4.4.20}$$

$$\mathrm{Cat}(n) = \sum_{k=1}^{n} \mathrm{Nar}(n,k). \tag{4.4.21}$$

One combinatorial interpretation of this refinement is by Dyck paths: $\mathrm{Nar}(n, k)$ counts the number of Dyck paths with $k - 1$ peaks, where a peak is an N-step followed immediately by an E-step.

Corollary 4.4.4. If W has rank n, then for $0 \leq k \leq n$,

$$\dim(\mathrm{FDR}_W)_{k,n-k} = \mathrm{Nar}(n+1, k+1) \tag{4.4.22}$$

and so

$$\sum_{k=0}^{n} \dim(\mathrm{FDR}_W)_{k,n-k} = \mathrm{Cat}(n+1). \tag{4.4.23}$$

Proof. Starting from the Theorem 4.4.2,

$$\begin{aligned}
dim(\mathrm{FDR}_W)_{k,n-k} &= \binom{n}{k}\binom{n}{n-k} - \binom{n}{k-1}\binom{n}{n-k-1} && (4.4.24)\\
&= \binom{n}{k}^2 - \binom{n}{k-1}\binom{n}{k+1} && (4.4.25)\\
&= \frac{n-k+1}{k}\binom{n}{k}\binom{n}{k-1} - \frac{n-k}{k+1}\binom{n}{k-1}\binom{n}{k} && (4.4.26)\\
&= \frac{n+1}{k(k+1)}\binom{n}{k}\binom{n}{k-1} = \frac{1}{n+1}\binom{n+1}{k+1}\binom{n+1}{k} && (4.4.27)\\
&= \mathrm{Nar}(n+1, k+1). && (4.4.28)
\end{aligned}$$

□

For any reflection group W, there are Catalan and Narayana numbers attached to W. However, the numbers appearing in our results are their type A instances independent of the choice of W.

Another interpretation of the Catalan and Narayana numbers is by *noncrossing partitions*. Recently, Jesse Kim and Rhoades used the top degree of FDR_{S_n} as a model for resolving a set partitions of $\{1, \ldots, n\}$ into a $\mathbb{C}$-algebra of noncrossing partitions such that the Narayana refinement agrees, see [39].

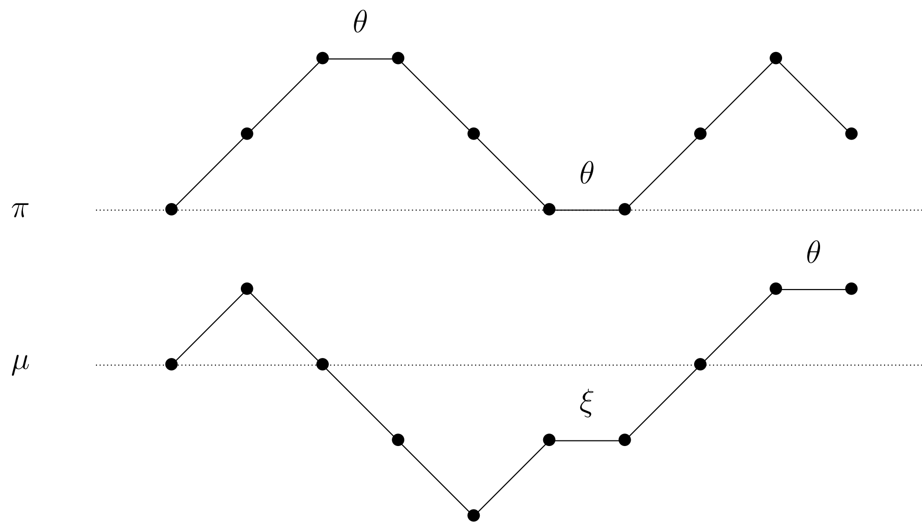


Figure 4.1: Two paths in $\Pi(9)$.

4.5 Motzkin Paths and Standard Monomial Basis

In this section we describe the standard monomial basis of FDR_W in terms of certain family of lattice paths. A *Motzkin path* is a lattice path in $\mathbb{Z}^2$ starting from the origin $(0,0)$, consisting of up-steps $(1,1)$, down-steps $(1,-1)$ and horizontal steps $(1,0)$ which end on and stay weakly above the x-axis. We consider a variant of Motzkin paths which allows decoration of horizontal steps and has no condition on the x-axis (need not end on nor stay weakly above the x-axis).

Let $\Pi(n)$ be the family of n-step lattice paths $\pi = (\pi_1, \ldots, \pi_n)$ in $\mathbb{Z}^2$ which start at the origin consisting of up-steps $(1,1)$, down-steps $(1,-1)$ and decorated horizontal steps $(1,0)^\theta$ and $(1,0)^\xi$. Let $\Pi(n)_{\geq 0} \subset \Pi(n)$ be the subfamily of paths which stay weakly above the x-axis. Two paths in $\Pi(9)$ are shown in Figure 4.1 where the top path lies in $\Pi(n)_{\geq 0}$ but the bottom path does not.

We define some quantities associated to each path. The *depth* $d(\pi)$ of a path $\pi \in \Pi(n)$ is the minimum y-coordinate reached by π. Two paths in Figure 4.1 have $d(\pi) = 0$ and $d(\mu) = -2$. A path is in $\Pi(n)_{\geq 0}$ if and only if $d(\pi) = 0$ and for every path π, we have $d(\pi) \leq 0$ as any path starts at the origin. Using set notation,

$$\Pi(n)_{\geq 0} = \{\pi \in \Pi(n) \mid d(\pi) = 0\}, \tag{4.5.1}$$

$$\Pi(n) - \Pi(n)_{\geq 0} = \{\pi \in \Pi(n) \mid d(\pi) < 0\}. \tag{4.5.2}$$

Let $\pi = (\pi_1, \ldots, \pi_n) \in \Pi(n)$. The *weight* of the i-th step π_i of π is defined

$$\mathrm{wt}(\pi_i) = \begin{cases} 1 & \text{if } \pi_i = (1,1) \text{ is an up-step} \\ \theta_i & \text{if } \pi_i = (1,0)^\theta \text{ is a } \theta\text{-decorated horizontal-step} \\ \xi_i & \text{if } \pi_i = (1,0)^\xi \text{ is a } \xi\text{-decorated horizontal-step} \\ \theta_i\xi_i & \text{if } \pi_i = (1,-1) \text{ is a down-step} \end{cases} \tag{4.5.3}$$

and the weight of π is the product of the weight of its steps in the order in which they appear

$$\mathrm{wt}(\pi) = \mathrm{wt}(\pi_1)\mathrm{wt}(\pi_2)\cdots\mathrm{wt}(\pi_n). \tag{4.5.4}$$

The paths in Figure 4.1 have weight

$$\mathrm{wt}(\pi) = \theta_3 \cdot \theta_4\xi_4 \cdot \theta_5\xi_5 \cdot \theta_5 \cdot \theta_9\xi_9, \tag{4.5.5}$$

$$\mathrm{wt}(\xi) = \theta_2\xi_2 \cdot \theta_3\xi_3 \cdot \theta_4\xi_4 \cdot \xi_6 \cdot \theta_9. \tag{4.5.6}$$

It's clear that $|\Pi(n)| = 4^n$ which is the dimension of $\wedge\{\Theta_n, \Xi_n\}$. The map

$$\pi \mapsto \mathrm{wt}(\pi) \tag{4.5.7}$$

is easily seen to be a bijection from $\Pi(n)$ to the set of monomials in $\{\Theta_n, \Xi_n\}$ ignoring the sign. So we will identify paths π with their weight monomial $\mathrm{wt}(\pi)$ and vice-versa.

The *degree* of a path π is the degree of its weight $\mathrm{wt}(\pi)$. In terms of path data,

$$\deg(\pi) = n - (\text{the terminal } y\text{-coordinate of } \pi). \tag{4.5.8}$$

Bidegree is also defined similarly. The θ-degree of $\deg_\theta(\pi)$ and $\deg_\xi(\pi)$ are simply θ-degree and ξ-degree of $\mathrm{wt}(\pi)$. In terms of path data,

$$\deg_\theta(\pi) = (\text{number of down-steps}) + (\text{number of } \theta\text{-decorated horizontal steps}), \tag{4.5.9}$$

$$\deg_\xi(\pi) = (\text{number of down-steps}) + (\text{number of } \xi\text{-decorated horizontal steps}). \tag{4.5.10}$$

The paths in Figure 4.1 have degrees

$$\begin{cases} \deg(\pi) = 8 \\ \deg_\theta(\pi) = 5 \\ \deg_\xi(\pi) = 3 \end{cases} \quad \text{and} \quad \begin{cases} \deg(\mu) = 8 \\ \deg_\theta(\mu) = 4 \\ \deg_\xi(\mu) = 4 \end{cases}. \tag{4.5.11}$$

We introduce the total order $\prec$ on paths in $\Pi(n)$ or on their monomial weights in $\wedge\{\Theta_n, \Xi_n\}$ which we will show to be a term order. We first define the lexicographical order $<_{lex}$ on the paths by declaring the step order

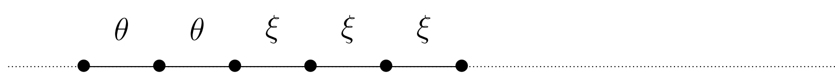
$$(1,1) < (1,0)^\theta < (1,0)^\xi < (1,-1). \tag{4.5.12}$$

Note that the above order follows the degree order on the weight of steps except that we arbitrarily made a choice that ξ-decoration is larger than θ-decoration for horizontal steps. Given $\pi, \pi' \in \Pi(n)$, we define

$$\pi \prec \pi' \iff \begin{cases} \deg(\pi) < \deg(\pi') & \text{or} \\ \deg(\pi) = \deg(\pi') \text{ and } d(\pi) > d(\pi') & \text{or} \\ \deg(\pi) = \deg(\pi') \text{ and } d(\pi) = d(\pi') \text{ and } \pi <_{lex} \pi'. \end{cases} \tag{4.5.13}$$

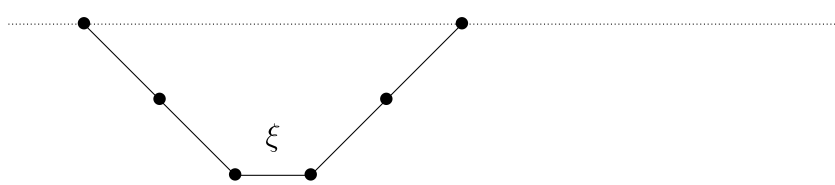
In Figure 4.1, we have $\deg(\pi) = \deg(\mu)$ and $d(\pi) > d(\mu)$, so $d(\pi) \prec d(\mu)$. The collection of paths with a given bidegree (i, j) form a subinterval of $\prec$ for all $0 \leq i, j \leq n$. For example, for $\Pi(5)$ with $(i, j) = (2, 3)$, the path

$((1,0)^\theta, (1,0)^\theta, (1,0)^\xi, (1,0)^\xi, (1,0)^\xi)$



is the minimum and

$((1,-1), (1,-1), (1,0)^\xi, (1,1), (1,1))$



is the maximum in the corresponding subinterval. Drawing the two lattice paths, one may observe that any path in $\Pi(5)$ of bidegree $(2,3)$ must fit in between the minimum and the maximum paths.

Lemma 4.5.1. The total order $\prec$ is a term order for $\wedge\Theta_n, \Xi_n$ and $\Pi(n)$.

Proof. The unique path consisting of $(1,1)$ path only has degree 0 is the minimum monomial under $\prec$ by the first condition on deg in the definition of $\prec$.

For the second requirement, observe that in terms of path, degree which is the terminal y-coordinate and depth which is the minimum y-coordinate and lexicographical order are all respected by inserting replacing up-steps by horizontal or down-steps and horizontal-steps by down-steps. $\square$

The set of monomials $\{\mathrm{wt}(\pi) \mid \pi \in \Pi(n)_{\geq 0}\}$ descends to a vector space basis of FDR_W. We prove this by using the exterior Groebner theory that we set up

Theorem 4.5.2. The set $\{\mathrm{wt}(\pi) \mid \pi \in \Pi(n)_{\geq 0}\}$ is the standard monomial basis of FDR_W with respect to the term order $\prec$.

A key idea in the proof is that one may obtain all paths in $\Pi(n)$ by attaching an extra step to paths in $\Pi(n-1)$. Moreover, an extension of a path in $\Pi(n-1)_{\geq 0}$ is a path in $\Pi(n)_{\geq 0}$ except for the case where a path of degree $n-1$ is extended by a down-step. On the other hand, removal of the last step of a path in $\Pi(n)_{\geq 0}$ yields a path in $\Pi(n-1)_{\geq 0}$.

Let $I_n = \langle \delta_n \rangle \subset \wedge\{\Theta_n, \Xi_n\}$ be the defining ideal of FDR_W. Identifying the paths with their monomial weight, the goal is to show that $N(I_n) = \Pi(n)_{\geq 0}$ with respect to $\prec$. We first prove a lemma that inductively proves $(\Pi(n) - \Pi(n)_{\geq 0}) \cap N(I_n) = \emptyset$.

Lemma 4.5.3. Let $\pi \in \Pi(n) - \Pi(n)_{\geq 0}$. The monomial weight of π lies in $LM(I_n)$

or else $\mathrm{wt}(\pi) = 0$ in the quotient FDR_W.

Proof. Let $\pi' \in \wedge\{\Theta_{n-1}, \Xi_{n-1}\}$ be the monomial or path obtained by removing the last step π_n of π. We do some case analysis depending on the step π_n.

Case 1: *The last step π_n is a horizontal step.*

Without loss of generality, assume that $\pi_n = (1, 0)^\theta$. In this case, $d(\pi') = d(\pi)$ so $\pi' \in \Pi(n-1) - \Pi(n-1)_{\geq 0}$ and $\pi = \pi'\pi_n$. By the induction hypothesis, $\pi' \in LM(I_{n-1})$ so that $\pi' = LM(f \cdot \delta_{n-1})$ for some $f \in \wedge\{\Theta_{n-1}, \Xi_{n-1}\}$. Note that

$$f \cdot \delta_n \cdot \theta_n = f \cdot (\delta_{n-1} + \theta_n \xi_n) \cdot \theta_n = f \cdot \delta_{n-1} \cdot \theta_n. \tag{4.5.14}$$

So we conclude that $f \cdot \delta_{n-1} \cdot \theta_n = -(f \cdot \theta_n) \cdot \delta_n \in I_n$. Taking the leading monomials,

$$LM(f \cdot \delta_{n-1} \cdot \theta_n) = LM(f \cdot \delta_{n-1}) \cdot \theta_n = \pi' \cdot \theta_n = \pi. \tag{4.5.15}$$

proving the first case.

Case 2: *The last step π_n is a down-step $(1, -1)$.*

If $\pi_n = (1, -1)$, then $d(\pi')$ may be 0 or not. If $d(\pi') < 0$, that is, $\pi' \in \Pi(n-1) - \Pi(n-1)_{\geq 0}$, the proof is similar to the Case 1 where we use $\theta_n \xi_n$ instead of θ_n. Assume $d(\pi') = 0$. In this case, $\pi' \in \Pi(n-1)$ but $\pi \in \Pi(n) - \Pi(n)_{\geq 0}$, so π' must end on the x-axis. Hence, $\deg(\pi') = n - 1$ and $\deg(\pi) = \deg(\pi') + \deg(1, -1) = n + 1$. By Theorem 4.4.2, we know that π vanishes in the quotient FDR_W, proving the second case.

Case 3: *The last step π_n is an up-step $(1, 1)$.*

In this case, $\mathrm{wt}(\pi) = \mathrm{wt}(\pi')$ and $d(\pi') = d(\pi) < 0$ which means that $\pi' \in \Pi(n-1) - \Pi(n-1)_{\geq 0}$. Similar to the other cases, by the induction hypothesis, there is $f \in \wedge\{\Theta_{n-1}, \Xi_{n-1}\}$ such that $\pi' = LM(f \cdot \delta_{n-1})$. We consider

$$f \cdot \delta_n = f \cdot \delta_{n-1} + f \cdot \theta_n x \xi_n \in I_n. \tag{4.5.16}$$

Note that π' is the leading monomial of the product $f \cdot \delta_{n-1}$, so we may remove some terms in f and still keep π' the leading monomial in $f \cdot \delta_{n-1}$. Assume that f is bi-homogeneous. Since $\pi_n = (1,1)$, $\mathrm{wt}(\pi)$ does not involve θ_n or ξ_n and $f \in \wedge\{\Theta_{n-1}, \Xi_{n-1}\}$, the path π does not appear in $f \cdot \theta_n \xi_n$. As the monomials appearing in $f \cdot \delta_n$ appear in only one of $f \cdot \delta_{n-1}$ or $f \cdot \theta_n \xi_n$, we still have $\pi = LM(f \cdot \delta_n)$ unless some monomial μ appearing in $f \cdot \theta_n \xi_n$ satisfies $\pi \prec \mu$.

Let μ be the largest element of $f \cdot \theta_n \xi_n$ under $\prec$ and assume $\pi \prec \mu$. Let $\mu' \in \Pi(n-1)$ be the path obtained by removing the last step of μ. Note that the last step of μ must be a down-step as μ appears in $f \cdot \theta_n \xi_n$. Since $\pi \prec \mu$, the bihomogeneity of f forces $d(\mu) \leq d(\pi) < 0$.

If $d(\mu') = 0$ or equivalently $\mu' \in \Pi(n-1)_{\geq 0}$, then $d(\mu) < 0$ implies that μ' ended on the x-axis at $(n-1, 0)$. Since the last step of μ is $(1, -1)$, we now have $\deg(\mu) = \deg(\mu') + 2 = n + 1$. By Theorem 4.4.2, we're forced to have $\mu \in I_n$. Hence, we may remove the term involving μ from $f \cdot \delta_n$ and still have an element of I_n involving π.

If $d(\mu') < 0$ or equivalently $\mu' \in \Pi(n-1) - \Pi(n-1)_{\geq 0}$, then by the induction hypothesis, there is an element $g \in \wedge\{\Theta_{n-1}, \Xi_{n-1}\}$ such that $\mu' = LM(g \cdot \delta_{n-1})$.

Since $g \cdot \delta_{n-1}$ does not involve any of θ_n or ξ_n,

$$LM(g \cdot \delta_n \cdot \theta_n \xi_n) = LM(g \cdot \delta_{n-1} \cdot \theta_n \xi_n) \tag{4.5.17}$$

$$= LM(g \cdot \delta_{n-1}) \cdot \theta_n \xi_n \tag{4.5.18}$$

$$= \mu' \cdot \theta_n \xi_n = \mu. \tag{4.5.19}$$

All the terms appearing in $g \cdot \delta_n \cdot \theta_n \xi_n$ involve θ_n and ξ_n. Since π does not involve θ_n or ξ_n, it does not appear in $g \cdot \delta_n \cdot \theta_n \xi_n = g \cdot \delta_{n-1} \cdot \theta_n \xi_n$. By removing $g \cdot \delta_n \cdot \theta_n \xi_n \in I_n$ from $f \cdot \delta_n \in I_n$, we obtain another element in I_n,

$$f \cdot \delta_n - g \cdot \delta_n \cdot \theta_n \xi_n = f \cdot \delta_{n-1} + (f - g \cdot \delta_{n-1}) \cdot \theta_n \xi_n \in I_n \tag{4.5.20}$$

such that π is still involved in the first term and now only involves monomials $\prec \mu$. Iterating the arguments so far, since $\prec$ is a term order, we eventually reach the point where π is the leading monomial of $f \cdot \delta_n$ and thus $\pi \in LM(I_n)$ proving the last case and this lemma. □

Now we prove Theorem 4.5.2 with some help of Lemma 4.5.3.

Proof of Theorem 4.5.2. Identifying paths with their monomial weights, we want to show that $N(I_n) = \Pi(n)_{\geq 0}$ with respect to $\prec$. By Lemma 4.5.2, we have a subset relation $N(I_n) = LM(I_n)^c \subseteq \Pi(n)_{\geq 0}$. By the exterior Groebner theory, $N(I_n)$ descends to the standard monomial basis of FDR_W with repsect to $\prec$. So to force equality, it suffices to verify

$$\dim \mathrm{FDR}_W = |\Pi(n)_{\geq 0}|. \tag{4.5.21}$$

In fact we show the dimension at the bigraded Hilbert series level. Define a generating function

$$P_n(q,t) = \sum_{\pi \in \Pi(n)_{\geq 0}} q^{\deg_\theta(\pi)} t^{\deg_\xi(\pi)}. \tag{4.5.22}$$

Let $\Pi(n)_{=0} \subset \Pi(n)_{\geq 0}$ denote the subset of paths that end on the x-axis or equivalently of degree n. Similarly, define

$$P'_n(q,t) = \sum_{\pi \in \Pi(n)_{=0}} q^{\deg_\theta(\pi)} t^{\deg_\xi(\pi)}. \tag{4.5.23}$$

As already discussed extensively, the addition of an extra step to a path yields a recursion

$$P_{n+1}(q,t) = (1+q+t+qt) \cdot P_n(q,t) - (qt) \cdot P'_n(q,t). \tag{4.5.24}$$

On the other hand, adopting the notation $W(n)$ whenver W has rank n, Theorem 4.4.2 yields

$$\dim(\mathrm{FDR}_{W(n+1)})_{i,j} = \begin{cases} \binom{n+1}{i} \cdot \binom{n+1}{j} - \binom{n+1}{i-1} \cdot \binom{n+1}{j-1} & \text{if } i,j > 0 \text{ and } i+j \leq n+1 \\ \binom{n+1}{i} \cdot \binom{n+1}{j} & \text{if } i = 0 \text{ or } j = 0 \\ 0 & \text{if } i+j > n+1 \end{cases}. \tag{4.5.25}$$

This can be interpreted at the generating function level as

$$\mathrm{Hilb}(\mathrm{FDR}_{W(n+1)}; q,t) = (1+q+t+qt) \cdot \mathrm{Hilb}(\mathrm{FDR}_{W(n)}; q,t) \tag{4.5.26}$$

$$- (qt) \cdot \sum_{i+j=n} \dim(\mathrm{FDR}_{W(n)})_{i,j} \cdot q^i t^j, \tag{4.5.27}$$

which matches the recursion for $P_n(q,t)$. So we proved that

$$\mathrm{Hilb}(\mathrm{FDR}_{W(n)};q,t) = P_n(q,t).$$

$\square$

In the proof of Theorem 4.5.2, we found a combinatorial expression for the Hilbert series of FDR_W.

Corollary 4.5.4. If W has rank n, its Hilbert series is

$$\mathrm{Hilb}(\mathrm{FDR}_W;q,t) = \sum_{\pi\in\Pi(n)_{\geq 0}} q^{\deg_\theta(\pi)} t^{\deg_\xi(\pi)}. \tag{4.5.28}$$

4.6 The case of S_n

So far our discussion built upon the action of W on its reflection representation of dimension equal to the rank of W. However, the S_n action on $R_{n,0,2}$ is induced by the permutation representation U of dimension $n = 1+\mathrm{rank}(S_n)$. Writing V for its reflection representation as usual, it is well known that

$$U = V \oplus U^{S_n} \quad \text{ and } U^* = V^* \oplus (U^*)^{S_n}. \tag{4.6.1}$$

Explicitly, $U^{S_n} = \mathrm{span}\{e_1+e_2+\cdots+e_n\}$ and $V = \mathrm{span}\{e_1-e_2, e_2-e_3, \cdots, e_{n-1}-e_n\}$. Using the properties of exterior product and direct sum, it follows that

$$\wedge(U\oplus U^*) \cong \wedge[(V\oplus U^{S_n}) \oplus (V^*\oplus (U^*)^{S_n})] \tag{4.6.2}$$

$$\cong \wedge[(V\oplus V^*) \oplus (U^{S_n}\oplus (U^*)^{S_n})] \tag{4.6.3}$$

$$\cong [\wedge(V\oplus V^*)] \oplus [\wedge(U^{S_n}\oplus (U^*)^{S_n})]. \tag{4.6.4}$$

Modding out by the ideals generated by S_n-invariants with vanishing constant term,

$$\wedge(U \oplus U^*)/\langle \wedge(U \oplus U^*)_+^{S_n} \rangle \cong \wedge(V \oplus V^*)/\langle \wedge(V \oplus V^*)_+^{S_n} \rangle. \tag{4.6.5}$$

Expressing the left hand side of the above equation in terms of coordinates Θ_n and Ξ_n, we have the following translation of Theorem 4.4.2 and its corollaries.

Theorem 4.6.1. Let $R_{n,0,2}$ be the Fermionic diagonal coinvariants

$$R_{n,0,2} = \wedge\{\Theta_n, \Xi_n\}/\langle \wedge\{\Theta_n, \Xi_n\}_+^{S_n} \rangle. \tag{4.6.6}$$

The S_n-structure of bigraded components of $R_{n,0,2}$ are

$$\mathrm{Frob}((R_{n,0,2})_{i,j}) = \begin{cases} s_{(n-i,1^i)} \otimes s_{(n-j,1^j)} - s_{(n-i+1,1^{i-1})} \otimes s_{(n-j+1,1^{j-1})} & \text{if } i+j < n \\ 0 & \text{if } i+j \geq n \end{cases}, \tag{4.6.7}$$

where we interpret $s_{(n+1,-1)} = 0$. The dimension of $R_{n,0,2}$ is

$$\dim(R_{n,0,2}) = \binom{2n-1}{n} \tag{4.6.8}$$

and for $1 \leq k \leq n$, the top degree components have

$$\dim(R_{n,0,2})_{k-1,n-k} = \mathrm{Nar}(n,k) \tag{4.6.9}$$

exhibiting the Catalan into Narayana refinement

$$\sum_{k=1}^{n} \dim(R_{n,0,2})_{k-1,n-k} = \mathrm{Cat}(n). \tag{4.6.10}$$

Work of Rosas, [44], implies that for $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$, the multiplicity of s_λ in $R_{n,0,2}$ is

$$\langle \mathrm{Frob}(R_{n,0,2}; q, t), s_\lambda \rangle = 0 \tag{4.6.11}$$

unless $\lambda_3 \leq 2$. These multiplicities involve formulas for the tensor product of hooks. In case when λ is a hook, these multiplicities are nice. In our discussion of the Carlsson-Oblomkov basis, we defined q-numbers. Define a *q, t-number*

$$[n]_{q,t} = \frac{q^n - t^n}{q - t} = q^{n-1} + q^{n-2}t + \cdots + qt^{n-2} + t^{n-1}. \tag{4.6.12}$$

Proposition 4.6.1. The graded multiplicities of the trivial and sign representation in $R_{n,0,2}$ are given by

$$\langle \mathrm{Frob}(R_{n,0,2}; q, t), s_{(n)} \rangle = 1 \quad \text{and} \quad \langle \mathrm{Frob}(R_{n,0,2}; q, t), s_{(1^n)} \rangle = [n]_{q,t}. \tag{4.6.13}$$

More specificially, if $0 < k < n-1$, then the graded multiplicities of the hooks are

$$\langle \mathrm{Frob}(R_{n,0,2}; q, t), s_{(s_{n-k}, 1^k} \rangle = [k+1]_{q,t} + qt \cdot [k]_{q,t}. \tag{4.6.14}$$

Proof. The multiplicity of $s_{(n)}$ is immediate since $R_{n,0,2}$ is defined by modding out $\wedge\{\Theta_n, \Xi_n\}$ by the S_n-invariants with vanishing constant term.

By Theorem 4.6.1 and the fact that

$$\langle s_\lambda \otimes s_\mu, s_{(1^n)} \rangle = \delta_{\lambda', \mu}, \tag{4.6.15}$$

we have the multiplicity of $s_{(1^n)}$.

Similar to the Kronecker delta, for any statement P, let the indicator function $\chi(P) = 1$ if P is true and $\chi(P) = 0$ if P is false. Rosas describes the multiplicity of a hook in the Kronecker product of two other hooks, that is,

$$\langle s_{(n-a,1^a)} \otimes s_{(n-b,1^b)}, s_{(n-c,1^c)} \rangle = \chi(|b-a| \leq c) \times \chi(c \leq a+b \leq 2n-c-2) \tag{4.6.16}$$

whenever $0 < a, b < n$ and $0 < c < n-1$.

For any $0 < k < n-1$, and $i + j < n$, we have

$$\begin{aligned} \langle \mathrm{Frob}(R_{n,0,2})_{i,j}, s_{(s_{n-k},1^k)} \rangle =& \langle s_{(n-i,1^i)} \otimes s_{(n-j,1^j)}, s_{(n-k,1^k)} \rangle \\ &+ \langle s_{(n-i+1,1^{i-1})} \otimes s_{(n-j+1,1^{j-1})}, s_{(n-k,1^k)} \rangle. \end{aligned} \tag{4.6.17}$$

Applying the hook multiplicity formula (4.6.15), we obtain

$$\langle \mathrm{Frob}(R_{n,0,2})_{i,j}, s_{(s_{n-k},1^k)} \rangle = \begin{cases} 1 & \text{if } i+j=k \\ 1 & \text{if } i+j=k+1 \text{ and } i,j>0 \\ 0 & \text{otherwise} \end{cases}, \tag{4.6.18}$$

which is equivalent to Equation (4.6.13). □

In order to state the translation of Theorem 4.5.2 to $R_{n,0,2}$, we introduce a new basis that replaces Ξ_n. For a path $\pi = (\pi_1, \pi_2, \ldots, \pi_n) \in \Pi(n)$, define the *primed* *weight* $\mathrm{wt}'(\pi_i)$ of the i-th step of π to be

$$\mathrm{wt}'(\pi_i) = \begin{cases} 1 & \text{if } \pi_i = (1,1) \\ \\ \theta_i & \text{if } \pi_i = (1,0)^\theta \\ \\ \xi_i' & \text{if } \pi_i = (1,0)^\xi \\ \\ \theta_i\xi_i' & \text{if } \pi_i = (1,-1) \end{cases}, \tag{4.6.19}$$

where $\xi_i' = \xi_i + \sum_{j=2}^n \xi_j$. The primed weight of path π is again the product

$$\mathrm{wt}'(\pi) = \mathrm{wt}'(\pi_1)\mathrm{wt}'(\pi_2)\cdots\mathrm{wt}'(\pi_n). \tag{4.6.20}$$

Let $\Pi(n)_{>0} \subseteq \Pi(n)_{\geq 0}$ consist of those paths that stay strictly above the x-axis once it leaves the origin $(0,0)$.

Theorem 4.6.2. The set of monomials $\{\mathrm{wt}'(\pi) \mid \pi \in \Pi(n)_{>0}\}$ descends to a basis of $R_{n,0,2}$, from which we get a combinatorial expression for the Hilbert series

$$\mathrm{Hilb}(R_{n,0,2}; q, t) = \sum_{\pi \in \Pi(n)_{>0}} q^{\deg_\theta(\pi)} t^{\deg_\xi(\pi)}. \tag{4.6.21}$$

Proof. Consider the three generators of $I_{n,0,2}$

$$p_1[\Theta_n] = \theta_1 + \theta_2 + \cdots + \theta_n, \tag{4.6.22}$$

$$p_1[\Xi_n] = \xi_1 + \xi_2 + \cdots + \xi_n, \tag{4.6.23}$$

$$p_1[\Theta\Xi_n] = \theta_1\xi_1 + \theta_2\xi_2 + \cdots + \theta_n\xi_n. \tag{4.6.24}$$

We express $R_{n,0,2}$ as a successive quotient

$$R_{n,0,2} = \wedge\{\Theta_n, \Xi_n\}/\langle p_1[\Theta_n], p_1[\Xi_n], p_1[\Theta\Xi_n]\rangle \tag{4.6.25}$$

$$= \left(\wedge\{\Theta_n\}/\langle p_1[\Theta_n]\rangle \otimes \wedge\{\Xi_n\}/\langle p_1[\Xi_n]\rangle\right)/\langle p_1[\Theta\Xi_n]\rangle. \tag{4.6.26}$$

Identifying $\theta_1 = -\theta_2 - \theta_3 - \cdots - \theta_n$ and $\xi_1 = -\xi_2 - \xi_3 - \cdots - \xi_n$, we have

$$R_{n,0,2} \cong (\wedge\{\theta_2 \ldots, \theta_n\} \otimes \wedge\{\xi_2, \ldots, \xi_n\})/\langle(-\sum_{j=2}^{n} \theta_j) \otimes (-\sum_{j=2}^{n} \xi_j) + \sum_{i=2}^{n} \theta_i \otimes \xi_i\rangle \tag{4.6.27}$$

$$\cong (\wedge\{\theta_2 \ldots, \theta_n\} \otimes \wedge\{\xi_2, \ldots, \xi_n\})/\langle\sum_{i=2}^{n} \theta_i \otimes (\xi_i + \sum_{j=2}^{n} \xi_j)\rangle. \tag{4.6.28}$$

The transition matrix from the set $\{\xi'_2 + \ldots + \xi'_n\} = \{\xi_2 + \sum_{j=2}^{n} \xi_j, \ldots, \xi_n + \sum_{j=2}^{n} \xi_j\}$ to the standard basis $\{\xi_2, \ldots, \xi_n\} \subseteq \wedge\{\xi_2, \ldots, \xi_n\}_1$ is

$$\begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdot & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{pmatrix} \tag{4.6.29}$$

which is easily checked to be invertible. (Clearly $(1, \ldots, 1)$ is in column span, from which we conclude that the matrix is full rank.) Therefore, the set $\{\xi'_2, \ldots, \xi'_n\}$ is also a basis of $\wedge\{\xi_2, \ldots, \xi_n\}$ and we write

$$R_{n,0,2} \cong \wedge\{\theta_2, \ldots, \theta_n, \xi_2, \ldots, \xi_n\}/\left\langle \sum_{i=2}^{n} \theta_i \xi'_i \right\rangle. \tag{4.6.30}$$

Theorem 4.5.2 applies and translates to the claim. □

4.7 Problems

Existence of the Lefschetz element was the key to main results in this Chapter. Given the importance of Lefschetz elements in geometry, it is natural to ask whether there is a geometric interpretation behind the Fermionic diagonal coinvariants.

Another natural question to ask is about other Boson-Fermionic diagonal coinvariants. One example is $R_{n,0,3}$ in three sets of Fermionic variables. Empirically, it has been unclear what the total dimension should be for general n and also what the top degree is.

Another example is $R_{n,2,2}$ of the Theta conjecture. Recently, Iraci, Rhoades and Romero proved in [36] that the symmetric function side of the Theta conjecture at $q = t = 0$ equals the character of $R_{n,0,2}$, providing an evidence for the Theta conjecture.

Theorem 4.7.1 (Iraci-Rhoades-Romero)**.** For $i + j < n$,

$$\mathrm{Frob}((R_{n,0,2})_{i,j}) = \Theta_{e_i}\Theta_{e_j}\nabla e_{n-i-j}\,|_{q=t=0}\,. \tag{4.7.1}$$

The idea of their proof is to obtain a recursive structure on each side of the identity by applying the skewing operator $h_d^\perp$ to Kronecker products $s_\lambda \otimes s_\mu$ and applying the results of D'Adderio and Romero on Theta operator identities, [18].

Here is a question to ask about the combinatorial side: is there a way to incorporate our Motzkin paths into parking functions? An answer to this question could provide a helpful evidence to the Theta conjecture.

Chapter 5

Another Kronecker Product

We discuss another S_n-module which involves the tensor product of two S_n-modules and also exhibits the Narayana refinement of Catalan numbers. The motivation comes from a symmetric function formula for the character of the type B diagonal coinvariants which was conjectured by Haiman. The conjectured formula, although verified to be false, involves a plethystic substitution by $(q+t)X$ in $\nabla e_n[X]$. We study some properties of the plethystic tranformation $(\nabla e_n)[(q+t)X]$.

5.1 Some properties of the Kronecker product

Early in this thesis, we defined the Kronecker product and some of its properties. We define the *Kronecker coefficient*.

Definition 5.1.1. Let λ, μ, ν be partitions. The the following are equivalent definitions of the *Kronecker coefficient*, denoted by $g^{\lambda}_{\mu\nu}$:

1. $\langle s_\mu[X] \otimes s_\nu[X], s_\lambda[X] \rangle$,

2. $\langle V^\mu \otimes V^\nu, V^\lambda \rangle$,

3. The coefficient of $s_\mu[X]s_\nu[Y]$ in $s_\lambda[XY] = \sum_{\mu,\nu} g^\lambda_{\mu\nu} s_\mu[X]s_\nu[Y]$.

The equivalence of the first two definitions comes from the equivalence of Λ and the Grothendieck ring of S_n-representaions. The formula in the last definition is the coproduct formla on Λ as a Hopf algebra. It is well-known that the coefficient $g^\lambda_{\mu\nu}$ is invariant under permuting λ, μ and ν.

Recall the *Cauchy kernel*

$$\Omega[X] = \sum_n h_n[X] = \prod_i \frac{1}{1 - x_i}, \tag{5.1.1}$$

$$\Omega[XY] = \sum_n h_n[XY] = \prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda s_\lambda[X]s_\mu[Y]. \tag{5.1.2}$$

Since h_n and s_λ are homogeneous of degree n, $h_n[XY]$ is homogeneous of degree $2n$ and so

$$s_{(n)}[XY] = h_n[XY] = \sum_{\lambda \vdash n} s_\lambda[X]s_\lambda[Y]. \tag{5.1.3}$$

Lemma 5.1.2. Let λ be a partition of n and X, Y be alphabets. Then we have

$$\langle s_\lambda[XY], s_{(n)}[X] \rangle = s_\lambda[Y], \tag{5.1.4}$$

$$\langle s_\lambda[XY], s_{(1^n)}[X] \rangle = s_{\lambda'}[Y]. \tag{5.1.5}$$

Proof. By definition,

$$\langle s_\lambda[XY], s_{(n)}[X]\rangle = \sum_\nu g^\lambda_{(n)\nu} s_\nu[Y], \tag{5.1.6}$$

$$\langle s_{(n)}[XY], s_\lambda[X]\rangle = \sum_\nu g^{(n)}_{\lambda\nu} s_\nu[Y]. \tag{5.1.7}$$

The lemma follows from the invariance of the Kronecker coefficient and the Cauchy kernel.

The second identity follows exactly the same as the first identity using the dual Cauchy kernel instead. We have

$$\langle s_\lambda[XY], s_{(1^n)}[X]\rangle = \langle s_{(1^n)}[XY], s_\lambda[X]\rangle \tag{5.1.8}$$

from which the dual Cauchy kernel $\sum_n e_n[XY] = \sum_\lambda s_\lambda[X]s_{\lambda'}[Y]$ concludes the proof. $\square$

5.2 m-ary strings and tensor product

We first prove fact stated as an easy exercise in [30].

Lemma 5.2.1. If V is a virtual S_n-representation such that $\mathrm{Frob}(V) = h_n[AX]$ for some A, then for any other virtual S_n-representation W,

$$\mathrm{Frob}(V \otimes W) = \mathrm{Frob}(W)[AX]. \tag{5.2.1}$$

Proof. By linearity, we may assume $W = V^\lambda$ is an irreducible representation corresponding to some λ. Then on the left hand side,

$$\begin{aligned}
\mathrm{Frob}(V \otimes W) &= \mathrm{Frob}(V) \otimes \mathrm{Frob}(W) && (5.2.2)\\
&= h_n[AX] \cdot s_\lambda[X] && (5.2.3)\\
&= \left(\sum_\mu \frac{1}{z_\mu} p_\mu[A] p_\mu[X]\right) \cdot \left(\sum_\nu \frac{\chi_\nu^\lambda}{z_\nu} p_\nu[X]\right) && (5.2.4)\\
&= \sum_\mu \frac{\chi_\mu^\lambda}{z_\mu} p_\mu[A] p_\mu[X] && (5.2.5)
\end{aligned}$$

which equals the right hand side by definition. □

Viewing as $\mathbb{C}$-vector space only, the exterior algebra $\wedge\{\Theta_n\}$ can be thought of as the span of the set of binary strings of length n. Similarly, we define S_n-modules $M_{n,m}$, the space of m-ary strings of length n. Let $n, m \geq 0$ and define $M_{n,m}$ to be the span

$$M_{n,m} = \mathrm{span}_{\mathbb{C}}\{(z_1, \ldots, z_n) \mid z_i \in \{a_i\}_{i=1}^m\}, \tag{5.2.6}$$

where $A^{(m)} = a_1 + a_2 + \cdots + a_m$ is a length m alphabet. The number of appearances of each a_i grades $M_{n,m}$ naturally. (Note that this is different from the usual degree grading on polynomial rings or exterior algebras.) For example, when $n = 2$ and $m = 3$, the space $M_{2,3}$ has six graded pieces with degrees $a_1^2, a_2^2, a_3^2, a_1a_2, a_1a_3$ and a_2a_3. It is clear that each graded piece is a S_n-submodule. The space $M_{n,m}$ is defined this way to satisfy the assumption of Lemma 5.2.1.

Proposition 5.2.1. Let $A = A^{(m)} = a_1 + \cdots + a_m$ and X be alphabets. Then

$$\mathrm{Frob}(M_{n,m}) = h_n[AX]. \tag{5.2.7}$$

Proof. On the right hand side, by the plethystic expansion

$$h_n[X] = \sum_{k=0}^{n} h_k[A^{(m-1)}X]h_{n-k}[a_m X]. \tag{5.2.8}$$

On the $M_{n,m}$ side, group the strings by the number of appearances of a_m, counted by $n-k$. Consider a S_k-module $M_{k,m-1}$ in the alphabet $A^{(m-1)}$ and a S_{n-k}-module $M_{n-k,1}$ in alphabet a_m. Let $M_{n,m} = \oplus_k (M_{n,m})_k$ be a decomposition by degree in a_m. Then we have

$$(M_{n,m})_k = \mathrm{Ind}_{S_k \times S_{n-k}}^{S_n} M_{k,m-1} \otimes M_{n-k,1} \tag{5.2.9}$$

which proves the claim. □

Combining the Proposition 5.2.1 and Lemma 5.2.1, we have a following transformation of the diagonal coinvariants and its Frobenius image

Theorem 5.2.2. For $n, m \geq 1$,

$$\mathrm{Frob}(\mathrm{DR}_n \otimes M_{n,m}) = (\nabla e_n)[A^{(m)}X], \tag{5.2.10}$$

where $\mathrm{DR}_n = R_{n,2,0}$ and $M_{n,m}$ are modules defined above.

The diagonal coinvariants as discussed in Chapter 2 and 3 have many interesting combinatorial properties. We describe some consequences when $m = 2$.

5.3 $t = 1$ case

In the early work on the diagonal coinvariants, Haiman and Garsia used the q-Lagrange inversion to show the following theorem [24]

Theorem 5.3.1. For a Dyck path $\pi \in D(n)$, define $\beta(\pi)$ to be the partition rearrangement of the column sizes (consecutive N-step sequences). Then at $t = 1$,

$$(\nabla e_n)[X; q, 1] = \sum_{\pi \in D(n)} q^{\mathrm{area}(\pi)} e_{\beta(\pi)}[X]. \tag{5.3.1}$$

From the Lemma 5.1.2, we have the following evaluation

Lemma 5.3.2. For $n, m \geq 1$,

$$\langle e_\lambda[A^{(m)}X], s_{(n)}\rangle = \begin{cases} \prod_{i=1}^{\ell(\lambda)} \left(\sum\limits_{\substack{A \subseteq A^{(m)} \\ |A| = \lambda_i}} \prod_{a \in A} a \right) & \text{if } \lambda_1 \leq m \\ \\ 0 & \text{otherwise.} \end{cases} \tag{5.3.2}$$

Proof. Schur functions λ form a basis, so Lemma 5.1.2 applies linearly to give

$$\langle e_\lambda[A^{(m)}X], s_{(n)}\rangle = e_\lambda[A^{(m)}] = \prod_{i=1}^{\ell(\lambda)} e_{\lambda_i}[A^{(m)}] \tag{5.3.3}$$

and the lemma follows from specializing $s_{(1^{\lambda_i})}$ at length m alphabet. $\square$

Combining the above two identities, we obtain a combinatorial interpretation of the trivial component at $t = 1$

Proposition 5.3.1. For $n \geq 1$,

$$\langle (\nabla e_n \mid_{t=1})[(q+t)X], s_{(n)}\rangle \mid_{t=1} = \sum_{\substack{\pi \in D(n) \\ \beta(\pi)_1 \leq 2}} q^{\mathrm{area}(\pi)} \cdot q^{m_2(\beta(\pi))} (1+q)^{m_1(\beta(\pi))}, \tag{5.3.4}$$

where $m_2(\beta(\pi))$ and $m_1(\beta(\pi))$ are the multiplicities of 2 and 1 in $\beta(\pi)$.

Pattern avoidance in permutations is an important topic in combinatorics. However, in the theory of the diagonal coinvariants and Macdonaly polynomials, pattern avoidance does not appear so often. We relate the multiplicity of the trivial component to 321-avoiding permutations.

A permutation $w = w_1 \cdots w_n \in S_n$ is 321-*avoiding* if there is no subsequence $w_i w_j w_k$ such that $w_i > w_j > w_k$ where $1 \leq i < j < k \leq n$. We call w_i in w a *left-to-right maximum* if $w_i = \max\{w_1, \ldots, w_i\}$. Let $\mathrm{Lrm}(w) = \{w_i \mid w_i \text{ is a left-to-right maximum}\}$ and $\mathrm{lrm}(w) = |\mathrm{Lrm}(w)|$. An *inversion* is a pair $i < j$ such that $w_i > w_j$ in w. Let $\mathrm{inv}(w)$ denote the number of inversion pairs in w. Let $\mathrm{Av}_n(321)$ denote the set of 321-avoiding permutations in S_n. Define the generating function

$$I_n(q,t) = \sum_{w \in \mathrm{Av}_n(321)} q^{\mathrm{inv}(w)} t^{\mathrm{lrm}(w)}. \tag{5.3.5}$$

The generating function $I_n(q,t)$ satisfies a recurrence relation [15].

Theorem 5.3.3. [15] For $n \geq 1$,

$$I_n(q,t) = tI_{n-1}(q,t) + \sum_{k=0}^{n-2} q^{k+1} I_k(q,t) I_{n-k-1}(q,t). \tag{5.3.6}$$

Call the trivial component at $t = 1$ by $L_n(q)$

$$L_n(q) = \sum_{\substack{\pi \in D(n) \\ \beta(\pi)_1 \leq 2}} q^{\mathrm{area}(\pi)} \cdot q^{m_2(\beta(\pi))} (1+q)^{m_1(\beta(\pi))} \tag{5.3.7}$$

We show that $L_n(q)$ satisfies the same recursion as $I_{n+1}(q,t)$ at $t = 1$.

Lemma 5.3.4. For $n \geq 1$,

$$L_n(q) = (1+q) \cdot L_{n-1}(q) + \sum_{k=1}^{n-1} q^{k+1} \cdot L_{k-1}(q) L_{n-k-1}(q) \tag{5.3.8}$$

with the boundary condition $L_0(q) = 1$.

Proof. This recursion comes from the usual recursion for Catalan numbers based on the first return to the diagonal. □

So Theorem 5.3.3 and Lemma 5.3.4 relate to give

Corollary 5.3.5. For $n \geq 1$,

$$L_n(q) = \langle (\nabla e_n)[(q+t)X], s_{(n)} \rangle \mid_{t=1} = I_{n+1}(q, 1). \tag{5.3.9}$$

One remark is that we did not use the t-statistic for $I_n(q,t)$ in our discussion. This is because the left-to-right maximum is not jointly nor symmetrically distributed with the inversion. Results in [15] relate the 321-avoiding permutations to a variant of Motzkin paths and parallelogram polyominoes. We will relate $\mathrm{DR}_n \otimes M_{n,2}$ with q,t-Narayana numbers and thus with parallelogram polyominoes.

5.4 Trivial and sign component at $m = 2$

By definition and discussions in previous sections, we have dimensions of the space $\mathrm{DR}_n \otimes M_{n,2}$ and its isotypic trivial and sign components.

$$\langle \mathrm{DR}_n \otimes M_{n,2}, V^{(n)} \rangle = \mathrm{Cat}(n+1).$$

Lemma 5.1.2 says that for a partition λ,

$$\langle s_\lambda[(q+t)X], s_{(n)} \rangle = \begin{cases} (qt)^j [i-j+1]_{q,t} & \text{if } \lambda = (i,j) \\ 0 & \text{otherwise} \end{cases}. \tag{5.4.1}$$

Using this fact, we can decompose the trivial component into the q,t-Narayana of [3], demonstrating the Narayana refinement of the Catalan again.

A *parallelogram polyomino with $m \times n$ bounding box* is a pair of lattice paths consisting of N-steps and E-steps from $(0,0)$ to (m,n) such that two paths only meet at the origin $(0,0)$ and (m,n). Let Polyo(m,n) denote the set of parallelogram polyominoes with $m \times n$ bounding box. An *area* of a parallelogram polyomino P is simply the number of cells in between two paths. There are two other statistics called *bounce* and *dinv* reminiscent of the bound and dinv on Dyck paths. Then [3] shows that the two generating functions "equal"

$$\mathrm{Nara}_{m,n}(q,t) = \sum_{P \in \mathrm{Polyo(m,n)}} q^{\mathrm{area}(P)} t^{\mathrm{bounce}(P)} = \sum_{P \in \mathrm{Polyo(n,m)}} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)}. \quad (5.4.2)$$

Moreover, they obtain a symmetric function interpretation of q,t-Narayana numbers

Theorem 5.4.1. [3] For all $m,n \geq 1$, we have

$$\mathrm{Nara}_{m,n}(q,t) = (qt)^{m+n-1} \langle \nabla e_{m+n-2}, h_{m-1} h_{n-1} \rangle. \quad (5.4.3)$$

Parallelogram polyominoes are interesting also because they are in bijection with *primitive Motzkin paths* which are very similar to our Moztkin paths in that the horizontal steps are decorated by two colors [2], [20]. Primitive Motzkin paths are lattice paths from $(0,0)$ to $(n,0)$ consisting of the up-steps, down-steps and horizontal steps decorated by two colors that lie strictly above the x-axis except at endpoints.

We briefly describe the bijection. Suppose the horizontal steps of Motzkin paths are colored by red or blue and identify a parallelogram polyomino by a pair of paths (π^1, π^2) where π^1 is in red and π^2 is in blue. If the bounding box is of dimension $m \times n$, then π^1 and π^2 are lattice paths consisting of $m+n$ steps each. We map (π^1, π^2) to a primitive Motzkin path by the following map on each pair of step: for $1 \leq i \leq m+n$,

$$(\pi_i^1, \pi_i^2) \mapsto \begin{cases} (1,1) & \text{if } \pi_i^1 = (1,0) \text{ and } \pi_i^2 = (0,1) \\ (1,-1) & \text{if } \pi_i^1 = (0,1) \text{ and } \pi_i^2 = (1,0) \\ (1,0)^{\text{red}} & \text{if } \pi_i^1 = (1,0) \text{ and } \pi_i^2 = (1,0) \\ (1,0)^{\text{blue}} & \text{if } \pi_i^1 = (0,1) \text{ and } \pi_i^2 = (0,1) \end{cases}. \tag{5.4.4}$$

We note that the three statistics area, bounce and dinv all map to primitive Motzkin paths naturally. In terms of Narayana numbers, $|\text{Polyo}(m,n)| = \text{Nar}(m+n-1, m)$. So for fixed N, Narayana refinement of Catalan number becomes

$$\text{Cat}(N) = \sum_{m+n=N+1} \text{Nar}(m+n-1, m). \tag{5.4.5}$$

Combinatorially, we enumerate parallelogram polyominoes with each path consisting of $N+1$ steps. Hence the paths have varying bounding box dimension. On the other hand, in terms of primitive Motzkin paths, bounding box dimension translates to the difference between the number of blue steps and red steps, leaving the endpoints of paths fixed. This stability may provide a better insights in some cases. For example, for $i, j \geq 1$, we have $s_{(i,j)} = h_{(i,j)} - h_{(i+1,j-1)}$. By Theorem 5.4.1 and representation theoretically, all three coefficients appearing below are positive

$$\langle \nabla e_n, s_{(i,j)} \rangle = \langle \nabla e_n, h_{(i,j)} \rangle - \langle \nabla e_n, h_{(i+1,j-1)} \rangle. \tag{5.4.6}$$

However, it's not clear from the paralleolgram polyomino enumeration why the difference above has to be positive. Perhaps there's a way to construct an injection between the subsets of primitive Motzkin paths to show positivity combinatorially.

Using the identification from Theorem 5.4.1, we have a following decomposition of the Catalan dimensional trivial components into the Narayana numbers

Theorem 5.4.2. For $n \geq 1$,

$$\langle (\nabla e_n)[(q+t)X], s_{(n)} \rangle = \sum_{i+j=n} \langle \nabla e_n, h_{(i,j)} \rangle q^i t^j. \tag{5.4.7}$$

Proof. On the left hand side, expanding in Schur basis first and appying (5.4.1),

$$\begin{aligned} \langle (\nabla e_n)[(q+t)X], s_{(n)} \rangle &= \langle \sum_{\lambda \vdash n} \langle \nabla e_n[X], s_\lambda[X] \rangle s_\lambda[(q+t)X], s_{(n)} \rangle & (5.4.8) \\ &= \sum_{\lambda \vdash n} \langle \nabla e_n[X], s_\lambda[X] \rangle \cdot \langle s_\lambda[(q+t)X], s_{(n)} \rangle & (5.4.9) \\ &= \sum_{\substack{i+j=n \\ i \geq j}} \langle \nabla e_n[X], s_{(i,j)}[X] \rangle \cdot (qt)^j [i-j+1]_{q,t}. & (5.4.10) \end{aligned}$$

Recall that $s_{(i,j)} = h_{(i,j)} - h_{(i+1,j-1)}$ where $h_{(n+1,-1)} = 0$. Replacing $s_{(i,j)}$ by $h_{(i,j)} - h_{(i+1,j-1)}$ and collecting the coefficient of $\langle \nabla e_n, h_{(i,j)} \rangle$ we obtain

$$(qt)^j [i-j+1]_{q,t} - (qt)^{j+1} [i-j-1]_{q,t} = q^i t^j + q^j t^i \tag{5.4.11}$$

except when $i = j$, in which case we simply get $q^i t^j$. $\square$

By using the same argument, we obtain a parallel decomposition for the sign component

$$\langle (\nabla e_n)[(q+t)X], s_{(1^n)} \rangle = \sum_{i+j=n} \langle \nabla e_n, e_{(i,j)} \rangle q^i t^j. \tag{5.4.12}$$

In fact, the same argument applies to bigger m to give an expression for the multiplicity of trivial and sign components.

5.5 Hook multiplicity for general m

Ignoring the grading, we can obtain the multiplicity of hook components of the module $\mathrm{DR}_n \otimes M_{n,m}$, thus its trivial and sign components. Recall that classical parking function module, which is isomorphic to $\mathrm{DR}_n \otimes \epsilon$ where ϵ is the sign representation, has the following Frobenius character

$$\mathrm{Frob}(\mathrm{DR}_n \otimes \epsilon) = \sum_{\lambda \vdash n} (n+1)^{\ell(\lambda)-1} \frac{p_\lambda[X]}{z_\lambda} \tag{5.5.1}$$

$$= \frac{1}{n+1} \Omega[Xz]^{n+1} \Big| z^n. \tag{5.5.2}$$

From the Cauchy kernel, we have

$$\mathrm{Frob}(\mathrm{DR}_n \otimes \epsilon) = \frac{1}{n+1} \sum_{\lambda \vdash n} s_\lambda[1^{n+1}] s_\lambda[X]. \tag{5.5.3}$$

Tensoring with $M_{n,m}$, we get

$$\mathrm{Frob}(\mathrm{DR}_n \otimes \epsilon \otimes M_{n,m}) = \left(\sum_{\lambda \vdash n} (n+1)^{\ell(\lambda)-1} \frac{p_\lambda[X]}{z_\lambda} \right) \otimes h_n[A^{(m)}X] \tag{5.5.4}$$

$$= \left(\sum_{\lambda \vdash n} (n+1)^{\ell(\lambda)-1} \frac{p_\lambda[X]}{z_\lambda} \right) \otimes \left(\sum_{\mu \vdash n} \frac{p_\mu[A^{(m)}X]}{z_\mu} \right) \tag{5.5.5}$$

setting $A^{(m)} = 1^m$,

$$= \sum_{\lambda \vdash n} (n+1)^{\ell(\lambda)-1} m^{\ell(\lambda)} \frac{p_\lambda[X]}{z_\lambda} \tag{5.5.6}$$

$$= \frac{1}{n+1} \Omega[Xz]^{m(n+1)} \Big| z^n. \tag{5.5.7}$$

Similarly, we have

$$\operatorname{Frob}(\mathrm{DR}_n \otimes M_{n,m}) = \frac{1}{n+1} \sum_{\lambda \vdash n} s_{\lambda'}[1^{m(n+1)}] s_\lambda[X]. \tag{5.5.8}$$

Taking $\lambda = (n-k+1, 1^{k-1})$ and using the principal evaluation of Schur function indexed by a hook, we obtain the following multiplicities

Proposition 5.5.1. For $n, m \geq 1$ and $1 \leq k \leq n$,

$$\langle \operatorname{Frob}(\mathrm{DR}_n \otimes M_{n,m}), s_{(n-k+1,1^{k-1})} \rangle = \frac{1}{n+1} s_{(k,1^{n-k})}[1^{m(n+1)}] \tag{5.5.9}$$

$$= \frac{1}{n+1} \binom{n-1}{k-1} \binom{m(n+1)+k-1}{n}. \tag{5.5.10}$$

These numbers are known as *Fuss-Catalan numbers*.

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