ANDERSON-BERNOULLI LOCALIZATION ON 2D AND 3D LATTICE

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ABSTRACT

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The Anderson model describes the behaviour of electrons inside a piece of metal with uniform impurity. The Anderson-Bernoulli model is a special case of the Anderson model where the potential has Bernoulli distribution. We consider Anderson-Bernoulli localization on $\mathbb{Z}^d$ for $d = 2, 3$. For $d = 2$, we prove that, if the potential has symmetric Bernoulli distribution and the disorder is large, then localization happens outside a small neighborhood of finitely many energies. For $d = 3$, we prove that localization happens at the bottom of the spectrum.

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Chapter 1

Introduction

The Anderson models are widely used to describe spectral and transport properties of disordered media, such as moving quantum mechanical particles, or electrons in a metal with impurities. The mathematical study of their localization phenomena can be traced back to the 1980s (see e.g. [KS80]), and since then there have been many results in models on both discrete and continuous spaces. The mathematical definition for Anderson model (and Anderson localization) on discrete spaces is as follows. Let $d \in \mathbb{Z}_+$, $\delta > 0$ and $V : \mathbb{Z}^d \to \mathbb{R}_{\geq 0}$ such that $\{V(a) : a \in \mathbb{Z}^d\}$ is a family of i.i.d. bounded random variables. Define the Anderson Hamiltonian $H = -\Delta + \delta V$ where Δ is the discrete Laplacian. Given a subset $I \subset \mathbb{R}$, we say Anderson localization happens in I if following holds: For any function $u : \mathbb{Z}^d \to \mathbb{R}$ and energy $\lambda \in I$, if $Hu = \lambda u$ and $\inf_{n \geq 0} \sup_{a \in \mathbb{Z}^d} (1 + |a|)^{-n} |u(a)| < \infty$, we have $\inf_{t>0} \sup_{a \in \mathbb{Z}^d} \exp(t|a|)|u(a)| < \infty$.

If I is a union of closed intervals, Anderson localization in I implies that H has pure point spectrum in I (see e.g. [Kir08, Section 7]). Note that this is related to but different from "dynamical localization" (see e.g. discussions in [AW15, Section 7.1]).

In most early works, some regularity conditions on the distribution of the random potential are needed. In [FS83], Fröhlich and Spencer used a multi-scale analysis argument to show that if $\{V(a) : a \in \mathbb{Z}^d\}$ are i.i.d. bounded random variables with bounded probability density, then the resolvent decays exponentially when δ is large enough or energy is sufficiently small. Then in [FMSS85], together with Martinelli and Scoppola, they proved Anderson localization under the same condition. This result was strengthened later by [CKM87], where the same results were proved under the condition that the distribution of $\{V(a) : a \in \mathbb{Z}^d\}$ are i.i.d., bounded, and Hölder continuous.

It remains an interesting problem to remove these regularity conditions. As described at the beginning of [DSS02], when using the Anderson models to study alloy type materials, it is natural to expect the random potential to take only finitely many values. A particular case is where the random potential are i.i.d. Bernoulli variables.

For the particular case of $d = 1$, in the above mentioned paper [CKM87] the authors proved that for the discrete model on $\mathbb{Z}$, Anderson localization holds for the full spectrum when the i.i.d. random potential is non-degenerate and has some finite moment. This includes the Bernoulli case. In [BDF$^+$19] a new proof is given for the case where the random potential has bounded support. In [DSS02], the continuous model on $\mathbb{R}$ was studied, and Anderson localization was proved for the full spectrum when the i.i.d. random potential is non-degenerate and has bounded support.

For higher dimensions, a breakthrough was then made by Bourgain and Kenig. In [BK05], they studied the continuous model $\mathbb{R}^d$, for $d \geq 2$, and proved Anderson-Bernoulli localization near the bottom of the spectrum. An important ingredient is the *unique continuation principle* in $\mathbb{R}^d$, i.e. [BK05, Lemma 3.10]. It roughly says that, if $u : \mathbb{R}^d \to \mathbb{R}$ satisfies $\Delta u = Vu$ for some bounded V on $\mathbb{R}^d$, and u is also bounded, then u can not be too small on any ball with positive radius. Using this unique continuation principle together with the Sperner lemma, they proved a Wegner estimate, which was used to prove the exponential decay of the resolvent. In doing this, many aspects of the usual multi-scale analysis framework were adapted; and in particular, they introduced the idea of "free sites". See [Bou05] for some more discussions. Later, Germinet and Klein [GK12] incorporated the new ideas of [BK05] and proved localization (in a strong form) near the bottom of the spectrum in the continuous model, for any non-degenerate potential with bounded support.

The Anderson-Bernoulli localization on lattices in higher dimensions remained open. There were efforts toward this goal by relaxing the condition that V only takes two values (see [Imb21]). Recently, the work of Ding and Smart [DS20] proved Anderson-Bernoulli localization near the edge of the spectrum on the 2D lattice. As discussed in [BK05, Section 1], the approach there cannot be directly applied to the lattice model, due to the lack of a discrete version of the unique continuation principle. A crucial difference between the lattice $\mathbb{Z}^d$ and $\mathbb{R}^d$ is that one could construct a function $u : \mathbb{Z}^d \to \mathbb{R}$, such that $\Delta u = Vu$ holds for some bounded V, but u is supported on a lower dimensional set (see Remark 2.1.6 below for an example on 3D lattice). Hence, a suitable “discrete unique continuation principle” in $\mathbb{Z}^d$ would state that, if a function u satisfies $-\Delta u + Vu = 0$ in a finite (hyper)cube, then u can not be too small (compared to its value at the origin) on a substantial portion of the (hyper)cube. In [DS20], a randomized version of the discrete unique continuation principle on $\mathbb{Z}^2$ was proved. The proof was inspired by [BLMS17], where unique continuation principle was proved for harmonic functions (i.e. $V = 0$) on $\mathbb{Z}^2$. An important observation exploited in [BLMS17] is that the harmonic function has a polynomial structure.

The rest of the thesis is organized as follows: In Chapter 2, we consider the Anderson-Bernoulli model on 3D lattice and prove localization near the bottom of the spectrum. In Chapter 3, we consider the Anderson-Bernoulli model on 2D lattice and prove localization at large disorder on the whole spectrum except a union of small intervals. Chapter 2 is based on the article [LZ22] joint with Lingfu Zhang and Chapter 3 is based on the article [Li20]. We refer the reader to original articles for more details.

Chapter 2

3D Anderson-Bernoulli localization near the edge

2.1 Introduction

2.1.1 Main result and background

In the 3D Anderson-Bernoulli model on the lattice, we study the random Schrödinger operator $H := -\Delta + \delta V$, acting on the space $\ell^2(\mathbb{Z}^3)$. Here $\delta > 0$ is the *disorder strength*, Δ is the discrete Laplacian:

$$\Delta u(a) = -6u(a) + \sum_{b \in \mathbb{Z}^3, |a-b|=1} u(b), \ \forall u \in \ell^2(\mathbb{Z}^3), a \in \mathbb{Z}^3, \tag{2.1.1}$$

and $V : \mathbb{Z}^3 \to \{0, 1\}$ is the Bernoulli random potential; i.e. for each $a \in \mathbb{Z}^3$, $V(a) = 1$ with probability $\frac{1}{2}$ independently. Here and throughout this chapter, $|\cdot|$ denotes the Euclidean norm.

Our main result is as follows.

Theorem 2.1.1. *There exists $\lambda_* > 0$, depending on δ, such that almost surely the following holds. For any function $u : \mathbb{Z}^3 \to \mathbb{R}$ and $\lambda \in [0, \lambda_*]$, if $Hu = \lambda u$ and $\inf_{n \geq 0} \sup_{a \in \mathbb{Z}^3} (1 + |a|)^{-n} |u(a)| < \infty$, we have $\inf_{t>0} \sup_{a \in \mathbb{Z}^3} \exp(t|a|)|u(a)| < \infty$.*

Our Theorem 2.1.1 settles the Anderson-Bernoulli localization near the edge of the spectrum on the 3D lattice. Our proof follows the framework of [BK05] and [DS20]. Our main contribution is the proof of a 3D discrete unique continuation principle. Unlike the 2D case, where some randomness is required (see Chapter 1), in 3D our discrete unique continuation principle is deterministic, and allows the potential V to be an arbitrary bounded function. It is also robust, in the sense that certain "sparse set" can be removed and the result still holds; and this makes it stand for the multi-scale analysis framework (see Theorem 2.3.4 below). The most innovative part of our proof is to explore the geometry of the 3D lattice.

Let us also mention that Anderson localization is not expected through the whole spectrum in $\mathbb{Z}^3$, when the potential is small and it is conjectured that there is a localization-delocalization transition. To be more precise, it is conjectured that there exists $\delta_0 > 0$ such that, for any $\delta < \delta_0$, $-\Delta + \delta V$ has purely absolutely continuous spectrum in some spectrum range (see e.g. [Sim00]). Localization and delocalization phenomenons are also studied for other models, see e.g. [AW15, Chapter 16] and [AS19] for regular tree graphs and expander graphs, and see [BYY20, BYYY18, YY21] and [SS17, SS21] for random band matrices.

2.1.2 An outline of the proof of the 3D discrete unique continuation principle

In this subsection we explain the most important ideas in the proof of the 3D discrete unique continuation principle.

The formal statement of the 3D discrete unique continuation principle is Theorem 2.3.4 below. It is stated to fit the framework of [BK05] and [DS20]. To make a clear outline, we state a simplified version here.

Definition 2.1.2. For any $a \in \mathbb{Z}^3$, and $r \in \mathbb{R}_+$, the set $a + ([-r, r] \cap \mathbb{Z})^3$ is called a *cube*, or $2r$-*cube*, and we denote it by $Q_r(a)$. Particularly, we also denote $Q_r := Q_r(\mathbf{0})$.

Theorem 2.1.3. *There exists constant $p > \frac{3}{2}$ such that the following holds. For each $K > 0$, there is $C_1 > 0$, such that for any large enough $n \in \mathbb{Z}_+$, and functions $u, V : \mathbb{Z}^3 \to \mathbb{R}$ with*

$$\Delta u = Vu \tag{2.1.2}$$

in Q_n and $\|V\|_\infty \leq K$, we have that

$$|\{a \in Q_n : |u(a)| \geq \exp(-C_1 n)|u(\mathbf{0})|\}| \geq n^p. \tag{2.1.3}$$

Remark 2.1.4. The power of $\frac{3}{2}$ should not be optimal. We state it this way because it is precisely what we need (in the proof of Lemma 2.3.5 below).

To prove Theorem 2.1.3, we first prove a version with a loose control on the magnitude of the function but with a two-dimensional support. It is a simplified version of Theorem 2.5.1 below.

Theorem 2.1.5. *For each $K > 0$, there is C_2 depending only on K, such that for any $n \in \mathbb{Z}_+$ and functions $u, V : \mathbb{Z}^3 \to \mathbb{R}$ with*

$$\Delta u = Vu \tag{2.1.4}$$

in Q_n and $\|V\|_\infty \leq K$, we have that

$$\left|\left\{a \in Q_n : |u(a)| \geq \exp(-C_2 n^3)|u(\mathbf{0})|\right\}\right| \geq C_3 n^2 (\log_2 n)^{-1}. \tag{2.1.5}$$

Here C_3 is a universal constant.

Remark 2.1.6. The power of n^2 can not be improved. Consider the case where $V \equiv 0$, and $u : (x, y, z) \mapsto (-1)^x \exp(sz)\mathbb{1}_{x=y}$, where $s \in \mathbb{R}_+$ is the constant satisfying $\exp(s) + \exp(-s) = 6$. One can check that $\Delta u_0 \equiv 0$, while $|\{a \in Q_n : u_0(a) \neq 0\}| = |\{(x, y, z) \in Q_n : x = y\}| = (2n + 1)^2$.

To prove Theorem 2.1.3, we find many disjoint translations of $Q_{n^{1/3}}$ inside Q_n, and use Theorem 2.1.5 on each of these translations. This is made precise by Theorem

2.6.1 in Section 2.6. The foundation of the arguments there is the "cone property", given in Section 2.2, which says that from any point in $\mathbb{Z}^3$, we can find a chain of points, where $|u|$ decays at most exponentially. Such property is also used in other parts of the chapter.

The proof of Theorem 2.1.5 is based on geometric arguments on $\mathbb{Z}^3$. We consider four collections of planes in $\mathbb{R}^3$.

Definition 2.1.7. Let $\mathbf{e}_1 := (1,0,0)$, $\mathbf{e}_2 := (0,1,0)$, and $\mathbf{e}_3 := (0,0,1)$ to be the standard basis of $\mathbb{R}^3$, and denote $\boldsymbol{\lambda}_1 := \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, $\boldsymbol{\lambda}_2 := -\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, $\boldsymbol{\lambda}_3 := \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3$, $\boldsymbol{\lambda}_4 := -\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3$. For any $k \in \mathbb{Z}$, and $\tau \in \{1,2,3,4\}$, denote $\mathcal{P}_{\tau,k} := \{a \in \mathbb{R}^3 : a \cdot \boldsymbol{\lambda}_\tau = k\}$.

We note that the intersection of $\mathbb{Z}^3$ with each of these planes is a *2D triangular lattice*. Besides, there is a family of regular tetrahedrons in $\mathbb{R}^3$, whose four faces are orthogonal to $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3, \boldsymbol{\lambda}_4$, respectively. Using these tetrahedrons, we construct some polyhedrons $\mathfrak{P} \subset \mathbb{R}^3$, called *pyramid*. For each of these pyramid $\mathfrak{P}$, the boundary $\partial\mathfrak{P}$ consists of subsets of some of the planes $\mathcal{P}_{\tau,k}$ (where $\tau \in \{1,2,3,4\}$ and $k \in \mathbb{Z}$). See Figure 2.7 for an illustration. Using these observations, we lower bound $|\{a \in Q_n : |u(a)| \geq \exp(-C_2 n^3)|u(\mathbf{0})|\} \cap \partial\mathfrak{P}|$.

To be more precise, we define such 2D triangular lattice as follows.

Definition 2.1.8. In $\mathbb{R}^2$, denote $\xi := (-1,0)$ and $\eta := \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Define the *triangular* *lattice* as $\Lambda := \{s\xi + t\eta : s, t \in \mathbb{Z}\}$. For $a \in \Lambda$ and $n \in \mathbb{Z}_{\geq 0}$, denote

$$T_{a;n} := \{a + s\xi + t\eta : t, s \in \mathbb{Z}, -n \leq t \leq 2n, t - n \leq s \leq n\}. \tag{2.1.6}$$

Then $T_{a;n}$ is an equilateral triangle of lattice points with center a, such that on each side there are $3n + 1$ lattice points.

Now we state the bound we need.

Theorem 2.1.9. *There exist constants $C_4 > 5$ and $\epsilon_1 > 0$ such that the following is true. For any $n \in \mathbb{Z}_+$ and any function $u : T_{\mathbf{0};n} \to \mathbb{R}$, if $|u(a) + u(a - \xi) + u(a + \eta)| < C_4^{-n}|u(\mathbf{0})|$ for any $a \in T_{\mathbf{0};\lfloor \frac{n}{2} \rfloor}$, then*

$$\left|\{a \in T_{\mathbf{0};n} : |u(a)| > C_4^{-n}|u(\mathbf{0})|\}\right| > \epsilon_1 n^2. \tag{2.1.7}$$

This theorem can be seen as a triangular version of [BLMS17, Theorem(A)]. Our proof is also similar to the arguments there, using the fact that the function u has an approximate polynomial structure.

Organization of remaining chapter

In Section 2.2, we state and prove the "cone properties". In Section 2.3, we introduce our discrete unique continuation (Theorem 2.3.4), and explain how to prove the resolvent estimate (Theorem 2.3.1) from it, by adapting the framework from [BK05] and [DS20]. The next three sections are devoted to the proof of our discrete unique continuation (Theorem 2.3.4): in Section 2.4 we prove the estimates on triangular lattice, i.e. Theorem 2.1.9 and its corollaries, using arguments similar to those in [BLMS17, Section 3]; in Section 2.5, we state and prove Theorem 2.5.1 (a stronger version of Theorem 2.1.5) by constructing pyramids and using Theorem 2.1.9; finally, in Section 2.6 we do induction on scales, and deduce Theorem 2.3.4 from Theorem 2.5.1.

We have three subsections with proofs of auxiliary lemmas. In Section 2.7.1 we state some auxiliary results from [DS20] that are used in the general framework. Section 2.7.2 is devoted to the base case of the multi-scale analysis in the general framework. In Section 2.7.3 we give some details on deducing Anderson localization (Theorem 2.1.1) from decay of the resolvent (Theorem 2.3.1), following existing arguments (from [BK05, Bou05, GK12]).

2.2 Cone properties

In this section we state and prove the "cone properties", which are widely used throughout the rest of this chapter.

Definition 2.2.1. For each $a \in \mathbb{Z}^3$, and $\tau \in \{1, 2, 3\}$, denote the cone

$$\mathcal{C}_a^\tau := \left\{ b \in \mathbb{Z}^3 : |(b-a) \cdot \mathbf{e}_\tau| \geq \sum_{\tau' \in \{1,2,3\} \setminus \{\tau\}} |(b-a) \cdot \mathbf{e}_{\tau'}| \right\}. \tag{2.2.1}$$

For each $k \in \mathbb{Z}$, let $\mathcal{C}_a^\tau(k) := \mathcal{C}_a^\tau \cap \{b \in \mathbb{Z}^3 : (b-a) \cdot \mathbf{e}_\tau = k\}$ be a section of the cone.

We also denote $\mathcal{C} := \mathcal{C}_{\mathbf{0}}^3$, for simplicity of notations.

First, we have the "local cone property".

Lemma 2.2.2. *For any $u : \mathbb{Z}^3 \to \mathbb{R}$, $a \in \mathbb{Z}^3$, and $v \in \{\pm\mathbf{e}_1, \pm\mathbf{e}_2, \pm\mathbf{e}_3\}$, if $|\Delta u(a+v)| \leq K|u(a+v)|$, we have*

$$\max_{b \in a+v+\{\mathbf{0},\pm\mathbf{e}_1,\pm\mathbf{e}_2,\pm\mathbf{e}_3\}\setminus\{a\}} |u(b)| \geq (K+11)^{-1}|u(a)|. \tag{2.2.2}$$

Proof. Without loss of generality we assume that $v = \mathbf{e}_1$. We have

$$\begin{aligned} |u(a)| &\leq (6+K)|u(a+\mathbf{e}_1)| + |u(a+2\mathbf{e}_1)| + |u(a+\mathbf{e}_1-\mathbf{e}_2)| + |u(a+\mathbf{e}_1+\mathbf{e}_2)| \\ &+ |u(a+\mathbf{e}_1+\mathbf{e}_3)| + |u(a+\mathbf{e}_1-\mathbf{e}_3)| \leq (K+11) \max_{b \in a+\mathbf{e}_1+\{\mathbf{0},\pm\mathbf{e}_1,\pm\mathbf{e}_2,\pm\mathbf{e}_3\}\setminus\{a\}} |u(b)|, \end{aligned} \tag{2.2.3}$$

and our conclusion follows. □

With Lemma 2.2.2, we can inductively construct an oriented "chain" from $\mathbf{0}$ to the boundary of a cube, and inside a cone.

Lemma 2.2.3. *Let $K \in \mathbb{R}_+$, and $u, V : \mathbb{Z}^3 \to \mathbb{R}$, such that $\|V\|_\infty \leq K$, and $\Delta u = Vu$ in Q_n for some $n \in \mathbb{Z}_+$. For any $a \in Q_{n-2}$, $\tau \in \{1,2,3\}$, $\iota \in \{1,-1\}$, and $k \in \mathbb{Z}_{\geq 0}$, if $\mathcal{C}_a^\tau(\iota k) \subset Q_n$, then there exists $w \in \mathbb{Z}_{\geq 0}$, and a sequence of points $a = a_0, a_1, \cdots, a_w \in \mathcal{C}_a^\tau \cap Q_n$, such that for any $1 \leq i \leq w$, we have $a_i - a_{i-1} \in (\iota\mathbf{e}_\tau + \{\mathbf{0}, \pm\mathbf{e}_1, \pm\mathbf{e}_2, \pm\mathbf{e}_3\}) \setminus \{\mathbf{0}\}$, $|u(a_i)| \geq (K+11)^{-1}|u(a_{i-1})|$; and $(a_w - a) \cdot (\iota\mathbf{e}_\tau) \in \{k-1, k\}$.*

Proof. We prove the case where $\iota = 1$, and the other case follows the same arguments.

We define the sequence inductively. Let $a_0 := a$. Suppose we have $a_i \in \mathcal{C}_a^\tau$, with $0 \leq (a_i - a) \cdot \mathbf{e}_\tau < k - 1$. Then $a_i + \mathbf{e}_\tau + \{\mathbf{0}, \pm\mathbf{e}_1, \pm\mathbf{e}_2, \pm\mathbf{e}_3\} \subset Q_n$. Let

$$a_{i+1} := \operatorname{argmax}_{b \in a_i + \mathbf{e}_\tau + \{\mathbf{0}, \pm\mathbf{e}_1, \pm\mathbf{e}_2, \pm\mathbf{e}_3\} \setminus \{a_i\}} |u(b)|. \tag{2.2.4}$$

Then we have that $a_{i+1} - a_i \in \mathbf{e}_\tau + \{\mathbf{0}, \pm\mathbf{e}_1, \pm\mathbf{e}_2, \pm\mathbf{e}_3\} \setminus \{\mathbf{0}\}$, $0 \leq (a_{i+1} - a) \cdot \mathbf{e}_\tau \leq k$, and $a_{i+1} \in \mathcal{C}_a^\tau$. By Lemma 2.2.2, we also have that $|u(a_{i+1})| \geq (K + 11)^{-1}|u(a_i)|$. This process will terminate when $(a_i - a) \cdot \mathbf{e}_\tau \geq k - 1$ for some $i \in \mathbb{Z}_{\geq 0}$. Then we let $w = i$; and from the construction we know that $(a_i - a) \cdot \mathbf{e}_\tau \in \{k - 1, k\}$. Thus we get the desired sequence of lattice points. $\square$

We also have a Dirichlet boundary version, whose proof is similar.

Lemma 2.2.4. *Take any $n \in \mathbb{Z}_+$, $K \in \mathbb{R}_+$, and $u, V : Q_n \to \mathbb{R}$, such that $\|V\|_\infty \leq K$ and $\Delta u = Vu$ with Dirichlet boundary condition. For any $a \in Q_n$, $\tau \in \{1, 2, 3\}$, $\iota \in \{1, -1\}$, and $k \in \mathbb{Z}_{\geq 0}$, if $\mathcal{C}_a^\tau(\iota k) \cap Q_n \neq \emptyset$, then the result of Lemma 2.2.3 still holds. In particular, we have $a_w \in (\mathcal{C}_a^\tau(\iota(k-1)) \cup \mathcal{C}_a^\tau(\iota k)) \cap Q_n$ and $|u(a_w)| \geq (K+11)^{-k}|u(a)|$.*

Proof. Again we only prove the case where $\iota = 1$, and define the sequence inductively. The only difference is that, given some $a_i \in \mathcal{C}_a^\tau$, if $0 \leq (a_i - a) \cdot \mathbf{e}_\tau < k - 1$, now we let

$$a_{i+1} := \operatorname{argmax}_{b \in (a_i + \mathbf{e}_\tau + \{\mathbf{0}, \pm\mathbf{e}_1, \pm\mathbf{e}_2, \pm\mathbf{e}_3\} \setminus \{a_i\}) \cap Q_n} |u(b)|. \tag{2.2.5}$$

By the Dirichlet boundary condition, we still have that

$$a_{i+1} - a_i \in \mathbf{e}_\tau + \{\mathbf{0}, \pm\mathbf{e}_1, \pm\mathbf{e}_2, \pm\mathbf{e}_3\} \setminus \{\mathbf{0}\},$$

$0 \le (a_{i+1} - a) \cdot \mathbf{e}_\tau \le k$, $a_{i+1} \in \mathcal{C}_a^\tau \cap Q_n$, and $|u(a_{i+1})| \ge (K+11)^{-1}|u(a_i)|$. □

2.3 General framework

This section is about the framework, based on the arguments in [DS20]. We formally state the discrete unique continuation principle (Theorem 2.3.4), and explain how to deduce Theorem 2.1.1 from it. For some results from [DS20] that are used in this section, we record them in Section 2.7.1 for easy reference purpose.

As in [DS20], these arguments essentially work for any i.i.d. potential V that is bounded and nontrivial. For simplicity we only study the $\frac{1}{2}$-Bernoulli case with disorder strength $\delta = 1$. Borrowing the formalism from [BK05] and [DS20], we allow V to take values in the interval $[0, 1]$, for the purpose of controlling the number of eigenvalues in proving the Wegner estimate (in the proof of Claim 2.3.9 below). In other words, we study the operator $H = -\Delta + V$, where V takes value in the space $[0, 1]^{\mathbb{Z}^3}$, equipped with the usual Borel sigma-algebra, and the distribution is given by the product of the $\frac{1}{2}$-Bernoulli measure (which is supported on $\{0, 1\}^{\mathbb{Z}^3}$).

We let $\mathrm{sp}(H)$ be the spectrum of H, then it is well known that, almost surely $\mathrm{sp}(H) = [0, 13]$ (see, e.g. [AW15, Corollary 3.13]). For any cube $Q \subset \mathbb{Z}^3$, let P_Q :

$\ell^2(\mathbb{Z}^3) \to \ell^2(Q)$ be the projection operator onto cube Q, i.e. $P_Q u = u|_Q$. Define $H_Q := P_Q H P_Q^\dagger$, where $P_Q^\dagger$ is the adjoint of P_Q. Then $H_Q : \ell^2(Q) \to \ell^2(Q)$ is the restriction of H on Q with Dirichlet boundary condition.

Throughout this section, by "dyadic", we mean a number being an integer power of 2.

The following result on decay of the resolvent is a 3D version of Theorem [DS20, Theorem 1.4], and it directly implies Theorem 2.1.1.

Theorem 2.3.1. *There exist $\kappa_0 > 0$, $0 < \lambda_* < 1$ and $L_* > 1$ such that*

$$\mathbb{P}\left[\left|(H_{Q_L} - \lambda)^{-1}(a,b)\right| \leq \exp\left(L^{1-\lambda_*} - \lambda_*|a-b|\right),\ \forall a,b \in Q_L\right] \geq 1 - L^{-\kappa_0} \quad (2.3.1)$$

for any $\lambda \in [0, \lambda_*]$ and dyadic scale $L \geq L_*$.

From Theorem 2.3.1, the arguments in [BK05, Section 7] prove Anderson localization in $[0, \lambda_*]$ (Theorem 2.1.1). See Section 2.7.3 for the details.

To prove Theorem 2.3.1, we will prove a 3D analog of [DS20, Theorem 8.3], i.e. Theorem 2.3.10 below. Except for replacing all 2D objects by 3D objects, the essential differences are:

1. We need more information on the *the frozen sites* defined in [DS20], rather than only knowing they're "η_k-regular" (see [DS20, Definition 3.4]).

2. We need a 3D Wegner estimate, an analog of [DS20, Lemma 5.6].

We now set up some geometric notations.

Definition 2.3.2. For any sets $A, B \subset \mathbb{R}^3$, let

$$\mathrm{dist}(A, B) := \inf_{a \in A, b \in B} |a - b|, \tag{2.3.2}$$

and

$$\mathrm{diam}(A) := \sup_{a, b \in A} |a - b|. \tag{2.3.3}$$

If $A = \{b \in \mathbb{R}^3 : |a - b| < r\}$, for some $r > 0$ and $a \in \mathbb{R}^3$, we call A a *(open) ball* and denote its *radius* as $\mathrm{radi}(A) := r$.

The following definitions are used to describe the frozen sites, and are stronger than being "η_k-regular" in [DS20].

Definition 2.3.3. Let $d \in \mathbb{Z}_{\geq 0}$, $N \in \mathbb{Z}_+$, and $C, \varepsilon > 0$, $l \geq 1$. A set $Z \subset \mathbb{R}^3$ is called (N, l, ε)*-scattered* if $Z = \bigcup_{j \in \mathbb{Z}_+, 1 \leq t \leq N} Z^{(j,t)}$ is a union of open balls such that,

1. for each $j \in \mathbb{Z}_+$ and $t \in \{1, \cdots, N\}$, $\mathrm{radi}(Z^{(j,t)}) = l$;
2. for any $j \neq j' \in \mathbb{Z}_+$ and $t \in \{1, 2, \cdots, N\}$, $\mathrm{dist}(Z^{(j,t)}, Z^{(j',t)}) \geq l^{1+\varepsilon}$.

A set $Z \subset \mathbb{R}^3$ is called C*-unitscattered*, if we can write $Z = \bigcup_{j \in \mathbb{Z}_+} Z^{(j)}$, where each $Z^{(j)} \subset \mathbb{R}^3$ is an open unit ball with center in $\mathbb{Z}^3$ and

$$\forall j \neq j' \in \mathbb{Z}_+, \mathrm{dist}(Z^{(j)}, Z^{(j')}) \geq C. \tag{2.3.4}$$

Let $l_1, \cdots l_d > 1$, we say that the vector $\vec{l} = (l_1, l_2, \cdots, l_d)$ is ε*-geometric* if for each $2 \leq i \leq d$, we have $l_{i-1}^{1+2\varepsilon} \leq l_i$. Given a vector of positive reals $\vec{l} = (l_1, l_2, \cdots, l_d)$, a set $E \subset \mathbb{R}^3$ is called an $(N, \vec{l}, C, \varepsilon)$-*graded set* if there exist sets $E_0, \cdots, E_d \subset \mathbb{R}^3$, such that $E = \bigcup_{i=0}^{d} E_i$ and the following holds:

1. $\vec{l}$ is ε-geometric,
2. E_0 is a C-unitscattered set,
3. for any $1 \leq i \leq d$, E_i is an (N, l_i, ε)-scattered set.

For each $1 \leq i \leq d$, we say that l_i is the *i-th scale length* of E. In particular, l_1 is called the *first scale length*. We also denote $l_0 := 1$.

Let $A \subset \mathbb{R}^3$, and E be an $(N, \vec{l}, C, \varepsilon)$-graded set and $\overline{C}, \overline{\varepsilon} > 0$. Then E is said to be $(\overline{C}, \overline{\varepsilon})$-*normal in* A, if $E_0 \cap A \neq \emptyset$ implies $\overline{C} \leq \mathrm{diam}(A)$, and $E_i \cap A \neq \emptyset$ implies $l_i \leq \mathrm{diam}(A)^{1-\frac{\overline{\varepsilon}}{2}}$ for any $i \in \{1, \cdots, d\}$.

In [DS20], a 2D Wegner estimate [DS20, Lemma 5.6] is proved and used in the multi-scale analysis. We will prove the 3D Wegner estimate based on our 3D discrete unique continuation, and we need to accommodate the frozen sites which emerge from the multi-scale analysis. For this we refine Theorem 2.1.3 as follows.

Theorem 2.3.4. *There exists a constant $p > \frac{3}{2}$, such that for any $N \in \mathbb{Z}_+$, $K \in \mathbb{R}_+$, and small enough $\varepsilon \in \mathbb{R}_+$, there exist $C_{\varepsilon,K}, C_{\varepsilon,N} > 0$ to make the following statement hold.*

Take $n \in \mathbb{Z}_+$ with $n > C_{\varepsilon,N}^4$ and functions $u, V : \mathbb{Z}^3 \to \mathbb{R}$ satisfying

$$\Delta u = Vu, \tag{2.3.5}$$

and $\|V\|_\infty \leq K$ *in* Q_n*. Let* $\vec{l}$ *be a vector of positive reals, and* $E \subset \mathbb{Z}^3$ *be an* $(N, \vec{l}, \varepsilon^{-1}, \varepsilon)$*-graded set with the first scale length* $l_1 > C_{\varepsilon,N}$ *and be* $(1, \varepsilon)$*-normal in* Q_n*. Then we have that*

$$|\{a \in Q_n \setminus E : |u(a)| \geq \exp(-C_{\varepsilon,K} n)|u(\mathbf{0})|\}| \geq n^p. \tag{2.3.6}$$

Assuming Theorem 2.3.4, we can prove the 3D Wegner estimate. For simplicity of notations, for any $A \subset \mathbb{Z}^3$, we denote $V_A := V|_A$, the restriction of the potential function V on A.

Lemma 2.3.5 (3D Wegner estimate)**.** *There exists* $\varepsilon_0 > 0$ *such that, if*

1. $\varepsilon > \delta > 0$*,* ε *is small enough, and* $\overline{\lambda} \in \mathrm{sp}(H) = [0, 13]$*,*
2. $N_1 \geq 1$ *is an integer and* $\vec{l}$ *is a vector of positive reals,*
3. $L_0 > \cdots > L_5 \geq C_{\varepsilon,\delta,N_1}$ *with* $L_j^{1-2\delta} \geq L_{j+1} \geq L_j^{1-\frac{1}{2}\varepsilon}$ *for* $j = 0, 1, 2, 3, 4$*, where* $C_{\varepsilon,\delta,N_1}$ *is a (large enough) constant, and* L_0*,* L_3 *are dyadic,*
4. $Q \subset \mathbb{Z}^3$ *and* Q *is an* L_0*-cube,*
5. $Q'_1, Q'_2, \cdots, Q'_{N_1} \subset Q$*, and* Q'_k *is an* L_3*-cube for each* $k = 1, 2, \cdots, N_1$ *(we call them "defects"),*
6. $G \subset \bigcup_{k=1}^{N_1} Q'_k$ *with* $0 < |G| < L_0^\delta$*,*
7. E *is a* $(1000N_1, \vec{l}, \varepsilon^{-1}, \varepsilon)$*-graded set with the first scale length* $l_1 \geq C_{\varepsilon,\delta,N_1}$ *and* $\mathscr{V} : E \cap Q \to \{0, 1\}$*,*

8. *for any L_3-cube $Q' \subset Q \setminus \bigcup_{k=1}^{N_1} Q'_k$, E is $(1,\varepsilon)$-normal in Q',*

9. *for any $V : \mathbb{Z}^3 \to [0,1]$ with $V_{E\cap Q} = \mathscr{V}$, $|\lambda - \overline{\lambda}| \le \exp(-L_5)$ and $H_Q u = \lambda u$, we have*

$$\exp(L_4)\|u\|_{\ell^2(Q\setminus\bigcup_k Q'_k)} \le \|u\|_{\ell^2(Q)} \le (1 + L_0^{-\delta})\|u\|_{\ell^2(G)}. \tag{2.3.7}$$

Then

$$\mathbb{P}\left[\|(H_Q - \overline{\lambda})^{-1}\| \le \exp(L_1)\middle|\, V_{E\cap Q} = \mathscr{V}\right] \ge 1 - L_0^{C\varepsilon-\varepsilon_0}, \tag{2.3.8}$$

where C is a universal constant, and $\|\cdot\|$ denotes the operator norm.

The proof is similar to that of [DS20, Lemma 5.6], after changing 2D notations to corresponding 3D notations. The major difference is in Claim 2.3.7 and 2.3.8 (corresponding to [DS20, Claim 5.9 5.10]), where Theorem 2.3.4 is used. This is also the reason why we need the constant $p > \frac{3}{2}$ in Theorem 2.3.4.

Proof of Lemma 2.3.5. Let $\varepsilon_0 < p - \frac{3}{2}$ where $p > \frac{3}{2}$ is the constant in Theorem 2.3.4. In this proof, we will use c, C to denote small and large universal constants.

We let $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{(L_0+1)^3}$ be the eigenvalues of H_Q. For each $1 \le k \le (L_0+1)^3$, choose eigenfunctions u_k such that $\|u_k\|_{\ell^2(Q)} = 1$ and $H_Q u_k = \lambda_k u_k$. We may think of λ_k and u_k as deterministic functions of the potential $V_Q \in [0,1]^Q$.

Let $E' = \left(\bigcup_{k=1}^{N_1} Q'_k\right) \cup (E \cap Q)$, then for any event $\mathcal{E}$,

$$\mathbb{P}\left[\mathcal{E}\middle|\, V_{E\cap Q} = \mathscr{V}\right] = 2^{-|E'\setminus E|} \sum_{\mathscr{V}' : E' \to \{0,1\}, \mathscr{V}'|_{E\cap Q} = \mathscr{V}} \mathbb{P}\left[\mathcal{E}\middle|\, V_{E'} = \mathscr{V}'\right]. \tag{2.3.9}$$

By the simple fact that the average is bounded from above by the maximum, we only need to prove

$$\mathbb{P}\left[\|(H_Q-\overline{\lambda})^{-1}\| > \exp(L_1)\middle|\, V_{E'}=\mathscr{V}'\right] \leq L_0^{C\varepsilon-\varepsilon_0}, \tag{2.3.10}$$

for any $\mathscr{V}': E' \to \{0,1\}$ with $\mathscr{V}'|_{E\cap Q}=\mathscr{V}$.

Claim 2.3.6. *There is a constant C_{N_1} such that the following is true. Suppose u satisfies $H_Q u=\lambda u$ for some $\lambda\in[0,13]$. Then there is $a'\in\mathbb{Z}^3$, such that $Q_{\frac{L_3}{2}}(a')\subset Q\setminus\bigcup_k Q'_k$, and*

$$|u(a')| \geq \exp(-C_{N_1}L_3)\|u\|_{\ell^\infty(Q)}. \tag{2.3.11}$$

Proof. Without loss of generality, we assume $Q=Q_{\frac{L_0}{2}}(\mathbf{0})$. Take $a_0\in Q$ such that $|u(a_0)|=\|u\|_{\ell^\infty(Q)}$. We assume without loss of generality that $a_0\cdot\mathbf{e}_\tau\leq 0$, for each $\tau\in\{1,2,3\}$. Since each Q'_k is an L_3-cube, by the Pigeonhole principle, there is $x'_0\in[a_0\cdot\mathbf{e}_1+100N_1L_3, a_0\cdot\mathbf{e}_1+200N_1L_3]$, such that

$$\{b\in Q: b\cdot\mathbf{e}_1\in[x'_0-16L_3, x'_0+16L_3]\}\cap\bigcup_{k=1}^{N_1}Q'_k=\emptyset. \tag{2.3.12}$$

Now we iteratively apply the cone property Lemma 2.2.4 with $K=13$. Recall the notations of cones from Definition 2.2.1, and note that $(K+11)<\exp(5)$. We find

$$a_1\in(\mathcal{C}^1_{a_0}(x'_0-a_0\cdot\mathbf{e}_1)\cup\mathcal{C}^1_{a_0}(x'_0-a_0\cdot\mathbf{e}_1+1))\cap Q \tag{2.3.13}$$

with

$$|u(a_1)| \geq \exp(-1000N_1L_3)|u(a_0)|, \tag{2.3.14}$$

and $a_2 \in (\mathcal{C}^2_{a_1}(4L_3) \cup \mathcal{C}^2_{a_1}(4L_3+1)) \cap Q$ with

$$|u(a_2)| \geq \exp(-(1000N_1+20)L_3)|u(a_0)|, \tag{2.3.15}$$

and $a_3 \in (\mathcal{C}^3_{a_2}(2L_3) \cup \mathcal{C}^3_{a_2}(2L_3+1)) \cap Q$ with

$$|u(a_3)| \geq \exp(-(1000N_1+30)L_3)|u(a_0)|. \tag{2.3.16}$$

By (2.3.13), we have $|a_1 \cdot \mathbf{e}_1 - x_0'| \leq 1$ and $-\frac{L_0}{2} \leq a_1 \cdot \mathbf{e}_\tau \leq 200N_1L_3 + 1$ for $\tau = 2, 3$. Then $|a_2 \cdot \mathbf{e}_1 - x_0'| \leq 4L_3 + 2$, and $-\frac{L_0}{2} + 4L_3 \leq a_2 \cdot \mathbf{e}_2 \leq (200N_1 + 4)L_3 + 2$, and $-\frac{L_0}{2} \leq a_2 \cdot \mathbf{e}_3 \leq (200N_1 + 4)L_3 + 2$. Finally, we have $|a_3 \cdot \mathbf{e}_1 - x_0'| \leq 6L_3 + 3$, and $-\frac{L_0}{2} + 2L_3 - 1 \leq a_3 \cdot \mathbf{e}_2 \leq (200N_1+6)L_3+3$, and $-\frac{L_0}{2} + 2L_3 \leq a_3 \cdot \mathbf{e}_3 \leq (200N_1+6)L_3+3$. This implies $Q_{\frac{L_3}{2}}(a_3) \subset Q \setminus \bigcup_{k=1}^{N_1} Q_k'$ and the claim follows by letting $a' = a_3$ and $C_{N_1} = 1000N_1 + 30$. □

Claim 2.3.7. *For any $\lambda \in [0, 13]$, $H_Q u = \lambda u$ implies*

$$\left|\left\{a \in Q : |u(a)| \geq \exp\left(-\frac{L_2}{4}\right) \|u\|_{\ell^2(Q)}\right\} \setminus E'\right| \geq \left(\frac{L_3}{2}\right)^p. \tag{2.3.17}$$

Proof. By applying Claim 2.3.6 to u, we can find a cube $Q_{\frac{L_3}{2}}(a') \subset Q \setminus \bigcup_k Q_k'$ for some $a' \in \mathbb{Z}^3$, such that $|u(a')| \geq \exp(-C_{N_1}L_3)\|u\|_{\ell^\infty(Q)} \geq \exp(-C_{N_1}L_3)(L_0+1)^{-\frac{3}{2}}\|u\|_{\ell^2(Q)}$.

By Condition 8, E is $(1,\varepsilon)$-normal in $Q_{\frac{L_3}{2}}(a')$. Applying Theorem 2.3.4 to cube $Q_{\frac{L_3}{2}}(a')$ with graded set E, function u, and $K = 13$, and letting $\frac{1}{4}C_{\varepsilon,\delta,N_1}^{2\delta} > C_{N_1} + C_{\varepsilon,K}$ where $C_{\varepsilon,K}$ is the constant in Theorem 2.3.4, the claim follows. □

Claim 2.3.8. *Let $s_i = \exp(-L_1 + (L_2 - L_4 + C)i)$ for each $i \in \mathbb{Z}$. For $1 \le k_1 \le k_2 \le (L_0+1)^3$ and $0 \le \ell \le CL_0^\delta$, we have*

$$\mathbb{P}\left[\mathcal{E}_{k_1,k_2,\ell} \middle| V_{E'} = \mathscr{V}'\right] \le CL_0^{\frac{3}{2}} L_3^{-p} \tag{2.3.18}$$

where $\mathcal{E}_{k_1,k_2,\ell}$ denotes the event

$$|\lambda_{k_1} - \overline{\lambda}|, |\lambda_{k_2} - \overline{\lambda}| < s_\ell,\ |\lambda_{k_1-1} - \overline{\lambda}|, |\lambda_{k_2+1} - \overline{\lambda}| \ge s_{\ell+1}. \tag{2.3.19}$$

Proof. For $i = 0, 1$, we let $\mathcal{E}_{k_1,k_2,\ell,i}$ denote the event

$$\mathcal{E}_{k_1,k_2,\ell} \cap \left\{ \left| \left\{ a \in Q : |u_{k_1}(a)| \ge \exp\left(-\frac{L_2}{4}\right), V(a) = i \right\} \setminus E' \right| \ge \frac{L_3^p}{8} \right\} \cap \{V_{E'} = \mathscr{V}'\}. \tag{2.3.20}$$

Since we are under the event $V_{E'} = \mathscr{V}'$, we can view $\mathcal{E}_{k_1,k_2,\ell,0}$ and $\mathcal{E}_{k_1,k_2,\ell,1}$ as subsets of $\{0,1\}^{Q\setminus E'}$. Observe that $\mathcal{E}_{k_1,k_2,\ell} \cap \{V_{E'} = \mathscr{V}'\} \subset \mathcal{E}_{k_1,k_2,\ell,0} \cup \mathcal{E}_{k_1,k_2,\ell,1}$ by Claim 2.3.7.

Fix $i \in \{0,1\}$. For each $\omega \in \mathcal{E}_{k_1,k_2,\ell,i}$, we denote

$$S_1(\omega) := \{a \in Q \setminus E' : \omega(a) = 1 - i\}, \tag{2.3.21}$$

and

$$S_2(\omega) := \left\{ a \in Q \setminus E' : \omega(a) = i, |u_{k_1}(a)| \geq \exp\left(-\frac{L_2}{4}\right) \right\}. \tag{2.3.22}$$

By definition of $\mathcal{E}_{k_1,k_2,\ell,i}$, we have $|S_2(\omega)| \geq \frac{L_3^p}{8}$. For each $\omega \in \mathcal{E}_{k_1,k_2,\ell,i}$, $a \in S_2(\omega)$, we define ω^a as

$$\omega^a(a) := 1 - \omega(a),\ \omega^a(a') := \omega(a'),\ \forall a' \in Q \setminus E', a' \neq a. \tag{2.3.23}$$

We claim that $\omega^a \notin \mathcal{E}_{k_1,k_2,\ell,i}$. In the case where $i = 0$, because of Condition 9 and $a \notin \bigcup_k Q'_k$, we have $\sum_{|\lambda_k - \overline{\lambda}| < \exp(-L_5)} u_k(a)^2 < \exp(-cL_4)$. Now we apply Lemma 2.7.2 to $H_Q - \overline{\lambda} + s_\ell$ with $r_1 = 2s_\ell$, $r_2 = s_{\ell+1}$, $r_3 = \exp(-\frac{1}{2}L_2)$, $r_4 = \exp(-cL_4)$ and $r_5 = \exp(-L_5)$. Then λ_{k_1} moves out of interval $(\overline{\lambda} - s_\ell, \overline{\lambda} + s_\ell)$ when $\omega(a)$ is changed from 0 to 1. Thus we have $\omega^a \notin \mathcal{E}_{k_1,k_2,\ell,0}$. The case where $i = 1$ is similar.

From this, we know that for any two $\omega, \omega' \in \mathcal{E}_{k_1,k_2,\ell,i}$, $S_1(\omega) \subset S_1(\omega')$ implies $S_1(\omega') \cap S_2(\omega) = \emptyset$. Since $|Q \setminus E'| \leq (L_0 + 1)^3 - (L_3 + 1)^3 \leq L_0^3$, we can apply Theorem 2.7.3 with set $\{S_1(\omega) : \omega \in \mathcal{E}_{k_1,k_2,\ell,i}\}$ and $\rho = \frac{1}{8} L_0^{-3} L_3^p$, and we conclude that $\mathbb{P}[\mathcal{E}_{k_1,k_2,\ell,i} | \, V_{E'} = \mathscr{V}'] \leq C L_0^{\frac{3}{2}} L_3^{-p}$. $\square$

Claim 2.3.9. *There is a set $K \subset \{1, 2, \cdots, (L_0 + 1)^3\}$ depending only on E' and $\mathscr{V}'$, such that $|K| \leq C L_0^\delta$ and*

$$\{\|(H_Q - \overline{\lambda})^{-1}\| > \exp(L_1)\} \cap \{V_{E'} = \mathscr{V}'\} \subset \bigcup_{\substack{k_1, k_2 \in K \\ k_1 \leq k_2}} \bigcup_{0 \leq \ell \leq C L_0^\delta} \mathcal{E}_{k_1,k_2,\ell}. \tag{2.3.24}$$

Proof. Conditioning on $V_{E'} = \mathscr{V}'$, we view λ_k and u_k as functions on $[0,1]^{Q\setminus E'}$. Let $1 \leq k_1 < \cdots < k_m \leq (L_0+1)^3$ be all indices k_i such that there is at least one $\omega \in [0,1]^{Q\setminus E'}$ with $|\lambda_{k_i}(w) - \overline{\lambda}| \leq \exp(-L_2)$. To prove the claim, it suffices to prove that $m \leq CL_0^\delta$. Indeed, then we can always find an $0 \leq \ell \leq m$ such that the annulus $\left[\overline{\lambda} - s_{\ell+1}, \overline{\lambda} + s_{\ell+1}\right] \setminus \left[\overline{\lambda} - s_\ell, \overline{\lambda} + s_\ell\right]$ contains no eigenvalue of H_Q.

Since $\bigcup_k Q'_k \subset E'$, Condition 9 implies that for any $\omega \in [0,1]^{Q\setminus E'}$ with $|\lambda_{k_i}(\omega) - \overline{\lambda}| \leq \exp(-L_5)$, we have $\|u_{k_i}(\omega)\|_{\ell^\infty(Q\setminus E')} \leq \exp(-L_4)$. In particular, if there is $\omega_0 \in [0,1]^{Q\setminus E'}$ such that $|\lambda_{k_i}(\omega_0) - \overline{\lambda}| \leq \exp(-L_2)$, then by eigenvalue variation,

$$|\lambda_{k_i}(\omega) - \overline{\lambda}| \leq \exp(-L_4) \tag{2.3.25}$$

holds for all $\omega \in \{0,1\}^{Q\setminus E'}$. Indeed, let $\omega_t = (1-t)\omega_0 + t\omega$ for $t \in [0,1]$. We compute

$$\begin{aligned} |\lambda_{k_i}(\omega_t) - \overline{\lambda}| \leq & |\lambda_{k_i}(\omega_0) - \overline{\lambda}| + \int_0^t \|u_{k_i}(\omega_s)\|^2_{\ell^2(Q\setminus E')} ds \\ \leq & \exp(-L_2) + \int_0^t |Q| \exp(-2L_4) + \mathbb{1}_{|\lambda_{k_i}(\omega_s) - \overline{\lambda}| \geq \exp(-L_5)} ds \\ \leq & \exp(-L_4) + \mathbb{1}_{\max_{0\leq s\leq t} |\lambda_{k_i}(\omega_s) - \overline{\lambda}| \geq \exp(-L_5)} \end{aligned} \tag{2.3.26}$$

and conclude by continuity. By (2.3.25) and Condition 9, for all $\omega \in \{0,1\}^{Q\setminus E'}$, we have $1 = \|u_{k_i}(\omega)\|_{\ell^2(Q)} \geq \|u_{k_i}(\omega)\|_{\ell^2(G)} \geq 1 - CL_0^{-\delta}$. In particular, we have $|\langle u_{k_i}(\omega), u_{k_j}(\omega)\rangle_{\ell^2(G)} - \mathbb{1}_{i=j}| \leq CL_0^{-\delta} \leq (5|G|)^{-\frac{1}{2}}$. By Lemma 2.7.1 we have that $m \leq C|G| \leq CL_0^\delta$. $\square$

Finally,

$$\mathbb{P}[\|(H_Q - \overline{\lambda})^{-1}\| > \exp(L_1)|\, V_{E'} = \mathscr{V}'] \le \sum_{k_1,k_2\in K} \sum_{1\le \ell \le CL_0^\delta} \mathbb{P}[\mathcal{E}_{k_1,k_2,\ell}|\, V_{E'} = \mathscr{V}'] \quad (2.3.27)$$

and thus

$$\mathbb{P}[\|(H_Q - \overline{\lambda})^{-1}\| > \exp(L_1)|\, V_{E'} = \mathscr{V}'] \le CL_0^{\frac{3}{2}+3\delta} L_3^{-p} \le L_0^{C\varepsilon-\varepsilon_0}, \quad (2.3.28)$$

so our conclusion follows. □

We now prove Theorem 2.3.1 by a multi-scale analysis argument.

In the remaining part of this section, by "dyadic cube", we mean a cube $Q_{2^n}(a)$ for some $a \in 2^{n-1}\mathbb{Z}^3$ and $n \in \mathbb{Z}_+$. For each $k, m \in \mathbb{Z}_+$ and each $2k$-cube Q, we denote by mQ the $2mk$-cube with the same center as Q.

Theorem 2.3.10 (Multi-scale Analysis)**.** *There exists $\kappa > 0$, such that for any $\varepsilon_* > 0$, there are*

1. $\varepsilon_* > \varepsilon > \nu > \delta > 0$,
2. $M, N \in \mathbb{Z}_+$,
3. *dyadic scales* L_k, *for* $k \in \mathbb{Z}_{\ge 0}$, *with* $\lfloor \log_2 L_{k+1}^{1-6\varepsilon} \rfloor = \log_2 L_k$,
4. *decay rates* $1 \ge m_k \ge L_k^{-\delta}$ *for* $k \in \mathbb{Z}_{\ge 0}$,

such that for any $0 \le \overline{\lambda} \le \exp(-L_M^\delta)$, *we have random sets* $\mathcal{O}_k \subset \mathbb{R}^3$ *for* $k \in \mathbb{Z}_{\ge 0}$ *with*

$\mathcal{O}_k \subset \mathcal{O}_{k+1}$ (depending on the Bernoulli potential V), and the following six statements hold for any $k \in \mathbb{Z}_{\geq 0}$:

1. *When $k \leq M$, $\mathcal{O}_k \cap \mathbb{Z}^3 = \lceil \varepsilon^{-1} \rceil \mathbb{Z}^3$.*

2. *When $k \geq M+1$, $\mathcal{O}_k$ is an $(N, \vec{l}, (2\varepsilon)^{-1}, 2\varepsilon)$-graded random set with $\vec{l} = (L_{M+1}^{1-2\varepsilon}, L_{M+2}^{1-2\varepsilon}, \cdots, L_k^{1-2\varepsilon})$.*

3. *For any L_k-cube Q, the set $\mathcal{O}_k$ is $(1, 2\varepsilon)$-normal in Q.*

4. *For any $i \in \mathbb{Z}_{\geq 0}$ and any dyadic $2^i L_k$-cube Q, the set $\mathcal{O}_k \cap Q$ is $V_{\mathcal{O}_{k-1} \cap 3Q}$-measurable.*

5. *For any dyadic L_k-cube Q, it is called* good *(otherwise* bad*), if for any potential $V' : \mathbb{Z}^3 \to [0,1]$ with $V'_{\mathcal{O}_k \cap Q} = V_{\mathcal{O}_k \cap Q}$, we have*

$$|(H'_Q - \overline{\lambda})^{-1}(x,y)| \leq \exp(L_k^{1-\varepsilon} - m_k|x-y|), \ \forall x, y \in Q. \tag{2.3.29}$$

   Here H'_Q is the restriction of $-\Delta + V'$ on Q with Dirichlet boundary condition. Then Q is good with probability at least $1 - L_k^{-\kappa}$.

6. *$m_k = m_{k-1} - L_{k-1}^{-\nu}$ when $k \geq M+1$.*

Proof. Throughout the proof, we use c, C to denote small and large universal constants.

Let κ be any number with $0 < \kappa < \varepsilon_0$, where ε_0 is from Lemma 2.3.5. Let small reals ε, δ, ν satisfy Condition 1 and to be determined. Let $M \in \mathbb{Z}_+$ satisfy

$\frac{3}{5}\delta < (1-6\varepsilon)^M < \frac{4}{5}\delta$; such M must exist as long as $\varepsilon < \frac{1}{24}$. Leave N to be determined, and let L_0 be large enough with $L_0 \geq \max\{C_{\delta,\varepsilon}, C_{\varepsilon,\delta,N}\}$, where $C_{\delta,\varepsilon}$ is the constant in Proposition 2.7.10 and $C_{\varepsilon,\delta,N}$ is the constant in Lemma 2.3.5 (with $N_1 = N$). For $k > 0$, let L_k be dyadic numbers satisfying Condition 3. Fix $\overline{\lambda} \in \left[0, \exp(-L_M^\delta)\right]$.

When $k = 0, 1, \cdots, M$, let $\mathcal{O}_k = \bigcup_{a \in \lceil \varepsilon^{-1} \rceil \mathbb{Z}^3} o_a$, where o_a is the open unit ball centered at a. Then Statement 1, 3, 4 hold. Let $m_k := L_k^{-\delta}$. Proposition 2.7.10 implies Statement 5 for $k = 1, 2, \cdots, M$.

We now prove by induction for $k > M$. Assume that Statement 1 to 6 hold for all $k' < k$.

For any $0 < k' < k$, by Lemma 2.7.6, any bad dyadic $L_{k'}$-cube Q must contain a bad $L_{k'-1}$-cube. For any $0 < i \leq k$, and a bad L_{k-i}-cube $Q' \subset Q$, we call Q' a *hereditary bad L_{k-i}-subcube of Q*, if there exists a sequence $Q' = \overline{Q}_i \subset \overline{Q}_{i-1} \subset \cdots \subset \overline{Q}_1 \subset Q$, where for each $j = 1, \cdots, i$, $\overline{Q}_j$ is a bad L_{k-j}-cube. We also call such sequence $\{\overline{Q}_j\}_{1 \leq j \leq i}$ a hereditary bad chain of length i. Note that the set of hereditary bad chains of Q is $V_{\mathcal{O}_{k-1} \cap Q}$-measurable.

Claim 2.3.11. *When ε is small enough, there exists N' depending on $M, \kappa, \delta, \varepsilon$, such that, for any dyadic L_k-cube Q,*

$$\mathbb{P}\left[Q \text{ has no more than } N' \text{ hereditary bad chain of length } M\right] \geq 1 - L_k^{-10}. \quad (2.3.30)$$

Proof. Writing $N' = (N'')^M$, we have

$$
\begin{aligned}
&\mathbb{P}[Q \text{ has more than } N' \text{ hereditary bad chain of length } M] \\
\leq &\sum_{\substack{Q' \subset Q \\ Q' \text{ is a dyadic } L_j\text{-cube} \\ k-M<j\leq k}} \mathbb{P}[Q' \text{ contains more than } N'' \text{ bad } L_{j-1}\text{-subcubes}].
\end{aligned}
\tag{2.3.31}
$$

We can use inductive hypothesis to bound this by

$$
\begin{aligned}
&\sum_{\substack{Q' \subset Q \\ Q' \text{ is a dyadic } L_j\text{-cube} \\ k-M<j\leq k}} \left(\frac{L_j}{L_{j-1}}\right)^{CN''} (L_{j-1}^{-\kappa})^{cN''} \\
\leq &\sum_{k-M<j\leq k} \left(\frac{L_k}{L_j}\right)^{C} \left(\frac{L_j}{L_{j-1}}\right)^{CN''} (L_{j-1}^{-\kappa})^{cN''} \\
\leq &CML_k^C \max_{k-M<j\leq k} L_{j-1}^{(C\varepsilon-c\kappa)N''} \leq CML_k^C(L_k^{(C\varepsilon-c\kappa)N''} + L_k^{(C\varepsilon-c\kappa)\delta N''}).
\end{aligned}
\tag{2.3.32}
$$

Here we used that $L_{k-M} > L_k^{\frac{\delta}{2}}$ in the last inequality. The claim follows by taking the ε sufficiently small (depending on κ) and N'' large enough (depending on $M, \kappa, \delta, \varepsilon$). $\square$

Now we let $N := 1000N'$. We call a dyadic L_k-cube Q *ready* if Q has no more than N' hereditary bad chain of length M. The event that Q is ready is $V_{\mathcal{O}_{k-1}\cap Q}$-measurable.

Suppose Q is an L_k-cube and is ready. Let $Q_1''', \cdots, Q_{N'}''' \subset Q$ be a complete list of all hereditary bad L_{k-M}-subcubes of Q. Let $Q_1'', \cdots, Q_{N'}'' \subset Q$ be the corresponding bad L_{k-1}-cubes, such that $Q_i''' \subset Q_i''$ for each $i = 1, 2, \cdots, N'$. These cubes are chosen in a way such that $\{Q_1'', \cdots, Q_{N'}''\}$ contains all the bad L_{k-1}-cubes in Q.

By Lemma 2.7.4, we can choose a dyadic scale L' satisfying

$$L_k^{1-3\varepsilon} \leq L' \leq L_k^{1-2\varepsilon} \tag{2.3.33}$$

and disjoint L'-cubes $Q'_1, \cdots, Q'_{N'} \subset Q$ such that, for every Q''_i, there is a Q'_j such that $Q''_i \subset Q'_j$ and $\operatorname{dist}(Q''_i, Q \setminus Q'_j) \geq \frac{L'}{8}$. For each $i = 1, 2, \cdots, N'$, we let $O_{Q,i}$ be the ball in $\mathbb{R}^3$, with the same center as Q'_i and with radius $L_k^{1-2\varepsilon}$. We can choose $O_{Q,i}, Q''_i, Q'''_i$ in a $V_{\mathcal{O}_{k-1}\cap Q}$-measurable way.

Now we let $\mathcal{O}_k$ be the union of $\mathcal{O}_{k-1}$ and balls $O_{Q,1}, \cdots, O_{Q,N'}$, for each ready L_k-cube Q; i.e.

$$\mathcal{O}_k := \mathcal{O}_{k-1} \cup \left(\bigcup_{Q \text{ is an } L_k\text{-cube and is ready}} \left(\bigcup_{i=1}^{N'} O_{Q,i} \right) \right), \tag{2.3.34}$$

and let $m_k = m_{k-1} - L_{k-1}^{-\nu}$. From induction hypothesis we have $m_k \geq L_{k-1}^{-\delta} - L_{k-1}^{-\nu} \geq L_k^{-\delta}$.

We now verify Statement 2 to 6. First note that Statement 4 and 6 hold for k by the above construction.

Claim 2.3.12. *Statement* 2 *and* 3 *hold for* k*.*

Proof. From (2.3.34), we let $\tilde{\mathcal{O}}_{k'} := \bigcup_{Q \text{ is an } L_{k'}\text{-cube and is ready}} \bigcup_{i=1}^{N'} O_{Q,i}$ for $k' > M$. Then we have that $\mathcal{O}_k = \mathcal{O}_M \cup \left(\bigcup_{k'=M+1}^{k} \tilde{\mathcal{O}}_{k'}\right)$, and we claim that

1. $\mathcal{O}_M$ is $(2\varepsilon)^{-1}$-unitscattered,

2. $\tilde{\mathcal{O}}_{k'}$ is an $(N, L_{k'}^{1-2\varepsilon}, 2\varepsilon)$-scattered set for each $k' > M$.

By these two claims, Statement 2 holds by Condition 3.

Now we check these two claims. For the first one, just note that we have $\mathcal{O}_M = \bigcup_{a\in\lceil\varepsilon^{-1}\rceil\mathbb{Z}^3} o_a$, then use Definition 2.3.3. For the second one, when $k' > M$ the set $\tilde{\mathcal{O}}_{k'}$ is the union of N' balls $O_{Q,1}, O_{Q,2}, \cdots, O_{Q,N'}$ for each ready $L_{k'}$-cube Q, and each ball $O_{Q,i}$ has radius $L_{k'}^{1-2\varepsilon}$. Denote the collection of dyadic $L_{k'}$-cubes by $\mathcal{Q}_{k'} := \left\{Q_{\frac{L_{k'}}{2}}(a) : a \in \frac{L_{k'}}{4}\mathbb{Z}^3\right\}$. We can divide $\mathcal{Q}_{k'}$ into at most 1000 subsets $\mathcal{Q}_{k'} = \bigcup_{t=1}^{1000} \mathcal{Q}_{k'}^{(t)}$, such that any two $L_{k'}$-cubes in the same subset have distance larger than $L_{k'}$, i.e.

$$\operatorname{dist}(Q, Q') \geq L_{k'} \text{ for any } t \in \{1, 2, \cdots, 1000\} \text{ and any } Q \neq Q' \in \mathcal{Q}_{k'}^{(t)}. \tag{2.3.35}$$

For each $1 \leq t \leq 1000$ and $1 \leq j \leq N'$, let

$$\mathfrak{O}_{k'}^{(t,j)} = \left\{O_{Q,j} : Q \text{ is ready and } Q \in \mathcal{Q}_{k'}^{(t)}\right\}.$$

Then for any two $O \neq O' \in \mathfrak{O}_{k'}^{(t,j)}$, by (2.3.35), we have

$$\operatorname{dist}(O, O') \geq L_{k'} - 2L_{k'}^{1-2\varepsilon} \geq L_{k'}^{1-4\varepsilon^2} = (\operatorname{radi}(O))^{1+2\varepsilon} = (\operatorname{radi}(O'))^{1+2\varepsilon}. \tag{2.3.36}$$

From Definition 2.3.3, we have that

$$\tilde{\mathcal{O}}_{k'} = \bigcup_{1\leq t\leq 1000, 1\leq j\leq N'} \left(\bigcup \mathfrak{O}_{k'}^{(t,j)}\right)$$

is an $(N, L_{k'}^{1-2\varepsilon}, 2\varepsilon)$-scattered set since $N = 1000N'$. Thus the second claim holds.

Finally, since $\mathrm{radi}(O_{Q,i}) = L_{k'}^{1-2\varepsilon} < \mathrm{diam}(Q)^{1-\varepsilon}$ for any ready $L_{k'}$-cube Q and $1 \le i \le N'$, we have that $\mathcal{O}_k$ is $(1, 2\varepsilon)$-normal in any L_k-cube. Hence Statement 3 holds. $\square$

Now it remains to check Statement 5 for k.

Claim 2.3.13. *If Q is an L_k-cube and Q is ready, then for any $1 \le i \le N'$, we have*

$$\exp(cL_{k-1}^{1-\delta})\|u\|_{\ell^\infty\left(Q_i'\setminus\bigcup_{j=1}^{N'} Q_j''\right)} \le \|u\|_{\ell^2(Q_i')} \le (1+\exp(-cL_{k-M}^{1-\delta}))\|u\|_{\ell^2\left(Q_i'\cap\bigcup_{j=1}^{N'} Q_j'''\right)}, \tag{2.3.37}$$

for any $\lambda \in \mathbb{R}$ with $|\lambda - \overline{\lambda}| \le \exp(-2L_{k-1}^{1-\varepsilon})$, and any $u : Q_i' \to \mathbb{R}$ with $H_{Q_i'}u = \lambda u$.

Proof. If $a \in Q_i' \setminus \bigcup_{j=1}^{N'} Q_j'''$, then there is a $j' = 1, \cdots, M$ and a good $L_{k-j'}$-cube $Q'' \subset Q_i'$ with $a \in Q''$ and $\mathrm{dist}(a, Q_i' \setminus Q'') \ge \frac{1}{8}L_{k-j'}$. Moreover, if $a \in Q_i' \setminus \bigcup_{j=1}^{N'} Q_j''$, then we can take $j' = 1$. By the definition of good and Lemma 2.7.5,

$$|u(a)| \le 2\exp\left(L_{k-j'}^{1-\varepsilon} - \frac{1}{8}m_{k-j'}L_{k-j'}\right)\|u\|_{\ell^1(Q_i')} \le \exp(-cL_{k-j'}^{1-\delta})\|u\|_{\ell^2(Q_i')}. \tag{2.3.38}$$

In particular, we see that

$$\|u\|_{\ell^\infty\left(Q_i'\setminus\bigcup_{j=1}^{N'} Q_j''\right)} \le \exp(-cL_{k-1}^{1-\delta})\|u\|_{\ell^2(Q_i')} \tag{2.3.39}$$

and

$$\|u\|_{\ell^\infty(Q_i'\setminus\bigcup_{j=1}^{N'} Q_j''')} \le \exp(-cL_{k-M}^{1-\delta})\|u\|_{\ell^2(Q_i')}. \tag{2.3.40}$$

These together imply the claim. $\square$

Claim 2.3.14. *If Q is an L_k-cube, and for any $1 \leq i \leq N'$, $\mathcal{E}_i(Q)$ denotes the event that*

$$Q \text{ is ready and } \|(H_{Q_i'} - \overline{\lambda})^{-1}\| \leq \exp(L_k^{1-4\varepsilon}), \tag{2.3.41}$$

then $\mathbb{P}[\mathcal{E}_i(Q)] \geq 1 - L_k^{C\varepsilon-\varepsilon_0}$.

Proof. Recall that the event where Q is ready is $V_{\mathcal{O}_{k-1}\cap Q}$-measurable, and subcubes Q_i''s are also $V_{\mathcal{O}_{k-1}\cap Q}$-measurable. Assuming $\varepsilon > 5\delta$, we apply Lemma 2.3.5 with $2\varepsilon > \delta > 0$, $N_1 = N'$, and to the cube Q_i' with scales $L' \geq L_k^{1-4\varepsilon} \geq L_k^{1-5\varepsilon} \geq L_{k-1} \geq L_{k-1}^{1-2\delta} \geq 2L_{k-1}^{1-\varepsilon}$ (recall that L' is the scale chosen above satisfying (2.3.33)), defects $\left\{Q_j'' : Q_j'' \subset Q_i'\right\}$, $G = \bigcup_{1\leq j\leq N':Q_j'''\subset Q_i'} Q_j'''$, and $E = \mathcal{O}_{k-1}$. Note that $L_k^{\frac{\delta}{2}} < L_{k-M} < L_k^{\frac{9\delta}{10}}$. Condition 9 of Lemma 2.3.5 is given by Claim 2.3.13. By Claim 2.3.11 this claim follows. $\square$

Claim 2.3.15. *If Q is an L_k-cube and $\mathcal{E}_1(Q), \cdots, \mathcal{E}_{N'}(Q)$ hold, then Q is good.*

Proof. We apply Lemma 2.7.6 to the cube Q with small parameters $\varepsilon > \nu > 0$, scales $L_k \geq L_k^{1-\varepsilon} \geq L' \geq L_k^{1-3\varepsilon} \geq L_k^{1-4\varepsilon} \geq L_{k-1} \geq L_{k-1}^{1-\varepsilon}$, and defects $Q_1', \cdots, Q_{N'}'$. We conclude that

$$|(H_Q - \overline{\lambda})^{-1}(a,b)| \leq \exp(L_k^{1-\varepsilon} - m_k|a-b|). \tag{2.3.42}$$

Since $Q_i' \subset \mathcal{O}_k$ when Q is ready, the events $\mathcal{E}_i(Q)$ are $V_{\mathcal{O}_k\cap Q}$-measurable, thus Q is good. $\square$

By combining Claim 2.3.14, Claim 2.3.15, and letting $C\varepsilon < \varepsilon_0 - \kappa$, we have that Statement 5 holds for k. Thus the induction principle proves the theorem. $\square$

Proof of Theorem 2.3.1. Apply Theorem 2.3.10 with any $\varepsilon_* < \frac{\kappa}{100}$, then there are $\{L_k\}_{k\in\mathbb{Z}_{\geq 0}}$, $\{m_k\}_{k\in\mathbb{Z}_{\geq 0}}$, ε, δ, ν, N and M such that the statements of Theorem 2.3.10 hold. Let $k_* \in \mathbb{Z}_+$ be large enough with $k_* \geq M+2$ and let $L_* = L_{k_*}$. Fix dyadic scale $L \geq L_*$, and let k be the maximal integer such that $L \geq L_{k+1}$. Then $L_k^{1+6\varepsilon} \leq L_{k+1} \leq L < L_{k+2} \leq L_k^{1+15\varepsilon}$. Denote

$$\mathcal{Q} := \{Q : Q \text{ is a dyadic } L_k\text{-cube and } Q \cap Q_L \neq \emptyset\}. \tag{2.3.43}$$

Then $Q_L \subset \bigcup_{Q\in\mathcal{Q}} Q$ and $|\mathcal{Q}| \leq 1000\left(\frac{L}{L_k}\right)^3 \leq L_k^{100\varepsilon} \leq L_k^{100\varepsilon_*}$. By elementary observations, for any $a \in Q_L$, there is a $Q \in \mathcal{Q}$ such that $a \in Q$ and $\operatorname{dist}(a, Q_L \setminus Q) \geq \frac{1}{8}L_k$. Fix $\lambda \in [0, \exp(-L_M^{\delta})]$. For each $Q \in \mathcal{Q}$, define A_Q to be the following event:

$$|(H_Q - \lambda)^{-1}(a,b)| \leq \exp(L_k^{1-\varepsilon} - m_k|a-b|) \text{ for each } a,b \in Q. \tag{2.3.44}$$

By Lemma 2.7.6, $\bigcap_{Q\in\mathcal{Q}} A_Q$ implies

$$|(H_{Q_L} - \lambda)^{-1}(a,b)| \leq \exp(L^{1-\varepsilon} - m|a-b|), \forall a,b \in Q_L, \tag{2.3.45}$$

where $m = m_k - L_k^{-\delta}$. Note that for $k \geq k_* - 1 \geq M + 1$ we have

$$m = m_k - L_k^{-\delta} \geq L_{k_*-2}^{-\delta} - L_{k_*-2}^{-\nu} - \cdots - L_{k-1}^{-\nu} - L_k^{-\delta} > \delta_0 \tag{2.3.46}$$

for some $\delta_0 > 0$ independent of k. Here the inequalities are by Condition 4 and Statement 6 in Theorem 2.3.10, and the fact that L_k increases super-exponentially and k_* is large enough.

By Theorem 2.3.10, for each $Q \in \mathcal{Q}$ we have

$$\mathbb{P}[A_Q] \geq 1 - L_k^{-\kappa}. \tag{2.3.47}$$

Thus

$$\mathbb{P}\left[\bigcap_{Q \in \mathcal{Q}} A_Q\right] \geq 1 - |\mathcal{Q}| L_k^{-\kappa} \geq 1 - L_k^{-\kappa + 100\varepsilon_*}. \tag{2.3.48}$$

Hence our theorem follows by letting $\kappa_0 = \frac{\kappa - 100\varepsilon_*}{1+15\varepsilon}$ and $\lambda_* = \min\left\{\delta_0, \exp(-L_M^{\delta}), \varepsilon\right\}$.

$\square$

2.4 Polynomial arguments on triangular lattice

The goal of this section is to prove Theorem 2.1.9, which is a triangular lattice version of [BLMS17, Theorem (A)]. Our proof closely follows that in [BLMS17], which employs the polynomial structure of u and the Remez inequality, and a Vitalli covering argument.

2.4.1 Notations and basic bounds

Before starting the proof, recall Definition 2.1.8 for some basic geometric objects. Here we need more notations for geometric patterns in Λ.

Definition 2.4.1. We denote $\gamma := \xi + \eta = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. For each $b = s\xi + t\eta \in \Lambda$, we denote $\xi(b) := s$ and $\eta(b) := t$. For $a \in \Lambda$ and $m \in \mathbb{Z}_{\geq 0}$, we denote the ξ-*edge*, η-*edge*, and γ-*edge* of $T_{a;m}$ to be the sets

$$\begin{gathered}\{a - m\eta + s\xi : -2m \leq s \leq m\} \cap \Lambda, \\ \{a + m\xi + t\eta : -m \leq t \leq 2m\} \cap \Lambda, \\ \{a - m\xi + s\xi + s\eta : -m \leq s \leq 2m\} \cap \Lambda\end{gathered} \tag{2.4.1}$$

respectively, each containing $3m + 1$ points. In this section, an *edge* of $T_{a;m}$ means one of its ξ-edge, η-edge and γ-edge.

For $a \in \Lambda$ and $m, \ell \in \mathbb{Z}_{\geq 0}$, denote

$$P_{a;m,\ell} := \{a + s\xi + t\eta : -\ell \leq t \leq 0, -m + t \leq s \leq 0\} \cap \Lambda,$$

a trapezoid of lattice points. Especially, when $\ell = 0$, $P_{a;m,\ell} = \{a + s\xi : 0 \leq s \leq m\}$ is a segment parallel to ξ. The *lower edge* of $P_{a;m,\ell}$ is defined to be the set $P_{a-\ell\eta;m+\ell,0}$, and the *upper edge* of $P_{a;m,\ell}$ is defined to be the set $P_{a;m,0}$. The *left leg* of $P_{a;m,\ell}$ is the

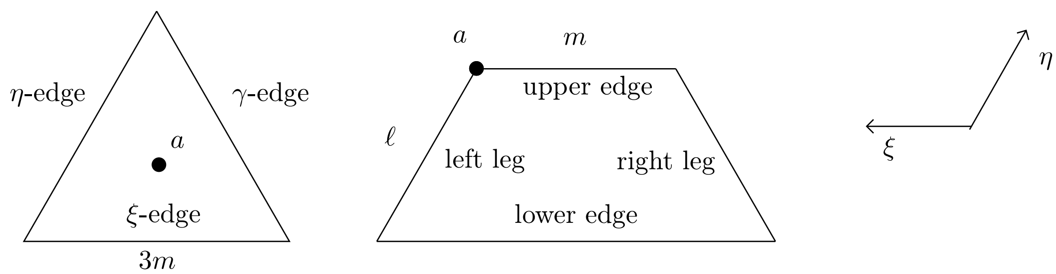

Figure 2.1: $T_{a;m}$ is the set of lattice points in the triangle region; $P_{a;m,\ell}$ is the set of lattice points in the trapezoid region.

set $\{a + t\eta : -\ell \leq t \leq 0\} \cap \Lambda$, and the *right leg* of $P_{a;m,\ell}$ is the set

$$\{a - m\xi - t\gamma : 0 \leq t \leq \ell\} \cap \Lambda.$$

See Figure 2.1 for an illustration of $T_{a;m}$ and $P_{a;m,\ell}$.

The following lemma can be proved using a straight forward induction.

Lemma 2.4.2. *Let $R, S \in \mathbb{R}_+$, $a \in \Lambda$, and $m \in \mathbb{Z}_+$. Suppose $u : \Lambda \to \mathbb{R}$ satisfies*

$$|u(b) + u(b - \xi) + u(b + \eta)| \leq R \tag{2.4.2}$$

for any $b \in T_{a;m}$ with $\eta(b) - \xi(b) < m$, and $|u| \leq S$ on one of three edges of $T_{a;m}$. Then $|u(b)| \leq 2^{3m}S + (2^{3m} - 1)R$ for each $b \in T_{a;m}$.

Proof. By symmetry, we only need to prove the result when $|u| \leq S$ on the ξ-edge of $T_{a;m}$. Without loss of generality we also assume that $a = \mathbf{0}$.

We claim that for each $k = 0, 1, \cdots, 3m$, $|u(b)| \leq 2^k S + (2^k - 1)R$ for any $b \in T_{\mathbf{0};m}$ with $\eta(b) = k - m$. We prove this claim by induction on k. The base case of $k = 0$ holds by the assumptions. We suppose that the statement is true for $0, 1, \cdots, k$. For any $b \in T_{\mathbf{0};m}$ with $\eta(b) = k - m$ and $\xi(b) > k - 2m$, we have $b, b - \xi \in T_{\mathbf{0};m}$ and $\eta(b) = \eta(b - \xi) = k - m$. By (2.4.2) and the induction hypothesis,

$$|u(b + \eta)| \leq |u(b)| + |u(b - \xi)| + R \leq 2(2^k S + (2^k - 1)R) + R = 2^{k+1} S + (2^{k+1} - 1)R. \tag{2.4.3}$$

Then our claim holds by induction, and the lemma follows from our claim. □

2.4.2 Key lemmas via polynomial arguments

In this subsection we prove two key results, Lemma 2.4.4 and 2.4.5 below, which are analogous to [BLMS17, Lemma 3.4] and [BLMS17, Lemma 3.6], respectively. We will use the Remez inequality [Rem36]. More precisely, we will use the following discrete version as stated and proved in [BLMS17].

Lemma 2.4.3 ([BLMS17, Corollary 3.2]). *Let $d, \ell \in \mathbb{Z}_+$, and p be a polynomial with degree no more than d. For $M \in \mathbb{R}_+$, suppose that $|p| \leq M$ on at least $d + \ell$ integer points on a closed interval I, then on I we have*

$$|p| \leq \left(\frac{4|I|}{\ell}\right)^d M. \tag{2.4.4}$$

Now we prove the following bound of $|u|$ in a trapezoid, given that $|u|$ is small on the upper edge and on a substantial fraction of the lower edge of the trapezoid.

Lemma 2.4.4. *Let $R, K \in \mathbb{R}_+$, $\ell, m \in \mathbb{Z}_+$ with $\ell \leq \frac{m}{10}$, and $a \in \Lambda$. There is a universal constant $C_5 > 1$ (independent of a, m, ℓ, K, R), such that the following is true. Suppose $u : P_{a;m,\ell} \to \mathbb{R}$ is a function satisfying that:*

1. *(2.4.2) holds for any $b \in P_{a-\eta;m,\ell-1}$,*
2. *$|u| \leq K$ on the upper edge of $P_{a;m,\ell}$,*
3. *$|u| \leq K$ for at least half of the points in the lower edge of $P_{a;m,\ell}$.*

Then $|u| \leq C_5^{\ell+m}(K+R)$ in $P_{a;m,\ell}$.

Proof. We assume without loss of generality that $a = \mathbf{0}$. We first claim that there is a function $v : P_{\mathbf{0};m,\ell} \to \mathbb{R}$ satisfying the following four conditions:

1. $v = 0$ on $\{-t\eta : 1 \leq t \leq \ell\}$.
2. $v = u$ on $P_{\mathbf{0};m,0}$.
3. For each point $b \in P_{-\eta;m,\ell-1}$,

$$v(b) + v(b-\xi) + v(b+\eta) = u(b) + u(b-\xi) + u(b+\eta). \tag{2.4.5}$$

4. $\|v\|_\infty \leq 4^{\ell+m}(K+R)$.

We construct the function v by first defining it on $\{-t\eta : 0 \leq t \leq \ell\}$ and $P_{\mathbf{0};m,0}$, then iterating (2.4.5) line by line. More precisely, for $-m \leq s \leq 0$, we let $v(s\xi) =$

$u(s\xi)$. For each $t = -1, -2 \cdots, -\ell$, we first define $v(t\eta) = 0$, then define

$$\begin{aligned} & v((s-1)\xi + t\eta) \\ :=& - v(s\xi + t\eta) - v(s\xi + (t+1)\eta) + u(s\xi + t\eta) \\ & + u((s-1)\xi + t\eta) + u(s\xi + (t+1)\eta) \end{aligned} \tag{2.4.6}$$

for all $-m + t + 1 \leq s \leq 0$. Then we have defined $v(s\xi + t\eta)$ for $-\ell \leq t \leq 0$ and $-m + t \leq s \leq 0$. By our construction, v satisfies Condition 1 to 3.

Now we prove v satisfies Condition 4. First, (2.4.5) implies that $|v(b) + v(b-\xi) + v(b+\eta)| \leq R$ for any $b \in P_{-\eta;m,\ell-1}$. Using this and $|v| \leq K$ on $P_{\mathbf{0};m,0}$, by an induction similar to that in the construction of v, we can prove that

$$|v(-s\xi - t\eta)| \leq 2^{s+t}K + (2^{s+t} - 1)R \tag{2.4.7}$$

for each $0 \leq t \leq \ell$ and $0 \leq s \leq m + t$. In particular, $|v| \leq (K + R)4^{\ell+m}$ on any point in trapezoid $P_{\mathbf{0};m,\ell}$, and v satisfies Condition 4.

Let $w := u - v$, then $w = 0$ on $P_{\mathbf{0};m,0}$ and $w(b) + w(b-\eta) + w(b-\gamma) = 0$ for each $b \in P_{\mathbf{0};m,\ell-1}$. Also, $|w| \leq (K+R)4^{\ell+m} + K \leq (K+R)5^{\ell+m}$ on at least half of points in the lower edge of $P_{\mathbf{0};m,\ell}$. Since $\ell \leq \frac{m}{10}$, we have

$$\left|\left\{0 \leq s \leq m + \ell : |w(-s\xi - \ell\eta)| \leq (K+R)5^{\ell+m}\right\}\right| \geq \frac{m+\ell}{2} \geq 5\ell. \tag{2.4.8}$$

We claim that for each $0 \le t \le \ell$, if we denote

$$g_t(s) = (-1)^s w(-s\xi - t\eta),\ \forall 0 \le s \le m+t, s \in \mathbb{Z}, \tag{2.4.9}$$

then g_t is a polynomial of degree at most t. We prove the claim by induction on t. For $t = 0$, this is true since $w = 0$ on the upper edge of $P_{\mathbf{0};m,\ell}$. Suppose the statement is true for t, then since

$$\begin{aligned} g_{t+1}(s) - g_{t+1}(s+1) &= (-1)^s w(-s\xi - (t+1)\eta) - (-1)^{s-1} w((-s-1)\xi - (t+1)\eta) \\ &= -(-1)^s w(-s\xi - t\eta) = -g_t(s), \end{aligned} \tag{2.4.10}$$

for all $0 \le s \le m+t, s \in \mathbb{Z}$, we have that g_{t+1} is a polynomial of degree at most $t+1$. Hence our claim holds.

In particular, $g_\ell(s) = (-1)^s w(-s\xi - \ell\eta)$ is a polynomial of degree at most ℓ. Hence by (2.4.8) and Lemma 2.4.3, there exists a constant $C > 0$ such that

$$|w(-s\xi - \ell\eta)| \le 5^{\ell+m} C^\ell (K+R) \tag{2.4.11}$$

for each $0 \le s \le m+\ell$. Thus on the lower edge of $P_{\mathbf{0};m,\ell}$,

$$|u| \le |w| + |v| \le 5^{\ell+m} C^\ell (K+R) + 4^{\ell+m}(K+R) \le (5C+4)^{\ell+m}(K+R), \tag{2.4.12}$$

Finally, by an inductive argument similar to the proof of Lemma 2.4.2, and letting

$C_5 = 10C + 8$, we get

$$|u| \leq 2^\ell (5C+4)^{\ell+m}(K+R) + (2^\ell - 1)R \leq C_5^{\ell+m}(K+R) \tag{2.4.13}$$

in $P_{\mathbf{0};m,\ell}$. $\square$

Our next lemma is obtained by applying Lemma 2.4.4 repeatedly.

Lemma 2.4.5. *Let $m, \ell \in \mathbb{Z}_+$ with $\ell \leq m \leq 2\ell$, $K, R \in \mathbb{R}_+$, and $a \in \Lambda$. Let $u : P_{a;m,\ell} \to \mathbb{R}$ be a function satisfying* (2.4.2) *for each $b \in P_{a-\eta;m,\ell-1}$. If $|u| \leq K$ on $P_{a;m,0}$ and $|\{b \in P_{a;m,\ell} : |u(b)| > K\}| \leq \frac{1}{10^5} m\ell$, then $|u| \leq (K+R)C_6^\ell$ in $P_{a;m,\lfloor \frac{\ell}{2} \rfloor}$, where $C_6 > 1$ is a universal constant.*

Proof. If $\ell \leq 120$, then the result holds trivially since $\frac{1}{10^5} m\ell \leq \frac{2}{10^5}\ell^2 < 1$. From now on we assume that $\ell \geq 120$, and let $C_6 = C_5^{1000}$ where C_5 is the constant in Lemma 2.4.4.

For each $k = 0, 1, \cdots, 29$, we choose an $l_k \in \left\{ \left\lfloor \frac{2k}{60}\ell \right\rfloor, \left\lfloor \frac{2k}{60}\ell \right\rfloor + 1, \cdots, \left\lfloor \frac{2k+1}{60}\ell \right\rfloor - 1 \right\}$ such that

$$|\{b : |u(b)| \leq K\} \cap P_{a-l_k\eta;m+l_k,0}| \geq \frac{1}{2}(m + l_k). \tag{2.4.14}$$

Such l_k must exist, since otherwise,

$$|\{b \in P_{a;m,\ell} : |u(b)| > K\}| > \frac{1}{2} \cdot \frac{1}{60} m\ell > \frac{1}{10^5} m\ell, \tag{2.4.15}$$

which contradicts with an assumption in the statement of this lemma. In particular, we can take $l_0 = 0$.

From the definition, we have $l_{k+1} - l_k \leq \frac{1}{20}\ell \leq \frac{1}{20}m$ and $l_{k+1} - l_k \geq \frac{1}{60}\ell \geq \frac{1}{120}m$. For each $k = 0, 1, \cdots, 28$, let $P_k = P_{a-l_k\eta;m+l_k,l_{k+1}-l_k}$, then we claim that $|u| \leq C_6^{l_{k+1}}(K+R)$ on P_k.

We prove this claim by induction on k. For $k = 0$, we use Lemma 2.4.4 for $P_{a;m,l_1}$ to get

$$|u| \leq (K+R)C_5^{l_1+m} \leq (K+R)C_5^{121l_1} \leq (K+R)C_6^{l_1} \tag{2.4.16}$$

in $P_0 = P_{a;m,l_1}$. Suppose the statement holds for k, then $|u| \leq (K+R)C_6^{l_{k+1}}$ in $P_{a-l_{k+1}\eta;m+l_{k+1},0}$ which is the upper edge of P_{k+1}. We use Lemma 2.4.4 again for P_{k+1}, and get $|u| \leq (K+R)C_6^{l_{k+2}}$ in P_{k+1}. Thus the claim follows.

Since $l_{29} \geq \frac{29}{30}\ell - 1 \geq \left\lfloor \frac{1}{2}\ell \right\rfloor + 1$ when $\ell \geq 120$, we have $P_{a;m,\left\lfloor \frac{\ell}{2} \right\rfloor} \subset \bigcup_{k=0}^{28} P_k$. Then the lemma is implied by this claim. $\square$

2.4.3 Proof of Theorem 2.1.9

In this subsection we finish the proof of Theorem 2.1.9. The key step is a triangular analogue of [BLMS17, Corollary 3.7] (Lemma 2.4.6 below); then we finish using a Vitalli covering argument.

Proof of Theorem 2.1.9. Let $\epsilon_1 = \frac{1}{10^{18}}$, and $C_4 = 6C_6 > 6$ where C_6 is the constant in Lemma 2.4.5. We note that now Theorem 2.1.9 holds trivially when $n < 10^9$, so below we assume that $n \geq 10^9$.

We argue by contradiction, i.e. we assume that

$$|\{b \in T_{\mathbf{0};n} : |u(b)| > K\}| \leq \epsilon_1 n^2, \tag{2.4.17}$$

where we take $K = C_4^{-n}|u(\mathbf{0})|$.

We first define a notion of triangles on which $|u|$ is "suitably bounded". For this, we let $R = C_4^{-n}|u(\mathbf{0})|$ as well, and we define a triangle $T_{a;m} \subset T_{\mathbf{0};\lfloor \frac{n}{2} \rfloor}$ as being *good* if m is even and $|u| \leq (K+R)\left(\frac{C_4}{3}\right)^{3m}$ on any point in $T_{a;m}$.

We choose points $a_i \in T_{\mathbf{0};\lfloor \frac{n}{20} \rfloor}$ for $1 \leq i \leq \left\lfloor \frac{n^2}{10^6} \right\rfloor$, such that each $T_{a_i,2} \subset T_{\mathbf{0};\lfloor \frac{n}{20} \rfloor}$, and $T_{a_i,2} \cap T_{a_j,2} = \emptyset$ for any $i \neq j$. Denote $S := \left\{T_{a_i,2} : 1 \leq i \leq \left\lfloor \frac{n^2}{10^6} \right\rfloor\right\}$. By (2.4.17), for at least half of the triangles in S, $|u| \leq K$ on each of them. Hence, there are at least $\frac{n^2}{10^7}$ good triangles in S. Denote

$$Q = \left\{a_i : 1 \leq i \leq \left\lfloor \frac{n^2}{10^6} \right\rfloor,\ T_{a_i,2} \text{ is good}\right\}. \tag{2.4.18}$$

For any $a \in Q$, let $l_a = \max\left\{l \in \mathbb{Z}_+ : T_{a,l} \text{ is good and } T_{a,l} \subset T_{\mathbf{0};\lfloor \frac{n}{2} \rfloor}\right\}$. Denote $X_a = T_{a;l_a}$ for each $a \in Q$.

If there exists $a \in Q$ with $l_a \geq \frac{n}{30}$, then this maximal triangle contains $\mathbf{0}$, and $|u(\mathbf{0})| \leq \left(\frac{C_4}{3}\right)^{3l_a}(K+R) \leq \left(\frac{C_4}{3}\right)^{n}(K+R) < |u(\mathbf{0})|$, which is impossible. Hence $l_a \leq \frac{n}{30}$ for any $a \in Q$. For any $a \in Q$, denote $Y_a := T_{a;4l_a}$. Then $Y_a \subset T_{\mathbf{0};\lfloor \frac{n}{2} \rfloor}$.

We need the following result on good triangles.

Lemma 2.4.6. *For any $m \in \mathbb{Z}_+$ and $a \in \Lambda$ the following is true. Let $T_1 = T_{a;2m}$,*

$T_2 = T_{a;5m}$ and $T_3 = T_{a;8m}$ (see Figure 2.2 for an illustration). If $T_3 \subset T_{\mathbf{0};\lfloor \frac{n}{2} \rfloor}$, and $|\{b \in T_3 : |u(b)| > K\}| \le \frac{m^2}{10^6}$, and T_1 is good, then T_2 is also good.

We assume this result for now and continue our proof of Theorem 2.1.9. We have that

$$|\{b \in Y_a : |u(b)| > K\}| \ge \frac{l_a^2}{10^7}, \quad \forall a \in Q, \tag{2.4.19}$$

since otherwise, by Lemma 2.4.6 with $T_1 = X_a$ and $T_3 = Y_a$, there is a good triangle strictly containing X_a and this contradicts with the maximal property of X_a.

Finally we will apply Vitalli's covering theorem to the collection of triangles $\{Y_a : a \in Q\}$. We can find a subset $\tilde{Q} \subset Q$ such that $\left|\bigcup_{a \in \tilde{Q}} Y_a\right| \ge \frac{1}{16} |\bigcup_{a \in Q} Y_a|$, and $Y_a \cap Y_{a'} = \emptyset$ for any $a \ne a' \in \tilde{Q}$. Hence

$$\left|\left\{a \in T_{\mathbf{0};\lfloor \frac{n}{2} \rfloor} : |u(a)| > K\right\}\right| \ge \frac{1}{10^7} \left|\bigcup_{a \in \tilde{Q}} Y_a\right| > \frac{1}{10^9} \left|\bigcup_{a \in Q} Y_a\right|. \tag{2.4.20}$$

Since $Q \subset \bigcup_{a \in Q} Y_a$, we have $\left|\bigcup_{a \in Q} Y_a\right| \ge |Q| > \frac{n^2}{10^7}$, so

$$\left|\left\{a \in T_{\mathbf{0};\lfloor \frac{n}{2} \rfloor} : |u(a)| > K\right\}\right| > \frac{1}{10^9} \cdot \frac{n^2}{10^7} = \frac{n^2}{10^{16}}.$$

This contradicts with our assumption (2.4.17) since $\epsilon_1 = \frac{1}{10^{18}}$. $\square$

It remains to prove Lemma 2.4.6.

Proof of Lemma 2.4.6. We first note that u satisfies (2.4.2) for any $b \in T_{\mathbf{0};\lfloor \frac{n}{2} \rfloor}$. Without loss of generality, we assume $a = \mathbf{0}$.

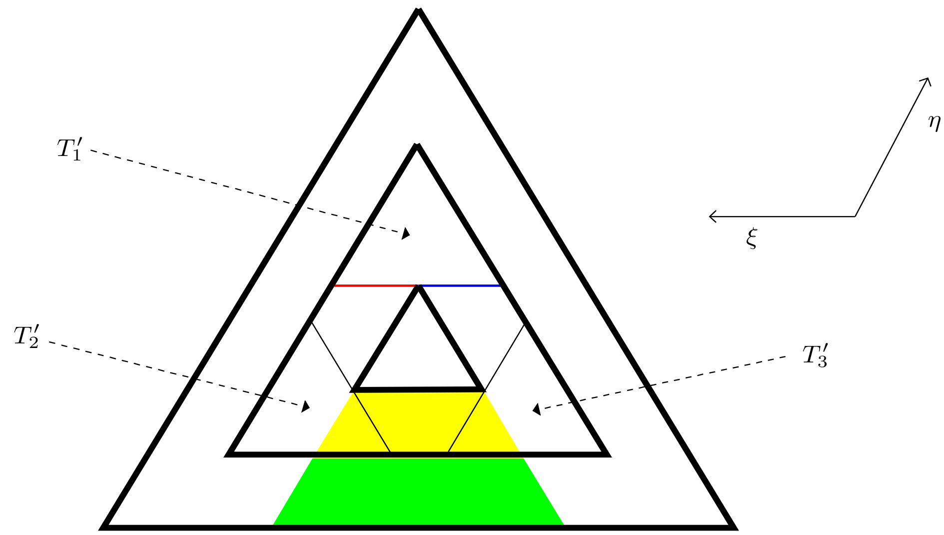

Figure 2.2: The thick lines indicate edges of T_1, T_2, and T_3. The blue segment indicates L_1 and the red segment indicates L_2. The yellow region indicates P_1' and the union of yellow region and green region indicates P_1.

Define $F : \Lambda \to \Lambda$ to be the counterclockwise rotation around $\mathbf{0}$ by $\frac{2\pi}{3}$, i.e.

$$F(s_1\xi + t_1\eta) = (t_1 - s_1)\xi - s_1\eta \tag{2.4.21}$$

for any $s_1, t_1 \in \mathbb{Z}$.

We first consider the trapezoid $P_1 := P_{2m\xi-2m\eta;6m,6m}$. The upper edge of P_1 is exactly the ξ-edge of T_1 and the lower edge of P_1 is contained in the ξ-edge of T_3. Denote $P_1' := P_{2m\xi-2m\eta;6m,3m}$, $K_1 := (K+R)(2C_6)^{6m}$ and $K_2 := (K_1+R)C_6^{6m}$. Then $|u| \le K_1$ in T_1 since T_1 is good. In particular, $|u| \le K_1$ on the upper edge of P_1. We also have $|\{b \in P_1 : |u(b)| > K\}| \le \frac{36}{10^5}m^2$, by $P_1 \subset T_3$ and the assumption of this lemma. Thus by Lemma 2.4.5, we deduce that $|u| \le K_2$ in P_1'.

Let $P_2 := F(P_1)$ and $P_3 := F^{-1}(P_1)$. A symmetric argument for P_2 and P_3 implies that $|u| \leq K_2$ also holds in $P_2' := F(P_1')$ and $P_3' := F^{-1}(P_1')$.

Consider the three triangles $T_1' := T_{3m\xi+6m\eta;2m}$, $T_2' := T_{3m\xi-3m\eta;2m}$ and $T_3' := T_{-6m\xi-3m\eta;2m}$ (see Figure 2.2). We have $T_2' = F(T_1')$ and $T_3' = F^{-1}(T_1')$. We claim that $|u| \leq (K_2+R)2^{6m}$ in $\bigcup_{i=1,2,3} T_i'$. By symmetry, we only need to prove the claim in T_1'. Denote $L_1 := \{s\xi + 4m\eta : -m \leq s \leq 2m\}$ and $L_2 := \{s\xi + 4m\eta : 2m \leq s \leq 5m\}$. Note that the ξ-edge of triangle T_1' is the set of points

$$\{s\xi + 4m\eta : -m \leq s \leq 5m\} = L_1 \cup L_2. \tag{2.4.22}$$

Since

$$F^{-1}(L_1) = \{-4m\xi + (s-4m)\eta : -m \leq s \leq 2m\} \subset P_1', \tag{2.4.23}$$

and

$$F(L_2) = \{(4m+t)\xi + t\eta : -5m \leq t \leq -2m\} \subset P_1', \tag{2.4.24}$$

we have $L_1 \subset F(P_1') = P_2'$ and $L_2 \subset F^{-1}(P_1') = P_3'$. Hence $|u| \leq K_2$ on $L_1 \cup L_2$, i.e. the ξ-edge of T_1'. By Lemma 2.4.2, $|u| \leq (K_2+R)2^{6m}$ in T_1', and our claim holds.

Since $\left(\bigcup_{i=1,2,3} T_i'\right) \cup \left(\bigcup_{i=1,2,3} P_i'\right) \cup T_1 = T_2$, we have $|u| \leq (K_2+R)2^{6m}$ in T_2. We also have that

$$2^{6m}(K_2+R) = 2^{12m}C_6^{12m}K + (2^{12m}C_6^{12m} + 2^{6m}C_6^{6m} + 2^{6m})R \leq \left(\frac{C_4}{3}\right)^{15m}(K+R), \tag{2.4.25}$$

so T_2 is good. $\square$

To apply Theorem 2.1.9 to prove Theorem 2.5.1 in the next section, we actually need the following two corollaries.

Corollary 2.4.7. *Let $a \in \Lambda$, and $m, \ell \in \mathbb{Z}_{\geq 0}$ with $m \geq 2\ell$. Take any nonempty*

$$L \subset \{a - t\xi : t \in \mathbb{Z}, \ell \leq t \leq m - \ell\}, \tag{2.4.26}$$

and function $u : P_{a;m,\ell} \to \mathbb{R}$ such that

$$|u(b) + u(b - \xi) + u(b + \eta)| \leq C_4^{-2\ell} \min_{c \in L} |u(c)|, \tag{2.4.27}$$

for any b with $\{b, b - \xi, b + \eta\} \subset P_{a;m,\ell}$. Then

$$\left| \left\{ b \in P_{a;m,\ell} : |u(b)| \geq C_4^{-2\ell} \min_{c \in L} |u(c)| \right\} \right| \geq \epsilon_2 (\ell + 1)^2 \tag{2.4.28}$$

whenever L contains at least one element; and

$$\left| \left\{ b \in P_{a;m,\ell} : |u(b)| \geq C_4^{-2\ell} \min_{c \in L} |u(c)| \right\} \right| \geq \epsilon_2 (m + 2)(\ell + 1) \tag{2.4.29}$$

if $m \geq 2\ell + 2$ and $L = \{a - t\xi : t \in \mathbb{Z}, \ell + 1 \leq t \leq m - \ell - 1\}$. Here ϵ_2 is a universal constant.

Proof. If $\ell \leq 10^9$ then the conclusion holds trivially by taking ϵ_2 small enough. From now on we assume $\ell > 10^9$. We denote $P := P_{a;m,\ell}$, for simplicity of notations. Without loss of generality, we assume that $\min_{c\in L} |u(c)| = 1$.

First we prove

$$\left|\left\{b \in P : |u(b)| \geq C_4^{-2\ell}\right\}\right| \geq \frac{\epsilon_1(\ell+1)^2}{100}, \tag{2.4.30}$$

which implies (2.4.28). We take $a' \in L$. By (2.4.27), for any $b \in P_{a-\xi;m-2,\ell-2}$ and $0 < k_1 < \ell$, if $|u(b)| \geq C_4^{-k_1}$, then $|u(b-\eta)| \geq C_4^{-k_1-1}$ or $|u(b-\gamma)| \geq C_4^{-k_1-1}$. Thus we can inductively pick $a_1 = a', a_2, \cdots, a_{\lfloor \frac{\ell}{3} \rfloor} \in P$, such that for each $i = 1, 2, \cdots, \lfloor \frac{\ell}{3} \rfloor$, $|u(a_i)| \geq C_4^{-i+1}$, and $a_i = a' - s_i\xi - i\eta$ with $s_i - s_{i-1} \in \{0, 1\}$ for each $2 \leq i \leq \lfloor \frac{\ell}{3} \rfloor$. In particular, we have $\left|u\left(a_{\lfloor \frac{\ell}{3} \rfloor}\right)\right| \geq C_4^{-\ell}$.

Denote $T' := T_{a_{\lfloor \frac{\ell}{3} \rfloor};2\lfloor \frac{\ell}{18} \rfloor}$. Then $T' \subset P$, and we can apply Theorem 2.1.9 in T' with $n = 2\lfloor \frac{\ell}{18} \rfloor$, thus (2.4.30) follows.

For the case where $L = \{a - t\xi : t \in \mathbb{Z}, \ell+1 \leq t \leq m-\ell-1\}$, we prove

$$\left|\left\{b \in P : |u(b)| \geq C_4^{-2\ell}\right\}\right| \geq \epsilon_1\left(\frac{(m+2)(\ell+1)}{800} - \frac{(\ell+1)^2}{100}\right). \tag{2.4.31}$$

When $m \leq 8\ell$, (2.4.31) is trivial. From now on we assume that $m > 8\ell$. Denote $l := \lceil \frac{m-2\ell-1}{4\ell} \rceil - 1$. We take $b_1 := a - (\ell+1)\xi$. Let $b_i := b_1 - 4\ell(i-1)\xi$ where $i = 2, \cdots, l$. For each $1 \leq i \leq l$, consider the trapezoid $P_i := P_{b_i;2\ell,\ell}$. We note that these trapezoids are disjoint, and $P_i \subset P$ for each $1 \leq i \leq l$ (see Figure 2.3 for an illustration). We apply the same arguments in the proof of (2.4.30), with P

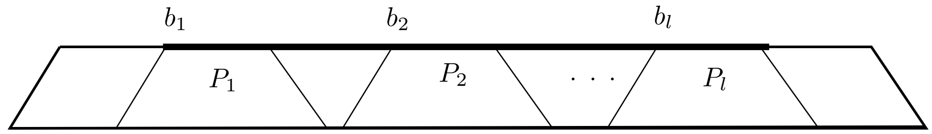

Figure 2.3: An illustration of P_i's. The thick line indicates L.

substituted by each P_i, and we get

$$\left|\left\{b \in P_i : |u(b)| \geq C_4^{-2\ell}\right\}\right| \geq \frac{\epsilon_1(\ell+1)^2}{100}, \tag{2.4.32}$$

for each $1 \leq i \leq l$. By summing over all i we get (2.4.31).

Finally, we can deduce (2.4.29) from (2.4.30) and (2.4.31). $\square$

For the next corollary, we set up notations for reversed trapezoids.

Definition 2.4.8. For any $a \in \Lambda$, $m, \ell \in \mathbb{Z}_{\geq 0}$ with $\ell \leq m$, we denote

$$P^r_{a;m,\ell} := \{a - t\xi - s\eta : s \leq t \leq m, 0 \leq s \leq \ell\} \cap \Lambda, \tag{2.4.33}$$

which is also a trapezoid, but its orientation is different from that of $P_{a;m,\ell}$ (see Figure 2.4 for an illustration). We also denote $\{a - t\xi : 0 \leq t \leq m\} \cap \Lambda$ to be the *upper edge* of $P^r_{a;m,\ell}$.

Corollary 2.4.9. *Let $a \in \Lambda$, and $m, \ell \in \mathbb{Z}_{\geq 0}$ with $m \geq \ell$. Let L be a nonempty subset of the upper edge of $P^r_{a;m,\ell}$. Take a function $u : P^r_{a;m,\ell} \to \mathbb{R}$ such that*

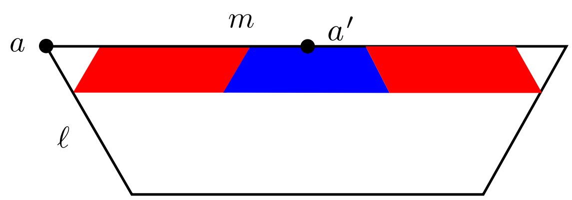

Figure 2.4: An illustration of Corollary 2.4.9: $P^r_{a;m,\ell}$ is the set of lattice points in the region surrounded by black lines. $P_{a'+(\lfloor\frac{\ell}{5}\rfloor+1)\xi;2\lfloor\frac{\ell}{5}\rfloor+2,\lfloor\frac{\ell}{5}\rfloor}$ is the blue region, and $P_{a-(\lfloor\frac{\ell}{5}\rfloor+2)\xi;m-2\lfloor\frac{\ell}{5}\rfloor-4,\lfloor\frac{\ell}{5}\rfloor}$ is the union of the blue and red regions.

$$|u(b)+u(b-\xi)+u(b+\eta)| \leq C_4^{-2\ell}\min_{c\in L}|u(c)|, \tag{2.4.34}$$

for any b with $\{b, b-\xi, b+\eta\}\subset P^r_{a;m,\ell}$. Then

$$\left|\left\{b\in P^r_{a;m,\ell} : |u(b)|\geq C_4^{-2\ell}\min_{c\in L}|u(c)|\right\}\right| \geq \epsilon_3(\ell+1)^2, \tag{2.4.35}$$

if $L=\left\{a-\left\lfloor\frac{m}{2}\right\rfloor\xi\right\}$ or $L=\left\{a-\left\lceil\frac{m}{2}\right\rceil\xi\right\}$. And

$$\left|\left\{b\in P^r_{a;m,\ell} : |u(b)|\geq C_4^{-2\ell}\min_{c\in L}|u(c)|\right\}\right| \geq \epsilon_3(m+2)(\ell+1), \tag{2.4.36}$$

if $L=\{a-t\xi : t\in\mathbb{Z}, 1\leq t\leq m-1\}$. Here ϵ_3 is a universal constant.

Proof. If $m\leq 10^9$, then the conclusion holds trivially by taking ϵ_3 small enough. From now on we assume that $m>10^9$. If $L=\left\{a-\left\lfloor\frac{m}{2}\right\rfloor\xi\right\}$ or $L=\left\{a-\left\lceil\frac{m}{2}\right\rceil\xi\right\}$, let $a'=a-\left\lfloor\frac{m}{2}\right\rfloor\xi$ or $a'=a-\left\lceil\frac{m}{2}\right\rceil\xi$ respectively. Consider $P_{a'+(\lfloor\frac{\ell}{5}\rfloor+1)\xi;2\lfloor\frac{\ell}{5}\rfloor+2,\lfloor\frac{\ell}{5}\rfloor}\subset P^r_{a;m,\ell}$ (blue region in Figure 2.4). Using Corollary 2.4.7 for this trapezoid, we get (2.4.35).

If $L = \{a - t\xi : t \in \mathbb{Z}, 1 \le t \le m-1\}$, consider $P_{a-(\lfloor \frac{\ell}{5} \rfloor + 2)\xi; m-2\lfloor \frac{\ell}{5} \rfloor - 4, \lfloor \frac{\ell}{5} \rfloor} \subset P^r_{a;m,\ell}$ (union of blue and red regions in Figure 2.4). Using Corollary 2.4.7 for this trapezoid, we get (2.4.36). $\square$

2.5 Geometric substructure on 3D lattice

In this section we state and prove the following stronger version of Theorem 2.1.5 which incorporates a graded set (which is defined in Definition 2.3.3).

Theorem 2.5.1. *For any $K \in \mathbb{R}_+$, $N \in \mathbb{Z}_+$, and small enough $\varepsilon \in \mathbb{R}_+$, we can find large $C_2 \in \mathbb{R}_+$ depending only on K and $C_{\varepsilon,N} \in \mathbb{R}_+$ depending only on ε, N, such that the following statement is true.*

Take integer $n > C_{\varepsilon,N}$ and functions $u, V : \mathbb{Z}^3 \to \mathbb{R}$, satisfying

$$\Delta u = Vu \tag{2.5.1}$$

in Q_n and $\|V\|_\infty \le K$. Let $\vec{l}$ be a vector of positive reals, and $E \subset \mathbb{Z}^3$ be any $(N, \vec{l}, \varepsilon^{-1}, \varepsilon)$-graded set, with the first scale length $l_1 > C_{\varepsilon,N}$. If E is $(1, 2\varepsilon)$-normal in Q_n, then we have that

$$\left|\left\{a \in Q_n : |u(a)| \ge \exp(-C_2 n^3)|u(\mathbf{0})|\right\} \setminus E\right| \ge C_3 n^2 (\log_2 n)^{-1}. \tag{2.5.2}$$

Here C_3 is a universal constant.

The first result we need is based on the "cone property" of the function u, as discussed in Section 2.2. We remind the reader of the notations $\mathbf{e}_\tau$, for $\tau = 1, 2, 3$; and $\boldsymbol{\lambda}_\tau$, $\mathcal{P}_{\tau,k}$, for $\tau \in \{1, 2, 3, 4\}$ and $k \in \mathbb{Z}$, from Definition 2.1.7; and the cones from Definition 2.2.1.

Proposition 2.5.2. *Let $K \in \mathbb{R}_+$, $n \in \mathbb{Z}_+$, and u, V satisfy* (2.5.1) *in Q_n, with $\|V\|_\infty \leq K$. Then there exists $\tau \in \{1, 2, 3, 4\}$, such that for any $0 \leq i \leq \frac{n}{10}$ there is*

$$a_i \in (\mathcal{P}_{\tau,i} \cup \mathcal{P}_{\tau,i+1}) \cap \mathcal{C} \cap Q_{\frac{n}{10}+1} \tag{2.5.3}$$

with $|u(a_i)| \geq (K + 11)^{-n}|u(\mathbf{0})|$.

Proof. We can assume that $n \geq 10$ since otherwise this proposition holds obviously. We argue by contradiction. Denote $\Upsilon := \{b \in Q_n : |u(b)| \geq (K + 11)^{-n}|u(\mathbf{0})|\}$. If the statement is not true, then for each $\tau \in \{1, 2, 3, 4\}$, there is $i_\tau \in \left[0, \frac{n}{10}\right]$, such that

$$(\mathcal{P}_{\tau,i_\tau} \cup \mathcal{P}_{\tau,i_\tau+1}) \cap \mathcal{C} \cap \Upsilon \cap Q_{\frac{n}{10}+1} = \emptyset. \tag{2.5.4}$$

Define $B_{in} := \bigcap_{\tau=1}^{4} \{a \in \mathcal{C} : a \cdot \boldsymbol{\lambda}_\tau < i_\tau\}$, $B_{bd} := \bigcap_{\tau=1}^{4} \{a \in \mathcal{C} : a \cdot \boldsymbol{\lambda}_\tau \leq i_\tau + 1\} \setminus B_{in}$, $B_{out} := \mathcal{C} \setminus (B_{in} \cup B_{bd})$. Then for any $a \in B_{in}$ and $b \in B_{out}$, we have $\|a - b\|_1 \geq 3$. Since $i_1, i_2, i_3, i_4 \leq \frac{n}{10}$, we have that

$$B_{bd} \subset \mathcal{C} \cap \left\{a \in \mathbb{Z}^3 : |a \cdot \mathbf{e}_1| + |a \cdot \mathbf{e}_2| + a \cdot \mathbf{e}_3 \leq \frac{n}{10} + 1\right\} \subset Q_{\frac{n}{10}+1}. \tag{2.5.5}$$

Then the condition (2.5.4) implies that $\Upsilon \cap B_{bd} = \emptyset$.

We now apply Lemma 2.2.3 to starting point $a_0 = \mathbf{0}$, in the $\mathbf{e}_3$ direction, and $k = n$. Let $\mathbf{0} = a_0, a_1, \cdots, a_w \in \mathcal{C} \cap \mathbb{Z}^3$ be the chain. Then $a_0 \in B_{in}$, and $a_w \cdot \mathbf{e}_3 \geq n-1$, which implies that $a_w \in B_{out}$ (since otherwise, $a_w \cdot \mathbf{e}_3 = \frac{1}{4}\sum_{\tau=1}^{4} a_w \cdot \boldsymbol{\lambda}_\tau \leq \frac{1}{4}\sum_{\tau=1}^{4} i_\tau + 1 \leq \frac{n}{10} + 1$). Thus $a_w \neq a_0$ and $w \geq 1$. Since $|u(a_i)| \geq (K+11)^{-1}|u(a_{i-1})|$ for each $i = 1, \cdots, w$, we also have that each $a_i \in \Upsilon$. As $\Upsilon \cap B_{bd} = \emptyset$, we can find $1 \leq i \leq w$, such that $a_{i-1} \in B_{in}$ and $a_i \in B_{out}$. This implies that $\|a_{i-1} - a_i\|_1 \geq 3$, which contradicts with the construction of the chain from Lemma 2.2.3. $\square$

Proposition 2.5.3. *For any $K \in \mathbb{R}_+$, $N \in \mathbb{Z}_+$, and small enough $\varepsilon > 0$, we can find $C_7, C_{\varepsilon,N} \in \mathbb{R}_+$, where C_7 depends only on K and $C_{\varepsilon,N}$ depends only on ε, N, such that following statement is true.*

Take integer $n > C_{\varepsilon,N}$, and let functions u, V satisfy (2.5.1) in Q_n, and $\|V\|_\infty \leq K$. Let $\vec{l}$ be a vector of positive reals, and E be an $(N, \vec{l}, \varepsilon^{-1}, \varepsilon)$-graded set with the first scale length $l_1 > C_{\varepsilon,N}$, and be $(1, 2\varepsilon)$-normal in Q_n. For any $\tau \in \{1,2,3,4\}$, $k \in \mathbb{Z}$, $0 \leq k \leq \frac{n}{10}$, and $a_0 \in \mathcal{P}_{\tau,k} \cap Q_{\frac{n}{4}}$, there exists $h \in \mathbb{Z}_+$, such that

$$\left|\left\{a \in Q_n \cap \bigcup_{i=0}^{h} \mathcal{P}_{\tau,k+i} : |u(a)| \geq \exp(-C_7 n^3)|u(a_0)|\right\} \setminus E\right| > C_8 h n (\log_2(n))^{-1}. \tag{2.5.6}$$

Here C_8 is a universal constant.

In Section 2.5.3, Theorem 2.5.1 is proved by applying Proposition 2.5.3 to each point a_i obtained from Proposition 2.5.2.

The next two subsections are devoted to the proof of Proposition 2.5.3. We will work with $\tau = 1$ only, and the cases where $\tau = 2, 3, 4$ follow the same arguments. Assuming the result does not hold, we can find many "gaps", i.e. intervals that do not intersect the set $\{|u(a)| : a \in Q_n \setminus E, a \cdot \boldsymbol{\lambda}_1 \geq k\}$. These gaps will allow us to construct geometric objects on $\mathbb{Z}^3$. We first find many "pyramids" in $\{a \in Q_n : a \cdot \boldsymbol{\lambda}_1 \geq k\}$ (see Lemma 2.5.5), then we prove Proposition 2.5.3 assuming a lower bound on the number of desired points in each "pyramid" (Proposition 2.5.11). In Section 2.5.2 we prove Proposition 2.5.11, by studying "faces" of each "pyramid", and using corollaries of Theorem 2.1.9.

2.5.1 Decomposition into pyramids

In this subsection we define pyramids in Q_n, and in the next subsection we study the structure of each of these pyramids.

We need some further geometric objects in $\mathbb{R}^3$.

Definition 2.5.4. For simplicity of notations we denote $\overline{\boldsymbol{\lambda}}_2 = \boldsymbol{\lambda}_2 = -\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, $\overline{\boldsymbol{\lambda}}_3 = \boldsymbol{\lambda}_3 = \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3$, and $\overline{\boldsymbol{\lambda}}_4 = -\boldsymbol{\lambda}_4 = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$. Then $\boldsymbol{\lambda}_1 \cdot \overline{\boldsymbol{\lambda}}_2 = \boldsymbol{\lambda}_1 \cdot \overline{\boldsymbol{\lambda}}_3 = \boldsymbol{\lambda}_1 \cdot \overline{\boldsymbol{\lambda}}_4 = 1$, and $\overline{\boldsymbol{\lambda}}_2 \cdot \overline{\boldsymbol{\lambda}}_3 = \overline{\boldsymbol{\lambda}}_2 \cdot \overline{\boldsymbol{\lambda}}_4 = \overline{\boldsymbol{\lambda}}_3 \cdot \overline{\boldsymbol{\lambda}}_4 = -1$.

For any $a \in \mathbb{R}^3$, $r \in \mathbb{Z}_+$, denote $\mathbf{t}_r(a) = a + r\mathbf{e}_1 + r\mathbf{e}_2 + 2r\mathbf{e}_3$. Then $\mathbf{t}_r(a) \cdot \overline{\boldsymbol{\lambda}}_2 = a \cdot \overline{\boldsymbol{\lambda}}_2 + 2r$, $\mathbf{t}_r(a) \cdot \overline{\boldsymbol{\lambda}}_3 = a \cdot \overline{\boldsymbol{\lambda}}_3 + 2r$, and $\mathbf{t}_r(a) \cdot \overline{\boldsymbol{\lambda}}_4 = a \cdot \overline{\boldsymbol{\lambda}}_4$. Denote

$$\mathcal{T}_{a,r} := \left\{ b \in \mathcal{P}_{1, a \cdot \boldsymbol{\lambda}_1} : b \cdot \overline{\boldsymbol{\lambda}}_\tau \leq \mathbf{t}_r(a) \cdot \overline{\boldsymbol{\lambda}}_\tau, \forall \tau \in \{2, 3, 4\} \right\}, \tag{2.5.7}$$

and let $\mathring{\mathcal{T}}_{a,r}$ be the interior of $\mathcal{T}_{a,r}$ in $\mathcal{P}_{1,a\cdot\boldsymbol{\lambda}_1}$. Respectively, $\mathring{\mathcal{T}}_{a,r}$ and $\mathcal{T}_{a,r}$ are the open and closed equilateral triangles with side length $2\sqrt{2}r$ in the plane $\mathcal{P}_{1,a\cdot\boldsymbol{\lambda}_1}$, and a is the midpoint of one side. When $a \in \mathbb{Z}^3$, there are $2r+1$ lattice points on each side of $\mathcal{T}_{a,r}$.

We also take

$$\mathfrak{T}_{a,r} := \left\{ b \in \mathbb{R}^3 : b \cdot \boldsymbol{\lambda}_1 \geq a \cdot \boldsymbol{\lambda}_1,\ b \cdot \overline{\boldsymbol{\lambda}}_\tau \leq \mathbf{t}_r(a) \cdot \overline{\boldsymbol{\lambda}}_\tau, \forall \tau \in \{2,3,4\} \right\}, \tag{2.5.8}$$

which is a (closed) regular tetrahedron, with four faces orthogonal to $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3, \boldsymbol{\lambda}_4$ respectively. The point $\mathbf{t}_r(a)$ is a vertex of $\mathfrak{T}_{a,r}$, and $\mathcal{T}_{a,r}$ is the face orthogonal to $\boldsymbol{\lambda}_1$. (See Figure 2.6 for an illustration)

For any $k \in \mathbb{Z}$, denote $\pi_k(a)$ to be the orthogonal projection of a onto $\mathcal{P}_{1,k}$.

The purpose of the following lemma is to find some triangles ($\mathcal{T}_{a_i,r_i}$ for a_i, r_i in Lemma 2.5.5) in $\mathcal{P}_{1,k} \cup \mathcal{P}_{1,k+1}$, and these triangles will be basements of pyramids to be constructed in the proof of Proposition 2.5.3.

Lemma 2.5.5. *Let $N \in \mathbb{Z}_+$, and $\varepsilon > 0$ and be small enough, then there exists $C_{\varepsilon,N} > 0$ such that the following statement is true. Suppose we have*

1. *a function $u : \mathbb{Z}^3 \to \mathbb{R}$,*
2. *$n, k \in \mathbb{Z}$, $n > C_{\varepsilon,N}$, $k \in \mathbb{Z} \cap \left[0, \frac{n}{10}\right)$, $a_0 \in \mathcal{P}_{1,k} \cap Q_{\frac{n}{4}}$,*
3. *a vector of positive reals $\vec{l}$, and an $(N, \vec{l}, \varepsilon^{-1}, \varepsilon)$-graded set E with the first scale length $l_1 > C_{\varepsilon,N}$, and E being $(1, 2\varepsilon)$-normal in Q_n,*

4. *$D \in \mathbb{R}_+$, and $0 < g_1, \cdots, g_{100n} < |u(a_0)|$, such that $g_i \leq g_{i+1}\exp(-Dn)$ for each $1 \leq i \leq 100n-1$.*

Then we can find $m \in \mathbb{Z}_+$, $r_1, r_2 \cdots, r_m \in \mathbb{Z} \cap \left[0, \frac{n}{32}\right)$,

$$a_1, a_2, \cdots, a_m \in (\mathcal{P}_{1,k} \cup \mathcal{P}_{1,k+1}) \cap Q_{\frac{n}{2}}$$

and $s_1, s_2, \cdots, s_m \in \{1, 2, \cdots, 100n\}$, satisfying the following conditions:

1. *$\sum_{i=1}^{m}(r_i + 1) \geq \frac{n}{100}$.*

2. *for each $1 \leq i \leq m$, we have $|u(a_i)| \geq \exp(Dn)g_{s_i}$, and $|u(b)| < g_{s_i}$ for any $b \in (\mathring{\mathcal{T}}_{\pi_k(a_i),r_i} \cup \mathring{\mathcal{T}}_{\pi_{k+1}(a_i),r_i}) \cap \mathbb{Z}^3$.*

3. *for any point $a \in \mathcal{P}_{1,k}$, we have $a \in \mathcal{T}_{\pi_k(a_i),r_i}$ for at most two $1 \leq i \leq m$.*

4. *E is $(\varepsilon^{-\frac{1}{2}}, \varepsilon)$-normal in $\mathfrak{T}_{a_i,r_i}$ for each $1 \leq i \leq m$.*

Proof. Denote $R := \left\{a \in (\mathcal{P}_{1,k} \cup \mathcal{P}_{1,k+1}) \cap Q_{\frac{n}{2}} : |u(a)| \geq \exp(Dn)g_1\right\}$. For each $a \in R$, denote

$$I(a) := \max\left\{i \in \{1, \cdots, 100n\} : |u(a)| \geq \exp(Dn)g_i\right\}, \tag{2.5.9}$$

and we let $r(a)$ be the largest integer, such that $0 \leq r(a) < \frac{n}{32}$, and

$$|u(b)| \leq g_{I(a)},\ \forall b \in \left(\mathring{\mathcal{T}}_{\pi_k(a),r(a)} \cup \mathring{\mathcal{T}}_{\pi_{k+1}(a),r(a)}\right) \cap \mathbb{Z}^3. \tag{2.5.10}$$

Suppose $\vec{l} = (l_1, l_2, \cdots, l_d)$. We write $E = \bigcup_{i=0}^{d} E_i$ where E_i is a (N, l_i, ε)-scattered set for $0 < i \leq d$, and E_0 is a ε^{-1}-unitscattered set. We write $E_i = \bigcup_{t=1}^{N} \bigcup_{j\in\mathbb{Z}_+} E_i^{(j,t)}$, where each $E_i^{(j,t)}$ is an open ball, and $\text{dist}(E_i^{(j,t)}, E_i^{(j',t)}) \geq l_i^{1+\varepsilon}$, $\forall j \neq j' \in \mathbb{Z}_+$. We also write $E_0 = \bigcup_{j\in\mathbb{Z}_+} o_j$ where each o_j is an open unit ball, such that $\forall j \neq j' \in \mathbb{Z}_+$ we have $\text{dist}(o_j, o_{j'}) \geq \varepsilon^{-1}$.

If $r(a) \geq \frac{n}{100}$ for any $a \in R$, then Condition 1 to 3 hold by letting $m = 1$, $a_1 = a$, $r_1 = r(a)$ and $s_1 = I(a)$. Now we show that Condition 4 also holds (when $C_{\varepsilon,N}$ is large enough). Since E is $(1, 2\varepsilon)$-normal in Q_n,

$$l_i < 4n^{1-\varepsilon}, \tag{2.5.11}$$

whenever $E_i \cap Q_n \neq \emptyset$. Then since $n > C_{\varepsilon,N}$, by taking $C_{\varepsilon,N}$ large enough we have $n > 300\varepsilon^{-\frac{1}{2}}$, and

$$l_i < 4n^{1-\varepsilon} < r(a)^{1-\frac{\varepsilon}{2}}. \tag{2.5.12}$$

Thus E is $(\varepsilon^{-\frac{1}{2}}, \varepsilon)$-normal in $\mathfrak{T}_{a_1,r_1}$. From now on, we assume $r(a) < \frac{n}{100}$ for each $a \in R$. We also assume that $n > 100$ by letting $C_{\varepsilon,N} > 100$.

For each $0 < i \leq d$, $1 \leq t \leq N$, and $j \in \mathbb{Z}_+$, denote $B_i^{(j,t)}$ to be the open ball with radius $l_i^{1+\frac{2}{3}\varepsilon}$ and the same center as $E_i^{(j,t)}$. Let $\tilde{B}_i^{(j,t)} := B_i^{(j,t)} \cap \mathcal{P}_{1,k}$, which is either a 2D open ball on the plane $\mathcal{P}_{1,k}$, or $\emptyset$. For each $j \in \mathbb{Z}_+$, let B_j be the open ball with radius $\varepsilon^{-\frac{2}{3}}$ and has the same center as o_j. Denote $\tilde{B}_j := B_j \cap \mathcal{P}_{1,k}$.

We define a graph G as follows. The set of vertices of G is

$$V(G) := \left\{\mathcal{T}_{\pi_k(a),r(a)+1} : a \in R\right\}$$
$$\cup \left\{\tilde{B}_i^{(j,t)} : 1 \leq i \leq d, 1 \leq t \leq N, j \in \mathbb{Z}_+, \tilde{B}_i^{(j,t)} \neq \emptyset\right\} \cup \left\{\tilde{B}_j : j \in \mathbb{Z}_+, \tilde{B}_j \neq \emptyset\right\}. \tag{2.5.13}$$

For any $v_1, v_2 \in V(G)$, there is an edge connecting v_1, v_2 if and only if $v_1 \cap v_2 \neq \emptyset$.

Claim 2.5.6. *There is $a_\infty \in R$, such that $\mathcal{T}_{\pi_k(a_0),r(a_0)+1}$ and $\mathcal{T}_{\pi_k(a_\infty),r(a_\infty)+1}$ are in the same connected component in G, and $\left(\mathcal{T}_{\pi_k(a_\infty),r(a_\infty)+1} \cup \mathcal{T}_{\pi_{k+1}(a_\infty),r(a_\infty)+1}\right) \cap \mathbb{Z}^3 \not\subset Q_{\frac{n}{2}}$.*

Proof. We let $b_0 := a_0$. For any $i \in \mathbb{Z}_{\geq 0}$, if $b_i \in R$, we choose

$$b_{i+1} \in \mathbb{Z}^3 \cap \left(\mathring{\mathcal{T}}_{\pi_k(b_i),r(b_i)+1} \cup \mathring{\mathcal{T}}_{\pi_{k+1}(b_i),r(b_i)+1}\right) \setminus \left(\mathring{\mathcal{T}}_{\pi_k(b_i),r(b_i)} \cup \mathring{\mathcal{T}}_{\pi_{k+1}(b_i),r(b_i)}\right), \tag{2.5.14}$$

with the largest $|u(b_{i+1})|$ (choose any one if not unique).

As $b_{i+1} \in \mathbb{Z}^3 \cap \left(\mathring{\mathcal{T}}_{\pi_k(b_i),r(b_i)+1} \cup \mathring{\mathcal{T}}_{\pi_{k+1}(b_i),r(b_i)+1}\right)$, we have that

$$b_{i+1} \cdot (-\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3) \geq b_i \cdot (-\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3) + 1. \tag{2.5.15}$$

By the definition of $r(b_i)$, we have that $|u(b_{i+1})| \geq g_{I(b_i)} \geq \exp(Dn) g_{I(b_i)-1}$, thus $I(b_{i+1}) \geq I(b_i) - 1$.

The construction terminates when we get some $q \in \mathbb{Z}_+$ such that $b_q \notin R$. We let $a_\infty := b_{q-1}$, and we show that it satisfies all the conditions.

From the construction, for each $i = 0, \cdots, q-1$ we have $\pi_k(b_{i+1}) \in \mathring{\mathcal{T}}_{\pi_k(b_i),r(b_i)+1}$, so there is an edge in G connecting $\mathcal{T}_{\pi_k(b_i),r(b_i)+1}$ and $\mathcal{T}_{\pi_k(b_{i+1}),r(b_{i+1})+1}$. This implies that $\mathcal{T}_{\pi_k(b_0),r(b_0)+1}$ and $\mathcal{T}_{\pi_k(b_{q-1}),r(b_{q-1})+1}$ are in the same connected component in G.

If $\left(\mathcal{T}_{\pi_k(b_{q-1}),r(b_{q-1})+1} \cup \mathcal{T}_{\pi_{k+1}(b_{q-1}),r(b_{q-1})+1}\right) \cap \mathbb{Z}^3 \subset Q_{\frac{n}{2}}$, we have $b_q \in Q_{\frac{n}{2}}$. By (2.5.15) we have that $b_q \cdot (-\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3) \geq b_0 \cdot (-\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3) + q$. Since $b_0, b_q \in Q_{\frac{n}{2}}$, we have $q \leq 4n$. This means that $I(b_q) \geq I(b_0) - q \geq 100n - 4n > 1$. Then we have that $b_q \in R$, which contradicts with its construction. This means that $a_\infty = b_{q-1}$ satisfies all the conditions stated in the claim. $\square$

We define a weight on the graph G, by letting each vertex in $\left\{\mathcal{T}_{\pi_k(a),r(a)+1} : a \in R\right\}$ (which are triangles) have weight 2, and each other vertex (which are balls) have weight 1. The weights are defined this way for the purpose of proving Condition 4. We then take a path $\gamma_{path} = \{v_1, v_2, \cdots, v_p\}$ such that $\pi_k(a_0) \in v_1$ and $\pi_k(a_\infty) \in v_p$, and has the least total weight (among all such paths). Then all these vertices are mutually different. For each $i = 1, 2, \cdots, p-1$ there is an edge connecting v_i and v_{i+1}, and these are all the edges in the subgraph induced by these vertices. Note that each v_i is either a ball or a triangle in $\mathcal{P}_{1.k}$. See Figure 2.5 for an illustration.

Suppose all the triangles in γ_{path} are $\left\{\mathcal{T}_{a_i,r(a_i)+1} : 1 \leq i \leq m\right\}$. Let $r_i := r(a_i)$ and $s_i := I(a_i)$. We claim that these a_i, r_i and s_i for $1 \leq i \leq m$ satisfy all the conditions.

Condition 2 follows from the definition of $r_i = r(a_i)$. As γ_{path} is a least weighted path, we have that $v_{i'} \cap v_{i''} = \emptyset$ whenever $|i' - i''| > 1$, thus Condition 3 follows as well.

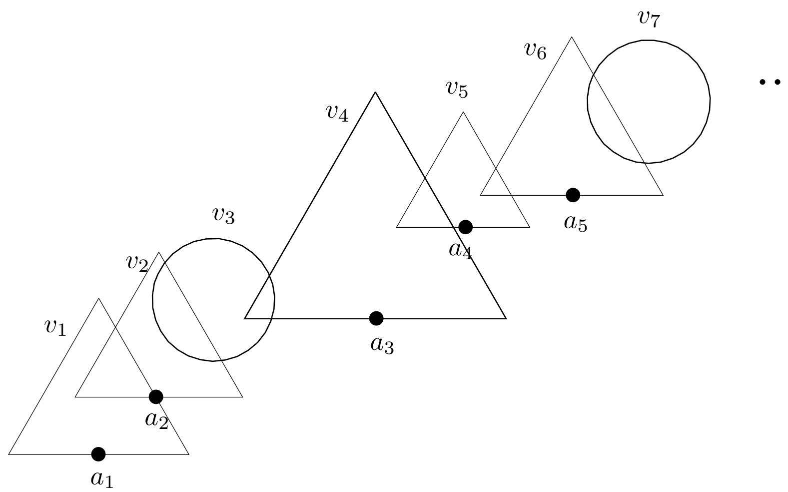

Figure 2.5: The path γ_{path}

We next verify Condition 1. For this, we need to show that in the path, triangles constitute a substantial fraction. This is incorporated in Claims 2.5.7 and 2.5.8 below. Denote $\ell_i := \operatorname{diam}(v_i)$, for each $1 \le i \le p$. As $r(a_\infty) < \frac{n}{100}$, we have $a_\infty \notin Q_{\frac{n}{2}-\frac{n}{20}}$; also note that $a_0 \in Q_{\frac{n}{4}}$, so we have

$$\ell_{total} := \sum_{i=1}^{p} \ell_i \ge \operatorname{dist}(Q_{\frac{n}{4}}, \mathbb{Z}^3 \setminus Q_{\frac{n}{2}-\frac{n}{20}}) \ge \frac{n}{5}. \tag{2.5.16}$$

For each $1 \le i \le d$ and $1 \le t \le N$, denote $\mathcal{V}_{i,t} := \left\{ v \in \gamma_{path} : \exists j \in \mathbb{Z}_+, v = \tilde{B}_i^{(j,t)} \right\}$.

Claim 2.5.7. *If $\mathcal{V}_{i,t} \neq \emptyset$, then $\sum_{i':v_{i'} \in \mathcal{V}_{i,t}} \ell_{i'} \le \ell_{total} l_i^{-\frac{\varepsilon}{4}}$, provided that ε is small enough and $C_{\varepsilon,N}$ is large enough.*

Proof. Since $\mathcal{V}_{i,t} \neq \emptyset$ and E is $(1, 2\varepsilon)$-normal in Q_n, we have $C_{\varepsilon,N} \le l_i \le n^{1-\varepsilon}$.

Case 1: $|\mathcal{V}_{i,t}| = 1$. Suppose $\{v_{i'}\} = \mathcal{V}_{i,t}$. Then by (2.5.16), when $C_{\varepsilon,N}$ is large enough we have

$$\ell_{i'} \leq 2l_i^{1+\frac{2}{3}\varepsilon} \leq \frac{nl_i^{-\frac{\varepsilon}{4}}}{5} \leq \ell_{total} l_i^{-\frac{\varepsilon}{4}}. \tag{2.5.17}$$

Case 2: $|\mathcal{V}_{i,t}| > 1$. Write $\mathcal{V}_{i,t} = \left\{v_{i_1}, v_{i_2}, \cdots, v_{i_q}\right\}$, where $1 \leq i_1 < i_2 < \cdots < i_q \leq p$, and $q \geq 2$. For each $w \in \{1, 2, \cdots, q-1\}$, consider the part of γ_{path} between v_{i_w} and $v_{i_{w+1}}$. By letting $C_{\varepsilon,N}$ large enough we have

$$\sum_{i'=i_w}^{i_{w+1}} \ell_{i'} \geq \operatorname{dist}(v_{i_w}, v_{i_{w+1}}) \geq l_i^{1+\varepsilon} - 2l_i^{1+\frac{2}{3}\varepsilon} \geq 2(\ell_{i_w} + \ell_{i_{w+1}}) l_i^{\frac{\varepsilon}{4}}. \tag{2.5.18}$$

Summing (2.5.18) through all $w \in \{1, 2, \cdots, q-1\}$, we get

$$\ell_{total} \geq \frac{1}{2} \sum_{w \in \{1,2,\cdots,q-1\}} \sum_{i'=i_w}^{i_{w+1}} \ell_{i'} \geq \left(\sum_{v_{i'} \in \mathcal{V}_{i,t}} \ell_{i'} \right) l_i^{\frac{\varepsilon}{4}}. \tag{2.5.19}$$

Then the claim follows as well. $\square$

Let $\mathcal{V}_0 := \left\{ v_{i'} \in \gamma_{path} : \exists j \in \mathbb{Z}_+, v_{i'} = \tilde{B}_j \right\}$.

Claim 2.5.8. *If $\mathcal{V}_0 \neq \emptyset$, then $\sum_{v_{i'} \in \mathcal{V}_0} \ell_{i'} \leq \varepsilon^{\frac{1}{4}} \ell_{total}$, provided that ε is small enough and $C_{\varepsilon,N}$ is large enough.*

This is by the same arguments as the proof of Claim 2.5.7.

From Claim 2.5.7 and Claim 2.5.8, by making ε small and $C_{\varepsilon,N}$ large enough, from

$l_1 > C_{\varepsilon,N}$ and $l_{i+1} \geq l_i^{1+2\varepsilon}$, we have

$$\sum_{i':v_{i'} \text{ is a 2D ball}} \ell_{i'} = \sum_{v_{i'} \in \mathcal{V}_0} \ell_{i'} + \sum_{1 \leq i \leq d, 1 \leq t \leq N} \sum_{v_{i'} \in \mathcal{V}_{i,t}} \ell_{i'} \leq \varepsilon^{\frac{1}{4}} \ell_{total} + N \ell_{total} \sum_{i=1}^{\infty} l_i^{-\frac{\varepsilon}{4}} \leq \frac{\ell_{total}}{100}. \tag{2.5.20}$$

Now we have that

$$\sum_{i=1}^{m} (r_i + 1) \geq (2\sqrt{2})^{-1} \sum_{i':v_{i'} \text{ is a triangle}} \ell_{i'} \geq (2\sqrt{2})^{-1} \frac{99}{100} \ell_{total} > \frac{n}{100}, \tag{2.5.21}$$

where the last inequality is due to (2.5.16). Then Condition 1 follows.

It remains to check Condition 4. We prove by contradiction. Suppose for some $1 \leq i' \leq m$, E is not $(\varepsilon^{-\frac{1}{2}}, \varepsilon)$-normal in $\mathfrak{T}_{a_{i'}, r_{i'}}$. There are only two cases:

Case 1: There exists $1 \leq i \leq d$ and $E_i^{(j,t)}$, such that

$$E_i^{(j,t)} \cap \mathfrak{T}_{a_{i'}, r_{i'}} \neq \emptyset \tag{2.5.22}$$

and

$$l_i > \operatorname{diam}(\mathfrak{T}_{a_{i'}, r_{i'}})^{1-\frac{\varepsilon}{2}}. \tag{2.5.23}$$

Recall that $B_i^{(j,t)}$ is the ball with radius $l_i^{1+\frac{2}{3}\varepsilon}$ and the same center as $E_i^{(j,t)}$. By (2.5.23) and letting $C_{\varepsilon,N}$ large enough, we have

$$\operatorname{radi}(B_i^{(j,t)}) - l_i = l_i^{1+\frac{2}{3}\varepsilon} - l_i > \operatorname{diam}(\mathfrak{T}_{a_{i'}, r_{i'}}) + 3. \tag{2.5.24}$$

This implies that $\mathcal{T}_{\pi_k(a_{i'}),r_{i'}+1} \subset B_i^{(j,t)}$ and $\mathcal{T}_{\pi_k(a_{i'}),r_{i'}+1} \subset \tilde{B}_i^{(j,t)}$. If we substitute $\mathcal{T}_{\pi_k(a_{i'}),r_{i'}+1}$ by $\tilde{B}_i^{(j,t)}$ in the path γ_{path}, then the new path has lower weight than γ_{path}. This contradicts with the fact that γ_{path} is a least weight path.

Case 2: $E_0 \cap \mathfrak{T}_{a_{i'},r_{i'}} \neq \emptyset$ and $\varepsilon^{-\frac{1}{2}} > \mathrm{diam}(\mathfrak{T}_{a_{i'},r_{i'}})$.

Then $\mathcal{T}_{\pi_k(a_{i'}),r_{i'}+1} \subset B_j$ and $\mathcal{T}_{\pi_k(a_{i'}),r_{i'}+1} \subset \tilde{B}_j$ for some $j \in \mathbb{Z}_+$, since $\mathrm{radi}(B_j)-1 = \varepsilon^{-\frac{2}{3}} - 1 > \varepsilon^{-\frac{1}{2}} + 3 > \mathrm{diam}(\mathfrak{T}_{a_{i'},r_{i'}}) + 3$. By the same reason as Case 1, we reach a contradiction. Thus Condition 4 holds and the conclusion follows. $\square$

Now we work on each tetrahedron $\mathfrak{T}_{a_i,r_i}$. We will construct a pyramid in each of them, and show that on the boundary of the pyramid, the number of points b such that $b \notin E$, $|u(b)| \geq \exp(-C_2 n^3)$, is at least in the order of $r_i^2 + 1$.

We start by defining a family of regular tetrahedrons. Recall that in Definition 2.5.4, we have defined the tetrahedron $\mathfrak{T}_{a,r}$ with one face being $\mathcal{T}_{a,r}$.

Definition 2.5.9. Let $a \in \mathbb{Z}^3$, $r \in \mathbb{Z}_+$. For each $b \in \mathfrak{T}_{a,r} \cap \mathbb{Z}^3$, we define a regular tetrahedron $\mathfrak{T}_{a,r,b}$ characterized by the following conditions. Its four faces are orthogonal to $\boldsymbol{\lambda}_1, \overline{\boldsymbol{\lambda}}_2, \overline{\boldsymbol{\lambda}}_3, \overline{\boldsymbol{\lambda}}_4$ respectively. For $\tau \in \{2,3,4\}$, we consider the distances between the faces of $\mathfrak{T}_{a,r}$ and $\mathfrak{T}_{a,r,b}$ that are orthogonal to $\overline{\boldsymbol{\lambda}}_\tau$, and they are the same for each τ. The point b is at the boundary of the face orthogonal to $\boldsymbol{\lambda}_1$. Formally, we denote

$$F_{a,r,b} := \max\left\{F : b \cdot \overline{\boldsymbol{\lambda}}_\tau \leq \mathbf{t}_r(a) \cdot \overline{\boldsymbol{\lambda}}_\tau - F,\ \forall \tau \in \{2,3,4\}\right\}. \tag{2.5.25}$$

Then $F_{a,r,b} \geq 0$ since $b \in \mathfrak{T}_{a,r}$, and $\frac{F_{a,r,b}}{\sqrt{3}}$ would be the distance between the faces of

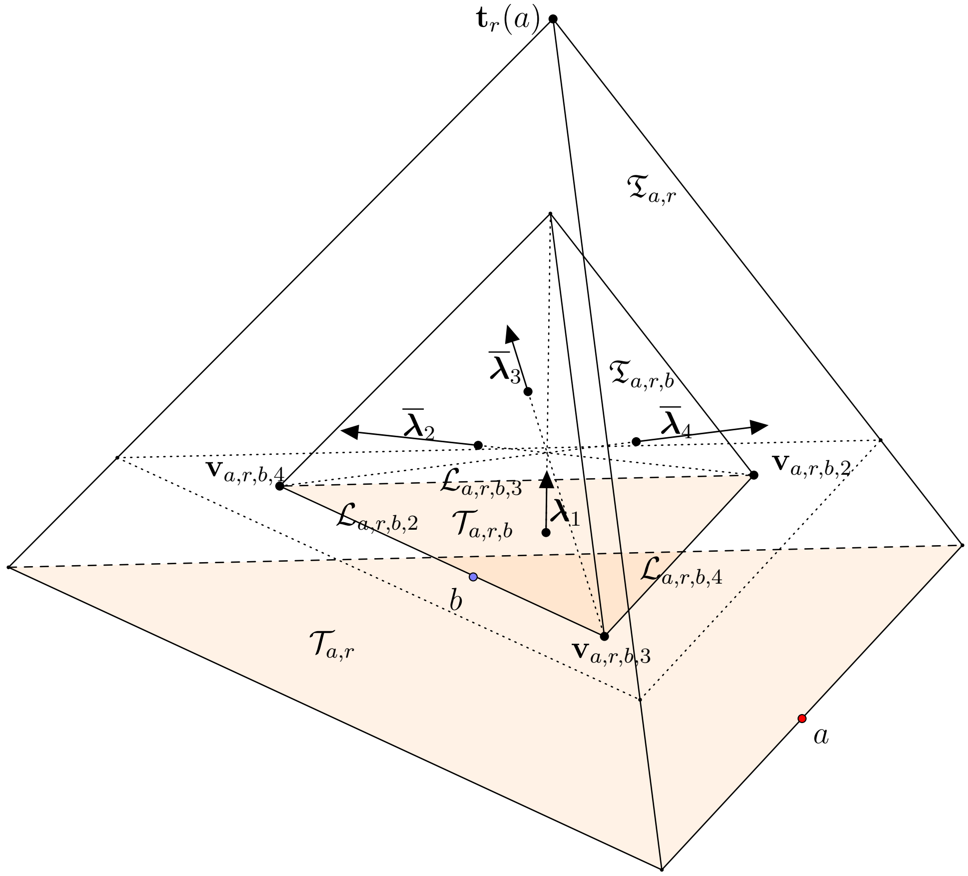

Figure 2.6: An illustration of the constructions in Definition 2.5.4 and 2.5.9. The colored triangles are $\mathcal{T}_{a,r}$ and $\mathcal{T}_{a,r,b}$.

$\mathfrak{T}_{a,r}$ and $\mathfrak{T}_{a,r,b}$ that are orthogonal to $\overline{\boldsymbol{\lambda}}_\tau$, for each $\tau \in \{2,3,4\}$. Define

$$\mathfrak{T}_{a,r,b} := \left\{ c \in \mathbb{R}^3 : c \cdot \boldsymbol{\lambda}_1 \geq b \cdot \boldsymbol{\lambda}_1,\ b \cdot \overline{\boldsymbol{\lambda}}_\tau \leq \mathbf{t}_r(a) \cdot \overline{\boldsymbol{\lambda}}_\tau - F_{a,r,b},\ \forall \tau \in \{2,3,4\} \right\}, \tag{2.5.26}$$

and let $\mathring{\mathfrak{T}}_{a,r,b}$ be the interior of $\mathfrak{T}_{a,r,b}$. We denote $\mathcal{T}_{a,r,b} := \mathfrak{T}_{a,r,b} \cap \mathcal{P}_{1,b\cdot\boldsymbol{\lambda}_1}$ to be the face of $\mathfrak{T}_{a,r,b}$ orthogonal to $\boldsymbol{\lambda}_1$, and we denote its three edges as

$$\mathcal{L}_{a,r,b,\tau} := \left\{ c \in \mathcal{T}_{a,r,b} : c \cdot \overline{\boldsymbol{\lambda}}_\tau = \mathbf{t}_r(a) \cdot \overline{\boldsymbol{\lambda}}_\tau - F_{a,r,b} \right\},\ \forall \tau \in \{2,3,4\}. \tag{2.5.27}$$

Then b is on one of these three edges. We denote the three vertices by

$$\mathbf{v}_{a,r,b,\tau} := \bigcap_{\tau' \in \{2,3,4\} \setminus \{\tau\}} \mathcal{L}_{a,r,b,\tau'},\ \tau \in \{2,3,4\}\,, \tag{2.5.28}$$

or equivalently, $\mathbf{v}_{a,r,b,\tau}$ is the unique point characterized by $\mathbf{v}_{a,r,b,\tau} \cdot \boldsymbol{\lambda}_1 = b \cdot \boldsymbol{\lambda}_1$, and $\mathbf{v}_{a,r,b,\tau} \cdot \overline{\boldsymbol{\lambda}}_{\tau'} = \mathbf{t}_r(a) \cdot \overline{\boldsymbol{\lambda}}_{\tau'} - F_{a,r,b}$ for $\tau' \in \{2,3,4\} \setminus \{\tau\}$. As $b \cdot \boldsymbol{\lambda}_1$ and each $\mathbf{t}_r(a) \cdot \overline{\boldsymbol{\lambda}}_{\tau'} - F_{a,r,b}$ are integers and have the same parity, we have $\mathbf{v}_{a,r,b,\tau} \in \mathbb{Z}^3$. We also denote the interior of these three edges by

$$\mathring{\mathcal{L}}_{a,r,b,\tau} := \mathcal{L}_{a,r,b,\tau} \setminus \{\mathbf{v}_{a,r,b,2}, \mathbf{v}_{a,r,b,3}, \mathbf{v}_{a,r,b,4}\}\,,\ \tau \in \{2,3,4\}\,. \tag{2.5.29}$$

We now define the pyramid using these tetrahedrons.

Definition 2.5.10. Take any $a \in \mathbb{Z}^3$, $r \in \mathbb{Z}_+$. For any $b \in \mathfrak{T}_{a,r} \bigcap \mathbb{Z}^3$ let

$$\mathring{\mathfrak{H}}_{a,r,b} := \left\{c \in \mathbb{R}^3 : c \cdot \boldsymbol{\lambda}_1 > b \cdot \boldsymbol{\lambda}_1\right\} \setminus \mathfrak{T}_{a,r,b}, \tag{2.5.30}$$

which is an open half space minus a regular tetrahedron. Let $\mathfrak{H}_{a,r,b}$ be the closure of $\mathring{\mathfrak{H}}_{a,r,b}$.

Let $\Gamma \subset \mathbb{Z}^3$, such that $a \in \Gamma$ and $\mathring{\mathcal{T}}_{a,r} \cap \Gamma = \emptyset$. We consider the collection of sets $\{\mathfrak{H}_{a,r,b}\}_{b \in \mathfrak{T}_{a,r} \cap \Gamma}$. They form a partially ordered set (POSET) by inclusion of sets. We take all the maximal elements in $\{\mathfrak{H}_{a,r,b}\}_{b \in \mathfrak{T}_{a,r} \cap \Gamma}$, and denote them as $\mathfrak{H}_{a,r,b_1}, \cdots, \mathfrak{H}_{a,r,b_m}$. In particular $\mathfrak{H}_{a,r,a} = \mathfrak{H}_{a,r}$ is maximal since $\mathring{\mathcal{T}}_{a,r} \cap \Gamma = \emptyset$, so we can assume that $b_1 = a$. (For each $2 \le i \le m$, the choice of each $b_i \in \mathfrak{T}_{a,r} \cap \Gamma$ may not be unique, but always gives the same $\mathfrak{H}_{a,r,b_i}$.) We note that since each $\mathfrak{H}_{a,r,b_i}$ is maximal, all the numbers $b_i \cdot \boldsymbol{\lambda}_1$ for $1 \le i \le m$ must be mutually different, so we can assume that $b_1 \cdot \boldsymbol{\lambda}_1 < \cdots < b_m \cdot \boldsymbol{\lambda}_1$.

The *pyramid* is defined as

$$\mathfrak{P}_{a,r,\Gamma} := \mathfrak{T}_{a,r,b_m} \cup \bigcup_{i=1}^{m-1} \left(\mathfrak{T}_{a,r,b_i} \cap \left\{c \in \mathbb{R}^3 : c \cdot \boldsymbol{\lambda}_1 \le b_{i+1} \cdot \boldsymbol{\lambda}_1\right\}\right), \tag{2.5.31}$$

and we let $\mathring{\mathfrak{P}}_{a,r,\Gamma}$ be the interior of $\mathfrak{P}_{a,r,\Gamma}$. Note that in this definition, $\mathfrak{P}_{a,0,\Gamma} := \{a\}$. Finally, let $\partial\mathfrak{P}_{a,r,\Gamma} := \mathfrak{P}_{a,r,\Gamma} \setminus (\mathring{\mathfrak{P}}_{a,r,\Gamma} \cup \mathring{\mathcal{T}}_{a,r})$ be the boundary of the pyramid (without the interior of its basement). See Figure 2.7 for an example of pyramid.

In words, we construct the pyramid $\mathfrak{P}_{a,r,\Gamma}$ by stacking together some "truncated" regular tetrahedrons $\mathfrak{T}_{a,r,b}$, for $b \in \Gamma$, so that $\mathfrak{P}_{a,r,\Gamma}$ intersects Γ only at its boundary. Indeed, for each $b \in \mathfrak{T}_{a,r} \cap \Gamma$ we have $b \in \mathfrak{H}_{a,r,b}$, and $\mathring{\mathfrak{P}}_{a,r,\Gamma} \cap \mathfrak{H}_{a,r,b} = \emptyset$.

Our key step towards proving Proposition 2.5.3 is the following estimate about points on the boundary of a pyramid.

Proposition 2.5.11. *There exists a constant C_9, such that for any $K \in \mathbb{R}_+$, $N \in \mathbb{Z}_+$, and any small enough $\varepsilon \in \mathbb{R}_+$, there are small $C_{10} \in \mathbb{R}_+$ depending only on K and large $C_{\varepsilon,N} \in \mathbb{R}_+$ depending only on ε, N, such that the following statement holds.*

Take any $g \in \mathbb{R}_+$, $n, r \in \mathbb{Z}_+$ with $0 \le r < \frac{n}{32}$, and functions u, V satisfying $\Delta u = Vu$ in Q_n and $\|V\|_\infty \le K$. Suppose we have that

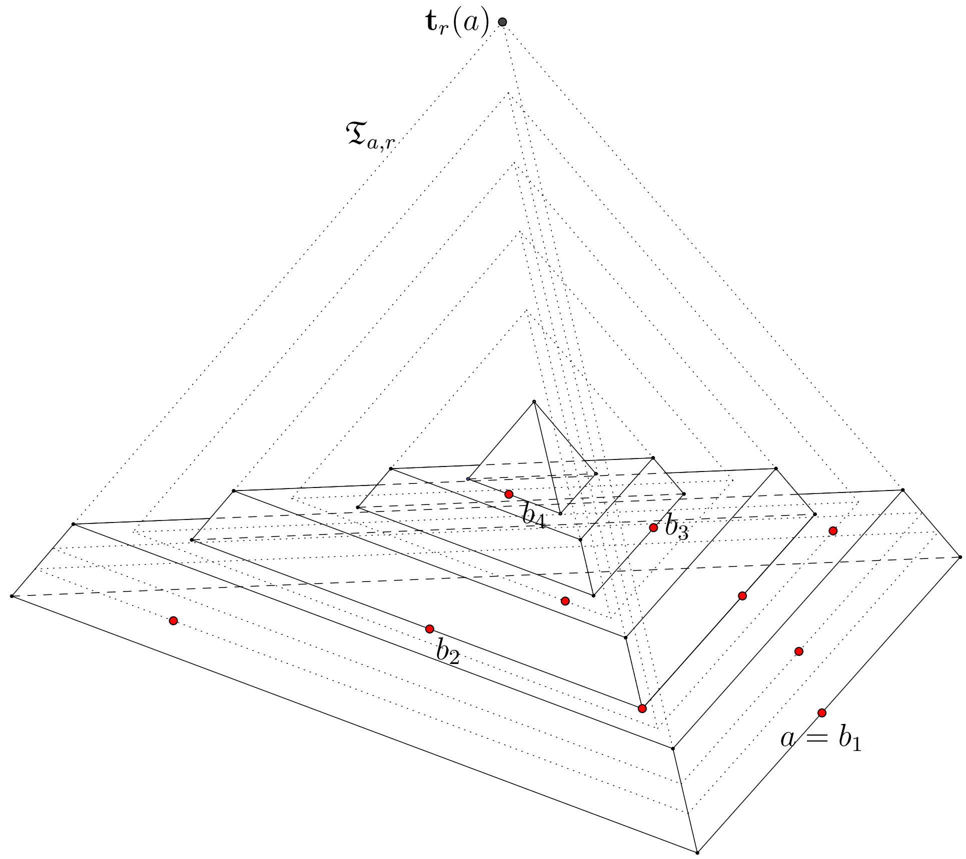

Figure 2.7: Pyramid $\mathfrak{P}_{a,r,\Gamma}$, where Γ is the collection of red points.

1. $\Gamma := \{b \in Q_n : |u(b)| \geq \exp(3C_{10}n)g\}$, *and* $a \in \Gamma \cap Q_{\frac{n}{2}}$*;*

2. $|u(b)| < g$ *for each* $b \in \mathring{\mathcal{T}}_{a,r} \cap \mathbb{Z}^3$, *and either* $|u(b)| < g$ *for each* $b \in \mathring{\mathcal{T}}_{a-\frac{\boldsymbol{\lambda}_1}{3},r} \cap \mathbb{Z}^3$ *or* $|u(b)| < g$ *for each* $b \in \mathring{\mathcal{T}}_{a+\frac{\boldsymbol{\lambda}_1}{3},r} \cap \mathbb{Z}^3$*;*

3. $\vec{l}$ *is a vector of positive reals,* E *is an* $(N, \vec{l}, \varepsilon^{-1}, \varepsilon)$*-graded set; in addition, the first scale length of* E *is* $l_1 > C_{\varepsilon,N}$*, and* E *is* $(\varepsilon^{-\frac{1}{2}}, \varepsilon)$*-normal in* $\mathfrak{T}_{a,r}$*;*

4. *for each* $b \in Q_n$ *with* $b \cdot \boldsymbol{\lambda}_1 \geq a \cdot \boldsymbol{\lambda}_1$*,* $g \leq |u(b)| \leq \exp(3C_{10}n)g$ *implies* $b \in E$*.*

Then

$$\left|\left\{b \in \partial\mathfrak{P}_{a,r,\Gamma} \cap \mathbb{Z}^3 : |u(b)| \geq \exp(C_{10}n)g\right\} \setminus E\right| \geq C_9(r^2+1). \tag{2.5.32}$$

The proof of Proposition 2.5.11 is left for the next subsection. We now finish the proof of Proposition 2.5.3 assuming it.

Proof of Proposition 2.5.3. The idea is to first apply Lemma 2.5.5 to find some triangles $\mathcal{T}_{a_i,r_i}$ in $\mathcal{P}_{1,k} \cup \mathcal{P}_{1,k+1}$, and build pyramids using these triangles as basements, then apply Proposition 2.5.11 to lower bound the number of desired points on the boundary of each pyramid and finally sum them up.

For the parameters, we take $C_7 = \max\{6C_{10}, \log(K+11)\}$ where C_{10} is the constant in Proposition 2.5.11. We leave C_8 to be determined. We require that ε is small as required by Lemma 2.5.5 and Proposition 2.5.11; and for each such ε we let $C_{\varepsilon,N}$ be large enough as required by Lemma 2.5.5 and Proposition 2.5.11.

Without loss of generality, we assume $\tau = 1$. We can also assume $n > 100$, by letting $C_{\varepsilon,N} > 100$. Denote

$$\Upsilon := \left\{a \in Q_n : |u(a)| \geq \exp(-C_7 n^3)|u(a_0)|, a \cdot \boldsymbol{\lambda}_1 \geq k\right\} \setminus E. \tag{2.5.33}$$

If $|\Upsilon| \geq n^2$, the conclusion follows by letting $h = 3n$ and $C_8 < \frac{1}{3}$. Now we assume that $|\Upsilon| < n^2$.

The interval $[\exp(-C_7 n^3)|u(a_0)|, |u(a_0)|)$ is the union of $2n^2$ disjoint intervals, which are

$$\left[\exp\left(-\frac{C_7(i+1)n}{2}\right)|u(a_0)|, \exp\left(-\frac{C_7 in}{2}\right)|u(a_0)|\right), \; i = 0, \cdots, 2n^2-1. \quad (2.5.34)$$

By the Pigeonhole principle, at least n^2 of these intervals do not intersect the set $\{|u(a)| : a \in \Upsilon\}$; i.e., we can find $\exp(-C_7 n^3)|u(a_0)| \leq g_1, \cdots, g_{n^2} \leq |u(a_0)|$, such that $g_i \leq g_{i+1} \exp\left(-\frac{C_7 n}{2}\right)$, for each $1 \leq i \leq n^2 - 1$, and

$$\left\{a \in Q_n : |u(a)| \in \bigcup_{i=1}^{n^2} \left[g_i, g_i \exp\left(\frac{C_7 n}{2}\right)\right), a \cdot \boldsymbol{\lambda}_1 \geq k\right\} \subset E. \quad (2.5.35)$$

We remark that actually we just need $g_1, \cdots, g_{100n}$ to apply Lemma 2.5.5, rather than n^2 numbers; but we cannot get a better quantitative lower bound for $|u|$ by optimizing this, since applying the Pigeonhole principle to $2n^2$ parts or $n^2 + 100n$ parts does not make any essential difference.

As we assume that $a_0 \in \mathcal{P}_{1,k} \cap Q_{\frac{n}{4}}$ and $0 \leq k \leq \frac{n}{10}$, we can apply Lemma 2.5.5 with $D = \frac{C_7}{2}$. Then we can find some $a_1, \cdots, a_m$, $r_1, \cdots, r_m$ and $g_{s_1}, \cdots, g_{s_m}$, satisfying the conditions there. In particular, we have $|u(a_i)| \geq g_{s_i} \exp\left(\frac{C_7 n}{2}\right) > \exp(-C_7 n^3)|u(a_0)|$, for each $1 \leq i \leq m$.

If $m > n$, we can just take $h = 2$, and (2.5.6) holds by taking C_8 small. Now assume that $m \leq n$. We argue by contradiction, assuming that (2.5.6) does not hold.

As $C_7 \geq 6C_{10}$, we can apply Proposition 2.5.11 to $a = a_i$, $r = r_i$ and $g = g_{s_i}$ for each $i = 1, 2, \cdots, m$, and get that

$$|\Upsilon \cap \mathfrak{T}_{a_i,r_i}| \geq \left|\left\{b \in \mathfrak{T}_{a_i,r_i} \cap \mathbb{Z}^3 : |u(b)| \geq \exp(C_{10}n)g_{s_i}\right\} \setminus E\right| \geq C_9(r_i^2 + 1). \quad (2.5.36)$$

As we have assumed that (2.5.6) does not hold, for each $h \in \mathbb{Z}_+$,

$$\begin{aligned} C_9 \sum_{i=1}^{m} \mathbb{1}_{h>4r_i}(r_i^2 + 1) &\leq \sum_{i=1}^{m} \mathbb{1}_{h>4r_i} |\Upsilon \cap \mathfrak{T}_{a_i,r_i}| \\ &\leq 2\left|\left(\bigcup_{i=0}^{h} \mathcal{P}_{1,k+i}\right) \cap \Upsilon\right| \leq 2C_8 hn(\log_2 n)^{-1} \end{aligned} \quad (2.5.37)$$

where the second inequality is due to the fact that any point is contained in at most two tetrahedrons $\mathfrak{T}_{a_i,r_i}$, by Conclusion 3 in Lemma 2.5.5.

Take $l := \lfloor \log_2 n \rfloor - 5$. For each $0 \leq j \leq l$, let

$$M_j = |\left\{i : 1 \leq i \leq m, 2^j \leq r_i + 1 < 2^{j+1}\right\}|.$$

Then we have that

$$\sum_{j=0}^{l} 2^j M_j \geq \frac{1}{2}\sum_{i=1}^{m}(r_i + 1) \geq \frac{n}{200}, \quad (2.5.38)$$

by Lemma 2.5.5. For each $0 \leq s \leq l$, by taking $h = 2^{s+3}$ in equation (2.5.37) we get

$$C_9 \sum_{j=0}^{s} 2^{2j} M_j \leq C_8 2^{s+4} n(l + 5)^{-1}. \quad (2.5.39)$$

Multiplying both sides of (2.5.39) by 2^{-s} and summing over all $s \in \mathbb{Z}_{\geq 0}$, we get

$$\sum_{j=0}^{l} 2^j M_j \leq \sum_{s=0}^{l}\sum_{j=0}^{s} 2^{2j-s} M_j \leq \sum_{s=0}^{l} 2^4 C_8 (C_9)^{-1} n(l+5)^{-1} < 2^4 C_8 (C_9)^{-1} n. \quad (2.5.40)$$

This contradicts with (2.5.38) whenever $C_8 < (200 \cdot 2^4)^{-1} C_9$. $\square$

2.5.2 Multi-layer structure of the pyramid and estimates on the boundary

The purpose of this subsection is to prove Proposition 2.5.11. We first show that, under slightly different conditions, there are many points in Γ on the boundary of a pyramid without removing the graded set.

Proposition 2.5.12. *There exists a constant C_9', so that for any $K \in \mathbb{R}_+$, there is $C_{10} > K + 11$, relying only on K, and the following is true.*

Take any $g \in \mathbb{R}_+$, $n, r \in \mathbb{Z}$ with $0 \leq r < \frac{n}{32}$, and functions u, V satisfying $\Delta u = Vu$ in Q_n and $\|V\|_\infty \leq K$. Suppose we have

1. *$\Gamma := \{b \in Q_n : |u(b)| \geq \exp(3C_{10}n)g\}$, and $a \in \Gamma \cap Q_{\frac{n}{2}}$;*

2. *$|u(b)| < g$ for each $b \in \mathring{\mathcal{T}}_{a,r} \cap \mathbb{Z}^3$, and either $|u(b)| < g$ for each $b \in \mathring{\mathcal{T}}_{a-\frac{\boldsymbol{\lambda}_1}{3},r} \cap \mathbb{Z}^3$ or for each $b \in \mathring{\mathcal{T}}_{a+\frac{\boldsymbol{\lambda}_1}{3},r} \cap \mathbb{Z}^3$;*

3. *$h := \max\{a \cdot \boldsymbol{\lambda}_1\} \cup \left\{b \cdot \boldsymbol{\lambda}_1 : b \in \mathring{\mathfrak{P}}_{a,r,\Gamma} \cap \mathbb{Z}^3, |\mathcal{L}_{a,r,b,2} \cap \mathbb{Z}^3| \geq \frac{r}{4}\right\}$, and $|u(b)| \leq \exp(C_{10}n)g$ for each $b \in \mathring{\mathfrak{P}}_{a,r,\Gamma} \cap \mathbb{Z}^3$ with $b \cdot \boldsymbol{\lambda}_1 \leq h$.*

Then

$$|\{b \in \partial\mathfrak{P}_{a,r,\Gamma} : |u(b)| \geq \exp(C_{10}n)g\}| \geq C_9'(r^2+1). \tag{2.5.41}$$

To prove Proposition 2.5.12, we analyze the structure of the pyramid boundary $\partial\mathfrak{P}_{a,r,\Gamma}$. Specifically, we study faces of it and estimate the number of lattice points b with $|u(b)| \geq \exp(C_{10}n)g$ on each face. For some of the faces, we can show that the number of such points is proportional to the area of the face. This is by observing that the lattice $\mathbb{Z}^3$ restricted to the face is a triangular lattice, and then using results from Section 2.4. Finally we sum up the points on all the faces and get the conclusion.

Proof of Proposition 2.5.12. We can assume that $r > 100$, since otherwise the statement holds by taking $C_9' < 10^{-5}$.

We take $a = b_1, \cdots, b_m$ from the definition of $\mathfrak{P}_{a,r,\Gamma}$. As $\mathfrak{H}_{a,r,b_1}, \cdots, \mathfrak{H}_{a,r,b_m}$ are all the maximal elements in $\{\mathfrak{H}_{a,r,b}\}_{b\in\mathfrak{T}_{a,r}\cap\Gamma}$, we have that

$$\bigcup_{b\in\mathfrak{T}_{a,r}\cap\Gamma} \mathfrak{H}_{a,r,b} = \bigcup_{i=1}^{m} \mathfrak{H}_{a,r,b_i}. \tag{2.5.42}$$

We can also characterize $\mathring{\mathfrak{P}}_{a,r,\Gamma}$ as the half space $\{b \in \mathbb{R}^3 : b \cdot \boldsymbol{\lambda}_1 > a \cdot \boldsymbol{\lambda}_1\}$ minus $\bigcup_{i=1}^{m} \mathfrak{H}_{a,r,b_i}$.

For each $s \in \mathbb{Z}$, we take $m_s \in \{1, \cdots, m\}$ to be the maximum number such that $b_{m_s} \cdot \boldsymbol{\lambda}_1 \leq s$.

We first study the faces of $\partial\mathfrak{P}_{a,r,\Gamma}$ that are orthogonal to $\boldsymbol{\lambda}_1$. For $2 \leq i \leq m$, we denote

$$\mathring{\mathcal{R}}_i := \mathring{\mathfrak{T}}_{a,r,b_{i-1}} \cap \mathcal{P}_{1,b_i\cdot\boldsymbol{\lambda}_1} = \left\{b \in \mathcal{P}_{1,b_i\cdot\boldsymbol{\lambda}_1} : b\cdot\overline{\boldsymbol{\lambda}}_\tau < \mathbf{t}_r(a)\cdot\overline{\boldsymbol{\lambda}}_\tau - F_{a,r,b_{i-1}},\ \forall\tau\in\{2,3,4\}\right\}. \tag{2.5.43}$$

Let $\mathcal{R}_i$ be the closure of $\mathring{\mathcal{R}}_i$, then $\mathcal{R}_i \supset \mathcal{T}_{a,r,b_i}$ and it has the same center as $\mathcal{T}_{a,r,b_i}$. We denote the side length of $\mathcal{R}_i$ to be θ_i. Note that the three vertices of $\mathcal{R}_i$ are in $\frac{1}{2}\mathbb{Z}^3$, so $\frac{\theta_i}{\sqrt{2}} \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Further, for each $1 \leq i \leq m+1$, we denote the side length of $\mathcal{T}_{a,r,b_i}$ to be ϑ_i. Note that the vertices of $\mathcal{T}_{a,r,b_i}$ are $\mathbf{v}_{a,r,b_i,\tau}$, for $\tau \in \{2,3,4\}$, and each of them is in $\mathbb{Z}^3$. Thus we have $\frac{\vartheta_i}{\sqrt{2}} = |\mathcal{L}_{a,r,b_i,2} \cap \mathbb{Z}^3| - 1 \in \mathbb{Z}_{\geq 0}$. We also obviously have that $2\sqrt{2}r = \vartheta_1 > \theta_2 > \vartheta_2 > \cdots > \theta_m > \vartheta_m \geq 0$. For simplicity of notations, we also denote $b_{m+1} := \operatorname{argmax}_{b\in\mathfrak{P}_{a,r,\Gamma}} b\cdot\boldsymbol{\lambda}_1$, and $\theta_{m+1} = \vartheta_{m+1} = 0$.

The following results will be useful in analyzing the face $\mathcal{R}_i$, for $1 \leq i \leq m_{h+1}$.

Claim 2.5.13. *For any $2 \leq i \leq m_{h+1}$ and $b \in \mathring{\mathcal{R}}_i \cap \mathbb{Z}^3$, if $b+\mathbf{e}_1-\mathbf{e}_3, b+\mathbf{e}_2-\mathbf{e}_3 \in \mathring{\mathcal{R}}_i$, then we have*

$$|u(c)| < \exp(C_{10}n)g,\ \forall c \in \{b-\mathbf{e}_3, b-\mathbf{e}_1-\mathbf{e}_3, b-\mathbf{e}_2-\mathbf{e}_3, b-2\mathbf{e}_3\}. \tag{2.5.44}$$

Claim 2.5.14. *If $C_{10} > K+11$, then for each $2 \leq i \leq m_h$, there exists $\tau_i \in \{2,3,4\}$, such that $b_i \in \mathcal{L}_{a,r,b_i,\tau_i}$, and $|u(b)| \geq \exp(2C_{10}n)g$, $\forall b \in \mathring{\mathcal{L}}_{a,r,b_i,\tau_i} \cap \mathbb{Z}^3$.*

We continue our proof assuming these claims. Fix $2 \leq i \leq m_{h+1}$. For any $b \in \mathring{\mathcal{R}}_i \cap \mathbb{Z}^3$ with $b+\mathbf{e}_1-\mathbf{e}_3, b+\mathbf{e}_2-\mathbf{e}_3 \in \mathring{\mathcal{R}}_i$, since $\Delta u(b-\mathbf{e}_3) = (Vu)(b-\mathbf{e}_3)$, and

$|V(b-\mathbf{e}_3)| \le K$, by Claim 2.5.13 we have

$$
\begin{aligned}
&|u(b)+u(b+\mathbf{e}_1-\mathbf{e}_3)+u(b+\mathbf{e}_2-\mathbf{e}_3)| \\
&\qquad \le (K+6) \max_{c\in\{b-\mathbf{e}_3,b-\mathbf{e}_1-\mathbf{e}_3,b-\mathbf{e}_2-\mathbf{e}_3,b-2\mathbf{e}_3\}} |u(c)| \le (K+6)\exp(C_{10}n)g. \quad (2.5.45)
\end{aligned}
$$

We take $C_{10} > 2\ln(C_4(K+6))$ where C_4 is the constant in Theorem 2.1.9. Then if $i \le m_h$, using Claim 2.5.14 and $b_i \in \Gamma$, we have

$$
|u(b)+u(b+\mathbf{e}_1-\mathbf{e}_3)+u(b+\mathbf{e}_2-\mathbf{e}_3)| < C_4^{-2n} \min_{c\in\left(\mathring{\mathcal{L}}_{a,r,b_i,\tau_i}\cap\mathbb{Z}^3\right)\cup\{b_i\}} |u(c)|, \quad (2.5.46)
$$

where $\tau_i \in \{2,3,4\}$ is given by Claim 2.5.14. If $m_h < m_{h+1}$ and $i = m_{h+1}$, as $b_i \in \Gamma$ we have

$$
|u(b)+u(b+\mathbf{e}_1-\mathbf{e}_3)+u(b+\mathbf{e}_2-\mathbf{e}_3)| < C_4^{-2n}|u(b_i)|. \quad (2.5.47)
$$

Without loss of generality, we assume that $\tau_i = 2$ in the former case, and $b_i \in \mathcal{L}_{a,r,b_{m_{h+1}},2}$ in the later case. We consider the following trapezoid in $\mathcal{R}_i$:

$$
\mathring{\mathcal{W}}_i := \left\{b \in \mathring{\mathcal{R}}_i : b\cdot\overline{\boldsymbol{\lambda}}_2 \ge b_i\cdot\overline{\boldsymbol{\lambda}}_2\right\}, \quad (2.5.48)
$$

and let $\mathcal{W}_i$ be the closure of $\mathring{\mathcal{W}}_i$. See Figure 2.8 for an illustration of $\mathcal{W}_i$. Then $\mathring{\mathcal{W}}_i\cap\mathbb{Z}^3$ can be treated as $P_{\mathbf{0};\frac{\vartheta_i}{\sqrt{2}}+2\left\lceil\frac{\theta_i-\vartheta_i}{3\sqrt{2}}\right\rceil-2,\left\lceil\frac{\theta_i-\vartheta_i}{3\sqrt{2}}\right\rceil-1}$ (see Definition 2.4.1). We apply Corollary 2.4.7 to $\mathring{\mathcal{W}}_i\cap\mathbb{Z}^3$, with $L = \mathring{\mathcal{L}}_{a,r,b_i,2}\cap\mathbb{Z}^3$ if $\vartheta_i \ge 2\sqrt{2}$ (thus $\mathring{\mathcal{L}}_{a,r,b_i,2}\cap\mathbb{Z}^3$ is not empty) and $i \le m_h$, and with $L = \{b_i\}$ otherwise. If $i \le m_h$, we have $\frac{\epsilon_2(\theta_i-\vartheta_i)^2}{(3\sqrt{2})^2} \ge \frac{\epsilon_2(\theta_i+2\vartheta_i)(\theta_i-\vartheta_i)}{5\cdot 3\sqrt{2}\cdot 3\sqrt{2}}$

when $\vartheta_i = \sqrt{2}$ or 0, since $\theta_i - \vartheta_i \geq \frac{\sqrt{2}}{2}$. Thus we always have

$$\left|\left\{b \in \mathring{\mathcal{W}}_i \cap \mathbb{Z}^3 : |u(b)| \geq C_4^{-\frac{2(\theta_i - \vartheta_i)}{3\sqrt{2}}} \min_{c \in \left(\mathring{\mathcal{L}}_{a,r,b_i,2} \cap \mathbb{Z}^3\right) \cup \{b_i\}} |u(c)|\right\}\right| \geq \frac{\epsilon_2(\theta_i + 2\vartheta_i)(\theta_i - \vartheta_i)}{5 \cdot 3\sqrt{2} \cdot 3\sqrt{2}}. \quad (2.5.49)$$

Since $\frac{\theta_i - \vartheta_i}{3\sqrt{2}} < n$, and $C_4^{-2n} \min_{c \in \left(\mathring{\mathcal{L}}_{a,r,b_i,2} \cap \mathbb{Z}^3\right) \cup \{b_i\}} |u(c)| \geq \exp(C_{10} n) g$ by Claim 2.5.14, we have

$$\left|\left\{b \in \mathring{\mathcal{W}}_i \cap \mathbb{Z}^3 : |u(b)| \geq \exp(C_{10} n) g\right\}\right| \geq \frac{\epsilon_2(\theta_i + 2\vartheta_i)(\theta_i - \vartheta_i)}{5 \cdot 3\sqrt{2} \cdot 3\sqrt{2}} \geq \frac{\epsilon_2 \theta_i(\theta_i - \vartheta_i)}{5 \cdot 3\sqrt{2} \cdot 3\sqrt{2}}. \quad (2.5.50)$$

If $m_h < m_{h+1}$ and $i = m_{h+1}$, we have

$$\left|\left\{b \in \mathring{\mathcal{W}}_i \cap \mathbb{Z}^3 : |u(b)| \geq C_4^{-\frac{2(\theta_i - \vartheta_i)}{3\sqrt{2}}} |u(b_i)|\right\}\right| \geq \frac{\epsilon_2(\theta_i - \vartheta_i)^2}{(3\sqrt{2})^2}. \quad (2.5.51)$$

Since $\frac{\theta_i - \vartheta_i}{3\sqrt{2}} < n$, and $C_4^{-2n} |u(b_i)| \geq \exp(C_{10} n) g$, we have

$$\left|\left\{b \in \mathring{\mathcal{W}}_i \cap \mathbb{Z}^3 : |u(b)| \geq \exp(C_{10} n) g\right\}\right| \geq \frac{\epsilon_2(\theta_i - \vartheta_i)^2}{(3\sqrt{2})^2}. \quad (2.5.52)$$

For the cases where $\tau_i = 3, 4$, $i \leq m_h$, and the cases where $m_h < m_{h+1} = i$ and $b_i \in \mathcal{L}_{a,r,b_{m_{h+1}},3}$ or $\mathcal{L}_{a,r,b_{m_{h+1}},4}$, we can argue similarly and get (2.5.50) and (2.5.52) as well.

We then study other faces of $\mathfrak{P}_{a,r,\Gamma}$. Again we fix $2 \leq i \leq m_h$, and assume that

$\tau_i = 2$, for τ_i given by Claim 2.5.14. We define

$$\hat{\mathcal{S}}_i :=$$

$$\left\{ b \in \mathcal{P}_{2, b_i \cdot \boldsymbol{\lambda}_2} : b \cdot \overline{\boldsymbol{\lambda}}_\tau < \mathbf{t}_r(a) \cdot \overline{\boldsymbol{\lambda}}_\tau - F_{a,r,b_i}, \forall \tau \in \{3,4\},\ b_i \cdot \boldsymbol{\lambda}_1 \leq b \cdot \boldsymbol{\lambda}_1 < b_{i+1} \cdot \boldsymbol{\lambda}_1 \right\}. \tag{2.5.53}$$

Let $\mathring{\hat{\mathcal{S}}}_i := \left\{ b \in \hat{\mathcal{S}}_i : b \cdot \boldsymbol{\lambda}_1 < h + 1 \right\}$, and $\mathcal{S}_i$ be the closure of $\mathring{\hat{\mathcal{S}}}_i$. Then $\mathcal{S}_i \subset \mathcal{P}_{2, \boldsymbol{\lambda}_2 \cdot b_i}$ and is a trapezoid. It is a face of $\partial \mathfrak{P}_{a,r,\Gamma}$, for $2 \leq i < m_h$, and for $i = m_h$ when $m_{h+1} > m_h$; and it is part of a face of $\partial \mathfrak{P}_{a,r,\Gamma}$ for $i = m_h$ when $m_{h+1} = m_h$. See Figure 2.8 for an illustration.

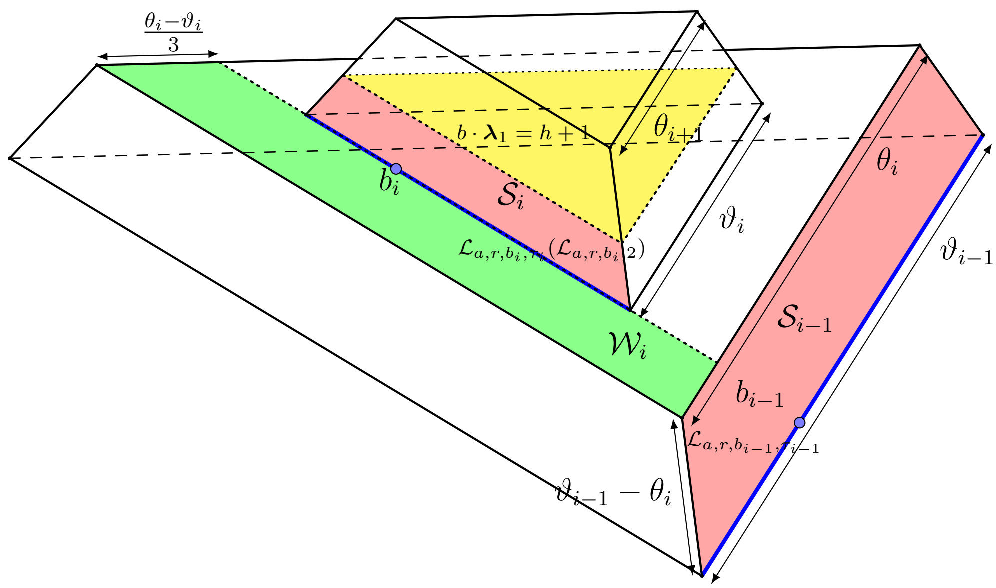

Figure 2.8: Faces $\mathcal{S}_i$, $\mathcal{W}_i$, and $\mathcal{S}_{i-1}$, in the pyramid boundary $\partial \mathfrak{P}_{a,r,\Gamma}$. The yellow triangle is the intersection of $\mathfrak{P}_{a,r,\Gamma}$ with the plane $b \cdot \boldsymbol{\lambda}_1 = h + 1$, and the blue lines are $\mathcal{L}_{a,r,b_i,\tau_i}$ and $\mathcal{L}_{a,r,b_{i-1},\tau_{i-1}}$.

Claim 2.5.15. *For $b \in \mathring{\mathcal{S}}_i \cap \mathbb{Z}^3$, if $b + \mathbf{e}_1 + \mathbf{e}_2, b + \mathbf{e}_1 + \mathbf{e}_3 \in \mathring{\mathcal{S}}_i$, then*

$$|u(c)| < \exp(C_{10}n)g, \ \forall c \in \{b + \mathbf{e}_1, b + \mathbf{e}_1 - \mathbf{e}_2, b + \mathbf{e}_1 - \mathbf{e}_3, b + 2\mathbf{e}_1\}. \tag{2.5.54}$$

We leave the proof of this claim for later as well. By Claim 2.5.15, and arguing as for (2.5.46) above, we conclude that $\forall b \in \mathring{\mathcal{S}}_i \cap \mathbb{Z}^3$ with $b + \mathbf{e}_1 + \mathbf{e}_2, b + \mathbf{e}_1 + \mathbf{e}_3 \in \mathring{\mathcal{S}}_i$,

$$|u(b) + u(b + \mathbf{e}_1 + \mathbf{e}_2) + u(b + \mathbf{e}_1 + \mathbf{e}_3)| < C_4^{-2n} \min_{c \in \left(\mathring{\mathcal{L}}_{a,r,b_i,2} \cap \mathbb{Z}^3\right) \cup \{b_i\}} |u(c)|. \tag{2.5.55}$$

Let's first assume that $\mathring{\mathcal{S}}_i \cap \mathbb{Z}^3 \neq \emptyset$. Then we have $\mathring{\mathcal{L}}_{a,r,b_i,2} \cap \mathbb{Z}^3 \neq \emptyset$, and $\vartheta_i \geq 2\sqrt{2}$. If $i < m_{h+1}$, then $b_{i+1} \cdot \boldsymbol{\lambda}_1 \leq h + 1$, so we treat $\mathring{\mathcal{S}}_i \cap \mathbb{Z}^3$ as $P^r_{\mathbf{0}; \frac{\vartheta_i}{\sqrt{2}} - 2, \left\lceil \frac{\vartheta_i - \theta_{i+1}}{\sqrt{2}} \right\rceil - 1}$ (from Definition 2.4.8), and $\mathcal{L}_{a,r,b_i,2} \cap \mathbb{Z}^3$ is its upper edge. If $i = m_h = m_{h+1} \geq 2$, then $b_{i+1} \cdot \boldsymbol{\lambda}_1 > h + 1$, and we treat $\mathring{\mathcal{S}}_i \cap \mathbb{Z}^3$ as $P^r_{\mathbf{0}; \frac{\vartheta_i}{\sqrt{2}} - 2, \left\lceil \frac{\vartheta_i - \theta_{i+1}}{\sqrt{2}} - \frac{b_{i+1} \cdot \boldsymbol{\lambda}_1 - (h+1)}{2} \right\rceil - 1}$. We apply Corollary 2.4.9 to the trapezoid, with $L = \mathring{\mathcal{L}}_{a,r,b_i,2} \cap \mathbb{Z}^3$ if it is not empty; similar to the study of $\mathcal{W}_i$, we conclude that

$$\left|\left\{b \in \mathcal{S}_i \cap \mathbb{Z}^3 : |u(b)| > \exp(C_{10}n)g\right\}\right| > \frac{\epsilon_3 \vartheta_i (\vartheta_i - \theta_{i+1})}{\sqrt{2} \cdot \sqrt{2}}, \tag{2.5.56}$$

if $2 \leq i < m_{h+1}$, and

$$\left|\left\{b \in \mathcal{S}_i \cap \mathbb{Z}^3 : |u(b)| > \exp(C_{10}n)g\right\}\right| > \frac{\epsilon_3 \vartheta_i}{\sqrt{2}} \left(\frac{\vartheta_i - \theta_{i+1}}{\sqrt{2}} - \frac{b_{i+1} \cdot \boldsymbol{\lambda}_1 - (h+1)}{2} \right), \tag{2.5.57}$$

if $i = m_h = m_{h+1} \geq 2$. In the case where $\mathring{\mathcal{S}}_i \cap \mathbb{Z}^3 = \emptyset$, we have $\vartheta_i \leq \sqrt{2}$, and these inequalities still hold, since $b_i \in \mathcal{S}_i \cap \mathbb{Z}^3$ and $|u(b_i)| > \exp(C_{10}n)g$.

When $\tau_i = 3, 4$, we can define $\mathcal{S}_i$ analogously, and obtain (2.5.56) and (2.5.57) as well.

In addition, we consider

$$\hat{\mathcal{S}}_1 := \left\{ b \in \mathcal{P}_{4,a\cdot\boldsymbol{\lambda}_4} : b \cdot \overline{\boldsymbol{\lambda}}_\tau < \mathbf{t}_r(a) \cdot \overline{\boldsymbol{\lambda}}_2,\ \forall \tau \in \{2,3\},\ a \cdot \boldsymbol{\lambda}_1 \le b \cdot \boldsymbol{\lambda}_1 < b_2 \cdot \boldsymbol{\lambda}_1 \right\}, \tag{2.5.58}$$

and let $\mathring{\mathcal{S}}_1 := \left\{ b \in \hat{\mathcal{S}}_1 : b \cdot \boldsymbol{\lambda}_1 < h+1 \right\}$, and $\mathcal{S}_1$ be the closure of $\mathring{\mathcal{S}}_1$. We treat $\mathcal{S}_1$ differently (from $\mathcal{S}_i$ for $2 \le i \le m_h$) because Claim 2.5.14 cannot be extended to $i = 1$. Also note that by taking $\mathcal{S}_1 \subset \mathcal{P}_{4,a\cdot\boldsymbol{\lambda}_4}$, $\mathcal{S}_1$ is defined as (possibly part of) the face in $\partial\mathfrak{P}_{a,r,\Gamma}$ that contains $a = b_1$.

Using arguments similar to those for $\mathcal{S}_i$, $2 \le i \le m_h$, we treat $\mathring{\mathcal{S}}_1 \cap \mathbb{Z}^3$ as $P^r_{\mathbf{0};\frac{\vartheta_1}{\sqrt{2}}-2,\left\lceil\frac{\vartheta_1-\theta_2}{\sqrt{2}}\right\rceil-1}$ if $m_{h+1} > 1$, and as $P^r_{\mathbf{0};\frac{\vartheta_1}{\sqrt{2}}-2,\left\lceil\frac{\vartheta_1-\theta_2}{\sqrt{2}}-\frac{b_2\cdot\boldsymbol{\lambda}_1-(h+1)}{2}\right\rceil-1}$ if $m_{h+1} = 1$. Then we apply Corollary 2.4.9 to it with $L = \{a\}$. We conclude that

$$\left|\left\{ b \in \mathcal{S}_1 \cap \mathbb{Z}^3 : |u(b)| > \exp(C_{10}n)g \right\}\right| > \begin{cases} \frac{\epsilon_3(\vartheta_1-\theta_2)^2}{(\sqrt{2})^2}, & m_{h+1} > 1, \\ \epsilon_3\left(\frac{\vartheta_1-\theta_2}{\sqrt{2}} - \frac{b_2\cdot\boldsymbol{\lambda}_1-(h+1)}{2}\right)^2 & m_{h+1} = 1. \end{cases} \tag{2.5.59}$$

We now put together the bounds we've obtained so far, for all $\mathcal{S}_i$ and $\mathcal{W}_i$ that are contained in $\{b \in \mathbb{R}^3 : b \cdot \boldsymbol{\lambda}_1 \le h+1\}$.

Case 1: $m_h = m_{h+1}$. In this case we consider $\mathcal{S}_i$ for $1 \le i \le m_h$ and $\mathcal{W}_i$ for $2 \le i \le m_h$.

We first show that $h \neq a \cdot \boldsymbol{\lambda}_1$. For this we argue by contradiction. Assume the contrary, i.e. $h = a \cdot \boldsymbol{\lambda}_1$. From the definition of h we have that $|\mathcal{L}_{a,r,c,2} \cap \mathbb{Z}^3| < \frac{r}{4}$, for any $c \in \mathring{\mathfrak{P}}_{a,r,\Gamma} \cap \mathbb{Z}^3$ with $c \cdot \boldsymbol{\lambda}_1 = h+1 = a \cdot \boldsymbol{\lambda}_1 + 1$. As we assumed that $r > 100$, we must have $b_2 \cdot \boldsymbol{\lambda}_1 = a \cdot \boldsymbol{\lambda}_1 + 1 = h+1$, and this implies $m_{h+1} = 2$. However, by $h = a \cdot \boldsymbol{\lambda}_1$ we have $m_h = 1$. This contradicts with the assumption that $m_h = m_{h+1}$.

We next show that

$$\frac{\theta_{m_h+1}}{\sqrt{2}} + \frac{b_{m_h+1} \cdot \boldsymbol{\lambda}_1 - (h+1)}{2} < \frac{r}{2}. \tag{2.5.60}$$

By the definition of h and $h \neq a \cdot \boldsymbol{\lambda}_1$, we can find $c \in \mathring{\mathfrak{P}}_{a,r,\Gamma} \cap \mathbb{Z}^3$ with $c \cdot \boldsymbol{\lambda}_1 = h$ and $|\mathcal{L}_{a,r,c,2} \cap \mathbb{Z}^3| \geq \frac{r}{4}$. Since we assumed that $r > 100$, using $m_h = m_{h+1}$ we have $\mathring{\mathfrak{P}}_{a,r,\Gamma} \cap \mathcal{P}_{1,h+1} \cap \mathbb{Z}^3 \neq \emptyset$. This implies that $b_{m_h+1} \cdot \boldsymbol{\lambda}_1 = b_{m_{h+1}+1} \cdot \boldsymbol{\lambda}_1 > h+1$ (since otherwise, by the definiton of m_{h+1}, we must have $m_{h+1} = m$ and $b_{m+1} \cdot \boldsymbol{\lambda}_1 \leq h+1$, implying $\mathring{\mathfrak{P}}_{a,r,\Gamma} \cap \mathcal{P}_{1,h+1} = \emptyset$). Also note that $b_{m_h} \cdot \boldsymbol{\lambda}_1 \leq h$, so we can find $b \in \mathbb{Z}^3$, and b in the closure of $\hat{\mathcal{S}}_{m_h}$, such that $b \cdot \boldsymbol{\lambda}_1 = h+1$ or $h+2$. As $|\mathcal{L}_{a,r,b,2} \cap \mathbb{Z}^3| = \frac{\theta_{m_h+1}}{\sqrt{2}} + 1 + \frac{(b_{m_h+1} - b) \cdot \boldsymbol{\lambda}_1}{2}$, we have $|\mathcal{L}_{a,r,b,2} \cap \mathbb{Z}^3| \geq \frac{\theta_{m_h+1}}{\sqrt{2}} + \frac{b_{m_h+1} \cdot \boldsymbol{\lambda}_1 - (h+1)}{2}$.

On the other hand, using $|\mathcal{L}_{a,r,c,2} \cap \mathbb{Z}^3| \geq \frac{r}{4}$ and $r > 100$ again, we have $\mathring{\mathfrak{P}}_{a,r,\Gamma} \cap \mathcal{P}_{1,b \cdot \boldsymbol{\lambda}_1} \cap \mathbb{Z}^3 \neq \emptyset$. By the maximum property of h, for any $b' \in \mathring{\mathfrak{P}}_{a,r,\Gamma} \cap \mathcal{P}_{1,b \cdot \boldsymbol{\lambda}_1} \cap \mathbb{Z}^3$, we have $|\mathcal{L}_{a,r,b',2} \cap \mathbb{Z}^3| < \frac{r}{4}$. Then $|\mathcal{L}_{a,r,b,2} \cap \mathbb{Z}^3| < \frac{r}{4} + 3 < \frac{r}{2}$, and (2.5.60) follows.

If $m_h = m_{h+1} = 1$, by (2.5.59) we have that

$$\left|\left\{b \in \mathfrak{P}_{a,r,\Gamma} \cap \mathbb{Z}^3 : |u(b)| > \exp(C_{10}n)g\right\}\right| > \epsilon_3 \left(\frac{\vartheta_1 - \theta_2}{\sqrt{2}} - \frac{b_2 \cdot \boldsymbol{\lambda}_1 - (h+1)}{2}\right)^2$$
$$> \epsilon_3 \left(2r - \frac{r}{2}\right)^2 > \epsilon_3 r^2, \quad (2.5.61)$$

where we use (2.5.60) and the fact that $\vartheta_1 = 2\sqrt{2}r$.

If $m_h = m_{h+1} > 1$, we note that for all $2 \leq i \leq m_h$, these $\mathcal{W}_i$ are mutually disjoint; and for all $1 \leq i \leq m_h$, these $\mathcal{S}_i$ are mutually disjoint. By equations (2.5.50),(2.5.56),(2.5.57),(2.5.59) and taking a small enough $\epsilon_4 > 0$, we have that

$$\begin{aligned}
&\left|\left\{b \in \mathfrak{P}_{a,r,\Gamma} \cap \mathbb{Z}^3 : |u(b)| > \exp(C_{10}n)g\right\}\right| \\
>&\epsilon_4 \left(\frac{(\vartheta_1 - \theta_2)^2}{2} + \sum_{i=2}^{m_h} \theta_i(\theta_i - \vartheta_i) + \vartheta_i(\vartheta_i - \theta_{i+1}) - \vartheta_{m_h} \frac{b_{m_h+1} \cdot \boldsymbol{\lambda}_1 - (h+1)}{\sqrt{2}}\right) \\
=&\epsilon_4 \left(\frac{\vartheta_1^2}{4} + \frac{(\vartheta_1 - 2\theta_2)^2}{4} + \frac{\sum_{i=2}^{m_h}(\theta_i - \vartheta_i)^2 + \sum_{i=2}^{m_h-1}(\vartheta_i - \theta_{i+1})^2}{2}\right. \\
&\left. + \frac{\left(\vartheta_{m_h} - \theta_{m_h+1} - \frac{b_{m_h+1}\cdot\boldsymbol{\lambda}_1 - (h+1)}{\sqrt{2}}\right)^2}{2} - \frac{\left(\theta_{m_h+1} + \frac{b_{m_h+1}\cdot\boldsymbol{\lambda}_1 - (h+1)}{\sqrt{2}}\right)^2}{2}\right) \\
\geq&\epsilon_4 \left(\frac{\vartheta_1^2}{4} - \frac{\left(\theta_{m_h+1} + \frac{b_{m_h+1}\cdot\boldsymbol{\lambda}_1 - (h+1)}{\sqrt{2}}\right)^2}{2}\right).
\end{aligned}$$

(2.5.62)

Using (2.5.60), we get

$$\left|\left\{b \in \mathfrak{P}_{a,r,\Gamma} \cap \mathbb{Z}^3 : |u(b)| > \exp(C_{10}n)g\right\}\right| > \epsilon_4 \left(2r^2 - \frac{r^2}{4}\right) > \epsilon_4 r^2. \quad (2.5.63)$$

Case 2: $m_h < m_{h+1}$. In this case we consider $\mathcal{S}_i$ for $1 \leq i \leq m_h$ and $\mathcal{W}_i$ for $2 \leq i \leq m_h + 1 = m_{h+1}$. Similar to the first case, by (2.5.50),(2.5.52),(2.5.56),(2.5.59) and taking a small enough $\epsilon_5 > 0$, we have

$$
\begin{aligned}
&\left|\left\{b \in \mathfrak{P}_{a,r,\Gamma} \cap \mathbb{Z}^3 : |u(b)| > \exp(C_{10}n)g\right\}\right| \\
\geq&\epsilon_5 \left(\frac{(\vartheta_1 - \theta_2)^2}{2} + \sum_{i=2}^{m_h} \theta_i(\theta_i - \vartheta_i) + \vartheta_i(\vartheta_i - \theta_{i+1}) + (\theta_{m_{h+1}} - \vartheta_{m_{h+1}})^2 \right) \\
=&\epsilon_5 \Bigg(\frac{\vartheta_1^2}{4} + \frac{(\vartheta_1 - 2\theta_2)^2}{4} \\
&+ \sum_{i=2}^{m_h} \frac{(\theta_i - \vartheta_i)^2 + (\vartheta_i - \theta_{i+1})^2}{2} + \frac{(\theta_{m_{h+1}} - 2\vartheta_{m_{h+1}})^2}{2} - \vartheta_{m_{h+1}}^2 \Bigg) \\
\geq&\epsilon_5 \left(\frac{\vartheta_1^2}{4} - \vartheta_{m_{h+1}}^2 \right).
\end{aligned}
\tag{2.5.64}
$$

We now show that $\vartheta_{m_{h+1}} < r$. Since $m_{h+1} > m_h$, we have $b_{m_{h+1}} \cdot \boldsymbol{\lambda}_1 = h + 1$. If $\vartheta_{m_{h+1}} \geq r$, then $|\mathcal{L}_{a,r,b_{m_{h+1}},2} \cap \mathbb{Z}^3| \geq \frac{r}{\sqrt{2}} + 1$, and we can find $b \in \mathring{\mathfrak{P}}_{a,r,\Gamma} \cap \mathcal{P}_{1,h+1}$, such that $|\mathcal{L}_{a,r,b,2} \cap \mathbb{Z}^3| \geq \frac{r}{\sqrt{2}} - 2 > \frac{r}{4}$. This contradicts with the definition of h.

With $\vartheta_{m_{h+1}} < r$, and note that $2\sqrt{2}r = \vartheta_1 \geq \theta_2$, we have

$$
\left|\left\{b \in \mathfrak{P}_{a,r,\Gamma} \cap \mathbb{Z}^3 : |u(b)| > \exp(C_{10}n)g\right\}\right| > \epsilon_5 \left(2r^2 - r^2\right) = \epsilon_5 r^2. \tag{2.5.65}
$$

By taking C_9' small enough, we get (2.5.41) by each of (2.5.61), (2.5.63), and (2.5.65).

$\square$

It remains to prove Claim 2.5.13, 2.5.14, and 2.5.15.

Proof of Claim 2.5.13. Take any $c \in \{b - \mathbf{e}_3, b - \mathbf{e}_1 - \mathbf{e}_3, b - \mathbf{e}_1 - \mathbf{e}_3, b - 2\mathbf{e}_3\}$. As

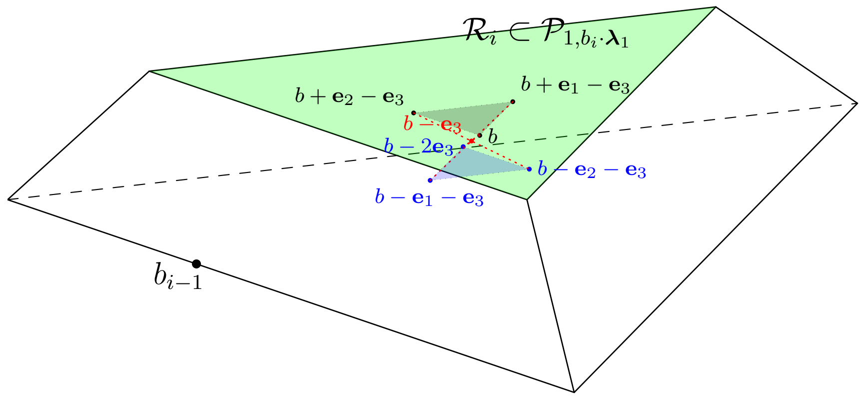

Figure 2.9: An illustration of points in Claim 2.5.13. The red point $(b-\mathbf{e}_3)$ is in $\mathcal{P}_{1,b_i\cdot\boldsymbol{\lambda}_1-1}$ and the blue points are in $\mathcal{P}_{1,b_i\cdot\boldsymbol{\lambda}_1-2}$. The point c is among the red and blue points.

$c\cdot\overline{\boldsymbol{\lambda}}_2 \le b\cdot\overline{\boldsymbol{\lambda}}_2$, $c\cdot\overline{\boldsymbol{\lambda}}_3 \le b\cdot\overline{\boldsymbol{\lambda}}_3$, and $c\cdot\overline{\boldsymbol{\lambda}}_4 \le (b+\mathbf{e}_1-\mathbf{e}_3)\cdot\overline{\boldsymbol{\lambda}}_4$, and $b, b+\mathbf{e}_1-\mathbf{e}_3 \in \mathring{\mathcal{R}}_i \subset \mathring{\mathfrak{T}}_{a,r}$, we have that

$$c\cdot\overline{\boldsymbol{\lambda}}_\tau < \mathbf{t}_r(a)\cdot\overline{\boldsymbol{\lambda}}_\tau - F_{a,r,b_{i-1}} \le \mathbf{t}_r(a)\cdot\overline{\boldsymbol{\lambda}}_\tau,\ \forall\tau\in\{2,3,4\}. \tag{2.5.66}$$

We first consider the case where $c \notin \mathring{\mathfrak{T}}_{a,r}$. Then we have that $a\cdot\boldsymbol{\lambda}_1 \ge c\cdot\boldsymbol{\lambda}_1 \ge b\cdot\boldsymbol{\lambda}_1 - 2 = b_i\cdot\boldsymbol{\lambda}_1 - 2 \ge a\cdot\boldsymbol{\lambda}_1 - 1$, where the last inequality is due to $b_i \in \mathring{\mathfrak{T}}_{a,r}$. If $c\cdot\boldsymbol{\lambda}_1 = a\cdot\boldsymbol{\lambda}_1$, we have $c \in \mathring{\mathcal{T}}_{a,r}$ by (2.5.66); and by the second condition of Proposition 2.5.12 we have that $|u(c)| < g$. If $c\cdot\boldsymbol{\lambda}_1 = a\cdot\boldsymbol{\lambda}_1 - 1$, we have $c \in \mathring{\mathcal{T}}_{a-\frac{\boldsymbol{\lambda}_1}{3},r}$ by (2.5.66). As $b_i\cdot\boldsymbol{\lambda}_1 > a\cdot\boldsymbol{\lambda}_1$, and $b_i\cdot\boldsymbol{\lambda}_1 = b\cdot\boldsymbol{\lambda}_1 \le c\cdot\boldsymbol{\lambda}_1 + 2$, we have that $b_i\cdot\boldsymbol{\lambda}_1 = a\cdot\boldsymbol{\lambda}_1 + 1$, thus $b_i \in \mathring{\mathcal{T}}_{a+\frac{\boldsymbol{\lambda}_1}{3},r}$. Since $|u(b_i)| \ge \exp(3C_{10}n)g$, by the second condition of Proposition 2.5.12 we have $|u(c)| < g$.

Now we assume that $c \in \mathring{\mathfrak{T}}_{a,r}$. For any j, if $i \le j \le m$, as $c\cdot\boldsymbol{\lambda}_1 < b_i\cdot\boldsymbol{\lambda}_1$, we have that $c\cdot\boldsymbol{\lambda}_1 < b_j\cdot\boldsymbol{\lambda}_1$, and thus $c \notin \mathfrak{H}_{a,r,b_j}$. If $1 \le j \le i-1$, we have $b_j\cdot\boldsymbol{\lambda}_1 \le b_{i-1}\cdot\boldsymbol{\lambda}_1$, so

$F_{a,r,b_j} \leq F_{a,r,b_{i-1}}$ (since otherwise $\mathfrak{H}_{a,r,b_{i-1}}$ is not maximal). By (2.5.66) we have that

$$c \cdot \overline{\boldsymbol{\lambda}}_\tau < \mathbf{t}_r(a) \cdot \overline{\boldsymbol{\lambda}}_\tau - F_{a,r,b_j},\ \forall \tau \in \{2,3,4\}, \tag{2.5.67}$$

thus $c \notin \mathfrak{H}_{a,r,b_j}$. Then by the definition of $\mathring{\mathfrak{P}}_{a,r,\Gamma}$, we have that $c \in \mathring{\mathfrak{P}}_{a,r,\Gamma}$. As $c \cdot \boldsymbol{\lambda}_1 \leq b_i \cdot \boldsymbol{\lambda}_1 - 1 \leq b_{m_{h+1}} \cdot \boldsymbol{\lambda}_1 - 1 \leq h$, we have $|u(c)| \leq \exp(C_{10}n)g$ by the third condition of Proposition 2.5.12. $\square$

Claim 2.5.15 can be proved in a similar way.

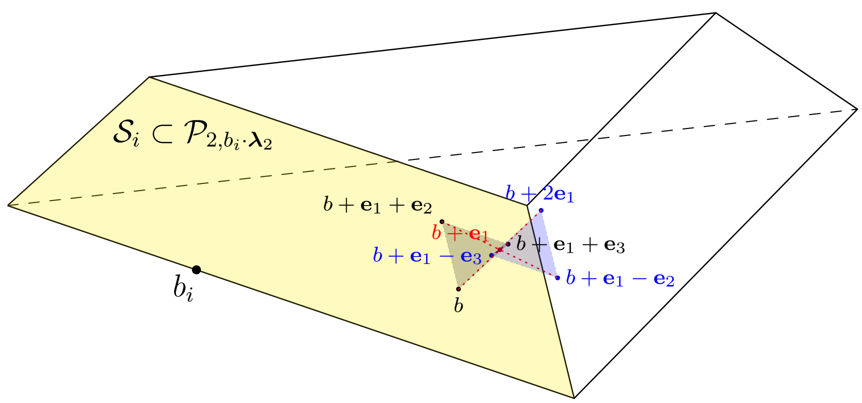

Figure 2.10: An illustration of points in Claim 2.5.15. The red point $(b + \mathbf{e}_1)$ is in $\mathcal{P}_{2,b_i\cdot\boldsymbol{\lambda}_2-1}$ and the blue points are in $\mathcal{P}_{2,b_i\cdot\boldsymbol{\lambda}_2-2}$. The point c is among the red and blue points.

Proof of Claim 2.5.15. We take $c \in \{b+\mathbf{e}_1, b+\mathbf{e}_1-\mathbf{e}_2, b+\mathbf{e}_1-\mathbf{e}_3, b+2\mathbf{e}_1\}$, then $c \cdot \overline{\boldsymbol{\lambda}}_2 < b \cdot \overline{\boldsymbol{\lambda}}_2 = b_i \cdot \overline{\boldsymbol{\lambda}}_2$, and $c \cdot \overline{\boldsymbol{\lambda}}_\tau \leq b \cdot \overline{\boldsymbol{\lambda}}_\tau + 2$ for $\tau \in \{3,4\}$. Since $b, b+\mathbf{e}_1+\mathbf{e}_2, b+\mathbf{e}_1+\mathbf{e}_3 \in \mathring{\mathcal{S}}_i$, we have that $b \cdot \overline{\boldsymbol{\lambda}}_3 + 2 = (b+\mathbf{e}_1+\mathbf{e}_3) \cdot \overline{\boldsymbol{\lambda}}_3 < \mathbf{t}_r(a) \cdot \overline{\boldsymbol{\lambda}}_3 - F_{a,r,b_i}$, and $b \cdot \overline{\boldsymbol{\lambda}}_4 + 2 = (b+\mathbf{e}_1+\mathbf{e}_2) \cdot \overline{\boldsymbol{\lambda}}_4 < \mathbf{t}_r(a) \cdot \overline{\boldsymbol{\lambda}}_4 + F_{a,r,b_i}$; then

$$c \cdot \overline{\boldsymbol{\lambda}}_\tau < \mathbf{t}_r(a) \cdot \overline{\boldsymbol{\lambda}}_\tau - F_{a,r,b_i} \leq \mathbf{t}_r(a) \cdot \overline{\boldsymbol{\lambda}}_\tau,\ \forall \tau \in \{2,3,4\}, \tag{2.5.68}$$

We claim that $c \notin \mathfrak{H}_{a,r,b_j}$ for any $1 \leq j \leq m$: for $j > i$, note that $b + \mathbf{e}_1 + \mathbf{e}_2 \in \mathring{\mathcal{S}}_i$, so $c \cdot \boldsymbol{\lambda}_1 \leq b \cdot \boldsymbol{\lambda}_1 + 2 = (b + \mathbf{e}_1 + \mathbf{e}_2) \cdot \boldsymbol{\lambda}_1 < b_{i+1} \cdot \boldsymbol{\lambda}_1$; for $j \leq i$, this is implied by (2.5.68). Thus $c \in \mathring{\mathfrak{P}}_{a,r,\Gamma} \cup \mathring{\mathcal{T}}_{a,r}$, since we also have $c \cdot \boldsymbol{\lambda}_1 \geq b \cdot \boldsymbol{\lambda}_1 \geq b_i \cdot \boldsymbol{\lambda}_1 \geq a \cdot \boldsymbol{\lambda}_1$. If $c \in \mathring{\mathcal{T}}_{a,r}$, by the second condition of Proposition 2.5.12, we have $|u(c)| \leq g < \exp(C_{10}n)g$. If $c \in \mathring{\mathfrak{P}}_{a,r,\Gamma}$, using the fact that $b + \mathbf{e}_1 + \mathbf{e}_2 \in \mathring{\mathcal{S}}_i$ again, we have $c \cdot \boldsymbol{\lambda}_1 \leq b \cdot \boldsymbol{\lambda}_1 + 2 = (b + \mathbf{e}_1 + \mathbf{e}_2) \cdot \boldsymbol{\lambda}_1 \leq h$, and this implies that $|u(c)| \leq \exp(C_{10}n)g$ by the third condition of Proposition 2.5.12. $\square$

Lastly, we prove Claim 2.5.14, using Claim 2.5.13 above and the local cone property (from Section 2.2).

Proof of Claim 2.5.14. Throughout this proof, we assume that $\left(\bigcup_{\tau \in \{2,3,4\}} \mathring{\mathcal{L}}_{a,r,b_i,\tau}\right) \cap \mathbb{Z}^3 \neq \emptyset$. We first show that we can find point $b \in \left(\bigcup_{\tau \in \{2,3,4\}} \mathring{\mathcal{L}}_{a,r,b_i,\tau}\right) \cap \mathbb{Z}^3$, such that

$$|u(b)| \geq (K + 11)^{-1} \exp(3C_{10}n)g. \tag{2.5.69}$$

This is obviously true if $b_i \in \bigcup_{\tau \in \{2,3,4\}} \mathring{\mathcal{L}}_{a,r,b_i,\tau}$; otherwise, by symmetry we assume that $b_i = \mathbf{v}_{a,r,b_i,4}$. By Lemma 2.2.2,

$$\max_{c \in \{b_i - \mathbf{e}_3, b_i - \mathbf{e}_3 + \mathbf{e}_1, b_i - \mathbf{e}_3 + \mathbf{e}_2, b_i - \mathbf{e}_3 - \mathbf{e}_1, b_i - \mathbf{e}_3 - \mathbf{e}_2, b_i - 2\mathbf{e}_3\}} |u(c)| \geq (K + 11)^{-1} \exp(3C_{10}n)g. \tag{2.5.70}$$

As $b_i, b_i - \mathbf{e}_3 + \mathbf{e}_1, b_i - \mathbf{e}_3 + \mathbf{e}_2 \in \mathring{\mathcal{R}}_i$, by Claim 2.5.13, we have

$$\max_{c \in \{b_i - \mathbf{e}_3 + \mathbf{e}_1, b_i - \mathbf{e}_3 + \mathbf{e}_2\}} |u(c)| \geq (K + 11)^{-1} \exp(3C_{10}n)g. \tag{2.5.71}$$

Note that $b_i - \mathbf{e}_3 + \mathbf{e}_1, b_i - \mathbf{e}_3 + \mathbf{e}_2 \in \bigcup_{\tau\in\{2,3,4\}} \mathring{\mathcal{L}}_{a,r,b_i,\tau}$, so we can choose $b \in \{b_i - \mathbf{e}_3 + \mathbf{e}_1, b_i - \mathbf{e}_3 + \mathbf{e}_2\}$ and the condition is satisfied.

Now by symmetry we assume that there is $b \in \mathring{\mathcal{L}}_{a,r,b_i,4} \cap \mathbb{Z}^3$ so that

$$|u(b)| \geq (K+11)^{-1} \exp(3C_{10}n)g. \tag{2.5.72}$$

We prove that, for any $b' \in \mathring{\mathcal{L}}_{a,r,b_i,4} \cap \mathbb{Z}^3$, we have $|u(b')| \geq \exp(2C_{10}n)g$. We argue by contradiction, and assume that there is $b' \in \mathring{\mathcal{L}}_{a,r,b_i,4} \cap \mathbb{Z}^3$ so that $|u(b')| < \exp(2C_{10}n)g$. Without loss of generality, we also assume that $b' \cdot \mathbf{e}_1 < b \cdot \mathbf{e}_1$. Consider the sequence of points in $\mathring{\mathcal{L}}_{a,r,b_i,4} \cap \mathbb{Z}^3$ between b and b'. We iterate this sequence from b to b', by adding $-\mathbf{e}_1 + \mathbf{e}_2$ at each step. We let c be the first one such that $|u(c - \mathbf{e}_1 + \mathbf{e}_2)| < (K+11)^{-1}|u(c)|$. The existence of such c is ensured by that $|u(b')| < (K+11)\exp(-C_{10}n)|u(b)|$, $|\mathring{\mathcal{L}}_{a,r,b_i,4} \cap \mathbb{Z}^3| < 2r < \frac{n}{16}$, and $C_{10} > K + 11$. For such c we also have $c, c - \mathbf{e}_1 + \mathbf{e}_2 \in \mathring{\mathcal{L}}_{a,r,b_i,4} \cap \mathbb{Z}^3$, and $|u(c)| \geq (K+11)^{-1-2r} \exp(3C_{10}n)g > \exp\left(\frac{5C_{10}n}{2}\right) g$. Since $c, c - \mathbf{e}_1 + \mathbf{e}_2, c - \mathbf{e}_1 + \mathbf{e}_3 \in \mathring{\mathcal{R}}_i$, by Claim 2.5.13 we have

$$\begin{aligned} |u(c')| <& \exp(C_{10}n)g \\ <& (K+11)^{-1}|u(c)|, \ \forall c' \in \{c - \mathbf{e}_1, c - \mathbf{e}_1 - \mathbf{e}_3, c - \mathbf{e}_1 - \mathbf{e}_2, c - 2\mathbf{e}_1\}. \end{aligned} \tag{2.5.73}$$

For $c - \mathbf{e}_1 + \mathbf{e}_3$, as $c - \mathbf{e}_1 + \mathbf{e}_3 \in \mathring{\mathfrak{P}}_{a,r,\Gamma}$ and $(c - \mathbf{e}_1 + \mathbf{e}_3) \cdot \boldsymbol{\lambda}_1 = c \cdot \boldsymbol{\lambda}_1 \leq h$, we have $|u(c - \mathbf{e}_1 + \mathbf{e}_3)| \leq \exp(C_{10}n)g < (K+11)^{-1}|u(c)|$ by the third condition of Proposition 2.5.12. Then we get a contradiction with Lemma 2.2.2. $\square$

The next step is to control the points in a graded set E.

Proposition 2.5.16. *For C_9' from Proposition 2.5.12, any small enough $\varepsilon > 0$, and any $N \in \mathbb{Z}_+$, there exists $C_{\varepsilon,N} > 0$ such that the following is true.*

Let $n \in \mathbb{Z}_+$, $r \in \mathbb{Z}$, $0 \leq r < \frac{n}{32}$. Let $\Gamma \subset Q_n$, $a \in \Gamma \cap Q_{\frac{n}{2}}$ such that $\mathring{\mathcal{T}}_{a,r} \cap \Gamma = \emptyset$. Suppose that $\vec{l}$ is a vector of positive reals, and E is an $(N, \vec{l}, \varepsilon^{-1}, \varepsilon)$-graded set with the first scale length $l_1 > C_{\varepsilon,N}$. If E is $(\varepsilon^{-\frac{1}{2}}, \varepsilon)$-normal in $\mathfrak{T}_{a,r}$, then

$$\left|E \cap \partial\mathfrak{P}_{a,r,\Gamma} \cap \mathbb{Z}^3\right| \leq \frac{C_9'}{2}(r^2 + 1). \tag{2.5.74}$$

Proof. If $r < \frac{1}{10\sqrt{\varepsilon}}$, since E is $(\varepsilon^{-\frac{1}{2}}, \varepsilon)$-normal in $\mathfrak{T}_{a,r}$, we have $E \cap \mathfrak{T}_{a,r} = \emptyset$ when $C_{\varepsilon,N}$ is large, and our conclusion holds.

From now on, we assume that $r \geq \frac{1}{10\sqrt{\varepsilon}}$. Denote $\pi := \pi_{a\cdot\boldsymbol{\lambda}_1}$ for the simplicity of notations. Evidently, for any two $b_1, b_2 \in \partial\mathfrak{P}_{a,r,\Gamma}$,

$$\frac{1}{10}|b_1 - b_2| \leq |\pi(b_1) - \pi(b_2)| \leq |b_1 - b_2|. \tag{2.5.75}$$

Suppose $\vec{l} = (l_1, \cdots, l_d)$, where $l_{i+1} \geq l_i^{1+2\varepsilon}$ for each $1 \leq i \leq d-1$. Write $E = \bigcup_{i=0}^{d} E_i$, where E_0 is a ε^{-1}-unitscattered set and E_i is an (N, l_i, ε)-scattered set. It suffices to prove that there exists a universal constant C such that for each $1 \leq i \leq d$,

$$\left|E_i \cap \partial\mathfrak{P}_{a,r,\Gamma} \cap \mathbb{Z}^3\right| \leq CNl_i^{-\varepsilon}r^2, \tag{2.5.76}$$

and

$$\left|E_0 \cap \partial\mathfrak{P}_{a,r,\Gamma} \cap \mathbb{Z}^3\right| \leq C\varepsilon^2 r^2. \tag{2.5.77}$$

Then with (2.5.76) and (2.5.77) we can take ε small enough, such that $C\varepsilon^2 < \frac{C_9'}{4}$; and take $C_{\varepsilon,N}$ large enough, such that

$$\sum_{i=1}^{\infty} CNl_i^{-\varepsilon} \leq \sum_{i=1}^{\infty} CNl_1^{-\varepsilon(1+2\varepsilon)^{i-1}} \leq \frac{C_9'}{4}. \tag{2.5.78}$$

Thus we get (2.5.74).

We first prove (2.5.76). As in Definition 2.3.3, for each $1 \leq i \leq d$, we write $E_i = \bigcup_{j\in\mathbb{Z}_+, 1\leq t\leq N} E_i^{(j,t)}$ where each $E_i^{(j,t)}$ is a open ball with radius l_i, and

$$\mathrm{dist}(E_i^{(j,t)}, E_i^{(j',t)}) \geq l_i^{1+\varepsilon}$$

for each $j \neq j'$.

Claim 2.5.17. *For any $1 \leq i \leq d$, $\left|\left\{(j,t) : E_i^{(j,t)} \cap \partial\mathfrak{P}_{a,r,\Gamma} \neq \emptyset\right\}\right| < CNl_i^{-2-\varepsilon}r^2$, where C is a universal constant.*

Proof. The proof is via a simple packing argument. Assume that $E_i \cap \mathfrak{T}_{a,r} \neq \emptyset$ (since otherwise the claim obviously holds). Denote $\tilde{\mathcal{T}}_{a,r}$ to be the closed equilateral triangle in $\mathcal{P}_{1,a\cdot\boldsymbol{\lambda}_1}$, such that it has the same center and orientation as $\mathcal{T}_{a,r}$, and its side length is $100r$. For any j, t, let $B_i^{(j,t)}$ be the open ball with radius $l_i^{1+\frac{\varepsilon}{2}}$ and with the same center as $E_i^{(j,t)}$. Since E is $(\varepsilon^{-\frac{1}{2}}, \varepsilon)$-normal in $\mathfrak{T}_{a,r}$, we have $\mathrm{diam}(B_i^{(j,t)}) \leq 10r^{1-\frac{\varepsilon^2}{4}}$.

Suppose $E_i^{(j,t)} \cap \partial \mathfrak{P}_{a,r,\Gamma} \neq \emptyset$, we then have $\pi(B_i^{(j,t)}) \subset \tilde{\mathcal{T}}_{a,r}$. In addition, if for some $j' \neq j$ we have $E_i^{(j',t)} \cap \partial \mathfrak{P}_{a,r,\Gamma} \neq \emptyset$ as well, by $\operatorname{dist}(E^{(j,t)}, E^{(j',t)}) \geq l_i^{1+\varepsilon}$ and (2.5.75), we have that (when $C_{\varepsilon,N}$ is large enough) $\pi(B_i^{(j,t)}) \cap \pi(B_i^{(j',t)}) = \emptyset$. Thus for any t,

$$\left|\left\{j : E_i^{(j,t)} \cap \partial \mathfrak{P}_{a,r,\Gamma} \neq \emptyset\right\}\right| l_i^{2+\varepsilon} < \operatorname{Area}(\tilde{\mathcal{T}}_{a,r}), \tag{2.5.79}$$

since $\operatorname{Area}(\pi(B_i^{(j,t)})) > l_i^{2+\varepsilon}$ for any j, t. Our claim follows by observing the fact that $\operatorname{Area}(\tilde{\mathcal{T}}_{a,r}) \leq Cr^2$. □

Claim 2.5.18. *There exists some universal constant C such that for any $j \in \mathbb{Z}_+$, $t \in \{1, 2, \cdots, N\}$ and $i \in \{1, 2, \cdots, d\}$,* $\left|E_i^{(j,t)} \cap \partial \mathfrak{P}_{a,r,\Gamma} \cap \mathbb{Z}^3\right| \leq C l_i^2$.

Proof. By (2.5.75), π is a injection from $\partial \mathfrak{P}_{a,r,\Gamma}$, so we only need to show

$$\left|\pi(E_i^{(j,t)}) \cap \pi(\mathbb{Z}^3)\right| \leq C l_i^2. \tag{2.5.80}$$

We note that $\pi(\mathbb{Z}^3)$ is a triangular lattice on $\mathcal{P}_{1,a\cdot\boldsymbol{\lambda}_1}$, with constant lattice length $\frac{\sqrt{6}}{3}$ and $\pi(E_i^{(j,t)})$ is a 2D ball with radius at least $C_{\varepsilon,N}$. Assuming $C_{\varepsilon,N} > 10$, we have

$$\left|\pi(E_i^{(j,t)}) \cap \pi(\mathbb{Z}^3)\right| \leq 10 \operatorname{Area}(\pi(E_i^{(j,t)})) \tag{2.5.81}$$

and our claim follows. □

Now by Claim 2.5.18,

$$\begin{aligned}\left|E_i \cap \partial\mathfrak{P}_{a,r,\Gamma} \cap \mathbb{Z}^3\right| &\leq \sum_{j,t} \left|E_i^{(j,t)} \cap \partial\mathfrak{P}_{a,r,\Gamma} \cap \mathbb{Z}^3\right| \\ &\leq \sum_{j,t} \left|\left\{(j,t) : E_i^{(j,t)} \cap \partial\mathfrak{P}_{a,r,\Gamma} \neq \emptyset\right\}\right| Cl_i^2.\end{aligned} \tag{2.5.82}$$

Then by Claim 2.5.17, we get (2.5.76).

As for (2.5.77), since by (2.5.75) π is a injection on $\partial\mathfrak{P}_{a,r,\Gamma}$, we only need to show

$$\left|\pi\left(E_0 \cap \partial\mathfrak{P}_{a,r,\Gamma} \cap \mathbb{Z}^3\right)\right| \leq C\varepsilon^2 r^2 \tag{2.5.83}$$

for some universal constant C. By (2.5.75) and the fact that E_0 is ε^{-1}-unitscattered, we have

$$|\pi(b) - \pi(b')| \geq \frac{\varepsilon^{-1}}{10} \tag{2.5.84}$$

for any $b \neq b' \in E_0 \cap \partial\mathfrak{P}_{a,r,\Gamma} \cap \mathbb{Z}^3$ (since b and b' are centers of different unit balls in E_0). Thus (2.5.83) follows from $\mathrm{Area}(\pi(\mathfrak{P}_{a,r,\Gamma})) < 100r^2$. □

Proof of Proposition 2.5.11. We assume that $r > 1000$, since otherwise the statement holds by taking C_9 small enough.

To apply Proposition 2.5.12, we need to check its third condition. We argue by contradiction, and assume that there exists $b \in \mathring{\mathfrak{P}}_{a,r,\Gamma} \cap \mathbb{Z}^3$ with $b \cdot \boldsymbol{\lambda}_1 \leq h$, and $|u(b)| > \exp(C_{10}n)g$. Consider the triangle $\mathcal{P}_{1,b\cdot\boldsymbol{\lambda}_1} \cap \mathring{\mathfrak{P}}_{a,r,\Gamma}$, and write it as $\{c \in \mathcal{P}_{1,b\cdot\boldsymbol{\lambda}_1} : c \cdot \overline{\boldsymbol{\lambda}}_\tau < \mathbf{t}_r(a) \cdot \overline{\boldsymbol{\lambda}}_\tau - F', \forall \tau = 2,3,4\}$ for some $F' \geq 0$. From the definition of

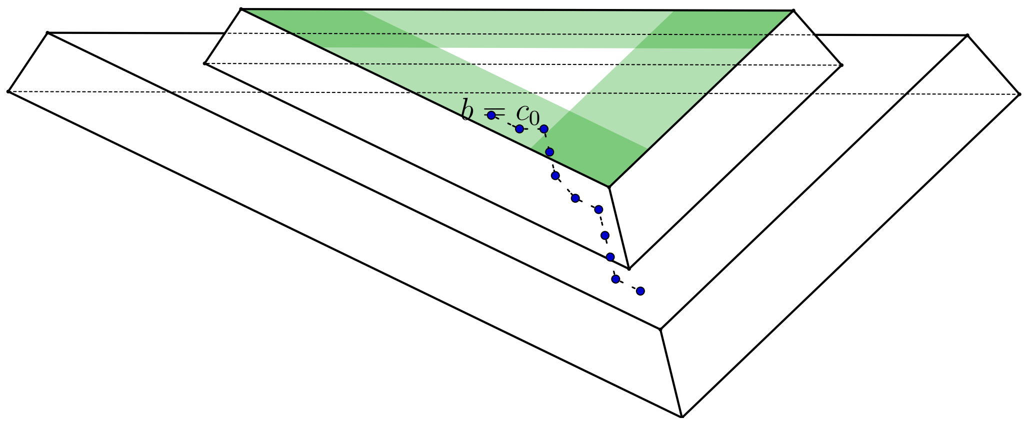

Figure 2.11: The three green areas are given by (2.5.85) and do not have common intersection, so $b = c_0 \in \mathcal{P}_{1,b\cdot\boldsymbol{\lambda}_1} \cap \mathring{\mathfrak{P}}_{a,r,\Gamma}$ is outside one of them, and we can construct a path in $\mathring{\mathfrak{P}}_{a,r,\Gamma}$ from it by using the cone property.

h, the its side length is at least $\sqrt{2}\left(\frac{r}{4}-1\right)$. Consider the three sets

$$\left\{c \in \mathcal{P}_{1,b\cdot\boldsymbol{\lambda}_1} : c \cdot \overline{\boldsymbol{\lambda}}_\tau > \mathbf{t}_r(a) \cdot \overline{\boldsymbol{\lambda}}_\tau - F' - \frac{r}{10}\right\} \tag{2.5.85}$$

where $\tau \in \{2,3,4\}$ (see Figure 2.11). The intersection of all three of them is empty, so by symmetry, we can assume that b is not in the first one, i.e.

$$b \cdot \overline{\boldsymbol{\lambda}}_2 \leq \mathbf{t}_r(a) \cdot \overline{\boldsymbol{\lambda}}_2 - F' - \frac{r}{10}. \tag{2.5.86}$$

Now we apply Lemma 2.2.3, starting from b and in the $-\mathbf{e}_1$ direction. Since $r < \frac{n}{32}$ and $a \in Q_{\frac{n}{2}}$, we can find a sequence of points $b = c_0, c_1, \cdots, c_r$, such that for any $1 \leq i \leq r$, we have $|u(c_i)| \geq (K+11)^{-1}|u(c_{i-1})|$, and $c_i - c_{i-1} \in \{-\mathbf{e}_1, -\mathbf{e}_1+\mathbf{e}_2, -\mathbf{e}_1+\mathbf{e}_3, -\mathbf{e}_1-\mathbf{e}_2, -\mathbf{e}_1-\mathbf{e}_3, -2\mathbf{e}_1\}$. Then we have that $c_i \cdot \overline{\boldsymbol{\lambda}}_2 \leq$

$c_{i-1} \cdot \overline{\boldsymbol{\lambda}}_2 + 2$, $c_i \cdot \overline{\boldsymbol{\lambda}}_3 \leq c_{i-1} \cdot \overline{\boldsymbol{\lambda}}_3$ and $c_i \cdot \overline{\boldsymbol{\lambda}}_4 \leq c_{i-1} \cdot \overline{\boldsymbol{\lambda}}_4$. This means that for $1 \leq i \leq \frac{r}{30}$,

$$
\begin{aligned}
&c_i \cdot \overline{\boldsymbol{\lambda}}_2 \leq b \cdot \overline{\boldsymbol{\lambda}}_2 + \frac{r}{15} < \mathbf{t}_r(a) \cdot \overline{\boldsymbol{\lambda}}_2 - F', \\
&c_i \cdot \overline{\boldsymbol{\lambda}}_\tau \leq b \cdot \overline{\boldsymbol{\lambda}}_\tau \leq \mathbf{t}_r(a) \cdot \overline{\boldsymbol{\lambda}}_\tau - F', \ \forall \tau \in \{3, 4\},
\end{aligned}
\tag{2.5.87}
$$

Also, for $i \leq \frac{r}{30}$, we have

$$
|u(c_i)| \geq (K + 11)^{-\frac{r}{30}} |u(c_0)| > \exp\left(\frac{C_{10} n}{2}\right) g, \tag{2.5.88}
$$

when $C_{10} > K + 11$. Since $c_{i-1}\boldsymbol{\lambda}_1 - 2 \leq c_i \cdot \boldsymbol{\lambda}_1 \leq c_{i-1}\boldsymbol{\lambda}_1$, by the second condition of Proposition 2.5.11, we have that $a \cdot \boldsymbol{\lambda}_1 < c_i \cdot \boldsymbol{\lambda}_1 \leq b \cdot \boldsymbol{\lambda}_1$ for each $1 \leq i \leq \frac{r}{30}$. With (2.5.87) this implies that $c_i \in \mathring{\mathfrak{P}}_{a,r,\Gamma}$ for each $1 \leq i \leq \frac{r}{30}$. See Figure 2.11 for an illustration.

By the definition of the pyramid $\mathfrak{P}_{a,r,\Gamma}$, for $0 \leq i \leq \frac{r}{30}$ we have that $c_i \not\in \Gamma$, thus $c_i \in E$ by (2.5.88) and the fourth condition of Proposition 2.5.11.

For $l \in \mathbb{R}_+$ with $1 \leq l < (2\sqrt{2}r)^{1-\frac{\varepsilon}{2}}$, and any $(1, l, \varepsilon)$-scattered set Z, the number of balls in Z that intersect $\{c_i\}_{i=1}^{\lfloor \frac{r}{30} \rfloor}$ is at most $2 \left\lfloor \frac{r}{30} \right\rfloor l^{-1-\varepsilon} + 1$. This is because, otherwise, there must exist $1 \leq i_1 < i_2 \leq \left\lfloor \frac{r}{30} \right\rfloor$, such that $|i_1 - i_2| < \frac{l^{1+\varepsilon}}{2}$, and c_{i_1} and c_{i_2} are contained in different balls. By construction the distance between c_{i_1} and c_{i_2} is at most $2|i_1 - i_2|$, and this contradicts with the fact that Z is $(1, l, \varepsilon)$-scattered. For each ball in Z, it contains at most $2l$ points in $\{c_i\}_{i=1}^{\lfloor \frac{r}{30} \rfloor}$. This is because $c_i \cdot \mathbf{e}_1 \leq c_{i-1} \cdot \mathbf{e}_1 - 1$

for $1 < i \leq \left\lfloor \frac{r}{30} \right\rfloor$, and the diameter of each ball is $2l$. Thus we have

$$\left| Z \cap \{c_i\}_{i=1}^{\left\lfloor \frac{r}{30} \right\rfloor} \right| \leq 2l \cdot \left(2 \left\lfloor \frac{r}{30} \right\rfloor l^{-1-\varepsilon} + 1 \right) < rl^{-\varepsilon} + 2l. \tag{2.5.89}$$

Similarly, for any ε^{-1}-unitscattered set Z, we have

$$\left| Z \cap \{c_i\}_{i=1}^{\left\lfloor \frac{r}{30} \right\rfloor} \right| < r\varepsilon + 2. \tag{2.5.90}$$

For the set E which is $(\varepsilon^{-\frac{1}{2}}, \varepsilon)$-normal in $\mathfrak{P}_{a,r,\Gamma}$, using (2.5.89) and (2.5.90) we have

$$\left| E \cap \{c_i\}_{i=1}^{\left\lfloor \frac{r}{30} \right\rfloor} \right| < r\varepsilon + 2 + \sum_{1 \leq i \leq d : l_i < (2\sqrt{2}r)^{1-\frac{\varepsilon}{2}}} Nrl_i^{-\varepsilon} + 2Nl_i \tag{2.5.91}$$

We have that

$$\begin{aligned} Nr\sum_{i=1}^{d} l_i^{-\varepsilon} \leq Nr\sum_{i=1}^{\infty} l_1^{-\varepsilon(1+2\varepsilon)^{i-1}} < Nr\sum_{i=1}^{\infty} C_{\varepsilon,N}^{-\varepsilon(1+2\varepsilon)^{i-1}} < Nr\sum_{i=1}^{\infty} C_{\varepsilon,N}^{-\varepsilon} C_{\varepsilon,N}^{-2(i-1)\varepsilon^2} \\ = \frac{NrC_{\varepsilon,N}^{-\varepsilon}}{1 - C_{\varepsilon,N}^{-2\varepsilon^2}}, \end{aligned} \tag{2.5.92}$$

and when $C_{\varepsilon,N}$ is large enough this is less than $\frac{r}{100}$.

Also, when $(2\sqrt{2}r)^{1-\frac{\varepsilon}{2}} > C_{\varepsilon,N} > 100$, and $\varepsilon < \frac{1}{200}$, we have

$$\begin{aligned} \sum_{1 \leq i \leq d : l_i < (2\sqrt{2}r)^{1-\frac{\varepsilon}{2}}} 2Nl_i < 2 \left(\frac{\log\left(\frac{\log(2\sqrt{2}r)}{\log(C_{\varepsilon,N})}\right)}{\log(1+2\varepsilon)} + 1 \right) N(2\sqrt{2}r)^{1-\frac{\varepsilon}{2}} \\ < \frac{4\log(\log(2\sqrt{2}r))}{\varepsilon} N(2\sqrt{2}r)^{1-\frac{\varepsilon}{2}}, \end{aligned} \tag{2.5.93}$$

where the first inequality is due to that there are at most $\left\lceil \frac{\log\left(\frac{\log(2\sqrt{2}r)}{\log(C_{\varepsilon,N})}\right)}{\log(1+2\varepsilon)} \right\rceil$ terms in the summation, and each is at most $2N(2\sqrt{2}r)^{1-\frac{\varepsilon}{2}}$. We further have that (2.5.93) is less than $\frac{r}{100}$ when $C_{\varepsilon,N}$ is large enough. When $(2\sqrt{2}r)^{1-\frac{\varepsilon}{2}} \leq C_{\varepsilon,N}$, the left hand side of (2.5.93) is zero. Thus the left hand side of (2.5.91) is less than $\frac{3r}{100} + 2 < \frac{r}{30}$ when $\varepsilon < \frac{1}{100}$ and $C_{\varepsilon,N}$ is large enough. This contradicts with the fact that $c_i \in E$ for each $0 \leq i \leq \frac{r}{30}$.

Finally, the conclusion follows from Proposition 2.5.12 and 2.5.16, by taking $C_9 = \frac{1}{2}C_9'$ and the same C_{10} as in Proposition 2.5.12. $\square$

2.5.3 Proof of Theorem 2.5.1

In this subsection we assemble results in previous subsections together and finish the proof of Theorem 2.5.1.

Proof of Theorem 2.5.1. By taking $C_{\varepsilon,N}$ large we can assume that $n > 100$.

We prove the result for $C_3 = \frac{1}{60}C_8$ and $C_2 = \max\{2C_7, 2\log(K+11)\}$, where C_8, C_7 are the constants in Proposition 2.5.3. We let ε be small enough, and $C_{\varepsilon,N}$ be the same as required by Proposition 2.5.3.

By Proposition 2.5.2, there exists $\tau \in \{1,2,3,4\}$, and

$$a_i \in (\mathcal{P}_{\tau,i} \cup \mathcal{P}_{\tau,i+1}) \cap \mathcal{C} \cap Q_{\frac{n}{10}+1} \tag{2.5.94}$$

for $i = 0, 1, \cdots, \left\lfloor \frac{n}{10} \right\rfloor - 1$, such that $|u(a_i)| \geq (K+11)^{-n}|u(\mathbf{0})|$.

For each $i = 0, 1, \cdots, \left\lfloor \frac{n}{10} \right\rfloor - 1$, we apply Proposition 2.5.3 to a_i, and find $h_i \in \mathbb{Z}_+$, such that

$$\begin{aligned}&\Bigg|\Bigg\{a \in Q_n \cap \bigcup_{j=0}^{h_i} \mathcal{P}_{\tau, a_i \cdot \boldsymbol{\lambda}_1 + j} :\\ &\qquad |u(a)| \geq \exp(-C_7 n^3)|u(a_i)| \geq \exp(-C_2 n^3)|u(\mathbf{0})|\Bigg\} \setminus E\Bigg|\\ &\qquad\qquad > C_8 h_i n(\log_2(n))^{-1}. \quad (2.5.95)\end{aligned}$$

Now for some $m \in \mathbb{Z}_{\geq 0}$, we define a sequence of nonnegative integers $i_1 < \cdots < i_m$ inductively. Let $i_1 := 0$. Given i_k, if $a_{i_k} \cdot \boldsymbol{\lambda}_\tau + h_k + 1 \leq \left\lfloor \frac{n}{10} \right\rfloor - 1$, we let $i_{k+1} := a_{i_k} \cdot \boldsymbol{\lambda}_\tau + h_{i_k} + 1$; otherwise, let $m = k$ and the process terminates.

Obviously, the sets

$$\left\{a \in Q_n \cap \bigcup_{j=0}^{h_{i_k}} \mathcal{P}_{\tau, a_{i_k} \cdot \boldsymbol{\lambda}_1 + j} : |u(a)| \geq \exp(-C_2 n^3)|u(\mathbf{0})|\right\} \setminus E \qquad (2.5.96)$$

for $k = 1, \cdots, m$ are mutually disjoint. Besides, we have that $a_{i_1} \cdot \boldsymbol{\lambda}_\tau \leq 1$ and $a_{i_m} \cdot \boldsymbol{\lambda}_\tau + h_{i_m} \geq \left\lfloor \frac{n}{10} \right\rfloor - 1$; and for each $1 \leq k < m$, $a_{i_{k+1}} \cdot \boldsymbol{\lambda}_\tau - a_{i_k} \cdot \boldsymbol{\lambda}_\tau \leq h_{i_k} + 2$. This implies that $\sum_{j=1}^{m} (h_{i_k} + 2) \geq \left\lfloor \frac{n}{10} \right\rfloor - 2$, thus $\sum_{j=1}^{m} h_{i_k} > \frac{n}{60}$, and

$$\begin{aligned}\left|\left\{a \in Q_n : |u(a)| \geq \exp(-C_2 n^3)|u(\mathbf{0})|\right\} \setminus E\right| &\geq C_8 \left(\sum_{k=1}^{m} h_{i_k}\right) n(\log_2(n))^{-1}\\ &> C_3 n^2 (\log_2(n))^{-1} \quad (2.5.97)\end{aligned}$$

which is (2.5.2). $\square$

2.6 Recursive construction: proof of discrete unique continuation

We deduce Theorem 2.3.4 from Theorem 2.5.1 in this section. The key step is the following result.

Theorem 2.6.1. *There exist universal constants β and $\alpha > \frac{5}{4}$ such that for any positive integers $m \leq n$ and any positive real K, the following is true. For any $u, V : \mathbb{Z}^3 \to \mathbb{R}$ such that $\Delta u = Vu$ in Q_n and $\|V\|_\infty \leq K$, we can find a subset $\Theta \subset Q_n$ with $|\Theta| \geq \beta \left(\frac{n}{m}\right)^\alpha$, such that*

1. *$|u(b)| \geq (K+11)^{-12n}|u(\mathbf{0})|$ for each $b \in \Theta$.*
2. *$Q_m(b) \cap Q_m(b') = \emptyset$ for $b, b' \in \Theta$, $b \neq b'$.*
3. *$Q_m(b) \subset Q_n$ for each $b \in \Theta$.*

The proof of Theorem 2.6.1 is based on the cone property, i.e. Lemma 2.2.3, and induction on $\frac{n}{m}$. We first set up some notations.

Definition 2.6.2. A set $B \subset \mathbb{Z}^3$ is called a *cuboid* if there are integers $t_\tau \leq k_\tau$, for $\tau = 1, 2, 3$, such that

$$B = \left\{ b \in \mathbb{Z}^3 : t_\tau \leq b \cdot \mathbf{e}_\tau \leq k_\tau, \tau = 1, 2, 3 \right\}. \tag{2.6.1}$$

We denote $p^+(B) := k_1$, $p^-(B) := t_1$, and $q^+(B) := k_2$, $q^-(B) := t_2$. A cuboid is called *even* if t_τ, k_τ are even for each $\tau = 1, 2, 3$.

Proof of Theorem 2.6.1. Without loss of generality we assume that $u(\mathbf{0}) = 1$.

Take $\alpha = 1.251 > \frac{5}{4}$, and leave β to be determined. We denote $f_m(x) = \beta(\frac{x}{m})^\alpha$ for $x > 0$. Then we have the following two inequalities:

$$4 \cdot 4^{-\alpha} + 4 \cdot 8^{-\alpha} > 1,\ 6 \cdot 4^{-\alpha} > 1. \tag{2.6.2}$$

This implies that there exists universal $N_0 > 10^8$ such that, for any positive integers m, n with $n > N_0 m$ and any real $\beta > 0$, we have

$$4 f_m\left(\frac{n}{4} - 3\right) + 4 f_m\left(\frac{n}{8} - 2\right) > f_m(n+7) \tag{2.6.3}$$

and

$$4 f_m\left(\frac{n}{4} - 3\right) + 2 f_m\left(\frac{n}{4} - 2\right) > f_m(n+7). \tag{2.6.4}$$

We let $\beta = (N_0 + 7)^{-\alpha}$, and fix $m \in \mathbb{Z}_+$. We need to show that, when $n \geq m$, there is $\Theta \subset Q_n$, such that $|\Theta| \geq f_m(n)$, and Θ satisfies the three conditions in the statement. For this, we do induction on n. First, it holds trivially when $m \leq n \leq N_0 m + 7$ by the choice of β. For simplicity of notations below, we only work on n that divides 8. At each step, we take some $n > N_0 m \geq 10^8 m$ with $\frac{n}{8} \in \mathbb{Z}$ and suppose our conclusion holds for all smaller n. Then we show that we can find a subset $\Theta \subset Q_n$ with $|\Theta| \geq f_m(n+7)$, such that the conditions in the statement are satisfied. Thus the conclusion holds for $n, n+1, \cdots, n+7$.

By Lemma 2.2.3, and using the notations in Definition 2.2.1, we pick $a_1 \in \mathcal{C}^3_{\mathbf{0}}(\frac{n}{2}) \cup \mathcal{C}^3_{\mathbf{0}}(\frac{n}{2}+1)$ and $a_2 \in \mathcal{C}^3_{\mathbf{0}}(-\frac{n}{2}) \cup \mathcal{C}^3_{\mathbf{0}}(-\frac{n}{2}-1)$ such that $|u(a_1)|, |u(a_2)| \geq (K+11)^{-n}$. For simplicity of notations, we denote Q^1 as the even cuboid such that we have $Q_{\frac{n}{2}-2}(a_1) \subset Q^1 \subset Q_{\frac{n}{2}-1}(a_1)$; and Q^2 as the even cuboid such that $Q_{\frac{n}{2}-2}(a_2) \subset Q^2 \subset Q_{\frac{n}{2}-1}(a_2)$.

Then we use Lemma 2.2.3 again to pick

$$\begin{aligned} a_{11} &\in \mathcal{C}^3_{a_1}\left(\frac{n}{4}-1\right) \cup \mathcal{C}^3_{a_1}\left(\frac{n}{4}\right), \\ a_{12} &\in \mathcal{C}^3_{a_1}\left(-\frac{n}{4}+1\right) \cup \mathcal{C}^3_{a_1}\left(-\frac{n}{4}\right), \\ a_{21} &\in \mathcal{C}^3_{a_2}\left(\frac{n}{4}-1\right) \cup \mathcal{C}^3_{a_2}\left(\frac{n}{4}\right), \\ a_{22} &\in \mathcal{C}^3_{a_2}\left(-\frac{n}{4}+1\right) \cup \mathcal{C}^3_{a_2}\left(-\frac{n}{4}\right), \end{aligned} \tag{2.6.5}$$

such that $|u(a_{11})|, |u(a_{12})|, |u(a_{21})|, |u(a_{22})| \geq (K+11)^{-2n}$. For $i, j \in \{1, 2\}$, let Q^{ij} be an even cuboid such that $Q_{\frac{n}{4}-3}(a_{ij}) \subset Q^{ij} \subset Q_{\frac{n}{4}-2}(a_{ij})$. Comparing the coordinates of a_{ij}'s, we see Q^{ij}'s are pairwise disjoint.

By inductive hypothesis, we can find $4f(\frac{n}{4}-3)$ points in $Q^{11} \cup Q^{12} \cup Q^{21} \cup Q^{22}$, such that for each b among them,

$$|u(b)| \geq (K+11)^{-2n}(K+11)^{-12(\frac{n}{4}-3)} \geq (K+11)^{-12n} \tag{2.6.6}$$

and all $Q_m(b)$ are mutually disjoint, and contained in $Q^{11} \cup Q^{12} \cup Q^{21} \cup Q^{22}$.

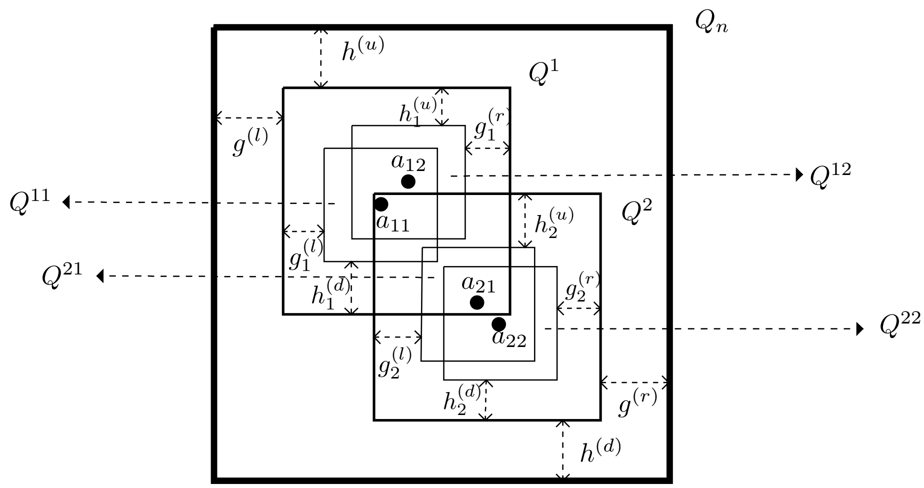

Figure 2.12: The projection onto the $\mathbf{e}_1\mathbf{e}_2$ plane.

Let B be the minimal cuboid containing $Q^1 \cup Q^2$, B_1 be the minimal cuboid containing $Q^{11} \cup Q^{12}$, and B_2 be the minimal cuboid containing $Q^{21} \cup Q^{22}$.

Let $g^{(r)} := p^+(Q_n) - p^+(B)$, $g^{(l)} := p^-(B) - p^-(Q_n)$, $g_1^{(r)} := p^+(Q^1) - p^+(B_1)$, $g_1^{(l)} := p^-(B_1) - p^-(Q^1)$, $g_2^{(r)} := p^+(Q^2) - p^+(B_2)$ and $g_2^{(l)} := p^-(B_2) - p^-(Q^2)$.

Similarly, in the $\mathbf{e}_2$-direction, let $h^{(u)} := q^+(Q_n) - q^+(B)$, $h^{(d)} := q^-(B) - q^-(Q_n)$, $h_1^{(u)} := q^+(Q^1) - q^+(B_1)$, $h_1^{(d)} := q^-(B_1) - q^-(Q^1)$, $h_2^{(u)} := q^+(Q^2) - q^+(B_2)$ and $h_2^{(d)} := q^-(B_2) - q^-(Q^2)$. See Figure 2.12 for an illustration of these definitions.

From the above definitions,

$$g^{(r)} + g^{(l)} + h^{(u)} + h^{(d)} = 4n - (p^+(B) - p^-(B)) - (q^+(B) - q^-(B)). \qquad (2.6.7)$$

Observe that

$$(p^+(B)-p^-(B))+(q^+(B)-q^-(B)) \leq |(a_1-a_2)\cdot\mathbf{e}_1|+|(a_1-a_2)\cdot\mathbf{e}_2|+4\left(\frac{n}{2}-1\right). \quad (2.6.8)$$

As $a_1 \in \mathcal{C}^3_{\mathbf{0}}(\frac{n}{2}) \cup \mathcal{C}^3_{\mathbf{0}}(\frac{n}{2}+1)$, we have $|a_1 \cdot \mathbf{e}_1| + |a_1 \cdot \mathbf{e}_2| \leq |a_1 \cdot \mathbf{e}_3| \leq \frac{n}{2}+1$; and similarly, we have $|a_2 \cdot \mathbf{e}_1| + |a_2 \cdot \mathbf{e}_2| \leq \frac{n}{2} + 1$. Using these and (2.6.8), and triangle inequality, we have

$$(p^+(B) - p^-(B)) + (q^+(B) - q^-(B)) \leq 3n - 2. \quad (2.6.9)$$

Thus with (2.6.7) we have

$$g^{(r)} + g^{(l)} + h^{(u)} + h^{(d)} \geq n + 2. \quad (2.6.10)$$

The same argument applying to smaller cubes Q^1 and Q^2, we have

$$g_1^{(r)} + g_1^{(l)} + h_1^{(u)} + h_1^{(d)} \geq \frac{n}{2} + 2 \quad (2.6.11)$$

and

$$g_2^{(r)} + g_2^{(l)} + h_2^{(u)} + h_2^{(d)} \geq \frac{n}{2} + 2. \quad (2.6.12)$$

Summing them together we get

$$g^{(r)}+g^{(l)}+g_1^{(r)}+g_1^{(l)}+g_2^{(r)}+g_2^{(l)}+h^{(u)}+h^{(d)}+h_1^{(u)}+h_1^{(d)}+h_2^{(u)}+h_2^{(d)} \geq 2n+6. \quad (2.6.13)$$

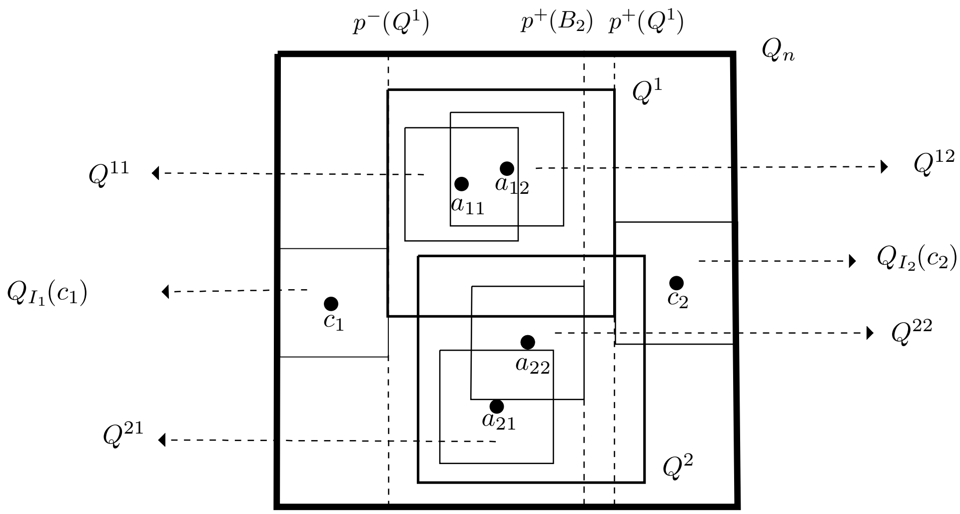

Figure 2.13: The projection onto the $\mathbf{e}_1\mathbf{e}_2$ plane in Case 1.

As these g's and h's are exchangeable, we assume without loss of generality that

$$g^{(r)} + g^{(l)} + g_1^{(r)} + g_1^{(l)} + g_2^{(r)} + g_2^{(l)} \geq n + 3. \tag{2.6.14}$$

By symmetry, we assume without loss of generality that $a_1 \cdot \mathbf{e}_1 \leq a_2 \cdot \mathbf{e}_1$; consequently $p^-(Q^1) \leq p^-(Q^2)$. We discuss two possible cases.

Case 1: $p^+(B_2) \leq p^+(Q^1)$ or $p^-(B_1) \geq p^-(Q^2)$. By symmetry again, it suffices to consider the scenario for $p^+(B_2) \leq p^+(Q^1)$. See Figure 2.13 for an illustration.

Consider cuboids

$$\begin{aligned} U_l &:= \left\{ b \in \mathbb{Z}^3 : |b \cdot \mathbf{e}_2|, |b \cdot \mathbf{e}_3| \leq n-1, -n+1 \leq b \cdot \mathbf{e}_1 \leq p^-(Q^1) - 1 \right\}, \\ U_r &:= \left\{ b \in \mathbb{Z}^3 : |b \cdot \mathbf{e}_2|, |b \cdot \mathbf{e}_3| \leq n-1, p^+(Q^1) + 1 \leq b \cdot \mathbf{e}_1 \leq n-1 \right\}. \end{aligned} \tag{2.6.15}$$

Then U_l, U_r, B_1, B_2 are mutually disjoint, since $p^+(B_2) \leq p^+(Q^1)$ and $p^-(Q^1) \leq p^-(Q^2)$. Now we use Lemma 2.2.3 to pick points

$$\begin{aligned} c_1 &\in \mathcal{C}^1_{\mathbf{0}}\left(\frac{1}{2}(p^-(Q^1)-n)\right) \cup \mathcal{C}^1_{\mathbf{0}}\left(\frac{1}{2}(p^-(Q^1)-n)+1\right), \\ c_2 &\in \mathcal{C}^1_{\mathbf{0}}\left(\frac{1}{2}(p^+(Q^1)+n)\right) \cup \mathcal{C}^1_{\mathbf{0}}\left(\frac{1}{2}(p^+(Q^1)+n)+1\right), \end{aligned} \tag{2.6.16}$$

such that $|u(c_1)|, |u(c_2)| \geq (K+11)^{-n}$. Denote $I_1 := \frac{p^-(Q^1)+n}{2} - 2, I_2 := \frac{n-p^+(Q^1)}{2} - 2$. Then $I_1, I_2 \leq \frac{n}{2}$. We also have

$$(p^-(Q^1)+n) + (n - p^+(Q^1)) = 2n + p^-(Q^1) - p^+(Q^1) \geq n+2, \tag{2.6.17}$$

so

$$I_1 + I_2 \geq \frac{n}{2} - 3. \tag{2.6.18}$$

We use inductive hypothesis on $Q_{I_1}(c_1) \subset U_l$, if $I_1 > m$; and on $Q_{I_2}(c_2) \subset U_r$, if $I_2 > m$. Note that U_l, U_r, B_1, B_2 are mutually disjoint. Thus we get $f_m(I_1)\mathbb{1}_{I_1>m} + f_m(I_2)\mathbb{1}_{I_2>m}$ points in $\mathbb{Z}^3$, such that for each point b among them,

- $|u(b)| \geq (K+11)^{-n}(K+11)^{-12\cdot\frac{n}{2}} \geq (K+11)^{-12n}$,
- $Q_m(b) \cap Q_m(b') = \emptyset$ for another $b' \neq b$ among them,
- $Q_m(b) \subset Q_n \setminus (Q^{11} \cup Q^{12} \cup Q^{21} \cup Q^{22})$.

We now show that

$$f_m(I_1)\mathbb{1}_{I_1>m} + f_m(I_2)\mathbb{1}_{I_2>m} \geq 2f_m\left(\frac{n}{4}-2\right). \tag{2.6.19}$$

If $I_1, I_2 > m$, (2.6.19) follows by convexity and monotonicity of the function f_m, and (2.6.18). If $I_1 \leq m$, by (2.6.18) and the assumption that $n > N_0 m \geq 10^8 m$, we have $I_2 \geq \frac{n}{2} - 3 - m > 10^7 m$. Then by monotonicity of f_m we have $f_m(I_2)\mathbb{1}_{I_2>m} = f_m(I_2) \geq f_m\left(\frac{n}{2}-3-m\right) \geq 2f_m\left(\frac{n}{4}-2\right)$, which implies (2.6.19). The case when $I_2 \leq m$ is symmetric.

Now together with the $4f_m\left(\frac{n}{4}-3\right)$ points we found in $Q^{11} \cup Q^{12} \cup Q^{21} \cup Q^{22}$, we have a set of at least $4f_m\left(\frac{n}{4}-3\right) + 2f_m\left(\frac{n}{4}-2\right)$ points in Q_n, satisfying all the three conditions.

Case 2: $p^+(B_2) > p^+(Q^1)$ and $p^-(B_1) < p^-(Q^2)$. See Figure 2.14 for an illustration.

Denote

$$\begin{aligned}
&U_1 := \left\{b \in \mathbb{Z}^3 : |b\cdot \mathbf{e}_2|, |b\cdot \mathbf{e}_3| \leq n-1, -n+1 \leq b\cdot\mathbf{e}_1 \leq p^-(B_1)-1\right\},\\
&U_2 := \left\{b \in \mathbb{Z}^3 : |b\cdot \mathbf{e}_2|, |b\cdot \mathbf{e}_3| \leq n-1, p^+(B_2)+1 \leq b\cdot\mathbf{e}_1 \leq n-1\right\},\\
&U_3 := \Big\{b \in \mathbb{Z}^3 :\\
&|b\cdot\mathbf{e}_2| \leq n-1, 1 \leq b\cdot\mathbf{e}_3 \leq n-1, p^+(B_1)+1 \leq b\cdot\mathbf{e}_1 \leq p^+(Q^1)-1\Big\},\\
&U_4 := \Big\{b \in \mathbb{Z}^3 :\\
&|b\cdot\mathbf{e}_2| \leq n-1, -n+1 \leq b\cdot\mathbf{e}_3 \leq -1, p^-(Q^2)+1 \leq b\cdot\mathbf{e}_1 \leq p^-(B_2)-1\Big\}.
\end{aligned} \tag{2.6.20}$$

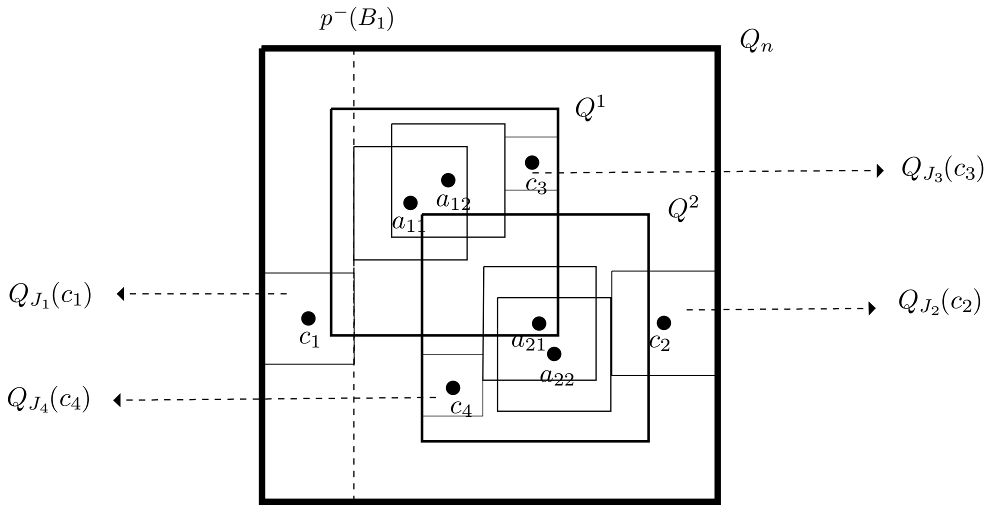

Figure 2.14: The projection onto the $\mathbf{e}_1\mathbf{e}_2$ plane in Case 2.

We note that U_1, U_2, U_3, U_4, B_1 and B_2 are mutually disjoint.

We use Lemma 2.2.3 to pick the following points:

$$
\begin{aligned}
c_1 \in & \mathcal{C}^1_{\mathbf{0}}\left(\frac{1}{2}\left(p^-\left(B_1\right)-n\right)\right) \cup \mathcal{C}^1_{\mathbf{0}}\left(\frac{1}{2}\left(p^-\left(B_1\right)-n\right)+1\right), \\
c_2 \in & \mathcal{C}^1_{\mathbf{0}}\left(\frac{1}{2}\left(p^+\left(B_2\right)+n\right)\right) \cup \mathcal{C}^1_{\mathbf{0}}\left(\frac{1}{2}\left(p^+\left(B_2\right)+n\right)+1\right), \\
c_3 \in & \mathcal{C}^1_{a_1}\left(\frac{1}{2}\left(p^+\left(B_1\right)+p^+\left(Q^1\right)\right)-a_1\cdot\mathbf{e}_1\right) \\
& \cup\mathcal{C}^1_{a_1}\left(\frac{1}{2}\left(p^+\left(B_1\right)+p^+\left(Q^1\right)\right)-a_1\cdot\mathbf{e}_1+1\right), \\
c_4 \in & \mathcal{C}^1_{a_2}\left(\frac{1}{2}\left(p^-\left(B_2\right)+p^-\left(Q^2\right)\right)-a_2\cdot\mathbf{e}_1\right) \\
& \cup\mathcal{C}^1_{a_2}\left(\frac{1}{2}\left(p^-\left(B_2\right)+p^-\left(Q^2\right)\right)-a_2\cdot\mathbf{e}_1+1\right),
\end{aligned}
\tag{2.6.21}
$$

such that $|u(c_i)| \geq (K+11)^{-3n}$ for each $i = 1,2,3,4$.

Denote $J_1 := \frac{p^-(B_1)+n}{2} - 2$, $J_2 := \frac{n-p^+(B_2)}{2} - 2$, $J_3 := \frac{p^+(Q^1)-p^+(B_1)}{2} - 2$, and

$J_4 := \frac{p^-(B_2)-p^-(Q^2)}{2} - 2$.

For each $i = 1, 2, 3, 4$, if $J_i > m$, we use inductive hypothesis on $Q_{J_i}(c_i) \subset U_i$ (note that $Q_{J_i}(c_i)$ is disjoint from Q_{11}, so $J_i \leq \frac{3n}{4}$). As the sets B_1, B_2, U_1, U_2, U_3 and U_4 are mutually disjoint, we can find $\sum_{i=1}^4 f_m(J_i)\mathbb{1}_{J_i>m}$ points in $\bigcup_{i=1}^4 U_i$, such that for each point b among them,

- $|u(b)| \geq (K+11)^{-3n}(K+11)^{-12\cdot\frac{3n}{4}} = (K+11)^{-12n}$,
- $Q_m(b) \cap Q_m(b') = \emptyset$ for another $b' \neq b$ among them,
- $Q_m(b) \subset Q_n \setminus (Q^{11} \cup Q^{12} \cup Q^{21} \cup Q^{22})$.

By (2.6.14), we have

$$\begin{aligned}(p^-(B_1)+n) + (n - p^+(B_2)) + (p^+(Q^1) - p^+(B_1)) + (p^-(B_2) - p^-(Q^2))\\ = g^{(r)} + g^{(l)} + g_1^{(r)} + g_1^{(l)} + g_2^{(r)} + g_2^{(l)} \geq n+3, \quad (2.6.22)\end{aligned}$$

thus $J_1 + J_3 + J_3 + J_4 \geq \frac{n}{2} - 7$. Similar to (2.6.19) above, by monotonicity and convexity of f_m, and $n > N_0 m \geq 10^8 m$, we have

$$\sum_{i=1}^{4} f_m(J_i)\mathbb{1}_{J_i>m} \geq 4 f_m\left(\frac{n}{8} - 2\right). \quad (2.6.23)$$

This implies that, together with the $4f_m\left(\frac{n}{4}-3\right)$ points we found in $Q^{11} \cup Q^{12} \cup Q^{21} \cup Q^{22}$, we have a set of at least $4f_m\left(\frac{n}{4}-3\right) + 4f_m\left(\frac{n}{8}-2\right)$ points in Q_n, satisfying all the three conditions.

In conclusion, by (2.6.3) and (2.6.4), in each case, we can always find a $\Theta \subset Q_n$ satisfying the three conditions, with $|\Theta| \geq f_m(n+7)$. Thus Theorem 2.6.1 follows from the principle of induction. $\square$

Now we prove Theorem 2.3.4.

Proof of Theorem 2.3.4. Let $p := \frac{1}{3}\alpha + \frac{13}{12}$, then $p > \frac{3}{2}$ since $\alpha > \frac{5}{4}$. Without loss of generality, we assume that $u(\mathbf{0}) = 1$.

Suppose $\vec{l} = (l_1, l_2, \cdots, l_d)$. Since E is $(N, \vec{l}, \varepsilon^{-1}, \varepsilon)$-graded, we can write $E = \bigcup_{i=0}^{d} E_i$ where E_i is an (N, l_i, ε)-scattered set for $i > 0$ and E_0 is a ε^{-1}-unitscattered set. We also write $E_i = \bigcup_{j\in\mathbb{Z}_+, 1\leq t\leq N} E_i^{(j,t)}$, where each $E_i^{(j,t)}$ is an open ball with radius l_i and

$$\operatorname{dist}(E_i^{(j,t)}, E_i^{(j',t)}) \geq l_i^{1+\varepsilon} \tag{2.6.24}$$

whenever $j \neq j'$.

We assume without loss of generality that $l_d \leq 4n^{1-\frac{\varepsilon}{2}}$. Otherwise, since E is $(1,\varepsilon)$-normal in Q_n, we can replace E by $E_0 \cup \left(\bigcup_{l_i \leq 4n^{1-\frac{\varepsilon}{2}}} E_i\right)$.

Let $n_k := \lfloor l_{d-k} \rfloor$ for $k = 0, 1, \cdots, d$.

Claim 2.6.3. *We can assume there is $M \in \mathbb{Z}_+$ such that $n^{\frac{1}{3}(1-4\varepsilon)} + 1 \leq n_M \leq n^{\frac{1}{3}}$.*

Proof. Suppose there is no such $M \in \mathbb{Z}_+$, we then add a level of empty set with scale length equal $n^{\frac{1}{3}(1-2\varepsilon)}$. More specifically, let k be the largest nonnegative integer satisfying $l_k \leq n^{\frac{1}{3}(1-4\varepsilon)}$, then $l_{k+1} > n^{\frac{1}{3}}$. We let $l_i' = l_i$ and $E_i' = E_i$ for each $0 \leq i \leq k$. Let $l_{k+1}' = n^{\frac{1}{3}(1-2\varepsilon)}$ and E_{k+1}' be any (N, l_k', ε)-scattered set that is disjoint from Q_n.

Let $l'_i = l_{i-1}$ and $E'_i = E_{i-1}$ for $i \geq k+2$. Then for each $1 \leq i \leq d+1$, we have $(l'_{i-1})^{1+2\varepsilon} \leq l'_i$, and E'_i is (N, l'_i, ε)-scattered. Also, as $n > C^4_{\varepsilon,N}$ we still have $l'_1 > C_{\varepsilon,N}$. Evidently, by replacing E with $\bigcup_{i=0}^{d+1} E'_i$, our claim holds with $M = k+1$. $\square$

Now we inductively construct subsets $\Theta_k \subset Q_n$ for $k = 0, 1, \cdots, M$, such that the following conditions hold.

1. $|\Theta_k| \geq \left(\frac{\beta}{2}\right)^{2k+2} \left(\frac{n}{n_k}\right)^{\alpha}$.

2. For any $a \in \Theta_k$, we have $|u(a)| \geq (K+11)^{-24(k+1)n}$.

3. For any $a, a' \in \Theta_k$ with $a \neq a'$, we have $Q_{n_k}(a) \cap Q_{n_k}(a') = \emptyset$.

4. For any $a \in \Theta_k$, we have $Q_{n_k}(a) \subset Q_n$.

5. When $k > 0$, for any $a \in \Theta_k$, there exists $a' \in \Theta_{k-1}$ such that $Q_{n_k}(a) \subset Q_{n_{k-1}}(a')$.

6. For any $a \in \Theta_k$ and $d-k \leq i \leq d$, we have $E_i \cap Q_{n_k}(a) = \emptyset$.

Let $n'_0 := \min\left\{\left\lfloor \frac{1}{4} n_0^{1+\varepsilon} \right\rfloor, n\right\}$. By using Theorem 2.6.1 for $m = n'_0$, we get a subset $\Theta'_0 \subset Q_n$ such that $|\Theta'_0| \geq \beta \left(\frac{n}{n'_0}\right)^{\alpha}$ and Θ'_0 satisfies Condition 1 to 3 in Theorem 2.6.1. For each fixed $t \in \{1, 2, \cdots, N\}$ and $j \neq j' \in \mathbb{Z}_+$, by definition we have $\mathrm{dist}(E_d^{(j,t)}, E_d^{(j',t)}) \geq 4n'_0$. This implies

$$\left|\left\{(j,t) : E_d^{(j,t)} \cap Q_{n'_0}(a) \neq \emptyset\right\}\right| \leq N, \tag{2.6.25}$$

for each $a \in \Theta_0'$. For each $a \in \Theta_0'$, by using Theorem 2.6.1 for $Q_{n_0'}(a)$ and $m = n_0$, we get a subset $\Theta_0^{(a)} \subset Q_{n_0'}(a)$ such that $|\Theta_0^{(a)}| \geq \beta(\frac{n_0'}{n_0})^\alpha$ and $\Theta_0^{(a)}$ satisfies Condition 1 to 3 in Theorem 2.6.1. For each j,t we have $\left|\left\{b \in \Theta_0^{(a)} : Q_{n_0}(b) \cap E_d^{(j,t)} \neq \emptyset\right\}\right| \leq 100$. This is because for each $b \in \Theta_0^{(a)}$ with $Q_{n_0}(b) \cap E_d^{(j,t)} \neq \emptyset$, the cube $Q_{n_0}(b)$ is contained in the closed ball of radius $2\sqrt{3}n_0 + l_d < (2\sqrt{3}+1)n_0 + 1$ with the same center as $E_d^{(j,t)}$. As we have $Q_{n_0}(b) \cap Q_{n_0}(b') = \emptyset$ for $b \neq b' \in \Theta_0^{(a)}$, the number of such $b \in \Theta_0^{(a)}$ is at most $\frac{(2(2\sqrt{3}+1)n_0+2)^3}{(2n_0+1)^3} < 100$. Thus by (2.6.25), we have

$$\left|\left\{b \in \Theta_0^{(a)} : Q_{n_0}(b) \cap E_d \neq \emptyset\right\}\right| \leq 100N. \tag{2.6.26}$$

Let $\tilde{\Theta}_0^{(a)} := \Theta_0^{(a)} \setminus \left\{b \in \Theta_0^{(a)} : Q_{n_0}(b) \cap E_d \neq \emptyset\right\}$ for each $a \in \Theta_0'$, and $\Theta_0 = \bigcup_{a \in \Theta_0'} \tilde{\Theta}_0^{(a)}$. Now we check the conditions. Condition 6 is from the definition, and Condition 5 automatically holds since $k = 0$. Condition 2 to 4 hold by the conditions in Theorem 2.6.1. For Condition 1, recall that $l_d \geq l_1 \geq C_{\varepsilon,N}$, and $l_d \leq 4n^{1-\frac{\varepsilon}{2}}$. By letting $C_{\varepsilon,N}$ large enough we have $n_0' > l_d^{1+\frac{\varepsilon}{2}}$, and then $\frac{1}{2}\beta(\frac{n_0'}{n_0})^\alpha > \frac{1}{2}\beta l_d^{\frac{1}{2}\alpha\varepsilon} \geq \frac{1}{2}\beta C_{\varepsilon,N}^{\frac{1}{2}\alpha\varepsilon} > 100N$. Thus for each $a \in \Theta_0'$ we have $|\tilde{\Theta}_0^{(a)}| \geq |\Theta_0^{(a)}| - 100N \geq \frac{1}{2}\beta(\frac{n_0'}{n_0})^\alpha$. This implies that

$$|\Theta_0| = \sum_{a \in \Theta_0'} |\tilde{\Theta}_0^{(a)}| \geq \left(\frac{1}{2}\beta\left(\frac{n_0'}{n_0}\right)^\alpha\right)\left(\beta\left(\frac{n}{n_0'}\right)^\alpha\right) > \left(\frac{\beta}{2}\right)^2\left(\frac{n}{n_0}\right)^\alpha. \tag{2.6.27}$$

Suppose we have constructed Θ_k, for some $0 \leq k < M$, we proceed to construct Θ_{k+1}. Note that as $l_{d-k-1}^{1+2\varepsilon} \leq l_{d-k}$, we have $n_k \geq n_{k+1}^{1+2\varepsilon} - 1$. Let $n_{k+1}' = \left\lfloor \frac{1}{4} n_{k+1}^{1+\varepsilon} \right\rfloor$. Take an arbitrary $a_0 \in \Theta_k$, use Theorem 2.6.1 for $Q_{n_k}(a_0)$ with $m = n_{k+1}'$, we get a subset

$\Theta'^{(a_0)}_{k+1} \subset Q_{n_k}(a_0)$ such that $|\Theta'^{(a_0)}_{k+1}| \geq \beta \left(\frac{n_k}{n'_{k+1}}\right)^\alpha$ and $\Theta'^{(a_0)}_{k+1}$ satisfies Condition 1 to 3 in Theorem 2.6.1. For each fixed $t \in \{1, 2, \cdots, N\}$ and $j \neq j' \in \mathbb{Z}_+$, by definition we have $\operatorname{dist}(E^{(j,t)}_{d-k-1}, E^{(j',t)}_{d-k-1}) \geq 4n'_{k+1}$. This implies, for each $a \in \Theta'^{(a_0)}_{k+1}$,

$$\left|\left\{(j,t) : E^{(j,t)}_{d-k-1} \cap Q_{n'_{k+1}}(a) \neq \emptyset\right\}\right| \leq N. \tag{2.6.28}$$

For each $a \in \Theta'^{(a_0)}_{k+1}$, by using Theorem 2.6.1 for $Q_{n'_{k+1}}(a)$ and $m = n_{k+1}$, we get a subset $\Theta^{(a)}_{k+1} \subset Q_{n'_{k+1}}(a)$ such that $|\Theta^{(a)}_{k+1}| \geq \beta \left(\frac{n'_{k+1}}{n_{k+1}}\right)^\alpha$ and $\Theta^{(a)}_{k+1}$ satisfies Condition 1 to 3 in Theorem 2.6.1. By (2.6.28),

$$\left|\left\{b \in \Theta^{(a)}_{k+1} : Q_{n_{k+1}}(b) \cap E_{d-k-1} \neq \emptyset\right\}\right| \leq 100N. \tag{2.6.29}$$

Let $\tilde{\Theta}^{(a)}_{k+1} := \Theta^{(a)}_{k+1} \setminus \left\{b \in \Theta^{(a)}_{k+1} : Q_{n_{k+1}}(b) \cap E_{d-k-1} \neq \emptyset\right\}$. Then $|\tilde{\Theta}^{(a)}_{k+1}| \geq |\Theta^{(a)}_{k+1}| - 100N \geq \frac{1}{2}\beta \left(\frac{n'_{k+1}}{n_{k+1}}\right)^\alpha$, when $C_{\varepsilon,N}$ is large enough; and for each $b \in \tilde{\Theta}^{(a)}_{k+1}$, $Q_{n_{k+1}}(b) \cap E_i \neq \emptyset$ implies $i \leq d-k-2$. Then

$$\left|\bigcup_{a \in \Theta'^{(a_0)}_{k+1}} \tilde{\Theta}^{(a)}_{k+1}\right| = \sum_{a \in \Theta'^{(a_0)}_{k+1}} |\tilde{\Theta}^{(a)}_{k+1}| \geq \left(\frac{\beta}{2}\right)^2 \left(\frac{n_k}{n_{k+1}}\right)^\alpha. \tag{2.6.30}$$

Now let $\Theta_{k+1} := \bigcup_{a_0 \in \Theta_k} \bigcup_{a \in \Theta'^{(a_0)}_{k+1}} \tilde{\Theta}^{(a)}_{k+1}$. Then Condition 2 to 6 hold for $k+1$

obviously. As for Condition 1,

$$|\Theta_{k+1}| = \sum_{a_0\in\Theta_k}\left|\bigcup_{a\in\Theta'^{(a_0)}_{k+1}}\tilde{\Theta}^{(a)}_{k+1}\right| \geq |\Theta_k|\left(\frac{\beta}{2}\right)^2\left(\frac{n_k}{n_{k+1}}\right)^\alpha \geq \left(\frac{\beta}{2}\right)^{2k+4}\left(\frac{n}{n_{k+1}}\right)^\alpha, \tag{2.6.31}$$

where the second inequality is true since Condition 1 holds for k.

Inductively, we have constructed Θ_M such that

1. $|\Theta_M| \geq \left(\frac{\beta}{2}\right)^{2M+2}\left(\frac{n}{n_M}\right)^\alpha$.

2. For any $a \in \Theta_M$, we have $|u(a)| \geq (K+11)^{-24(M+1)n}$.

3. For any $a, a' \in \Theta_M$ with $a \neq a'$, we have $Q_{n_M}(a) \cap Q_{n_M}(a') = \emptyset$.

4. For any $a \in \Theta_M$, we have $Q_{n_M}(a) \subset Q_n$.

5. For any $a \in \Theta_M$ and $d - M \leq i \leq d$, we have $E_i \cap Q_{n_M}(a) = \emptyset$.

As $l_{d-k-1}^{1+2\varepsilon} \leq l_{d-k}$ for each $0 \leq k < M$, we have $n_M \leq l_d^{\left(\frac{1}{1+2\varepsilon}\right)^M} \leq n^{\left(\frac{1}{1+2\varepsilon}\right)^M}$. Note that $n_M > n^{\frac{1}{3}(1-4\varepsilon)}$, thus $\left(\frac{1}{1+2\varepsilon}\right)^M \geq \frac{1}{3}(1-4\varepsilon)$. From this we have

$$M < 2\varepsilon^{-1}. \tag{2.6.32}$$

Since $l_{d-M-1}^{1+2\varepsilon} \leq l_{d-M}$ and $l_{d-M} \geq l_1 \geq C_{\varepsilon,N}$ we have $l_{d-M-1} < n_M^{1-\varepsilon}$ when $C_{\varepsilon,N}$ is large enough. Then for each $a \in \Theta_M$, by Condition 5 we have that E is $(1, 2\varepsilon)$-normal in

$Q_{n_M}(a)$. For any $a \in \Theta_M$, we apply Theorem 2.5.1 to $Q_{n_M}(a)$, then

$$\left|\left\{b \in Q_{n_M}(a) : |u(b)| \geq (K+11)^{-24(M+1)n} \exp(-C_2 n_M^3)\right\} \setminus E\right| \geq C_3 \frac{n_M^2}{\log(n_M)}. \tag{2.6.33}$$

Let $C_{\varepsilon,K} = C_2 + 96\log(K+11)\varepsilon^{-1}$. From (2.6.33), (2.6.32) and $n^{\frac{1}{3}(1-4\varepsilon)} < n_M < n^{\frac{1}{3}}$, we have

$$|\{b \in Q_{n_M}(a) : |u(b)| \geq \exp(-C_{\varepsilon,K} n)\} \setminus E| \geq C_3 \frac{n_M^2}{\log(n_M)}. \tag{2.6.34}$$

Since $Q_{n_M}(a) \cap Q_{n_M}(a') = \emptyset$ when $a \neq a' \in \Theta_M$, in total we have

$$\begin{aligned}|\{b \in Q_n : |u(b)| \geq \exp(-C_{\varepsilon,K} n)\} \setminus E| &\geq C_3 \frac{n_M^2}{\log(n_M)} |\Theta_M| \\ &\geq C_3 \left(\frac{\beta}{2}\right)^{2M+2} n^{\frac{2}{3}(1-4\varepsilon)+\frac{2}{3}\alpha} (\log(n_M))^{-1} \geq n^p, \end{aligned} \tag{2.6.35}$$

where the last inequality holds by taking ε small enough, and then $C_{\varepsilon,N}$ large enough (recall that we require $n > C_{\varepsilon,N}^4$). $\square$

2.7 Proofs of auxiliary lemmas

2.7.1 Auxiliary lemmas for the framework

In our general framework several results from [DS20] are used, and some of them are also used in Section 2.7.3 below as well. For the convenience of readers we record them here.

There are a couple of results from linear algebra. The first of them is an estimate on the number of almost orthonormal vectors, which appears in [Tao] as well as [DS20].

Lemma 2.7.1 ([Tao][DS20, Lemma 5.2]). *Assume $v_1, \cdots, v_m \in \mathbb{R}^n$ such that $|v_i \cdot v_j - \mathbb{1}_{i=j}| \leq (5n)^{-\frac{1}{2}}$, then $m \leq \frac{5-\sqrt{5}}{2} n$.*

The second one is about the variation of eigenvalues.

Lemma 2.7.2 ([DS20, Lemma 5.1]). *Suppose the real symmetric $n \times n$ matrix A has eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n \in \mathbb{R}$ with orthonormal eigenbasis $v_1, \cdots, v_n \in \mathbb{R}^n$. If*

1. $1 \leq i \leq j \leq n$, $1 \leq k \leq n$
2. $0 < r_1 < r_2 < r_3 < r_4 < r_5 < 1$
3. $r_1 \leq c \min\{r_3 r_5, r_2 r_3 / r_4\}$ *where* $c > 0$ *is a universal constant*
4. $0 < \lambda_j \leq \lambda_i < r_1 < r_2 < \lambda_{i-1}$
5. $v_{j,k}^2 \geq r_3$
6. $\sum_{r_2 < \lambda_\ell < r_5} v_{\ell,k}^2 \leq r_4$

then the i-th largest eigenvalue λ_i' (counting with multiplicity) of $A + e_k e_k^\dagger$ is at least r_1, where e_k is the k-th standard basis element and $e_k^\dagger$ is its transpose.

We then state the generalized Sperner's theorem, used in the proof of our 3D Wegner estimate (Lemma 2.3.5).

Theorem 2.7.3 ([DS20, Theorem 4.2])**.** *Suppose $\rho \in (0,1]$, and $\mathcal{A}$ is a set of subsets of $\{1,\cdots,n\}$ satisfying the following. For every $A \in \mathcal{A}$, there is a set $B(A) \subset \{1,\cdots,n\} \setminus A$ such that $|B(A)| \geq \rho(n-|A|)$, and $A' \cap B(A) = \emptyset$ for any $A \subset A' \in \mathcal{A}$. Then*

$$|\mathcal{A}| \leq 2^n n^{-\frac{1}{2}} \rho^{-1}. \tag{2.7.1}$$

For the next several results, in [DS20] they are stated and proved in the 2D lattice setting, but the proofs work, essentially verbatim, in the 3D setting.

The following covering lemma is used in the multi-scale analysis. Recall that by "dyadic" we mean an integer power of 2.

Lemma 2.7.4 ([DS20, Lemma 8.1])**.** *There is a constant $C > 1$ such that following holds. Suppose $K \geq 1$ is an integer, $\alpha \geq C^K$ is a dyadic scale, $L_0 \geq \alpha L_1 \geq L_1 \geq \alpha L_2 \geq L_2$ are dyadic scales, $Q \subset \mathbb{Z}^3$ is an L_0-cube, and $Q''_1, \cdots Q''_K \subset Q$ are L_2-cubes. Then there is a dyadic scale $L_3 \in [L_1, \alpha L_1]$ and disjoint L_3-cubes $Q'_1, \cdots, Q'_K \subset Q$, such that for each Q''_k there is Q'_j with $Q''_k \subset Q'_j$ and $\operatorname{dist}(Q''_k, Q \setminus Q'_j) \geq \frac{1}{8} L_3$.*

We need the following continuity of resolvent estimate. It is stated in a slightly different way from [DS20, Lemma 6.4], so we add a proof here.

Lemma 2.7.5 ([DS20, Lemma 6.4])**.** *If for $\lambda \in \mathbb{R}$, $\alpha > \beta > 0$, and a cube $Q \subset \mathbb{Z}^3$, we have*

$$|(H_Q - \lambda)^{-1}(a,b)| \leq \exp(\alpha - \beta|a-b|) \text{ for } a, b \in Q, \tag{2.7.2}$$

then for λ' *with* $|\lambda' - \lambda| \le \frac{1}{2}|Q|^{-1}\exp(-\alpha)$, *we have*

$$|(H_Q - \lambda')^{-1}(a,b)| \le 2\exp(\alpha - \beta|a-b|) \text{ for } a, b \in Q. \tag{2.7.3}$$

Proof. We first prove (2.7.3) assuming λ' is not an eigenvalue of H_Q. By resolvent identity we have,

$$(H_Q - \lambda')^{-1} = (H_Q - \lambda)^{-1} + (H_Q - \lambda')^{-1}(\lambda' - \lambda)(H_Q - \lambda)^{-1}. \tag{2.7.4}$$

Let $\gamma = \max_{a,b\in Q}\exp(\beta|a-b| - \alpha)|(H_Q - \lambda')^{-1}(a,b)|$. Then for any $a, b \in Q$,

$$\begin{aligned}
&|(H_Q - \lambda')^{-1}(a,b)| \\
\le&|(H_Q - \lambda)^{-1}(a,b)| + |\lambda' - \lambda|\sum_{c\in Q}|(H_Q - \lambda')^{-1}(a,c)||(H_Q - \lambda)^{-1}(c,b)| \\
\le&\exp(\alpha - \beta|a-b|) + |\lambda' - \lambda|\sum_{c\in Q}\exp(\alpha - \beta|a-c|)\exp(\alpha - \beta|c-b|)\gamma \\
\le&\exp(\alpha - \beta|a-b|) + |\lambda' - \lambda||Q|\exp(2\alpha - \beta|a-b|)\gamma \\
\le&\exp(\alpha - \beta|a-b|) + \frac{1}{2}\exp(\alpha - \beta|a-b|)\gamma.
\end{aligned} \tag{2.7.5}$$

This implies $\gamma \le 1 + \frac{1}{2}\gamma$ and thus $\gamma \le 2$ and (2.7.3) follows.

Now we can deduce that $|\det(H_Q - \lambda')^{-1}|$ is uniformly bounded for λ' that is not an eigenvalue of H_Q and satisfies $|\lambda' - \lambda| \le \frac{1}{2}|Q|^{-1}\exp(-\alpha)$. By continuity of the determinant (as a function of λ'), we conclude that H_Q has no eigenvalue in $\left[\lambda - \frac{1}{2}|Q|^{-1}\exp(-\alpha), \lambda + \frac{1}{2}|Q|^{-1}\exp(-\alpha)\right]$. Thus our conclusion follows. $\square$

We also need the following result to deduce exponential decay of the resolvent in a cube from the decay of the resolvent in subcubes.

Lemma 2.7.6 ([DS20, Lemma 6.2]). *Suppose*

1. $\varepsilon > \delta > 0$ *are small,*

2. $K \geq 1$ *is an integer and* $\lambda \in [0, 13]$*,*

3. $L_0 \geq \cdots \geq L_6$ *are large enough (depending on* ε, δ, K*) with* $L_k^{1-\varepsilon} \geq L_{k+1}$ *,*

4. $1 \geq m \geq 2L_5^{-\delta}$ *represents the exponential decay rate,*

5. $Q \subset \mathbb{Z}^3$ *is an* L_0*-cube,*

6. $Q'_1, \cdots, Q'_K \subset Q$ *are disjoint* L_2*-cubes with* $\|(H_{Q'_k} - \lambda)^{-1}\| \leq \exp(L_4)$*,*

7. *for all* $a \in Q$*, one of the following holds*

   - *there is* Q'_k *with* $a \in Q'_k$ *and* $\operatorname{dist}(a, Q \setminus Q'_k) \geq \frac{1}{8} L_2$

   - *there is an* L_5*-cube* $Q'' \subset Q$ *such that* $a \in Q''$, $\operatorname{dist}(a, Q \setminus Q'') \geq \frac{1}{8} L_5$*, and* $|(H_{Q''} - \lambda)^{-1}(b, b')| \leq \exp(L_6 - m|b - b'|)$ *for* $b, b' \in Q''$*.*

Then $|(H_Q - \lambda)^{-1}(a, a')| \leq \exp(L_1 - \tilde{m}|a - a'|)$ *for* $a, a' \in Q$ *where* $\tilde{m} = m - L_5^{-\delta}$*.*

2.7.2 The principal eigenvalue

This section sets up the base case in the induction proof of Theorem 2.3.10. We follow [DS20, Section 7], and generalize their result to higher dimensions. We take $d \in \mathbb{Z}$, $d > 2$, and denote $Q_n := \left\{a \in \mathbb{Z}^d : \|a\|_\infty \leq n\right\}$ instead.

Theorem 2.7.7. *Let $\overline{V} : Q_n \to [0,1]$ be any potential function, and $R > 0$ large enough, such that for any $a \in Q_n$, there exists $b \in Q_n$ with $\overline{V}(b) = 1$ and $|a - b| < R$. Let $\overline{H} : \ell^2(Q_n) \to \ell^2(Q_n)$, $\overline{H} = -\Delta + \overline{V}$, with Dirichlet boundary condition. Then its principal eigenvalue is no less than CR^{-d}, where C is a constant depending only on d.*

Proof. Let λ_0 denote the principal eigenvalue, then by e.g. [Eva10, Exercise 6.14] we have

$$\lambda_0 = \sup_{u:Q_n \to \mathbb{R}_+} \min_{Q_n} \frac{\overline{H}u}{u}. \tag{2.7.6}$$

Hence we lower bound λ_0 by constructing a function u. Let $\tilde{G} : \mathbb{Z}^d \to \mathbb{R}$ be the lattice Green's function; i.e. for any $a \in \mathbb{Z}^d$, $\tilde{G}(a)$ is the expected number of times that a (discrete time) simple random walk starting at $\mathbf{0}$ gets to a. Let $G := \tilde{G}/2d$. Then G is the only function such that $-\Delta G = \delta_{\mathbf{0}}$ (where $\delta_{\mathbf{0}}(\mathbf{0}) = 1$ and $\delta_{\mathbf{0}}(a) = 0$ for $a \neq \mathbf{0}$), and $0 \leq G(a) \leq G(\mathbf{0})$ for any $a \in \mathbb{Z}^d$. In addition, for any $a \in \mathbb{Z}^d$ with $a \neq \mathbf{0}$, by e.g. [LL10, Theorem 4.3.1] we have

$$G(a) = \frac{C_d}{|a|^{d-2}} + O\left(\frac{1}{|a|^d}\right), \tag{2.7.7}$$

where C_d is a constant depending only on d. Hence

$$\frac{4C_d}{5|a|^{d-2}} \leq G(a) \leq \frac{3C_d}{2|a|^{d-2}} \tag{2.7.8}$$

when $|a|$ is large enough.

We define $u : \mathbb{Z}^d \to \mathbb{R}_+$ as

$$u(a) := 1 + G(\mathbf{0}) - G(a) - \varepsilon_d R^{-d}|a|^2, \ \forall a \in \mathbb{Z}^d, \tag{2.7.9}$$

where $\varepsilon_d > 0$ is a small enough constant depending on d. Then

$$-\Delta u = -\delta_{\mathbf{0}} + 2d\varepsilon_d R^{-d}, \tag{2.7.10}$$

and for any $a \in \mathbb{Z}^d$ with $|a| < 3R$, we have $0 < u(a) \leq 1 + G(\mathbf{0})$.

Assume that R is large enough. For any a with $2R < |a| < 3R$, we have $u(a) \geq 1+G(\mathbf{0}) - \frac{3C_d}{2(2R)^{d-2}} - 9\varepsilon_d R^{-d+2}$; and for any a with $|a| < R$, $u(a) \leq 1+G(\mathbf{0}) - \frac{4C_d}{5R^{d-2}} \leq 1+G(\mathbf{0}) - \frac{3C_d}{2(2R)^{d-2}} - 9\varepsilon_d R^{-d+2}$, as long as $\varepsilon_d < \frac{C_d}{180}$ (also note that here we have $d > 2$). Thus

$$\min_{2R<|a|<3R} u(a) \geq \max_{|a|<R} u(a) \tag{2.7.11}$$

Now we define $u_0 : Q_n \to \mathbb{R}_+$, as $u_0(a) := \min_{|a-b|<3R, \overline{V}(b)=1} u(a-b), \ \forall a \in Q_n$. Pick an arbitrary $a' \in Q_n$, by (2.7.11) there is b' with $|a' - b'| \leq 2R$ such that $u_0(a') = u(a' - b')$ and $\overline{V}(b') = 1$. For any $a'' \in Q_n$ with $|a'' - a'| = 1$, since $|a'' - b'| \leq 2R + 1 < 3R$, we have

$$u_0(a'') = \min_{|a''-b|<3R, \overline{V}(b)=1} u(a'' - b) \leq u(a'' - b'). \tag{2.7.12}$$

Thus by (2.7.10), and Dirichlet boundary condition,

$$\begin{aligned}
\overline{H}u_0(a') =&2du_0(a') - \sum_{a''\in Q_n, |a'-a''|=1} u_0(a'') + \overline{V}(a')u_0(a') \\
\geq& 2du(a'-b') - \sum_{a''\in Q_n, |a'-a''|=1} u(a''-b') + \overline{V}(a')u(a'-b') \\
\geq& -\Delta u(a'-b') + \overline{V}(a')u(a'-b') \\
=& -\delta_{\mathbf{0}}(a'-b') + 2d\varepsilon_d R^{-d} + \overline{V}(a')u(a'-b') \\
\geq& 2d\varepsilon_d R^{-d}.
\end{aligned} \tag{2.7.13}$$

Since a' is arbitrary and $0 < u_0(a') \leq 1 + G(\mathbf{0})$, by (2.7.6) and letting $C = \frac{2d\varepsilon_d}{1+G(\mathbf{0})}$, we have $\lambda_0 \geq CR^{-d}$. □

Remark 2.7.8. The exponent in R^{-d} is optimal. Consider a potential $\overline{V}$ such that $\overline{V}(a) = 1$ only if $a \in \lceil R \rceil \mathbb{Z}^d \cap Q_n$ and $\overline{V}(a) = 0$ otherwise. In this case we have that $\lambda_0 \leq 8dR^{-d} + 4dn^{-1}$. To see this, consider the test function $\phi(a) = 1 - \overline{V}(a)$ for $a \in Q_n$ and use the variational principle $\lambda_0 \leq \frac{\langle \phi, \overline{H}\phi \rangle}{\|\phi\|_2^2}$.

Corollary 2.7.9. *Let $\overline{H}$, C be defined as in Theorem 2.7.7. Let $0 \leq \lambda < \frac{CR^{-d}}{2}$. Then $\|(\overline{H}-\lambda)^{-1}\| \leq \frac{2R^d}{C}$ and*

$$|(\overline{H}-\lambda)^{-1}(a,b)| \leq \frac{2R^d}{C} \exp\left(-\frac{CR^{-d}}{8d+2}|a-b|\right) \tag{2.7.14}$$

for any $a, b \in Q_n$.

Proof. As the principal eigenvalue of $\overline{H}$ is no less than CR^{-d}, we have $\|(\overline{H}-\lambda)^{-1}\| \leq$

$\frac{2R^d}{C}$. Let $T := I - \frac{1}{4d+1}(\overline{H} - \lambda)$. Since any eigenvalue of $\overline{H}$ is in $\left[CR^{-d}, 4d+1\right]$, the eigenvalues of T are in $\left[0, 1 - \frac{C}{8d+2}R^{-d}\right]$, so $\|T\| \leq 1 - \frac{C}{8d+2}R^{-d}$.

Note that for each $i > 0$ and $a, b \in Q_n$, $T^i(a,b) = 0$ if $|a - b| > i$. Then we have

$$|(\overline{H} - \lambda)^{-1}(a,b)| = (4d+1)^{-1}|(I-T)^{-1}(a,b)| \leq (4d+1)^{-1}\sum_{i \geq 0}|T^i(a,b)|$$
$$= (4d+1)^{-1}\sum_{i \geq |a-b|}|T^i(a,b)| \leq (4d+1)^{-1}\sum_{i \geq |a-b|}\|T\|^i \leq \frac{2R^d}{C}\exp\left(-\frac{CR^{-d}}{8d+2}|a-b|\right), \tag{2.7.15}$$

so the corollary follows. □

Finally, we have the following result, which implies the base case in the induction proof of Theorem 2.3.10.

Proposition 2.7.10. *Let $d = 3$, and V be the Bernoulli potential, i.e. $\mathbb{P}(V(a) = 0) = \mathbb{P}(V(a) = 1) = \frac{1}{2}$ for each $a \in \mathbb{Z}^3$ independently. For any $0 < \delta < \frac{1}{10}$ and $\varepsilon > 0$, there exists $C_{\delta,\varepsilon}$ such that for any $n > C_{\delta,\varepsilon}$ and $0 \leq \lambda < \frac{Cn^{-\frac{3\delta}{10}}}{2}$, with probability at least $1 - n^{-1}$ the following is true.*

Take any $V' : \mathbb{Z}^3 \to [0,1]$ such that $V'_{Q_n \cap \lceil \varepsilon^{-1} \rceil \mathbb{Z}^3} = V_{Q_n \cap \lceil \varepsilon^{-1} \rceil \mathbb{Z}^3}$. Let H'_{Q_n} be the restriction of $-\Delta + V'$ on Q_n with Dirichlet boundary condition. Then we have

$$\|(H'_{Q_n} - \lambda)^{-1}\| \leq \exp(n^{2\delta}), \tag{2.7.16}$$

and

$$|(H'_{Q_n} - \lambda)^{-1}(a,b)| \le n^{2\delta} \exp(-n^{-\delta}|a-b|) \textit{ for any } a,b \in Q_n. \tag{2.7.17}$$

Proof. Let $R := n^{\frac{\delta}{10}}$, and let A denote the following event:

$$\forall a \in Q_n, \exists b \in Q_n \cap \lceil \varepsilon^{-1} \rceil \mathbb{Z}^3, \text{ s.t. } |a-b| \le R, V(b) = 1. \tag{2.7.18}$$

Then A only depends on $V_{Q_n \cap \lceil \varepsilon^{-1} \rceil \mathbb{Z}^3}$.

Using Corollary 2.7.9 with $d = 3$, we have that (2.7.16) and (2.7.17) hold under the event A, when n is large enough.

Finally, since there are $(2n+1)^3$ points in Q_n, and inside each ball of radius R, there are at least $\frac{1}{8} n^{\frac{3\delta}{10}} \varepsilon^3$ points in $\lceil \varepsilon^{-1} \rceil \mathbb{Z}^3 \cap Q_n$, we have $\mathbb{P}(A^c) \le (2n+1)^3 2^{-\frac{1}{8} n^{\frac{3\delta}{10}} \varepsilon^3} \le n^{-1}$, when n is large enough. $\square$

2.7.3 Deducing Anderson localization from the resolvent estimate

The arguments in this section originally come from [BK05, Section 7] (see also [GK12, Section 6, 7] and [Bou05, Section 6]). These previous works are about the continuous space model. For completeness and for the reader's convenience, we adapt the arguments for the lattice model, thus deducing Theorem 2.1.1 from Theorem 2.3.1.

As in Section 2.3, in this section, by "dyadic" we mean an integer power of 2, and by "dyadic cube", we mean a cube $Q_{2^n}(a)$ for some $a \in 2^{n-1}\mathbb{Z}^3$ and $n \in \mathbb{Z}_+$.

For any $k \in \mathbb{Z}_+$, we define

$$\Omega_k := \{u : \mathbb{Z}^3 \to \mathbb{R} : |u(a)| \leq k(1+|a|)^k, \ \forall a \in \mathbb{Z}^3, \ \text{and} \ u(\mathbf{0}) = 1\}. \tag{2.7.19}$$

Since the law of H is invariant under translation, to prove Theorem 2.1.1, it suffices to show that for any $k \in \mathbb{Z}_+$, almost surely

$$\inf_{t>0} \sup_{a \in \mathbb{Z}^3} \exp(t|a|)|u(a)| < \infty, \tag{2.7.20}$$

for any $u \in \Omega_k$ and $\lambda \in [0, \lambda_*]$ with $Hu = \lambda u$.

Denote $\mathcal{I} = (0, \lambda_*)$. We first see that it suffices to prove (2.7.20) for any $u \in \Omega_k$ and $\lambda \in \mathcal{I}$ with $Hu = \lambda u$, by applying the following lemma to $\lambda = 0$ and $\lambda = \lambda_*$.

Lemma 2.7.11. *Suppose $\lambda \in [0, \lambda_*]$ and $k \in \mathbb{Z}_+$. Then almost surely, there is no $u \in \Omega_k$ with $Hu = \lambda u$.*

Proof. Let $L_i = 2^i$ for $i \in \mathbb{Z}_+$. By Theorem 2.3.1 and the Borel-Cantelli lemma, almost surely, there exists $i' > 0$, such that for any $i > i'$,

$$\left|(H_{Q_{L_i}} - \lambda)^{-1}(a,b)\right| \leq \exp\left(L_i^{1-\lambda_*} - \lambda_*|a-b|\right), \ \forall a, b \in Q_{L_i}. \tag{2.7.21}$$

Assume there exists $u \in \Omega_k$ with $Hu = \lambda u$. For each large enough i we have

$$
\begin{aligned}
|u(\mathbf{0})| = & \left| \sum_{\substack{a \in Q_{L_i}, a' \in \mathbb{Z}^3 \setminus Q_{L_i} \\ |a - a'| = 1}} (H_{Q_{L_i}} - \lambda)^{-1}(\mathbf{0}, a) u(a') \right| \\
\leq & 6 \cdot (2L_i + 1)^2 \exp\left(-\frac{\lambda_* L_i}{2} \right) k (1 + \sqrt{3} L_i)^k
\end{aligned}
\tag{2.7.22}
$$

which converges to zero as $i \to \infty$. Thus $u(\mathbf{0}) = 0$, which contradicts with the fact that $u \in \Omega_k$. □

Let us fix $k \in \mathbb{Z}_+$ and denote by $\sigma_k(H)$ the set of all $\lambda \in \mathcal{I}$, such that $Hu = \lambda u$ for some $u \in \Omega_k$. For each $L \in \mathbb{Z}_+$, denote by $\sigma(H_{Q_L})$ the set of eigenvalues of H_{Q_L}. The first key step is to prove that for any large enough L, with high probability, the distance between any $\lambda \in \sigma_k(H)$ and $\sigma(H_{Q_L})$ is small, exponentially in L.

Proposition 2.7.12. *There exist $\kappa', c_1 > 0$ such that for any dyadic L large enough, we can find a V_{Q_L}-measurable event $\mathcal{E}_{wloc}^{(L)}$, such that*

$$
\mathbb{P}\left[\mathcal{E}_{wloc}^{(L)} \right] \geq 1 - L^{-\kappa'},
\tag{2.7.23}
$$

and under the event $\mathcal{E}_{wloc}^{(L)}$, we have $\operatorname{dist}(\lambda, \sigma(H_{Q_L}) \cap \mathcal{I}) \leq \exp(-c_1 L)$ *for any*

$$
\lambda \in \sigma_k(H) \cap \left[\exp(-c_1 \sqrt{L}), \lambda_* - \exp(-c_1 \sqrt{L}) \right].
$$

The next key step is to strengthen Proposition 2.7.12 so that each $\lambda \in \sigma_k(H)$ is not only exponentially close to $\sigma(H_{Q_L})$, but also exponentially close to a finite subset $S \subset \sigma(H_{Q_L})$ with $|S| < L^{\delta'}$ for arbitrarily small δ'.

Proposition 2.7.13. *For any $\delta' > 0$, there exist $\kappa'', c_2 > 0$ such that for each dyadic L large enough (depending on δ'), we can find a V_{Q_L}-measurable event $\mathcal{E}^{(L)}_{sloc}$ with*

$$\mathbb{P}\left[\mathcal{E}^{(L)}_{sloc}\right] \geq 1 - L^{-\kappa''}, \tag{2.7.24}$$

and under the event $\mathcal{E}^{(L)}_{sloc}$, there exists a finite set $S \subset \sigma(H_{Q_L}) \cap \mathcal{I}$ with $|S| < L^{\delta'}$ such that $\operatorname{dist}(\lambda, S) \leq \exp(-c_2 L)$ *for any $\lambda \in \sigma_k(H) \cap [\exp(-L^{c_2}), \lambda_* - \exp(-L^{c_2})]$.*

Proposition 2.7.12 and 2.7.13 are discrete versions of [Bou05, Lemma 6.1] and [Bou05, Lemma 6.4] respectively. See also [GK12, Proposition 6.3, 6.9]. Now we leave the proofs of these two propositions to the next two subsections, and prove localization assuming them.

Proof of Theorem 2.1.1. We apply Proposition 2.7.13 with $\delta' < \kappa_0$ where κ_0 is the constant in Theorem 2.3.1. Take large enough dyadic L, and consider the annulus $A_L = Q_{5L} \setminus Q_{2L}$. We cover A_L by $2L$-cubes $\{Q^{(j)} : 1 \leq j \leq 1000\}$ that are disjoint with Q_L, such that for each $a \in A_L$ there is $1 \leq j \leq 1000$ with $a \in Q^{(j)}$ and $\operatorname{dist}(a, \mathbb{Z}^3 \setminus Q^{(j)}) \geq \frac{1}{8}L$. Apply Theorem 2.3.1 to each of $Q^{(j)}$'s and to each energy $\lambda \in S \subset \sigma(H_{Q_L}) \cap \mathcal{I}$, we have

$$\mathbb{P}\left[\mathcal{E}^{(L)}_{ann} \middle| \, \mathcal{E}^{(L)}_{sloc}\right] \geq 1 - 1000 L^{\delta' - \kappa_0} \tag{2.7.25}$$

where $\mathcal{E}_{ann}^{(L)}$ denotes the event:

$$\left|(H_{Q^{(j)}}-\lambda)^{-1}(a,b)\right| \le \exp\left(L^{1-\lambda_*}-\lambda_*|a-b|\right)$$
$$\forall 1\le j\le 1000,\ \forall a,b\in Q^{(j)},\ \text{and } \forall\lambda\in S. \tag{2.7.26}$$

Then by Proposition 2.7.13 we have

$$\mathbb{P}\left[\mathcal{E}_{ann}^{(L)}\cap\mathcal{E}_{sloc}^{(L)}\right]\ge(1-L^{-\kappa''})(1-1000L^{\delta'-\kappa_0})\ge 1-L^{-\kappa'''}, \tag{2.7.27}$$

for some constant $\kappa'''>0$ and large enough L.

Under the event $\mathcal{E}_{ann}^{(L)}\cap\mathcal{E}_{sloc}^{(L)}$, we take any $u\in\Omega_k$ with $Hu=\lambda u$ and $\lambda\in[\exp(-L^{c_2}),\lambda_*-\exp(-L^{c_2})]$, and $\lambda'\in S$ with $|\lambda-\lambda'|<\exp(-c_2L)$. Thus using Lemma 2.7.5, we have

$$\begin{aligned}\|u\|_{\ell^\infty(A_L)} &\le 2\exp\left(L^{1-\lambda_*}-\frac{1}{8}\lambda_*L\right)\|u\|_{\ell^1(Q_{6L})}\\ &\le 2\exp\left(L^{1-\lambda_*}-\frac{1}{8}\lambda_*L\right)k(6\sqrt{3}L+1)^k(12L+1)^3\le\exp(-c'L)\end{aligned} \tag{2.7.28}$$

for some constant $c'<\frac{\lambda_*}{8}$ and large enough L.

Now we consider the event

$$\mathcal{E}_{loc}=\bigcup_{i'\ge 0}\bigcap_{i\ge i'}(\mathcal{E}_{ann}^{(2^i)}\cap\mathcal{E}_{sloc}^{(2^i)}). \tag{2.7.29}$$

We have $\mathbb{P}[\mathcal{E}_{loc}] = 1$ by (2.7.27). Note that for any $\lambda \in \mathcal{I}$, we have

$$\lambda \in [\exp(-L^{c_2}), \lambda_* - \exp(-L^{c_2})]$$

for large enough L. We also have that $\bigcup_{i \geq i'} A_{2^i} = \mathbb{Z}^3 \setminus Q_{2^{i'+1}}$ for any $i' \in \mathbb{Z}_+$. By (2.7.28) we have that (2.7.20) holds under the event $\mathcal{E}_{loc}$. Then localization is proved. $\square$

2.7.4 The first spectral reduction

For simplicity of notations, for any $\lambda \in \mathbb{R}$, dyadic scale L, and $a \in \mathbb{Z}^3$, we say $Q_L(a)$ is λ-*good* if

$$\left|(H_{Q_L(a)} - \lambda)^{-1}(b, b')\right| \leq \exp\left(L^{1-\lambda_*} - \lambda_*|b - b'|\right), \ \forall b, b' \in Q_L(a). \tag{2.7.30}$$

Otherwise, we call it λ-*bad*. By Theorem 2.3.1, for any large enough dyadic scale L and $\lambda \in [0, \lambda_*]$, we have

$$\mathbb{P}[Q_L(a) \text{ is } \lambda\text{-bad}] \leq L^{-\kappa_0}. \tag{2.7.31}$$

Proof of Proposition 2.7.12. Throughout the proof, we use C to denote large universal constants. For a dyadic scale L, we construct a graph G_L whose vertices are all the dyadic $2L$-cubes. The edges are given as follows: for any $a \neq a' \in \frac{L}{2}\mathbb{Z}^3$, there is an edge connecting $Q_L(a)$ and $Q_L(a')$ if and only if $Q_L(a) \cap Q_L(a') \neq \emptyset$.

Fix large dyadic scale L. Take the dyadic scale $L_0 \in \left\{\sqrt{L}, \sqrt{2L}\right\}$. For any $\lambda \in \mathcal{I}$, denote by $\mathcal{E}^{\lambda}_{per}$ the event that there is a path of λ-bad $2L_0$-cubes $\overline{Q}_1, \cdots, \overline{Q}_m$ in G_{L_0} such that

$$\overline{Q}_1 \cap Q_{\frac{L}{2}} \neq \emptyset \text{ and } \overline{Q}_m \cap Q_L = \emptyset. \tag{2.7.32}$$

Under the event $\mathcal{E}^{\lambda}_{per}$, suppose that $\Gamma_0 = (\overline{Q}_1, \cdots, \overline{Q}_m)$ is such a path with the shortest length. Since $\mathrm{dist}(Q_{\frac{L}{2}}, \mathbb{Z}^3 \setminus Q_L) \geq \frac{L}{2}$, we have $m \geq \frac{L}{4\sqrt{3}L_0}$. By definition of dyadic cubes and that Γ_0 has the shortest length, there are at least $\frac{m}{1000}$ disjoint λ-bad cubes in Γ_0. Hence,

$$\mathbb{P}[\mathcal{E}^{\lambda}_{per}] \leq \sum_{m \geq \frac{L}{4\sqrt{3}L_0}} CL^3 1000^m (L_0^{-\kappa_0})^{\frac{m}{1000}} \leq 2CL^3 (1000 L_0^{-\frac{\kappa_0}{1000}})^{\frac{L}{4\sqrt{3}L_0}} \leq L_0^{-c' L_0} \tag{2.7.33}$$

for some $c' > 0$. Here the first inequality is by (2.7.31), and counting the total number of G_{L_0} paths with length m and one end intersecting $Q_{\frac{L}{2}}$.

Claim 2.7.14. *Under the event $(\mathcal{E}^{\lambda}_{per})^c$, any $\lambda' \in \sigma_k(H)$ with $|\lambda' - \lambda| \leq \exp(-L_0^{1-\frac{\lambda_*}{2}})$ satisfies $\mathrm{dist}(\lambda', \sigma(H_{Q_{\frac{3}{2}L}})) \leq \exp(-\epsilon' L_0)$ for a universal constant $\epsilon' > 0$.*

Proof. Denote the set of all the λ-bad L_0-cubes contained in $Q_{\frac{3}{2}L}$ by $\mathcal{S}$. We consider $\mathbb{Z}^3$ as a graph with edges between nearest neighbors. Consider the set $S_0 := (\bigcup \mathcal{S}) \cup Q_{\frac{L}{2}} \subset Q_{\frac{3}{2}L}$. Let S_1 be the maximal connected component of S_0 which contains $Q_{\frac{L}{2}}$. Then $(\mathcal{E}^{\lambda}_{per})^c$ implies $S_1 \subset Q_{L+2L_0}$. Denote

$$\partial^- S_1 = \{a \in S_1 : |a - a'| = 1 \text{ for some } a' \in \mathbb{Z}^3 \setminus S_1\}, \tag{2.7.34}$$

and

$$\partial^+ S_1 = \{a \in \mathbb{Z}^3 \setminus S_1 : |a - a'| = 1 \text{ for some } a' \in S_1\}. \tag{2.7.35}$$

Assume λ' satisfies the hypothesis in the claim, then there is $u \in \Omega_k$ such that $Hu = \lambda' u$. For any $a' \in \partial^- S_1 \cup \partial^+ S_1$, there is a dyadic L_0-cube Q' such that $a' \in Q'$ and $\mathrm{dist}(a', \mathbb{Z}^3 \setminus Q') \geq \frac{1}{8} L_0$. By maximality of S_1, we have Q' is λ-good. Thus by Lemma 2.7.5,

$$\begin{aligned} |u(a')| \leq & 2\exp(L_0^{1-\lambda_*} - \frac{1}{8}\lambda_* L_0)\|u\|_{\ell^1(Q_{L+4L_0})} \\ \leq & 2\exp(L_0^{1-\lambda_*} - \frac{1}{8}\lambda_* L_0)(2L + 8L_0 + 1)^3 k(\sqrt{3}L + 4\sqrt{3}L_0 + 1)^k \\ \leq & \exp(-\frac{1}{10}\lambda_* L_0) \end{aligned} \tag{2.7.36}$$

for large enough L_0. Let $u_* : Q_{\frac{3}{2}L} \to \mathbb{R}$ be defined by $u_* = u$ on S_1 and $u_* = 0$ on $Q_{\frac{3}{2}L} \setminus S_1$. Then

$$(H_{Q_{\frac{3}{2}L}} - \lambda')u_*(a) = \begin{cases} 0 & \text{if } a \in Q_{\frac{3}{2}L} \setminus (\partial^- S_1 \cup \partial^+ S_1), \\ \sum_{|a'-a|=1, a' \in \partial^+ S_1} u(a') & \text{if } a \in \partial^- S_1, \\ -\sum_{|a'-a|=1, a' \in \partial^- S_1} u(a') & \text{if } a \in \partial^+ S_1. \end{cases} \tag{2.7.37}$$

By (2.7.36), we have

$$\|(H_{Q_{\frac{3}{2}L}} - \lambda')u_*\|_{\ell^2(Q_{\frac{3}{2}L})} \leq 6(3L+1)^{\frac{3}{2}} \exp(-\frac{1}{10}\lambda_* L_0) \leq \exp(-\epsilon' L_0)\|u_*\|_{\ell^2(Q_{\frac{3}{2}L})} \tag{2.7.38}$$

for large enough L. Here, we used $\|u_*\|_{\ell^2(Q_{\frac{3}{2}L})} \geq 1$ since $\mathbf{0} \in S_1$ and $u(\mathbf{0}) = 1$. By expanding u_* into a linear combination of eigenvectors of $H_{Q_{\frac{3}{2}L}}$, (2.7.38) guarantees that there is an eigenvalue λ_0 of $H_{Q_{\frac{3}{2}L}}$ such that $|\lambda' - \lambda_0| \leq \exp(-\epsilon' L_0)$. Our claim follows. $\square$

Denote $\lambda^{(h)} = h \exp(-L_0)$ for $h \in \mathbb{Z}_+$ and let

$$\mathcal{E}^0_{trap} = \bigcap_{\lambda^{(h)} \in \mathcal{I}} (\mathcal{E}^{\lambda^{(h)}}_{per})^c. \tag{2.7.39}$$

Then by (2.7.33),

$$\mathbb{P}[\mathcal{E}^0_{trap}] \geq 1 - \lambda_* \exp(L_0) L_0^{-c' L_0} \geq 1 - L^{-10} \tag{2.7.40}$$

for large L.

Claim 2.7.15. *Under the event $\mathcal{E}^0_{trap}$, any $\lambda \in [0, \lambda_*] \cap \sigma_k(H)$ satisfies*

$$\operatorname{dist}(\lambda, \sigma(H_{Q_{\frac{3}{2}L}})) \leq \exp(-\epsilon' L_0). \tag{2.7.41}$$

Proof. For any $\lambda \in [0, \lambda_*]$, there exists an $h \in \mathbb{Z}_+$ such that $\lambda^{(h)} \in \mathcal{I}$ and $|\lambda - \lambda^{(h)}| \leq \exp(-L_0^{1 - \frac{\lambda_*}{2}})$. Our claim follows from Claim 2.7.14. $\square$

Let q be the smallest positive integer such that $2^{\frac{1}{q}} - 1 < \frac{\lambda_*}{2}$ and let $\tau = 2^{\frac{1}{q}} - 1$. Define $\tilde{L}_1 = L_0^{1+\tau}$ and $\tilde{L}_{i+1} = \tilde{L}_i^{1+\tau}$ for $i = 1, 2, \cdots, q-1$. Then $L \leq \tilde{L}_q = L_0^2 \leq 2L$. Let L_i be the (unique) dyadic scale such that $L_i \in [\tilde{L}_i, 2\tilde{L}_i)$ for each $i = 1, \cdots, q$.

Let $M_i = \frac{3}{2}L + C' \sum_{1\le j\le i} L_j$ for each $i = 1, \cdots, q$ and $M_0 = \frac{3}{2}L$. Here C' is a large constant to be determined. Then

$$M_i \le \frac{3}{2}L + 4C'iL \le \left(\frac{3}{2} + 4C'q\right) L \tag{2.7.42}$$

for each $0 \le i \le q$. In addition, we denote $M_{q+1} = 2^w L$ where w is the smallest integer with $2^w > 3 + 8C'q$, and let $L_{q+1} = L_q$.

For any $\lambda \in \mathcal{I}$ and any $j \in \{1, \cdots, q+1\}$, denote by $\mathcal{E}_{per}^{\lambda,j}$ the following event: there exists a path of λ-bad $2L_j$-cubes in G_{L_j}, say $\overline{Q}_1, \cdots, \overline{Q}_m$, such that

$$\begin{aligned} &\overline{Q}_i \subset Q_{M_j} \setminus Q_{M_{j-1}}, \ \forall i \in \{1, \cdots, m\}, \\ &\overline{Q}_1 \cap Q_{M_{j-1}+10L_j} \neq \emptyset, \\ &\overline{Q}_m \cap Q_{M_j - 10L_j} \neq \emptyset. \end{aligned} \tag{2.7.43}$$

Under the event $\mathcal{E}_{per}^{\lambda,j}$, suppose that $\Gamma_0 = (\overline{Q}_1, \cdots, \overline{Q}_m)$ in G_{L_j} is such a path with the shortest length. Since $\mathrm{dist}(Q_{M_{j-1}+10L_j}, \mathbb{Z}^3 \setminus Q_{M_j-10L_j}) \ge (C'-20)L_j$, we have $m \ge \frac{C'}{4}$ when C' is large enough. By definition of dyadic cubes and that Γ_0 has the shortest length, there are at least $\frac{m}{1000}$ disjoint λ-bad cubes in Γ_0. Hence,

$$\mathbb{P}[\mathcal{E}_{per}^{\lambda,j}] \le \sum_{m \ge \frac{C'}{4}} C(C'L)^3 1000^m (L_j^{-\kappa_0})^{\frac{m}{1000}} \le 2C(C'L)^3 (1000 L_j^{-\frac{\kappa_0}{1000}})^{\frac{C'}{4}} \le L^{-10}. \tag{2.7.44}$$

Here the first inequality is by (2.7.31) and counting the number of paths in G_{L_j} with length m and one end intersecting $Q_{M_{j-1}+10L_j}$, and the last inequality is by taking C' large enough.

By adapting the proof of Claim 2.7.14 we can get the following result.

Claim 2.7.16. *Under the event* $(\mathcal{E}_{per}^{\lambda,j})^c$*, any* $\lambda' \in \sigma_k(H)$ *with* $|\lambda' - \lambda| \leq \exp(-L_j^{1-\frac{\lambda_*}{2}})$ *satisfies* $\mathrm{dist}(\lambda', \sigma(H_{Q_{M_j}})) \leq \exp(-\epsilon'' L_j)$ *for a universal constant* $\epsilon'' > 0$.

Note that, given $\lambda \in \mathcal{I}$, the event $\mathcal{E}_{per}^{\lambda,j}$ is $V_{Q_{M_j} \setminus Q_{M_{j-1}}}$-measurable. Hence, the event $\mathcal{E}_{trap}^{j} := \left(\bigcup_{\lambda \in \sigma(H_{Q_{M_{j-1}}}) \cap \mathcal{I}} \mathcal{E}_{per}^{\lambda,j}\right)^c$ satisfies

$$\mathbb{P}[\mathcal{E}_{trap}^{j} | V_{Q_{M_{j-1}}}] \geq 1 - (M_{j-1}+1)^3 L^{-10} \geq 1 - L^{-6} \tag{2.7.45}$$

by (2.7.42) and (2.7.44) for large enough L. For each $0 \leq j \leq q+1$, $\mathcal{E}_{trap}^{j}$ is $V_{Q_{M_j}}$-measurable, thus the event $\mathcal{E}_{trap} := \bigcap_{0 \leq j \leq q+1} \mathcal{E}_{trap}^{j}$ is $V_{Q_{M_{q+1}}}$-measurable. By (2.7.40) and (2.7.45), we have

$$\mathbb{P}[\mathcal{E}_{trap}] \geq 1 - (q+2)L^{-6} \geq 1 - L^{-5}. \tag{2.7.46}$$

Claim 2.7.17. *Under the event* $\mathcal{E}_{trap}$*, any* $\lambda \in [\exp(-\epsilon''' L_0/2), \lambda_* - \exp(-\epsilon''' L_0/2)] \cap \sigma_k(H)$ *satisfies*

$$\mathrm{dist}(\lambda, \sigma(H_{Q_{M_{q+1}}})) \leq \exp(-\epsilon''' L) \tag{2.7.47}$$

for some $\epsilon''' > 0$.

Proof. Let $\epsilon''' = \min\{\epsilon', \epsilon''\}$. Let $\lambda \in [\exp(-\epsilon''' L_0/2), \lambda_* - \exp(-\epsilon''' L_0/2)] \cap \sigma_k(H)$.

We inductively prove that, $\mathrm{dist}(\lambda, \sigma(H_{Q_{L_j}})) \leq \exp(-\epsilon''' L_j)$ for any $0 \leq j \leq q+1$. Thus, in particular, we have

$$\mathrm{dist}(\lambda, \sigma(H_{Q_{M_{q+1}}})) \leq \exp(-\epsilon''' L_{q+1}) \leq \exp(-\epsilon''' L), \tag{2.7.48}$$

and the claim follows.

For the case $j = 0$, by Claim 2.7.15, $\mathrm{dist}(\lambda, \sigma(H_{Q_{M_0}})) \leq \exp(-\epsilon' L_0)$. Assume the conclusion holds for some $j < q+1$, then $|\lambda - \lambda_0| \leq \exp(-\epsilon''' L_j)$ for some $\lambda_0 \in \sigma(H_{Q_{M_j}})$. As $\lambda \in [\exp(-\epsilon''' L_0/2), \lambda_* - \exp(-\epsilon''' L_0/2)]$, we must have $\lambda_0 \in \mathcal{I}$. Since $\tau < \frac{\lambda_*}{2}$, for L large enough we have $\epsilon''' L_j > L_{j+1}^{1-\frac{\lambda_*}{2}}$ and $|\lambda - \lambda_0| \leq \exp(-L_{j+1}^{1-\frac{\lambda_*}{2}})$. Thus Claim 2.7.16 implies $\mathrm{dist}(\lambda, \sigma(H_{Q_{M_{j+1}}})) \leq \exp(-\epsilon'' L_{j+1})$. $\square$

Finally, since $M_{q+1} = 2^w L$ and w is a constant, the proposition follows from Claim 2.7.17 and (2.7.46). $\square$

2.7.5 The second spectral reduction

For any positive integers $L'' > L'$, we denote the annulus $A_{L'',L'} = Q_{L''} \setminus Q_{L'}$. Take any $\delta > 0$. For $\lambda \in \mathcal{I}$ and $L'' > 2L'$, let $\mathcal{E}^{(\lambda)}_{L'',L'}$ denote the following event: there exists a subset $G^{(\lambda)}_{L'',L'} \subset A_{L'',L'}$ with $|G^{(\lambda)}_{L'',L'}| \leq (L')^{\frac{\delta}{2}}$ such that, for any $a \in A_{L'',2L'} \setminus G^{(\lambda)}_{L'',L'}$, there is a λ-good cube $Q_{L'''}(b) \subset A_{L'',L'}$ such that $\mathrm{dist}(a, Q_{L''} \setminus Q_{L'''}(b)) \geq \frac{1}{8} L'''$, and $(L')^{\frac{\delta}{10}} \leq L''' \leq L'$. Note that, $\mathcal{E}^{(\lambda)}_{L'',L'}$ is $V_{A_{L'',L'}}$-measurable.

Lemma 2.7.18. *Let $\varepsilon, \delta > 0$ be small enough. Suppose L', L'' are dyadic, satisfying*

$(L')^{1+\frac{1}{2}\varepsilon} < L'' < (L')^{1+\varepsilon}$*, and* L' *is large enough (depending on* ε, δ*). Then for any* $\lambda \in \mathcal{I}$ *we have* $\mathbb{P}[\mathcal{E}^{(\lambda)}_{L'',L'}] \geq 1 - (L')^{-10}$.

Proof. Let $\tilde{L}^{(0)} = L'$, $\tilde{L}^{(i+1)} = (\tilde{L}^{(i)})^{1-\varepsilon}$, and $L^{(i)}$ be the (unique) dyadic scale with $L^{(i)} \in [\tilde{L}^{(i)}, 2\tilde{L}^{(i)})$, for $i \in \mathbb{Z}_{\geq 0}$. Let $M' \in \mathbb{Z}_+$ such that $\frac{1}{10}\delta < (1-\varepsilon)^{M'} < \frac{1}{6}\delta$. For any dyadic $2L^{(M')}$-cube $Q \subset A_{L'',L'}$, we call it hereditary bad if there are λ-bad dyadic cubes $Q^{(0)}, \cdots, Q^{(M')} = Q$ such that, $Q^{(i+1)} \subset Q^{(i)} \subset A_{L'',L'}$ for each $0 \leq i \leq M'-1$ and $Q^{(i)}$ is a dyadic $2L^{(i)}$-cube. By (2.7.31), and the same arguments in the proof of Claim 2.3.11, the following is true. For small enough ε, there exists $N \in \mathbb{Z}_+$ depending on ε, δ, such that with probability at least $1 - (L')^{-10}$,

$$|\{Q \subset A_{L'',L'} : Q \text{ is a hereditary bad } 2L^{(M')}\text{-cube}\}| < N. \tag{2.7.49}$$

Let $G^{(\lambda)}_{L'',L'} = \bigcup\{Q \subset A_{L'',L'} : Q \text{ is a hereditary bad } 2L^{(M')}\text{-cube}\}$. Then (2.7.49) implies $|G^{(\lambda)}_{L'',L'}| \leq N(2L^{(M')}+1)^3 \leq (L')^{\frac{\delta}{2}}$ for large enough L'. For each $a \in A_{L'',2L'} \setminus G^{(\lambda)}_{L'',L'}$, there is $0 \leq i' \leq M'$ and a λ-good cube $Q_{L^{(i')}}(b) \subset A_{L'',L'}$ such that $\operatorname{dist}(a, Q_{L''} \setminus Q_{L^{(i')}}(b)) \geq \frac{1}{8}L^{(i')}$. Since $(L')^{\frac{\delta}{10}} \leq L^{(i')} \leq L'$, our claim follows. $\square$

For any large enough dyadic scales L', L'' with $(L')^{1+\frac{1}{2}\varepsilon} < L'' < (L')^{1+\varepsilon}$, we denote $\mathcal{E}^{supp}_{L'',L'} = \bigcap_{\lambda \in \sigma(H_{Q_{L'}}) \cap \mathcal{I}} \mathcal{E}^{(\lambda)}_{L'',L'}$. Then by Lemma 2.7.18, as each $\mathcal{E}^{(\lambda)}_{L'',L'}$ is $V_{A_{L'',L'}}$-measurable, we have

$$\mathbb{P}[\mathcal{E}^{supp}_{L'',L'}] \geq 1 - (L')^{-6}. \tag{2.7.50}$$

Proof of Proposition 2.7.13. In this proof we let $\varepsilon > 0$ be a small universal constant, and $\delta > 0$ be a number depending on δ'. Both of them are to be determined.

Now we fix dyadic scale L large enough (depending on ϵ, δ and thus depending on δ'). Let $\tilde{L}_0 = L$, $\tilde{L}_{i+1} = \tilde{L}_i^{1-\frac{3}{4}\varepsilon}$, and L_i be the (unique) dyadic scale with $L_i \in [\tilde{L}_i, 2\tilde{L}_i)$, for $i \in \mathbb{Z}_{\geq 0}$. Pick $M \in \mathbb{Z}_+$ such that $\frac{1}{10}\delta < (1-\frac{3}{4}\varepsilon)^M < \frac{1}{6}\delta$. Write $\overline{L_i} = \frac{1}{16}L_i$ for $0 \leq i \leq M$ and let

$$\mathcal{E}^{supp} = \bigcap_{0 \leq i \leq M-1} \mathcal{E}^{supp}_{L_i, \overline{L_{i+1}}}. \tag{2.7.51}$$

Then by (2.7.50),

$$\mathbb{P}[\mathcal{E}^{supp}] \geq 1 - M\left(\frac{L_M}{16}\right)^{-6} \geq 1 - L^{-\frac{\delta}{2}} \tag{2.7.52}$$

as L is large enough. For $0 \leq i \leq M$, denote by Θ_i the set of eigenvalues $\lambda \in \sigma(H_{Q_{L_i}})$ such that,

$$\lambda \in [(M-i+1)\exp(-L^{\frac{\delta}{20}}), \lambda_* - (M-i+1)\exp(-L^{\frac{\delta}{20}})], \tag{2.7.53}$$

and

$$\operatorname{dist}(\lambda, \sigma(H_{Q_{\overline{L_j}}})), \operatorname{dist}(\lambda, \sigma(H_{Q_{L_j}})) \leq 2^i \exp(-c'L_j) \quad \forall j \in \{i, i+1, \cdots, M\}. \tag{2.7.54}$$

Here the constant $c' = \frac{c_1}{20}$ where c_1 is the constant from Proposition 2.7.12.

Claim 2.7.19. *Under the event $\mathcal{E}^{supp}$, for any $1 \leq i \leq M$ and $\lambda \in \Theta_i$, there exists $G^{(i-1)} \subset Q_{L_{i-1}}$ with $10 \leq |G^{(i-1)}| \leq L^{\frac{2}{3}\delta}$ such that the following holds. For any $\lambda' \in \sigma(H_{Q_{L_{i-1}}})$ and $u \in \ell^2(Q_{L_{i-1}})$ with $|\lambda - \lambda'| \leq 2^{i-1}\exp(-c'L_i)$ and $H_{Q_{L_{i-1}}}u = \lambda' u$,* *we have* $\|u\|_{\ell^2(G^{(i-1)})} \geq (1 - |G^{(i-1)}|^{-2})\|u\|_{\ell^2(Q_{L_{i-1}})}$.

Proof. Since $\lambda \in \Theta_i$, there are $\lambda^{(j)} \in \sigma(H_{Q_{\overline{L_j}}})$ such that $|\lambda - \lambda^{(j)}| \leq 2^i \exp(-c' L_j)$ for each $i \leq j \leq M$. Let

$$G_*^{(i-1)} = \bigcup_{i-1 \leq j \leq M-1} G_{L_j, \overline{L_{j+1}}}^{(\lambda^{(j+1)})}. \tag{2.7.55}$$

Then $|G_*^{(i-1)}| \leq M L^{\frac{\delta}{2}}$. Suppose λ' and u satisfy the hypothesis. Then

$$|\lambda' - \lambda^{(j)}| \leq |\lambda' - \lambda| + |\lambda - \lambda^{(j)}| \leq 2^{i-1}\exp(-c' L_i) + 2^i \exp(-c' L_j) \leq 2^{M+1}\exp(-c' L_j) \tag{2.7.56}$$

for each $i \leq j \leq M$. Denote $L_j' = \frac{1}{2} L_j$ for each $i \leq j \leq M-1$ and $L_{i-1}' = L_{i-1}$. Pick an arbitrary $a \in Q_{L_{i-1}} \setminus Q_{L_M}$, there exists $j' \in \{i-1, \cdots, M-1\}$ such that $a \in A_{L_{j'}', 2\overline{L_{j'+1}}}$. If $a \notin G_*^{(i-1)}$, by definition of $G_{L_{j'}, \overline{L_{j'+1}}}^{\lambda^{(j'+1)}}$, there exists a $\lambda^{(j'+1)}$-good cube $Q_{L'''}(b)$ such that $\overline{L_{j'+1}} \geq L''' \geq \overline{L_{j'+1}}^{\frac{\delta}{10}} \geq L^{\frac{\delta^2}{100}}$, and $\operatorname{dist}(a, Q_{L_{j'}} \setminus Q_{L'''}(b)) \geq \frac{1}{8} L'''$. Then since $a \in Q_{L_{j'}'}$, we have $\operatorname{dist}(a, Q_{L_{i-1}} \setminus Q_{L'''}(b)) \geq \frac{1}{8} L'''$. We also have that

$$|\lambda' - \lambda^{(j'+1)}| \leq 2^{M+1}\exp(-c' L_{j'+1}) \leq 2^{M+1}\exp(-16 c' L'''). \tag{2.7.57}$$

Then by Claim 2.7.5 we have,

$$|u(a)| \leq 2\exp\left((L''')^{1-\lambda_*} - \frac{1}{8}\lambda_* L'''\right)\|u\|_{\ell^1(Q_{L_{i-1}})} \leq L^{-10}\|u\|_{\ell^2(Q_{L_{i-1}})}. \tag{2.7.58}$$

Hence, by letting $G^{(i-1)} = G_*^{(i-1)} \cup Q_{L_M}$, we have $10 \leq |G^{(i-1)}| \leq |G_*^{(i-1)}| + |Q_{L_M}| \leq$

$ML^{\frac{\delta}{2}} + 100L^{\frac{\delta}{2}} \leq L^{\frac{2}{3}\delta}$, and

$$\|u\|_{\ell^2(G^{(i-1)})} \geq \left(1 - (2L_{i-1}+1)^3 L^{-20}\right)^{\frac{1}{2}} \|u\|_{\ell^2(Q_{L_{i-1}})} \geq (1 - |G^{(i-1)}|^{-2})\|u\|_{\ell^2(Q_{L_{i-1}})}. \tag{2.7.59}$$

Thus our claim follows. □

Claim 2.7.20. *Under the event $\mathcal{E}^{supp}$, for any $1 \leq i \leq M$ and $\lambda \in \Theta_i$, we have*

$$|\{\lambda' \in \sigma(H_{Q_{L_{i-1}}}) : |\lambda - \lambda'| \leq 2^{i-1}\exp(-c'L_i)\}| \leq 2L^{\frac{2}{3}\delta}. \tag{2.7.60}$$

Proof. Let $\lambda_1, \cdots, \lambda_p \in \sigma(H_{Q_{L_{i-1}}})$ be all the eigenvalues (counting with multiplicity) in the interval

$$[\lambda - 2^{i-1}\exp(-c'L_i), \lambda + 2^{i-1}\exp(-c'L_i)]. \tag{2.7.61}$$

Let $u_1, \cdots, u_p$ be the corresponding (mutually orthogonal) eigenvectors with properties that $H_{Q_{L_{i-1}}} u_s = \lambda_s u_s$ and $\|u_s\|_{\ell^2(Q_{L_{i-1}})} = 1$ for $1 \leq s \leq p$. By Claim 2.7.19, $\|u_s\|_{\ell^2(G^{(i-1)})} \geq 1 - |G^{(i-1)}|^{-2}$ for $1 \leq s \leq p$. Thus we have

$$|\langle u_{s_1}, u_{s_2}\rangle_{\ell^2(G^{(i-1)})} - \mathbb{1}_{s_1=s_2}| \leq 2|G^{(i-1)}|^{-2} \tag{2.7.62}$$

for $1 \leq s_1, s_2 \leq p$. By Lemma 2.7.1, we have $p \leq 2|G^{(i-1)}| \leq 2L^{\frac{2}{3}\delta}$. □

Claim 2.7.21. *We have $|\Theta_0| \leq L^{M\delta}$ under the event $\mathcal{E}^{supp}$.*

Proof. Suppose $\mathcal{E}^{supp}$ holds. For each $1 \leq i \leq M$ and $\lambda \in \Theta_{i-1}$, there are $\lambda^{(j)} \in$

$\sigma(H_{Q_{L_j}})$ and $\overline{\lambda^{(j)}} \in \sigma(H_{Q_{\overline{L_j}}})$ with $|\lambda - \lambda^{(j)}|, |\lambda - \overline{\lambda^{(j)}}| \leq 2^{i-1}\exp(-c'L_j)$, for $i \leq j \leq M$. In particular, $|\lambda - \lambda^{(i)}| \leq 2^{i-1}\exp(-c'L_i)$. Thus $|\lambda^{(i)} - \lambda^{(j)}| \leq 2^{i-1}(\exp(-c'L_j) + \exp(-c'L_i)) \leq 2^i \exp(-c'L_j)$ and similarly $|\lambda^{(i)} - \overline{\lambda^{(j)}}| \leq 2^i \exp(-c'L_j)$ for $i \leq j \leq M$. Moreover, $\lambda \in \Theta_{i-1}$ implies that $\lambda \in [(M-i+2)\exp(-L^{\frac{\delta}{20}}), \lambda_* - (M-i+2)\exp(-L^{\frac{\delta}{20}})]$ and thus

$$\lambda^{(i)} \in [(M-i+1)\exp(-L^{\frac{\delta}{20}}), \lambda_* - (M-i+1)\exp(-L^{\frac{\delta}{20}})]. \tag{2.7.63}$$

These imply $\lambda^{(i)} \in \Theta_i$. Hence, we have

$$\Theta_{i-1} \subset \{\lambda \in \sigma(H_{Q_{L_{i-1}}}) : \mathrm{dist}(\lambda, \Theta_i) \leq 2^{i-1}\exp(-c'L_i)\}. \tag{2.7.64}$$

Together with Claim 2.7.20, we have $|\Theta_{i-1}| \leq 2L^{\frac{2}{3}\delta}|\Theta_i|$ for $1 \leq i \leq M$. Since $|\Theta_M| \leq |\sigma(H_{Q_{L_M}})| \leq 10L_M^3 \leq 100L^{\frac{\delta}{2}}$, we have $|\Theta_0| \leq 100L^{\frac{\delta}{2}} \cdot 2^M L^{\frac{2}{3}M\delta} \leq L^{M\delta}$. $\square$

Now we denote $\mathcal{E}_{sloc}^{(L)} = \mathcal{E}^{supp} \cap \bigcap_{0 \leq i \leq M} \mathcal{E}_{wloc}^{(L_i)} \cap \mathcal{E}_{wloc}^{(\overline{L_i})}$. By Proposition 2.7.12 and (2.7.52),

$$\mathbb{P}[\mathcal{E}_{sloc}^{(L)}] \geq 1 - L^{-\frac{\delta}{2}} - 2(M+1)L^{-\frac{1}{20}\kappa'\delta} \geq 1 - L^{-\kappa''} \tag{2.7.65}$$

for some small $\kappa'' > 0$ depending on δ, M. Take $c_2 = \min\{\frac{\delta}{30}, c'\}$. Under the event $\mathcal{E}_{sloc}^{(L)}$, for any

$$\lambda \in \sigma_k(H) \cap [\exp(-L^{c_2}), \lambda_* - \exp(-L^{c_2})], \tag{2.7.66}$$

we claim that

$$\operatorname{dist}(\lambda, \Theta_0) \leq \exp(-c_2 L). \tag{2.7.67}$$

To see this, by definition of $\mathcal{E}_{wloc}^{(L_i)}$ and $\mathcal{E}_{wloc}^{(\overline{L_i})}$, (2.7.66) implies $\operatorname{dist}(\lambda, \sigma(H_{Q_{L_i}})) \leq \exp(-c_1 L_i)$ and $\operatorname{dist}(\lambda, \sigma(H_{Q_{\overline{L_i}}})) \leq \exp(-\frac{c_1}{16} L_i)$ for each $0 \leq i \leq M$. In particular, there is $\lambda_0 \in \sigma(H_{Q_L})$ such that $|\lambda - \lambda_0| \leq \exp(-c_1 L)$. Since $c' = \frac{c_1}{20}$, we have $\lambda_0 \in \left[(M+1)\exp(-L^{\frac{\delta}{20}}), \lambda_* - (M+1)\exp(-L^{\frac{\delta}{20}})\right]$ by (2.7.66), and also

$$\begin{aligned} \operatorname{dist}(\lambda_0, \sigma(H_{Q_{L_i}})) \leq& |\lambda - \lambda_0| + \operatorname{dist}(\lambda, \sigma(H_{Q_{L_i}})) \\ \leq& \exp(-c_1 L) + \exp(-c_1 L_i) \\ \leq& \exp(-c' L_i), \\ \operatorname{dist}(\lambda_0, \sigma(H_{Q_{\overline{L_i}}})) \leq& |\lambda - \lambda_0| + \operatorname{dist}(\lambda, \sigma(H_{Q_{\overline{L_i}}})) \\ \leq& \exp(-c_1 L) + \exp(-\frac{c_1}{16} L_i) \\ \leq& \exp(-c' L_i), \end{aligned} \tag{2.7.68}$$

for $0 \leq i \leq M$. Hence $\lambda_0 \in \Theta_0$ and (2.7.67) follows.

Finally, observe that $|\Theta_0| \leq L^{M\delta} \leq L^{\frac{\log(\frac{1}{10}\delta)}{\log(1-\frac{3}{4}\varepsilon)}\delta} \leq L^{\delta'}$ by taking δ small enough (depending on δ'), the proposition follows by letting $S = \Theta_0$. □

Chapter 3

2D Anderson-Bernoulli localization with large disorder

3.1 Introduction

3.1.1 Main result

Let $p \in (0,1)$ and $\bar{V} > 0$. Let $V : \mathbb{Z}^d \to \{0, \bar{V}\}$ be a random function such that $\{V(a) : a \in \mathbb{Z}^d\}$ is a family of independent Bernoulli random variables with $\mathbb{P}(V(a) = 0) = p$ and $\mathbb{P}(V(a) = \bar{V}) = 1 - p$ for each $a \in \mathbb{Z}^d$. Let Δ denote the Laplacian

$$\Delta u(a) = -2du(a) + \sum_{b \in \mathbb{Z}^d, |a-b|=1} u(b), \ \forall u : \mathbb{Z}^d \to \mathbb{R}, a \in \mathbb{Z}^d. \tag{3.1.1}$$

Here and throughout the chapter, $|a| = \|a\|_\infty$ for $a \in \mathbb{Z}^d$. We study the spectra property of the (random) Anderson Hamiltonian

$$H = -\Delta + V \tag{3.1.2}$$

when $\bar{V}$ is large enough.

It is known that (see e.g. [Pas80]), almost surely, the spectrum of $H = -\Delta + V$ is

$$\sigma(H) = [0, 4d] \cup \left[\bar{V}, \bar{V} + 4d\right] \tag{3.1.3}$$

which is a union of two disjoint intervals when $\bar{V} > 4d$. Here and throughout the chapter, we denote by $\sigma(A)$ the spectrum of a self-adjoint operator A. Our main theorem is the following

Theorem 3.1.1 (Main theorem)**.** *Let $d = 2$, $p = \frac{1}{2}$. There exist positive integer n and energies $\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(n)} \in [0, 8]$ such that the following holds.*

For each $\bar{V}$ large enough, suppose $\widetilde{\lambda^{(i)}} = \bar{V} + 8 - \lambda^{(i)}$ for $i = 1, \cdots, n$. Let

$$Y_{\bar{V}} = \bigcup_{i=1}^{n} \left[\lambda^{(i)} - \bar{V}^{-\frac{1}{4}}, \lambda^{(i)} + \bar{V}^{-\frac{1}{4}}\right]$$

and

$$\widetilde{Y_{\bar{V}}} = \bigcup_{i=1}^{n} \left[\widetilde{\lambda^{(i)}} - \bar{V}^{-\frac{1}{4}}, \widetilde{\lambda^{(i)}} + \bar{V}^{-\frac{1}{4}}\right].$$

Let H be defined as in (3.1.2)*. Then almost surely, for any $\lambda_0 \in \sigma(H) \setminus (Y_{\bar{V}} \cup \widetilde{Y_{\bar{V}}})$*

and $u : \mathbb{Z}^2 \to \mathbb{R}$, *if* $Hu = \lambda_0 u$ *and*

$$\inf_{m>0} \sup_{a\in\mathbb{Z}^2} (|a|+1)^{-m}|u(a)| < \infty, \tag{3.1.4}$$

then

$$\inf_{c>0} \sup_{a\in\mathbb{Z}^2} \exp(c|a|)|u(a)| < \infty. \tag{3.1.5}$$

Remark 3.1.2. The energies $\lambda^{(i)}$'s are defined in Definition 3.2.12 below and they do not depend on $\bar{V}$. In fact, $\lambda^{(i)}$'s are Dirichlet eigenvalues of $-\Delta$ restricted on finite subsets of $\mathbb{Z}^2$ and $\widetilde{\lambda^{(i)}}$'s are simply images of $\lambda^{(i)}$'s under the mapping $x \mapsto \bar{V} + 8 - x$.

Remark 3.1.3. Our proof and conclusions in Theorem 3.1.1 extend to $1 - p_c < p < p_c$ where $p_c > \frac{1}{2}$ is the site percolation threshold on $\mathbb{Z}^2$ (see Section 3.2.1). $p \in (1-p_c, p_c)$ is an essential assumption for our method to prove Theorem 3.1.1 (see Section 3.1.2 below). Thus it is an interesting question that whether a similar result can be proved for $p \in (0, 1-p_c] \cup [p_c, 1)$.

For simplicity, throughout this chapter, we restrict ourselves to the case $p = \frac{1}{2}$.

The result in Theorem 3.1.1 means Anderson localization happens in $\sigma(H) \setminus (Y_{\bar{V}} \cup \widetilde{Y_{\bar{V}}})$. In his seminal paper [And58], Anderson said,

> *The theorem is that at sufficiently low densities, transport does not take place; the exact wave functions are localized in a small region of space.*

Here, the density refers to the *density of states measure* (*DOS measure*). Intuitively, DOS measure in interval $[E_1, E_2]$ gives the "number of states per unit volume" with energy in $[E_1, E_2]$. More precisely, we restrict the operator H to the square centered at origin with edge length $2L$, and denote the (random) empirical distribution of the restricted operator's eigenvalues by μ_L. It is known that, almost surely, when L goes to infinity, μ_L converges weakly to some probability measure which is called the DOS measure (see e.g. [AW15, Chapter 3] by Aizenman and Warzel). The smallness of DOS measure was mathematically verified for several cases, in particular for the following two cases,

1. For any nontrivial distribution of V, the DOS measure is extremely small near the bottom of the spectrum. This is also called the "Lifshitz tail phenomenon". See e.g. [AW15, Chapter 4.4] and also [Kir08, Section 6.2].

2. Suppose $V = \delta V_0$ where V_0 has uniformly Hölder continuous distribution (see [AW15, Definition 4.5]). The DOS measure of any finite interval with given length becomes uniformly small when the disorder strength δ increases to infinity. See e.g. [AW15, Theorem 4.6].

In both cases, according to [And58], one expects Anderson localization to happen in the corresponding spectrum range, namely, near the bottom in the first case and throughout the whole spectrum in the second case. In fact, both cases have been studied extensively and Anderson localization was proved for several distributions of V.

For V with Hölder continuous distribution, Anderson localization was proved in both cases in any dimension, namely, near the bottom of the spectrum or throughout the spectrum when the disorder strength is large enough. As written in Chapter 1, this was first proved for distributions with bounded density in [FS83],[FMSS85] by Fröhlich, Martinelli, Scoppola and Spencer. Later on, the multi-scale method in [FS83],[FMSS85] was strengthened to prove the same result for general Hölder continuous distribution in [CKM87] by Carmona, Klein and Martinelli.

As for Bernoulli potential, Anderson-Bernoulli localization near the bottom of spectrum was verified in the continuous model $\mathbb{R}^d (d \geq 2)$ by Bourgain and Kenig in [BK05], and later in the discrete model $\mathbb{Z}^d$ by Ding and Smart in [DS20] for $d = 2$ and by Zhang and the author in [LZ22] (which is the arxiv version of Chapter 2) for $d = 3$.

For Bernoulli potential with large disorder (i.e. operator (3.1.2) with large $\bar{V}$), the total length of spectrum is always $8d$ by equation (3.1.3). When $\bar{V}$ increases, the DOS measure behaves completely different from the case when V has Hölder continuous distribution. When $d = 2$ and $p = \frac{1}{2}$, the DOS measure always has a constant lower bound in the sets $Y_{\bar{V}}$ and $\widetilde{Y_{\bar{V}}}$ defined in Theorem 3.1.1 for sufficiently large $\bar{V}$. On the other hand, the DOS measure is constantly small outside $Y_{\bar{V}} \cup \widetilde{Y_{\bar{V}}}$. Hence, Theorem 3.1.1 is again under the umbrella of prediction in [And58].

Let us also mention that, although smallness of DOS implies localization in many cases, the converse is not true. In fact, much stronger result for Anderson localization is expected in dimension one and two. For one dimension, it is proved that Anderson localization happens throughout the whole spectrum for any nontrivial distribution of

V with finite moment (see e.g. [CKM87]). It is a general belief among physicists that (see e.g. [Sim00] by Simon), in dimension two, Anderson localization also happens throughout the whole spectrum for any finite nontrivial distribution of V. Thus it is reasonable to conjecture that, in our model, localization also happens inside $Y_{\bar{V}} \cup \widetilde{Y_{\bar{V}}}$ and it is more of a technical limitation that we have to exclude $Y_{\bar{V}} \cup \widetilde{Y_{\bar{V}}}$.

In order to prove Theorem 3.1.1, we only need to consider the spectrum of H contained in $[0, 8]$ and prove the exponential decaying property of resolvent as follows.

Theorem 3.1.4. *Let $d = 2$, $p = \frac{1}{2}$. There exist positive integer n, constants $\kappa, \alpha, \varepsilon > 0$ and energies $\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(n)} \in [0, 8]$ such that the following holds.*

For any $\bar{V} > 0$, denote $Y_{\bar{V}} = \bigcup_{i=1}^{n} \left[\lambda^{(i)} - \bar{V}^{-\frac{1}{4}}, \lambda^{(i)} + \bar{V}^{-\frac{1}{4}}\right]$. Let H be defined as in (3.1.2). Then for each $\bar{V}, L > \alpha$, each $\lambda_0 \in [0, 8] \setminus Y_{\bar{V}}$ and each box $Q \subset \mathbb{Z}^2$ of side length L,

$$\mathbb{P}\left[|(H_Q - \lambda_0)^{-1}(a, b)| \leq \bar{V}^{L^{1-\varepsilon} - \varepsilon|a-b|} \text{ for } a, b \in Q\right] \geq 1 - L^{-\kappa}. \tag{3.1.6}$$

Here $H_Q : \ell^2(Q) \to \ell^2(Q)$ is the restriction of Hamiltonian H to the box Q with the Dirichlet boundary condition.

Proof of Theorem 3.1.1 assuming Theorem 3.1.4. Probability estimate (3.1.6) with the arguments in [BK05, Section 7] implies that Anderson localization happens in $[0, 8] \setminus Y_{\bar{V}}$. See also [GK12, Section 6,7] by Germinet and Klein, and Section 2.7.3 in Chapter 2.

Now we use symmetry to prove the Anderson localization for the spectrum range

$$[\bar{V}, \bar{V}+8] \setminus \bigcup_{i=1}^{n} \left[\widetilde{\lambda^{(i)}} - \bar{V}^{-\frac{1}{4}}, \widetilde{\lambda^{(i)}} + \bar{V}^{-\frac{1}{4}}\right],$$

where $\widetilde{\lambda^{(i)}} = \bar{V} + 8 - \lambda^{(i)}$. Define $\tilde{V} : \mathbb{Z}^2 \to \{0, \bar{V}\}$ by $\tilde{V}(a) = \bar{V} - V(a)(a \in \mathbb{Z}^2)$ and let $\tilde{H} = -\Delta + \tilde{V}$. Let $\tilde{\lambda} = \bar{V} + 8 - \lambda$ for every $\lambda \in \mathbb{R}$. For each $u : \mathbb{Z}^2 \to \mathbb{R}$, define $\tilde{u} : \mathbb{Z}^2 \to \mathbb{R}$ by $\tilde{u}(x, y) = (-1)^{x+y} u(x, y)$ for $x, y \in \mathbb{Z}$. This gives a bijection $u \mapsto \tilde{u}$ from functions on $\mathbb{Z}^2$ to themselves. The properties (3.1.4) and (3.1.5) in Theorem 3.1.1 are preserved under this bijection. Moreover, by direct calculations, we have

$$Hu = \lambda u \text{ if and only if } \tilde{H}\tilde{u} = \tilde{\lambda}\tilde{u}. \tag{3.1.7}$$

Since $\tilde{H}$ has the same distribution as H, Anderson localization happens in $\{\tilde{\lambda} : \lambda \in [0, 8] \setminus Y_{\bar{V}}\} = [\bar{V}, \bar{V}+8] \setminus \bigcup_{i=1}^{n} \left[\widetilde{\lambda^{(i)}} - \bar{V}^{-\frac{1}{4}}, \widetilde{\lambda^{(i)}} + \bar{V}^{-\frac{1}{4}}\right]$. Theorem 3.1.1 follows. $\square$

3.1.2 Outline

In order to prove localization, [DS20] and [BK05] used a multi-scale analysis to prove an estimate similar to (3.1.6) (see also Section 2.3 in Chapter 2). These two previous works considered the edge of spectrum where the Lifshitz tail phenomenon happens and used this phenomenon to prove the initial step of the induction in the multi-scale analysis. Then they used an eigenvalue variation argument to prove the Wegner estimate which is crucial to the inductive steps. The key to the eigenvalue variation argument is the unique continuation principle (see [DS20, Theorem 1.6] and [BK05, Lemma 3.10]).

Our method follows the multi-scale analysis framework in [DS20] and [BK05], and studies the spectrum range beyond the edge by taking advantage of site percolation. Informally, the condition $p = \frac{1}{2}$ implies that the sites with the same potential rarely form large connected components (see Section 3.2.1 and [Gri99, Chapter 1.6] for the former definition of site percolation). By this fact, the initial scale case (Proposition 3.2.21) for the multi-scale analysis is proved for energies away from $\lambda^{(i)}$'s which are eigenvalues of the minus Laplacian restricted on small finite subsets of $\mathbb{Z}^2$ (thus away from $\lambda^{(i)}$'s the DOS measure is small).

The most important and difficult part for the induction of multi-scale analysis is to prove the Wegner estimate (Proposition 3.3.18) which indicates log-Hölder continuity of the DOS measure (see e.g. [Bou05, Section 6]). Our Wegner estimate states that, for an interval of length less than $O(\bar{V}^{-L^{1-\varepsilon'}})$, the probability that it contains an eigenvalue of H_{Q_L} is less than $O(L^{-\kappa'})$ for some $\kappa', \varepsilon' > 0$.

In order to prove the Wegner estimate, we prove an upper bound and a lower bound on how far an eigenvalue of H_{Q_L} will move after perturbing the potential function V. Here, "perturb" means changing the value of V at some vertices from 0 to $\bar{V}$ or from $\bar{V}$ to 0.

The upper bound estimate requires to show that if the j-th smallest eigenvalue is close to a given real number λ_0, then one can perturb the potential V on a $(1-$

ε) portion of Q_L such that the j-th smallest eigenvalue will not move too far (less than $O(\bar{V}^{-L^{1-\varepsilon''}})$ with $\varepsilon'' > \varepsilon'$). Here and throughout this section, when we say j-th smallest eigenvalue, we always count with multiplicity. While this upper bound estimate was proved for λ_0 near the bottom of the spectrum in [DS20], it is simply not true for λ_0 away from the bottom. For example, suppose H_{Q_L} has $k > 0$ eigenvalues (with multiplicities) in $[0, 8]$. Pick an arbitrary $a \in Q_L$ with $V(a) = 0$ and let the perturbed operator H'_{Q_L} be obtained by changing the potential V from 0 to $\bar{V}$ only at vertex a. It can be shown that, the k-th smallest eigenvalue of H'_{Q_L} is in $[\bar{V}, \bar{V}+8]$ and thus is far from the k-th smallest eigenvalue of H_{Q_L} which is in $[0, 8]$. Hence we can not expect the upper bound estimate to hold in its original version.

It turns out that a different version of upper bound estimate still holds. In that version, we will not compare the j-th smallest eigenvalue of an operator with the j-th smallest eigenvalue of its perturbation. We will make another correspondence between eigenvalues of an operator and eigenvalues of its perturbation. To clarify, in the previous example, the k-th eigenvalue of H_{Q_L} will actually correspond to the $(k-1)$-th eigenvalue of H'_{Q_L} and the distance between these two eigenvalues will be shown to be small, provided one of them is close to λ_0. To rigorously find the correspondence between eigenvalues of an operator and eigenvalues of its perturbation, we will introduce the *"cutting procedure"* which continuously "transforms" the operator H_{Q_L} (and H'_{Q_L}) to a direct sum operator $\bigoplus_i H_{\Lambda_i}$ (and $\bigoplus_i H'_{\Lambda_i}$) respectively. Here, $\bigcup_i \Lambda_i = Q_L$ is a disjoint union. The j-th eigenvalue of the operator H_{Q_L} corresponds to j'-th eigenvalue of H'_{Q_L} only if the j-th eigenfunction of $\bigoplus_i H_{\Lambda_i}$ equals the j'-th eigenfunction of $\bigoplus_i H'_{\Lambda_i}$. Under this correspondence of eigenvalues, the upper bound estimate is stated as Claim 3.3.21. The formal definition of cutting procedure is given in Definition 3.2.9 and 3.2.16 by using percolation clusters.

The lower bound estimate requires to show that there are an enough portion of points in Q_L such that, when the potential increases on any of these points, a given eigenvalue will move a decent distance (at least $\Omega(\bar{V}^{-L^{1-\varepsilon'}})$). Based on the heuristic that increasing the potential at vertices where an eigenfunction u has large absolute values will increase the associated eigenvalue fast, one only needs to show that the eigenfunction u has a decent lower bound on an enough portion of points in Q_L. This is guaranteed by a discrete version of unique continuation principle Theorem 3.1.5 which is analogue of [DS20, Theorem 1.6]. However, under the new correspondence of eigenvalues, the j-th eigenvalue of H_{Q_L} may correspond to either the j-th eigenvalue or the $(j-1)$-th eigenvalue of the perturbation H'_{Q_L} (here H'_{Q_L} is obtained from H_{Q_L} by changing the potential V from 0 to $\bar{V}$ only at one vertex). If it corresponds to the j-th eigenvalue of H'_{Q_L}, then by monotonicity, the eigenvalue will increase. Otherwise if it corresponds to the $(j-1)$-th eigenvalue of H'_{Q_L}, then by Cauchy interlacing theorem, the eigenvalue will decrease. Either way the lower bound estimate can be proved for rank one perturbation, provided we can have a quantitative estimate on the difference. This is considered in Lemma 3.3.8 and Lemma 3.3.9.

In order to have a polynomial bound on the probability (i.e. the right hand side of

(3.1.6)), we need to consider the perturbation on a large set of vertices rather than only one vertex. For this purpose, the previous works [DS20] and [BK05] used the Sperner lemma which deals with monotone functions. However, as seen in the argument above, under the new correspondence, the eigenvalue is no longer a monotone function of the potential. Thus the original Sperner lemma ([DS20, Theorem 4.2]) can not be applied to our case. Instead, we generalize Sperner lemma to deal with directed graph products and prove Lemma 3.3.16 which is another new ingredient. The original Sperner lemma ([DS20, Theorem 4.2]) can be seen as a special case of Lemma 3.3.16 when each directed graph consists of two vertices and one directed edge. The details are given in Section 3.3.2.

3.1.3 Discrete unique continuation principle

We state the discrete unique continuation principle here which roughly says that, with high probability, any solution of $Hu = \lambda u$ in a box Q with side length L satisfies $|u| \geq (\bar{V}L)^{-L}$ on $\Omega(L^2)$ many points in Q.

Theorem 3.1.5. *For every small $\varepsilon > 0$, there exists $\alpha > 1$ such that the following holds. If $\lambda_0 \in [0, 8]$ is an energy, $\bar{V} \geq 2$ and $Q \subset \mathbb{Z}^2$ is a box of side length $L \geq \alpha$, then $\mathbb{P}[\mathcal{E}] \geq 1 - \exp(-\varepsilon L^{\frac{2}{3}})$, where $\mathcal{E}$ denotes the event that*

$$\left|\left\{a \in Q : |u(a)| \geq (\bar{V}L)^{-\alpha L} \|u\|_{\ell^\infty(\frac{Q}{100})}\right\}\right| \geq \varepsilon^3 L^2 \tag{3.1.8}$$

holds whenever $\lambda \in \mathbb{R}$, $u : \mathbb{Z}^2 \to \mathbb{R}$, $|\lambda - \lambda_0| \leq (\bar{V}L)^{-\alpha L}$ and $Hu = \lambda u$ in Q.

In fact, the multi-scale analysis framework requires us to prove a slightly stronger version Lemma 3.3.5 which accommodates a sparse "frozen set". An important feature of Theorem 3.1.5 and Lemma 3.3.5 is that the probability estimate does not depend on $\bar{V}$. This feature is one of the major reasons why the set of energies $\lambda^{(i)}$'s in Theorem 3.1.1 does not grow when $\bar{V}$ increases.

Theorem 3.1.5 generalizes [DS20, Theorem 1.6] to deal with large $\bar{V}$ and proves a $\Omega(L^2)$ lower bound on the cardinality of the support of u. This improves the previous $\Omega(L^{\frac{3}{2}}(\log L)^{-\frac{1}{2}})$ lower bound in [DS20, Theorem 1.6]. The price we pay is that the energy window needs to be $O((\bar{V}L)^{-\alpha L})$ (while it was $O(\exp(-\alpha(L\log L)^{\frac{1}{2}}))$ in [DS20]).

We refer the reader to the beginning of Section 3.5 for a proof outline and a comparison between proofs of [DS20, Theorem 1.6] and Theorem 3.1.5. Here, we only mention the main new ingredient Lemma 3.1.6 which is proved in Section 3.5.1. In fact, a weaker form of Lemma 3.1.6 will suffice for the proof of Theorem 3.1.5.

Lemma 3.1.6. *Given positive integers $k < n$, denote the n dimensional Boolean cube by $B^n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_i \in \{0, 1\}$ for each $1 \le i \le n\}$. Then for any k dimensional affine space $\Gamma \subset \mathbb{R}^n$,*

$$\#\{a \in B^n : \min_{b\in\Gamma} \|a - b\|_2 < \frac{1}{4}n^{-\frac{1}{2}}(n-k)^{-\frac{1}{2}}\} \le 2^{k+1}. \tag{3.1.9}$$

Lemma 3.1.6 can be seen as a quantitative version of Odlyzko Lemma (see e.g. [Odl88]). To prove it, we will find a subset $\mathcal{S} \subset \{1, \cdots, n\}$ with $|\mathcal{S}| = n - k - 1$

such that, the projection operator onto the orthogonal complement of Γ is "well invertible" when it is restricted on $\mathbb{R}^{\mathcal{S}}$. The existence of $\mathcal{S}$ is a direct consequence of the following "Restricted Invertibility Theorem" for matrices with isotropic columns which was previously proved in [MSS14] by Marcus, Spielman and Srivastava.

Lemma 3.1.7 (Theorem 3.1 in [MSS14])**.** *Suppose $v_1, v_2, \cdots, v_l \in \mathbb{C}^m$ are vectors with $\sum_{i=1}^{l} v_i v_i^\dagger = I_m$ where $v_i^\dagger$ is the dual vector of v_i and I_m is the identity matrix.*

Then for every $m' < m$ there is a subset $\mathcal{S} \subset \{1, 2, \cdots, m\}$ of size m' such that the m'-th largest eigenvalue of $\sum_{i \in \mathcal{S}} v_i v_i^\dagger$ is at least $\left(1 - \sqrt{\frac{m'}{m}}\right)^2 \frac{m}{l}$.

For general restricted invertibility principles and their history, we refer to [NY17] by Naor and Youssef.

3.1.4 Notations

We set up some notations in this subsection. Throughout the chapter, we regard $\mathbb{Z}^2$ as a graph with vertices $\{(x, y) : x, y \in \mathbb{Z}\}$ and there is an edge connecting $a, b \in \mathbb{Z}^2$ if and only if $|a - b| = 1$ (in this case, we also write $a \sim b$). We let

$$Q_l(a) = \left\{ a' \in \mathbb{Z}^2 : |a - a'| \leq \frac{l-1}{2} \right\} \tag{3.1.10}$$

for real number $l \geq 1$ and $a \in \mathbb{Z}^2$, and denote its side length $\ell(Q_l(a)) = 2\lfloor \frac{l-1}{2} \rfloor$. For simplicity, we denote $Q_l = Q_l(\mathbf{0})$. Given real number $k > 0$, we write $kQ_l(a) = Q_{kl}(a)$.

Given any subset $S \subset \mathbb{Z}^2$ and function $f : \mathbb{Z}^2 \to \mathbb{R}$, define the restriction $f|_S$:

$S \to \mathbb{R}$ by $f|_S(a) = f(a)$ for $a \in S$. We denote $P_S : \ell^2(\mathbb{Z}^2) \to \ell^2(S)$ to be the projection operator defined by $P_S f = f|_S$ for each $f \in \ell^2(\mathbb{Z}^2)$. For simplicity, we write $\|f\|_{\ell^2(S)} = \|P_S f\|_{\ell^2(S)}$. For an operator A on $\ell^2(\mathbb{Z}^2)$, we denote $A_S = P_S A P_S^\dagger$ where $P_S^\dagger$ is the adjoint operator of P_S.

Given $a \in \mathbb{Z}^2$, define $1_a(a) = 1$ and $1_a(a') = 0$ if $a' \neq a$. Given $S \subset \mathbb{Z}^2$, an operator A on $\ell^2(S)$ and $a, b \in S$, write $A(a,b) = \langle 1_a, A 1_b \rangle_{\ell^2(S)}$ where $\langle \cdot, \cdot \rangle_{\ell^2(S)}$ denotes the inner product in $\ell^2(S)$. We also denote by $\|A\|$ the Euclidean norm of the operator A.

Throughout the rest of the chapter, H always denotes the operator defined in (3.1.2). Given $\lambda \in \mathbb{C} \setminus \sigma(H_S)$, we write $G_S(a,b;\lambda) = (H_S - \lambda)^{-1}(a,b)$ for $S \subset \mathbb{Z}^2$ and $a, b \in S$.

For any real function u defined on a domain D and any real number c, we use $\{u \geq c\}$ as shorthand for the set $\{a \in D : u(a) \geq c\}$.

Organization of remaining chapter

In Section 3.2, we define the cutting procedure. Along this way, we prove the induction base case (Proposition 3.2.21) for multi-scale analysis. The sharpness of site percolation (Proposition 3.2.2) plays a key role there.

In Section 3.3, we prove the Wegner estimate Proposition 3.3.18. We will first collect all needed lemmas in Section 3.3.1 and prove a generalized Sperner lemma in Section 3.3.2. The proof of Wegner estimate is given in Section 3.3.3.

In Section 3.4, we perform the multi-scale analysis by using Wegner estimate and prove Theorem 3.1.4.

In Section 3.5, we prove the unique continuation theorem 3.1.5 and Lemma 3.3.5.

Among these four sections, Section 3.4 follows closely the existing framework in [DS20] and [BK05] while other sections contain the new ingredients as follows:

- A "cutting procedure" which allows us to match eigenvalues under different potential functions (Section 3.2).

- The use of sharpness of site percolation in the proof of initial case of multi-scale analysis (Section 3.2).

- A generalized Sperner lemma for directed graph products (Section 3.3).

- A 2D unique continuation theorem with an improved lower bound (see (3.1.8)) and a smaller energy window (Section 3.5).

3.2 Initial scale

In this section, we use site percolation (Section 3.2.1) to define the cutting procedure described in the introduction. We will first define r-*bits* which are boxes centered in a sublattice with certain edge length (Definition 3.2.4). We then define the cutting procedure for Hamiltonian restricted on r-bits by using percolation clusters (Definition 3.2.9). These r-bits will also be used as "basic units" for eigenvalue variation arguments in the proof of Wegner estimate Proposition 3.3.18 in Section 3.3. Then we will extend the cutting procedure to boxes with larger length scale (Definition

3.2.16). Finally, we will prove the induction base case for the multi-scale analysis (Proposition 3.2.21).

3.2.1 Site percolation

Consider the Bernoulli site percolation on $\mathbb{Z}^2$. Let $p \in (0,1)$, suppose each vertex in $\mathbb{Z}^2$ is independently occupied with probability p. It is well known that there exists a critical probability $p_c \in (0,1)$ such that, for $p > p_c$, almost surely, there exists an infinite connected subset of $\mathbb{Z}^2$ whose vertices are occupied; for $p < p_c$, almost surely, there does not exist an infinite connected subset of $\mathbb{Z}^2$ whose vertices are occupied. It is known that $p_c > \frac{1}{2}$, see e.g. [GS98] by Grimmett and Stacey.

Definition 3.2.1. For any $S \subset \mathbb{Z}^2$, denote

$$\partial^+ S = \{a \in \mathbb{Z}^2 \setminus S : a \sim b \text{ for some } b \in S\}$$

to be the outer boundary of S; and

$$\partial^- S = \{a \in S : a \sim b \text{ for some } b \in \mathbb{Z}^2 \setminus S\}$$

to be the inner boundary of S. Denote

$$\partial S = \big\{\{a,b\} : a \in \partial^+ S,\, b \in \partial^- S \text{ and } a \sim b\big\}$$

to be the set of edges connecting elements in $\partial^- S$ and $\partial^+ S$.

The following sharpness proposition follows directly from $p_c > \frac{1}{2}$ and [AB87, Theorem 7.3] by Aizenman and Barsky:

Proposition 3.2.2. *Suppose $V : \mathbb{Z}^2 \to \{0, \bar{V}\}$ is a random function such that $\{V(a) : a \in \mathbb{Z}^2\}$ is a family of i.i.d. random variables such that $\mathbb{P}[V(a) = 0] = \frac{1}{2}$ and $\mathbb{P}\left[V(a) = \bar{V}\right] = \frac{1}{2}$. There is a numerical constant $c_0 > 0$ such that, for each $l > 10$ and $b \in \mathbb{Z}^2$,*

$$\mathbb{P}\left[\mathcal{E}^l_{per}(b)\right] < \exp(-c_0 l). \tag{3.2.1}$$

Here, $\mathcal{E}^l_{per}(b)$ denotes the event that there is a path in $\mathbb{Z}^2$ joining b to some vertex in $\partial^- Q_l(b)$ such that V equals 0 on all vertices in this path.

3.2.2 r-bit

Let $\varepsilon_0 > 0$ be a fixed small constant such that

$$\varepsilon_0 < \varepsilon_1^{10} \tag{3.2.2}$$

where ε_1 is the numerical constant appeared in Lemma 3.3.5 below.

The inequality (3.2.2) will only be used in the proof of Proposition 3.3.18. At this moment, the reader can think of ε_0 as a small numerical constant.

Definition 3.2.3. For any large odd number r, denote $\dot{r} = \left\lceil (1 - \frac{\varepsilon_0}{2})(r-1) \right\rceil$. For any vertex $a \in \dot{r}\mathbb{Z}^2$ where $\dot{r}\mathbb{Z}^2 = \{(\dot{r}x, \dot{r}y) : x, y \in \mathbb{Z}\}$, let $\Omega_r(a) = Q_{(1-2\varepsilon_0)r}(a)$,

$\tilde{\Omega}_r(a) = Q_{(1-\frac{3}{2}\varepsilon_0)r}(a)$ and $F_r(a) = Q_r(a) \setminus \Omega_r(a)$.

Definition 3.2.4. Given a large odd number r, a vertex $a \in \dot{r}\mathbb{Z}^2$ and a potential function $V' : F_r(a) \to \{0, \bar{V}\}$, we call $(Q_r(a), V')$ an *r-bit*. We say $(Q_r(a), V')$ is *admissible* if the following two items hold:

- For each $x \in \partial^- Q_r(a)$ and $y \in F_r(a)$ with $|x - y| \geq \frac{\varepsilon_0}{30} r$, there is no path in $F_r(a)$ joining x to y such that V' equals 0 on all vertices in the path.

- There is no path in $F_r(a)$ joining some vertex in $\partial^+ \Omega_r(a)$ to some vertex in $\partial^- \tilde{\Omega}(a)$ such that V' equals 0 on all vertices in the path.

With a little abuse of notations, we also call $Q_r(a)$ an r-bit if $a \in \dot{r}\mathbb{Z}^2$. When $V' : F_r(a) \to \{0, \bar{V}\}$ is obviously given, we also say $Q_r(a)$ is admissible if $(Q_r(a), V')$ is admissible.

Given an r-bit $Q_r(a)$, we say it is *inside* some $S \subset \mathbb{Z}^2$ if $Q_r(a) \subset S$. We say it *does not affect* S if $\Omega_r(a) \cap S = \emptyset$.

Remark 3.2.5. We give here three remarks on r-bits, the first two are from Definition 3.2.3 and the third one is obvious by Definition 3.2.4. See also Figure 3.1 for an illustration.

1. For two different r-bits $Q_r(a_1)$ and $Q_r(a_2)$, we have

$$\tilde{\Omega}_r(a_1) \cap (\partial^+ Q_r(a_2) \cup Q_r(a_2)) = \emptyset.$$

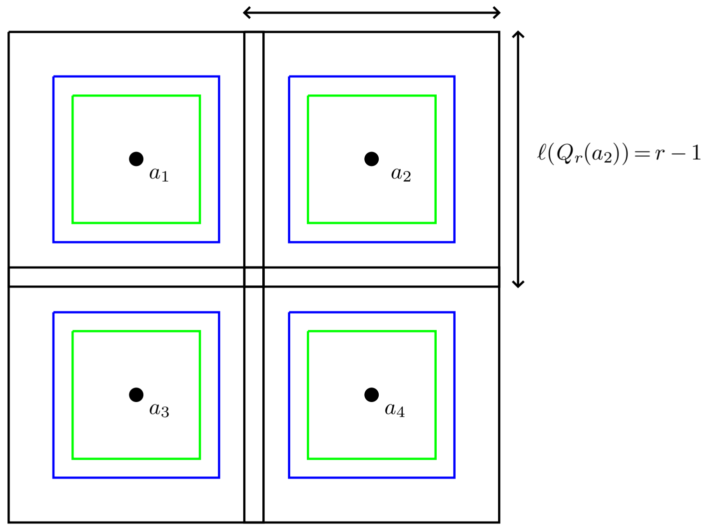

Figure 3.1: The black squares represent r-bits $Q_r(a_i)(i = 1, 2, 3, 4)$ with overlaps, the blue squares represent $\tilde{\Omega}_r(a_i)(i = 1, 2, 3, 4)$ and the green squares represent $\Omega_r(a_i)(i = 1, 2, 3, 4)$.

Note that, $\tilde{\Omega}_r(a_1)$ is a scaling image of r-bit $Q_r(a_1)$ with the scaling constant slightly smaller than 1. Thus the equation above means $\tilde{\Omega}_r(a_1)$ is disjoint from other r-bits and their outer boundaries.

2. For any $a \in \mathbb{Z}^2$, there exists an r-bit $Q_r(b)$ with $a \in Q_{(1-\frac{2}{5}\varepsilon_0)r}(b)$.

3. Suppose r-bits $(Q_r(a), V')$ and $(Q_r(a'), V'')$ satisfy $V'(b) = V''(b-a+a')$ for each $b \in F_r(a)$, then $(Q_r(a), V')$ is admissible if and only if $(Q_r(a'), V'')$ is admissible.

The following Proposition 3.2.6 is the place where we use the sharpness of site percolation (Proposition 3.2.2).

Proposition 3.2.6. *Suppose odd number r is large enough. Let $V : \mathbb{Z}^2 \to \{0, \bar{V}\}$ be the i.i.d. Bernoulli random potential with $\mathbb{P}(V(a) = 0) = \mathbb{P}(V(a) = \bar{V}) = \frac{1}{2}$ for each*

$a \in \mathbb{Z}^2$. *Then for each $a \in \dot{r}\mathbb{Z}^2$, we have*

$$\mathbb{P}\left[(Q_r(a), V|_{F_r(a)}) \text{ is admissible}\right] > 1 - \exp(-8c_1 r) \tag{3.2.3}$$

where c_1 is a numerical constant.

Proof. Let $\mathcal{E}_{nad}(a)$ be the event that $(Q_r(a), V|_{F_r(a)})$ is not admissible. Then by Definition 3.2.4,

$$\mathcal{E}_{nad}(a) \subset \bigcup_{b \in \partial^- \tilde{\Omega}(a) \cup \partial^- Q_r(a)} \mathcal{E}_{per}^{\frac{\varepsilon_0}{60} r}(b). \tag{3.2.4}$$

Here, the notation $\mathcal{E}_{per}^{l}(b)$ is defined in Proposition 3.2.2. Assume r is large enough, by Proposition 3.2.2,

$$\mathbb{P}\left[\mathcal{E}_{nad}(a)\right] \le 8r \exp(-\frac{c_0 \varepsilon_0}{60} r) < \exp(-8c_1 r), \tag{3.2.5}$$

where $c_1 < \frac{c_0 \varepsilon_0}{480}$ is a numerical constant. $\square$

Definition 3.2.7. For any r-bit $(Q_r(a), V|_{F_r(a)})$, we denote by $S_r(a)$ the maximal connected subset of $\Omega_r(a) \cup \{b \in F_r(a) : V(b) = 0\}$ that contains $\Omega_r(a)$.

Lemma 3.2.8. *Given $V_0 : Q_r(a) \to \{0, \bar{V}\}$, suppose $(Q_r(a), V_0|_{F_r(a)})$ is an admissible r-bit. Then we have the following properties:*

1. *$\Omega_r(a) \subset S_r(a) \subset \tilde{\Omega}_r(a) \setminus \partial^- \tilde{\Omega}_r(a)$.*

2. *$S_r(a)$ is $V_0|_{F_r(a)}$-measurable.*

3. $V_0(b) = \bar{V}$ *for each* $b \in \partial^+ S_r(a)$.

Proof. The first property is due to the second item in Definition 3.2.4. The second property follows directly from Definition 3.2.7. The third property follows from the maximality of $S_r(a)$. $\square$

We now define the "cutting procedure" on an admissible r-bit $Q_r(a)$. Intuitively, the cutting procedure on $Q_r(a)$ continuously modifies the edge weight of $\partial S_r(a)$ and finally splits $S_r(a)$ and $Q_r(a) \setminus S_r(a)$.

Definition 3.2.9. Given $V : Q_r(a) \to \{0, \bar{V}\}$, suppose $(Q_r(a), V|_{F_r(a)})$ is an admissible r-bit. For $t \in [0,1]$, define operator $H^t_{Q_r(a)} : \ell^2(Q_r(a)) \to \ell^2(Q_r(a))$ as follows: $H^t_{Q_r(a)}(b,c) = t - 1$ if $\{b,c\} \in \partial S_r(a)$; $H^t_{Q_r(a)}(b,c) = H_{Q_r(a)}(b,c)$ otherwise. Denote $G^t_{Q_r(a)}(b,c;\lambda) = (H^t_{Q_r(a)} - \lambda)^{-1}(b,c)$ for any $b, c \in Q_r(a)$.

Remark 3.2.10. From Definition 3.2.9, $H^t_{Q_r(a)}$ is self-adjoint for each t. We have $H^0_{Q_r(a)} = H_{Q_r(a)}$ and $H^1_{Q_r(a)} = H_{S_r(a)} \bigoplus H_{Q_r(a) \setminus S_r(a)}$.

Lemma 3.2.11. *Given* $V : Q_r(a) \to \{0, \bar{V}\}$*, suppose* $(Q_r(a), V|_{F_r(a)})$ *is an admissible* r*-bit. Then for each* $t \in [0,1]$ *and each connected subset* $S \subset Q_r(a)$*, we have*

$$\sigma\left(H^t_{Q_r(a)}\right) \subset [0,8] \cup \left[\bar{V}, \bar{V} + 8\right], \tag{3.2.6}$$

and

$$\sigma\left(H_S\right) \subset [0,8] \cup \left[\bar{V}, \bar{V} + 8\right]. \tag{3.2.7}$$

Proof. We first prove (3.2.6). Suppose $\lambda \in \sigma\left(H^t_{Q_r(a)}\right)$, let u be an eigenfunction with $H^t_{Q_r(a)}u = \lambda u$. Pick $b \in Q_r(a)$ with $|u(b)| \geq |u(b')|$ for each $b' \in Q_r(a)$. Then we have

$$(V(b) + 4 - \lambda)u(b) = -\sum_{\substack{b' \sim b \\ b' \in Q_r(a)}} H^t_{Q_r(a)}(b, b')u(b'). \tag{3.2.8}$$

Since $|H^t_{Q_r(a)}(b, b')| \leq 1$ for each $b \neq b'$, (3.2.8) implies

$$|(V(b) + 4 - \lambda)u(b)| \leq 4|u(b)|,$$

and thus $|(V(b) + 4 - \lambda)| \leq 4$. The conclusion follows from $V(b) \in \{0, \bar{V}\}$.

Finally, to prove (3.2.7), substitute $H^t_{Q_r(a)}$ by H_S and repeat the above argument.

□

3.2.3 Resolvent estimate on r-bits

Now we define the exceptional energies $\lambda^{(i)}$'s in Theorem 3.1.1 and Theorem 3.1.4. They are exactly the eigenvalues of the minus Laplacian restricted on connected subsets of Q_r. A small neighbourhood of them (the set $J^{\bar{V}}_r$ in Definition 3.2.12) is excluded so that the resolvent is bounded on admissible r-bits (Proposition 3.2.13).

Definition 3.2.12. Given an odd number r and a real number $U > 1$, let

$$Eig_r = \bigcup_{\substack{S \subset Q_r \\ S \text{ is connected}}} \sigma\big((-\Delta)_S\big)$$

and

$$J_r^U = \bigcup_{x \in Eig_r} \left[x - U^{-\frac{1}{4}}, x + U^{-\frac{1}{4}}\right].$$

Proposition 3.2.13. *Given r a large odd number, $a \in \dot{r}\mathbb{Z}^2$ and $V' : F_r(a) \to \{0, \bar{V}\}$, we assume $\bar{V} > \exp(r^2)$. Suppose r-bit $(Q_r(a), V')$ is admissible and $\lambda_0 \in [0,8] \setminus J_r^{\bar{V}}$. Then for each $V : Q_r(a) \to \{0, \bar{V}\}$ with $V|_{F_r(a)} = V'$, each $t \in [0,1]$ and each connected subset $S \subset Q_r(a)$, we have the following:*

- $\|(H^t_{Q_r(a)} - \lambda_0)^{-1}\| \le 2\bar{V}^{\frac{1}{4}}$.
- $\|(H_S - \lambda_0)^{-1}\| \le 2\bar{V}^{\frac{1}{4}}$.
- $|G^t_{Q_r(a)}(b, b'; \lambda_0)| \le \bar{V}^{-\frac{1}{4}}$ *for each* $b \in \partial^- Q_r(a)$, $b' \in Q_r(a)$ *such that* $|b - b'| \ge \frac{\varepsilon_0}{8} r$.

Proof. We first prove the first item. The strategy here is to prove that for any eigenvalue λ of $H^t_{Q_r(a)}$, there is some $W' \subset Q_r(a)$ such that λ is close to an eigenvalue of $H_{W'}$.

If there is no eigenvalue of $H^t_{Q_r(a)}$ in $[0,8]$, then by Lemma 3.2.11, $\|(H^t_{Q_r(a)} - \lambda_0)^{-1}\| \le (\bar{V} - 8)^{-1} < 2\bar{V}^{\frac{1}{4}}$ and the first item holds.

Now assume there is an eigenvalue λ of $H^t_{Q_r(a)}$ in $[0,8]$ and we need to prove $|\lambda - \lambda_0| \ge \frac{1}{2}\bar{V}^{-\frac{1}{4}}$. Let v be an $\ell^2(Q_r(a))$ normalised eigenfunction of $H^t_{Q_r(a)}$ with eigenvalue λ. Write $T = \{a' \in Q_r(a) : V(a') = \bar{V}\}$. For each $a' \in T$, we have

$$-\sum_{\substack{b' \sim a' \\ b' \in Q_r(a)}} H^t_{Q_r(a)}(a', b') v(b') = (\bar{V} + 4 - \lambda) v(a'). \tag{3.2.9}$$

Since $\|v\|_{\ell^2(Q_r(a))} = 1$ and $|H^t_{Q_r(a)}(b', b'')| \leq 1$ for any $b' \neq b''$, we have $|v(a')| \leq 4/(\bar{V} - 4)$ for $a' \in T$. This implies $\|v\|_{\ell^2(T)} \leq 4r/(\bar{V} - 4) < \frac{1}{2}$ since $\bar{V} > \exp(r^2)$. Consider all maximal connected subsets $W \subset Q_r(a) \setminus T$. The number of them is less than r^2, thus there exists one of these subsets $W' \subset Q_r(a)$ with

$$\|v\|_{\ell^2(W')} \geq \frac{1}{2r}. \tag{3.2.10}$$

Since $V = 0$ on W', by Lemma 3.2.8, $\partial^+ S_r(a) \cap W' = \emptyset$ and $(H^t_{Q_r(a)})_{W'} = H_{W'}$. Thus for each $b \in W'$,

$$(H_{W'} - \lambda)(v|_{W'})(b) = (H^t_{Q_r(a)} - \lambda)v(b) - \sum_{\substack{b' \sim b \\ b' \in \partial^+ W' \cap Q_r(a)}} H^t_{Q_r(a)}(b, b')v(b'). \tag{3.2.11}$$

By maximality of W', for each $a' \in \partial^+ W' \cap Q_r(a)$, $a' \in T$ and thus

$$|v(a')| \leq 4/(\bar{V} - 4).$$

Since $(H^t_{Q_r(a)} - \lambda)v = 0$ and $|H^t_{Q_r(a)}(b, b')| \leq 1$ when $b \neq b'$, (3.2.11) implies

$$|(H_{W'} - \lambda)(v|_{W'})(b)| \leq 16/(\bar{V} - 4)$$

for each $b \in W'$. Thus

$$\|(H_{W'} - \lambda)(v|_{W'})\|_{\ell^2(W')} \leq 16r/(\bar{V} - 4) \leq 32r^2/(\bar{V} - 4)\|v\|_{\ell^2(W')} \tag{3.2.12}$$

by (3.2.10). Writing $v|_{W'}$ as a linear combination of eigenfunctions of $H_{W'}$, (3.2.12) provides an eigenvalue λ' of $H_{W'}$ such that $|\lambda-\lambda'| \leq 32r^2/(\bar{V}-4)$. Since $\lambda' \in Eig_r$ and $\lambda_0 \notin J_r^{\bar{V}}$, by Definition 3.2.12,

$$|\lambda_0-\lambda| \geq |\lambda_0-\lambda'| - |\lambda'-\lambda| > \bar{V}^{-\frac{1}{4}} - 32r^2/(\bar{V}-4) > \frac{1}{2}\bar{V}^{-\frac{1}{4}}.$$

Here, we used $\bar{V} > \exp(r^2)$. The first item follows.

The proof of the second item is similar to the proof of the first item. Assume there is an eigenvalue λ_* of H_S in $[0,8]$. Let v_* be an $\ell^2(S)$ normalised eigenfunction of H_S with eigenvalue λ_*, we need to prove $|\lambda_*-\lambda_0| \geq \frac{1}{2}\bar{V}^{-\frac{1}{4}}$. For each $a' \in T \cap S$, we have

$$-\sum_{\substack{b'\sim a'\\ b'\in S}} H_S(a',b')v_*(b') = (\bar{V}+4-\lambda_*)v_*(a'). \tag{3.2.13}$$

Since $|H_S(b',b'')| \leq 1$ for any $b' \neq b''$, we have $|v_*(a')| \leq 4/(\bar{V}-4)$ for $a' \in T \cap S$. This implies $\|v_*\|_{\ell^2(T)} \leq 4r/(\bar{V}-4) < \frac{1}{2}$. Consider all maximal connected subsets $W \subset S \setminus T$. The number of them is less than r^2, thus there exists one of these subsets $W'' \subset S$ with $\|v_*\|_{\ell^2(W'')} \geq \frac{1}{2r}$. For each $b \in W''$,

$$(H_{W''}-\lambda_*)(v_*|_{W''})(b) = (H_S-\lambda_*)v_*(b) - \sum_{\substack{b'\sim b\\ b'\in\partial^+W''\cap S}} H_S(b,b')v_*(b'). \tag{3.2.14}$$

By maximality of W'', for each $a' \in \partial^+ W'' \cap S$, $a' \in T$ and thus

$$|v_*(a')| \leq 4/(\bar{V} - 4).$$

Since $(H_S - \lambda_*)v_* = 0$ and $|H_S(b, b')| \leq 1$ when $b \neq b'$, (3.2.14) implies

$$|(H_{W''} - \lambda_*)(v_*|_{W''})(b)| \leq 16/(\bar{V} - 4)$$

for each $b \in W''$. Thus

$$\|(H_{W''} - \lambda_*)(v_*|_{W''})\|_{\ell^2(W'')} \leq 16r/(\bar{V} - 4) \leq 32r^2/(\bar{V} - 4)\|v_*\|_{\ell^2(W'')}. \qquad (3.2.15)$$

Writing $v_*|_{W''}$ as a linear combination of eigenfunctions of $H_{W''}$, (3.2.15) provides an eigenvalue λ'_* of $H_{W''}$ such that $|\lambda_* - \lambda'_*| \leq 32r^2/(\bar{V} - 4)$. Since $\lambda'_* \in Eig_r$ and $\lambda_0 \notin J_r^{\bar{V}}$, by the same argument for the first item, we have

$$|\lambda_0 - \lambda_*| > \frac{1}{2}\bar{V}^{-\frac{1}{4}}$$

and the second item follows.

Now we prove the third item and the strategy here is to exploit the resolvent identity. Pick $b, b' \in Q_r(a)$ with $b \in \partial^- Q_r(a)$ and $|b - b'| \geq \frac{\varepsilon_0 r}{8}$. We claim that, there exists connected $S_0 \subset Q_r(a) \cap Q_{\frac{\varepsilon_0 r}{9}}(b)$ with $b \in S_0$ such that, for any $c \in S_0$ and $c' \in Q_r(a) \setminus S_0$ with $c \sim c'$, we have $c \in T$. To see this, if $V(b) = \bar{V}$, then simply let $S_0 = \{b\}$; otherwise, let S_1 be the maximal connected subset of $Q_r(a) \setminus T$ that contains b. Since $Q_r(a)$ is admissible, the first item in Definition 3.2.4 implies $S_1 \subset Q_r(a) \cap Q_{\frac{\varepsilon_0 r}{10}(b)}$. Let $S_0 = S_1 \cup (\partial^+ S_1 \cap Q_r(a))$ and our claim follows from the maximality of S_1.

By Lemma 3.2.8, $S_r(a) \subset \tilde{\Omega}_r(a)$ and thus $S_r(a) \cap (S_0 \cup \partial^+ S_0) = \emptyset$. By resolvent identity,

$$G^t_{Q_r(a)}(b, b'; \lambda_0) = \sum_{\substack{c \in S_0, c \sim c' \\ c' \in Q_r(a) \setminus S_0}} G_{S_0}(b, c; \lambda_0) G^t_{Q_r(a)}(c', b'; \lambda_0). \tag{3.2.16}$$

By definition of resolvent,

$$(V(c) - \lambda_0 + 4) G_{S_0}(b, c; \lambda_0) = \delta_{c,b} + \sum_{c'' \sim c, c'' \in S_0} G_{S_0}(b, c''; \lambda_0) \tag{3.2.17}$$

where $\delta_{c,b} = 1$ if $c = b$ and $\delta_{c,b} = 0$ otherwise. Hence

$$|G_{S_0}(b, c; \lambda_0)| \leq \frac{1}{|V(c) - \lambda_0 + 4|}(1 + 4\|(H_{S_0} - \lambda_0)^{-1}\|). \tag{3.2.18}$$

The second item of this proposition implies $\|(H_{S_0} - \lambda_0)^{-1}\| \leq 2\bar{V}^{\frac{1}{4}}$. Assume $c \sim c'$ for some $c \in S_0$ and $c' \in Q_r(a) \setminus S_0$, then the property of S_0 and inequality (3.2.18) together imply $V(c) = \bar{V}$ and

$$|G_{S_0}(b, c; \lambda_0)| \leq 20\bar{V}^{-\frac{3}{4}}. \tag{3.2.19}$$

Finally, in (3.2.16), by the first item in this proposition and inequality (3.2.19),

$$\begin{aligned}
|G^t_{Q_r(a)}(b,b';\lambda_0)| \leq &\|(H^t_{Q_r(a)}-\lambda_0)^{-1}\| \sum_{\substack{c\in S_0, c\sim c'\\ c'\in Q_r(a)\setminus S_0}} |G_{S_0}(b,c;\lambda_0)| \\
\leq & 2\bar{V}^{\frac{1}{4}} \sum_{\substack{c\in S_0, c\sim c'\\ c'\in Q_r(a)\setminus S_0}} |G_{S_0}(b,c;\lambda_0)| \\
\leq & 320 r^2 \bar{V}^{-\frac{1}{2}} \\
< & \bar{V}^{-\frac{1}{4}}.
\end{aligned}$$

□

3.2.4 Initial scale analysis

In this subsection, we extend the cutting procedure to larger boxes and prove the induction base case for multi-scale analysis (Proposition 3.2.21).

Definition 3.2.14. Suppose r is an odd number, $a \in \mathbb{Z}^2$ and $L \in \mathbb{Z}_+$. We say $Q_L(a)$ is *r-dyadic* if there exists $k \in \mathbb{Z}_+$ such that $a \in 2^k \dot{r}\mathbb{Z}^2$ and $L = 2^{k+1}\dot{r} + r$. In this case, L is called an r-dyadic scale.

Lemma 3.2.15. *Suppose $Q_L(a)$ is an r-dyadic box. Then we have*

$$Q_L(a) = \bigcup_{b\in \dot{r}\mathbb{Z}^2 \cap Q_L(a)} Q_r(b).$$

If r-bit $Q_r(b') \not\subset Q_L(a)$, then $\tilde{\Omega}_r(b') \cap Q_L(a) = \emptyset$.

Proof. We assume without loss of generality that $a = \mathbf{0}$. Write $L = 2^{k+1}\dot{r} + r$ with $k \in \mathbb{Z}_+$. By (3.1.10),

$$Q_L(\mathbf{0}) = \{(x, y) \in \mathbb{Z}^2 : -2^k\dot{r} - \frac{r-1}{2} \leq x, y \leq 2^k\dot{r} + \frac{r-1}{2}\}. \tag{3.2.20}$$

By Definition 3.2.3, $\dot{r} \geq (1 - \frac{\varepsilon_0}{2})(r-1) > \frac{r}{2}$ and thus

$$\dot{r}\mathbb{Z}^2 \cap Q_L(\mathbf{0}) = \{(\dot{r}x, \dot{r}y) : |x|, |y| \leq 2^k,\ x, y \in \mathbb{Z}\}. \tag{3.2.21}$$

Hence by (3.2.20) and $\dot{r} < r - 1$,

$$Q_L(\mathbf{0}) = \bigcup_{b \in \dot{r}\mathbb{Z}^2 \cap Q_L(\mathbf{0})} \{b' \in \mathbb{Z}^2 : |b - b'| \leq \frac{r-1}{2}\} = \bigcup_{b \in \dot{r}\mathbb{Z}^2 \cap Q_L(\mathbf{0})} Q_r(b).$$

Assume r-bit $Q_r(b') \not\subset Q_L(\mathbf{0})$, then $b' \notin \dot{r}\mathbb{Z}^2 \cap Q_L(\mathbf{0})$. Write $b' = (\dot{r}x', \dot{r}y')$ with $x', y' \in \mathbb{Z}$. By (3.2.21), without loss of generality, we assume $|x'| \geq 2^k + 1$. By (3.2.20),

$$\inf_{b'' \in Q_L(\mathbf{0})} |b' - b''| \geq \dot{r} - \frac{r-1}{2} \geq (1 - \frac{\varepsilon_0}{2})(r-1) - \frac{r-1}{2} > \frac{(1 - \frac{3}{2}\varepsilon_0)r - 1}{2}.$$

By Definition 3.2.3, $\tilde{\Omega}_r(b') \cap Q_L(\mathbf{0}) = \emptyset$. $\square$

We now extend the "cutting procedure" to r-dyadic boxes. It will be used in the proof of Proposition 3.3.18.

Definition 3.2.16. Given an r-dyadic box $Q_L(a)$ and $V : Q_L(a) \to \{0, \bar{V}\}$, let $\mathcal{R}$ be a subset of admissible r-bits inside $Q_L(a)$. For each $t \in [0,1]$, define $H^{\mathcal{R},t}_{Q_L(a)} : \ell^2(Q_L(a)) \to \ell^2(Q_L(a))$ as follows:

$$\begin{cases} H^{\mathcal{R},t}_{Q_L(a)}(b,c) = t-1 & \text{if } \{b,c\} \in \bigcup_{Q_r(a')\in\mathcal{R}} \partial S_r(a'); \\ H^{\mathcal{R},t}_{Q_L(a)}(b,c) = H_{Q_L(a)}(b,c) & \text{otherwise.} \end{cases}$$

Denote $G^{\mathcal{R},t}_{Q_L(a)}(b,c;\lambda) = (H^{\mathcal{R},t}_{Q_L(a)} - \lambda)^{-1}(b,c)$ for $b, c \in Q_L(a)$.

Definition 3.2.17. For any large odd number r, denote $\Theta^r = \cup_{a\in \dot{r}\mathbb{Z}^2} F_r(a)$. For simplicity, we also denote Θ^r by Θ_1 if r is already given in context.

The reason to define Θ^r is that, one only needs to know the value of V on Θ^r to decide whether each r-bit is admissible or not. The sub-index of "Θ_1" is for the consistency of notations in later multi-scale analysis Theorem 3.4.7.

Definition 3.2.18. Given an odd number r, an r-dyadic box $Q_L(a)$ and a potential function $V' : \Theta_1 \cap Q_L(a) \to \{0, \bar{V}\}$, we say $Q_L(a)$ is *perfect* if for any r-bit $Q_r(b) \subset Q_L(a)$, $(Q_r(b), V'|_{F_r(b)})$ is admissible.

Proposition 3.2.19. *Suppose odd number r is large enough and c_1 is the constant in Proposition 3.2.6. Given r-dyadic box $Q_L(a)$ with $L \le \exp(c_1 r)$, the event that $Q_L(a)$ is perfect only depends on $V|_{\Theta_1 \cap Q_L(a)}$ and*

$$\mathbb{P}[Q_L(a) \text{ is perfect}] \ge 1 - L^{-6}. \tag{3.2.22}$$

Proof. Since for each r-bit $Q_r(b) \subset Q_L(a)$, the event that it is admissible only depends on $V|_{F_r(b)}$, thus the event that $Q_L(a)$ is perfect only depends on $V|_{\Theta_1 \cap Q_L(a)}$.

By Proposition 3.2.6, we have

$$\mathbb{P}[Q_L(a) \text{ is perfect}] \geq 1 - L^2 \exp(-8c_1 r) \geq 1 - L^{-6} \tag{3.2.23}$$

since $L \leq \exp(c_1 r)$. □

Definition 3.2.20. Given $S_1, S_2 \subset \mathbb{Z}^2$, denote

$$\operatorname{dist}(S_1, S_2) = \inf_{a \in S_1, b \in S_2} |a - b|.$$

We now prove the exponential decaying property of resolvent for perfect r-dyadic boxes. It will serve as the induction base case for the multi-scale analysis in Section 3.4.

Proposition 3.2.21. *Suppose odd number r is large enough and $\bar{V} > \exp(r^2)$. If $V' : \Theta_1 \cap Q_L(a) \to \{0, \bar{V}\}$ such that $Q_L(a)$ is a perfect r-dyadic box, then for any $V : Q_L(a) \to \{0, \bar{V}\}$ with $V|_{\Theta_1 \cap Q_L(a)} = V'$, any $\lambda_0 \in [0,8] \setminus J_r^{\bar{V}}$, any subset $\mathcal{R}$ of r-bits inside $Q_L(a)$, any $t \in [0,1]$, and each $b, c \in Q_L(a)$, we have*

$$|G_{Q_L(a)}^{\mathcal{R},t}(b, c; \lambda_0)| \leq \bar{V}^{-\frac{|b-c|}{8r}+1}. \tag{3.2.24}$$

Proof. For simplicity of notations, we assume $a = \mathbf{0}$.

Let $g = \max_{b,c\in Q_L} |G^{\mathcal{R},t}_{Q_L}(b,c;\lambda_0)|\bar{V}^{\frac{|b-c|}{8r}}$ and assume that for some $b, b' \in Q_L$, $|G^{\mathcal{R},t}_{Q_L}(b,b';\lambda_0)|\bar{V}^{\frac{|b-b'|}{8r}} = g$.

Note that $Q_L = \bigcup_{a'\in\dot{r}\mathbb{Z}^2\cap Q_L} Q_r(a')$ by Lemma 3.2.15. By definition, we have $\dot{r} = \left\lceil \left(1-\frac{\varepsilon_0}{2}\right)(r-1)\right\rceil$. By elementary geometry, there is an r-bit $Q_r(c) \subset Q_L$ such that $b' \in Q_r(c)$ and $\mathrm{dist}(b', Q_L \setminus Q_r(c)) \geq \frac{\varepsilon_0 r}{7}$. By resolvent identity,

$$\begin{aligned}
&G^{\mathcal{R},t}_{Q_L}(b,b';\lambda_0)\\
&\quad= \sum_{\substack{b''\in Q_r(c)\\ b''\sim b'''\\ b'''\in Q_L\setminus Q_r(c)}} \widetilde{G_{Q_r(c)}}(b',b'';\lambda_0)G^{\mathcal{R},t}_{Q_L}(b''',b;\lambda_0) + 1_{b\in Q_r(c)}\widetilde{G_{Q_r(c)}}(b,b';\lambda_0). \quad (3.2.25)
\end{aligned}$$

Here, we have $\widetilde{G_{Q_r(c)}}(b',b'';\lambda_0) = G^t_{Q_r(c)}(b',b'';\lambda_0)$ if $Q_r(c) \in \mathcal{R}$ and $\widetilde{G_{Q_r(c)}}(b',b'';\lambda_0) = G_{Q_r(c)}(b',b'';\lambda_0)$ otherwise. Note that, if $b'' \sim b'''$ for some $b'' \in Q_r(c)$ and $b''' \in Q_L \setminus Q_r(c)$, then $|b'-b''| \geq \frac{\varepsilon_0 r}{7} - 1 \geq \frac{\varepsilon_0 r}{8}$. In this case, by Proposition 3.2.13, $|\widetilde{G_{Q_r(c)}}(b',b'';\lambda_0)| \leq \bar{V}^{-\frac{1}{4}} \leq \bar{V}^{-\frac{|b'-b''|}{4r}}$ since $|b'-b''| \leq r$. Thus, we can estimate the first term in the right hand side of (3.2.25) by

$$\left|\sum_{\substack{b''\in Q_r(c)\\ b''\sim b'''\\ b'''\in Q_L\setminus Q_r(c)}} \widetilde{G_{Q_r(c)}}(b',b'';\lambda_0)G^{\mathcal{R},t}_{Q_L}(b''',b;\lambda_0)\right| \quad (3.2.26)$$

$$\leq \sum_{\substack{b''\in Q_r(c)\\ b''\sim b'''\\ b'''\in Q_L\setminus Q_r(c)}} g\bar{V}^{-\frac{|b'-b''|}{4r}-\frac{|b-b'''|}{8r}} \quad (3.2.27)$$

$$< \frac{1}{2}\bar{V}^{-\frac{|b-b'|}{8r}} g \quad (3.2.28)$$

The second inequality is because, by triangle inequality with $|b''-b'''|=1$ and $|b'-b''|\geq\frac{\varepsilon_0 r}{8}$,

$$\bar{V}^{-\frac{|b'-b''|}{4r}-\frac{|b-b'''|}{8r}} \leq \bar{V}^{-\frac{|b-b'|}{8r}+\frac{1}{8r}-\frac{\varepsilon_0}{64}} \tag{3.2.29}$$

$$\leq \exp(\frac{1}{8}r-\frac{\varepsilon_0}{64}r^2)\bar{V}^{-\frac{|b-b'|}{8r}} \tag{3.2.30}$$

$$< \frac{1}{16}r^{-1}\bar{V}^{-\frac{|b-b'|}{8r}} \tag{3.2.31}$$

for large enough r, where (3.2.30) is due to $\bar{V}>\exp(r^2)$. Since

$$|G^{\mathcal{R},t}_{Q_L}(b,b';\lambda_0)| = \bar{V}^{-\frac{|b-b'|}{8r}}g,$$

by (3.2.25) and (3.2.28),

$$g \leq \frac{1}{2}g + \bar{V}^{\frac{|b-b'|}{8r}} 1_{b\in Q_r(c)}|\widetilde{G_{Q_r(c)}}(b,b';\lambda_0)|. \tag{3.2.32}$$

If $b\notin Q_r(c)$, then we have $g=0$. Otherwise, $|b-b'|\leq r$. By the first item in Proposition 3.2.13, $|\widetilde{G_{Q_r(c)}}(b,b';\lambda_0)|\leq 2\bar{V}^{\frac{1}{4}}$ and thus

$$g \leq 2\bar{V}^{\frac{|b-b'|}{8r}}|\widetilde{G_{Q_r(c)}}(b,b';\lambda_0)| \leq 4\bar{V}^{\frac{3}{8}} < \bar{V}, \tag{3.2.33}$$

which is equivalent to (3.2.24). □

3.3 Wegner Estimate

In this section we prove the Wegner estimate (Proposition 3.3.18) which will be used in multi-scale analysis Theorem 3.4.7. In Section 3.3.1, we collect several lemmas on unique continuation (Lemma 3.3.5), eigenvalue variation (Lemma 3.3.8 and Lemma 3.3.9) and almost orthonormal vectors (Lemma 3.3.10). In Section 3.3.2, a generalized Sperner lemma (Lemma 3.3.16) for directed graph products is proved. All these lemmas will be used in Section 3.3.3 to prove the Wegner estimate Proposition 3.3.18.

3.3.1 Auxiliary lemmas

We first need some geometry notations from [DS20]. The following Definition 3.3.1 to 3.3.4 are the same as Definition 3.1 to 3.4 in [DS20].

Definition 3.3.1. Given two subsets $I, J \subset \mathbb{Z}$, denote

$$R_{I,J} = \{(x, y) \in \mathbb{Z}^2 : x + y \in I \text{ and } x - y \in J\}. \tag{3.3.1}$$

We call $R_{I,J}$ a *tilted rectangle* if I, J are intervals. A *tilted square* $\tilde{Q}$ is a tilted rectangle $R_{I,J}$ with $|I| = |J|$. With a little abuse of notations, we denote $\ell(\tilde{Q}) = |I|$ for a tilted square $\tilde{Q} = R_{I,J}$.

Definition 3.3.2. Given $k \in \mathbb{Z}$, define the diagonals

$$\mathcal{D}_k^{\pm} = \{(x, y) \in \mathbb{Z}^2 : x \pm y = k\}. \tag{3.3.2}$$

Definition 3.3.3. Suppose $\Theta \subset \mathbb{Z}^2$, $\eta > 0$ a density, and R a tilted rectangle. Say that Θ is $(\eta, \pm)$-sparse in R if

$$|\mathcal{D}_k^{\pm} \cap \Theta \cap R| \leq \eta |\mathcal{D}_k^{\pm} \cap R| \text{ for all diagonals } \mathcal{D}_k^{\pm}. \tag{3.3.3}$$

We say that Θ is η-sparse in R if it is both $(\eta, +)$-sparse and $(\eta, -)$-sparse in R.

Definition 3.3.4. A subset $\Theta \subset \mathbb{Z}^2$ is called η-regular in a set $E \subset \mathbb{Z}^2$ if we have $\sum_k |Q_k| \leq \eta |E|$ whenever Θ is not η-sparse in each of the disjoint tilted squares $Q_1, Q_2, \cdots, Q_n \subset E$.

The following Lemma 3.3.5 and 3.3.6 are used to find an enough portion of the box where an eigenfunction has a decent lower bound. In particular, Lemma 3.3.5 is analogue of [DS20, Theorem 3.5] and its proof is given in Section 3.5.

Lemma 3.3.5. *There exists numerical constant $0 < \varepsilon_1 < \frac{1}{100}$ such that the following holds. For every $\varepsilon \leq \varepsilon_1$, there is a large $\alpha > 1$ depending on ε such that, if*

1. *$Q \subset \mathbb{Z}^2$ a box with $\ell(Q) \geq \alpha$*
2. *$\Theta \subset Q$ is ε-regular in Q*
3. *$\bar{V} \geq 2$ and $\lambda_0 \in [0, 8]$*
4. *$V' : \Theta \to \{0, \bar{V}\}$*

5. $\mathcal{E}_{uc}^{\varepsilon,\alpha}(Q,\Theta)$ *denotes the event that*

$$\begin{cases} |\lambda-\lambda_0|\le(\ell(Q)\bar{V})^{-\alpha\ell(Q)} \\ Hu=\lambda u \text{ in } Q \\ |u|\le 1 \text{ in a } 1-\varepsilon^3 \text{ fraction of } Q\setminus\Theta \end{cases} \tag{3.3.4}$$

implies $|u|\le(\ell(Q)\bar{V})^{\alpha\ell(Q)}$ *in* $\frac{1}{100}Q$,

then $\mathbb{P}[\mathcal{E}_{uc}^{\varepsilon,\alpha}(Q,\Theta)\big|V|_\Theta=V']\ge 1-\exp(-\varepsilon\ell(Q)^{\frac{2}{3}})$.

The following lemma is a rewrite of [DS20, Lemma 5.3] and its proof is the same as the proof of [DS20, Lemma 5.3].

Lemma 3.3.6. *For every integer $K\ge 1$, there exists $C_K>0$ depending on K such that the following holds. If*

1. $\bar{V}\ge 2$ *and* $\lambda\in[0,8]$
2. $L\ge C_K L'\ge L'\ge C_K$
3. *box* $Q\subset\mathbb{Z}^2$ *with* $\ell(Q)=L$
4. *boxes* $Q'_k\subset Q$ *with* $\ell(Q'_k)=L'$ *for* $k=1,2,\cdots,K$
5. $H_Q u=\lambda u$,

then,

$$\|u\|_{\ell^\infty(Q')}\ge\bar{V}^{-C_K L'}\|u\|_{\ell^\infty(Q)} \tag{3.3.5}$$

holds for some $2Q'\subset Q\setminus\cup_k Q'_k$ *with* $\ell(Q')=L'$.

Definition 3.3.7. Given a self-adjoint matrix A and $\lambda \in \mathbb{R}$, denote

$$n(A;\lambda) = \operatorname{trace} 1_{(-\infty,\lambda)}(A).$$

i.e. $n(A;\lambda)$ is the number of A's eigenvalues (with multiplicities) which are less than λ.

The following Lemma 3.3.8 and 3.3.9 will provide a lower bound of the eigenvalue variation under a rank one perturbation of an operator. Lemma 3.3.8 was proved in [DS20, Lemma 5.1].

Lemma 3.3.8. *Suppose real symmetric $n \times n$ matrix A has eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \in \mathbb{R}$ with orthonormal eigenbasis $v_1, v_2, \cdots, v_n \in \mathbb{R}^n$. Let integers $k \in \{1, 2, \cdots, n\}$ and $1 \leq j \leq i \leq n-1$. If*

1. $0 < r_1 < r_2 < r_3 < r_4 < r_5 < 1$
2. $r_1 \leq C \min\{r_3 r_5, r_2 r_3 / r_4\}$
3. $0 < \lambda_j \leq \lambda_i < r_1 < r_2 < \lambda_{i+1}$
4. $v_j^2(k) \geq r_3$
5. $\sum_{r_2 < \lambda_l < r_5} v_l^2(k) \leq r_4$

then $n(A; r_1) > n(A + tP_k; r_1)$ for $t \geq 1$, where P_k is the projection operator defined by $(P_k u)(i) = 0$ if $i \neq k$ and $(P_k u)(k) = u(k)$ for any $u \in \mathbb{R}^n$.

Proof. [DS20, Lemma 5.1] implies the conclusion for the case when $t = 1$. The conclusion still holds for $t \geq 1$ by monotonicity. $\square$

Lemma 3.3.9. *Let $k \in \{1, 2, \cdots, n\}$ and P_k be the projection operator defined by $(P_k u)(i) = 0$ if $i \neq k$ and $(P_k u)(k) = u(k)$ for any $u \in \mathbb{R}^n$. Suppose self-adjoint operator $A : \mathbb{R}^n \to \mathbb{R}^n$ has eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \in \mathbb{R}$ with orthonormal eigenbasis $v_1, v_2, \cdots, v_n \in \mathbb{R}^n$.*

If $\lambda \notin \sigma(A)$ and $\sum_{i=1}^n \frac{v_i(k)^2}{\lambda_i - \lambda} > 0 (< 0)$, then $\lambda \notin \sigma(A + tP_k)$ for each $t > 0 (< 0)$, respectively.

Proof. We only consider the case when $\sum_{i=1}^n \frac{v_i(k)^2}{\lambda_i - \lambda} > 0$, the other case follows the same argument.

For each $t \in \mathbb{R}$, let $v_1^t, v_2^t, \cdots, v_n^t$ be the orthonormal eigenbasis of $A + tP_k$ with eigenvalues $\lambda_1^t, \lambda_2^t \cdots, \lambda_n^t$. Then the resolvent of $A + tP_k$ at k with energy $\lambda_0 \notin \sigma(A + tP_k)$ is

$$G_t(k, k; \lambda_0) = \left\langle 1_k, (A + tP_k - \lambda_0)^{-1} 1_k \right\rangle = \sum_{i=1}^n \frac{v_i^t(k)^2}{\lambda_i^t - \lambda_0}. \tag{3.3.6}$$

Let i denote the imaginary unit. By resolvent identity, for each $t, \eta > 0$,

$$G_t(k, k; \lambda + \mathrm{i}\eta) = \frac{1}{t + G_0(k, k; \lambda + \mathrm{i}\eta)^{-1}}. \tag{3.3.7}$$

Since $G_0(k, k; \lambda) = \sum_{i=1}^n \frac{v_i(k)^2}{\lambda_i - \lambda} > 0$, (3.3.7) implies $\lim_{\eta \to 0} G_t(k, k; \lambda + \mathrm{i}\eta) > 0$ for any

$t > 0$. Since $G_t(k, k; \lambda + i\eta) = \sum_{i=1}^{n} \frac{v_i^t(k)^2}{\lambda_i^t - \lambda - i\eta}$,

$$\lim_{\eta \to 0} \sum_{i=1}^{n} \frac{v_i^t(k)^2}{\lambda_i^t - \lambda - i\eta} < \infty. \tag{3.3.8}$$

Assume $\lambda \in \sigma(A + tP_k)$, then there exists i_0 with $\lambda_{i_0}^t = \lambda$. Equation (3.3.8) implies $v_{i_0}^t(k) = 0$. Since $(A + tP_k)v_{i_0}^t = \lambda_{i_0}^t v_{i_0}^t$, we have $Av_{i_0}^t = \lambda v_{i_0}^t$. This contradicts with $\lambda \notin \sigma(A)$. $\square$

We also need the following bound on the number of almost orthonormal vectors which was proved in [DS20]. A similar version of the following lemma was also proved in [Tao].

Lemma 3.3.10 (Lemma 5.2 in [DS20]). *If $v_1, \cdots, v_m \in \mathbb{R}^n$ satisfy $|\langle v_i, v_j \rangle - \delta_{ij}| \leq (5n)^{-\frac{1}{2}}$, then $m \leq (5 - \sqrt{5})n/2$.*

3.3.2 Sperner Lemma

We prove a generalization of [DS20, Theorem 4.2] which will be used in an eigenvalue variation argument in the proof of Proposition 3.3.18.

Definition 3.3.11. Suppose $\rho \in (0, 1]$. A set $\mathcal{A}$ of subsets of $\{1, 2, \cdots, n\}$ is ρ-Sperner if, for every $A \in \mathcal{A}$, there is a set $B(A) \subset \{1, 2, \cdots, n\} \setminus A$ such that $|B(A)| \geq \rho(n - |A|)$ and $A \subset A' \in \mathcal{A}$ implies $A' \cap B(A) = \emptyset$.

The following lemma is proved in [DS20, Theorem 4.2].

Lemma 3.3.12 (Theorem 4.2 in [DS20])**.** *If $\rho \in (0, 1]$ and $\mathcal{A}$ is a ρ-Sperner set of subsets of $\{1, 2, \cdots, n\}$, then*

$$|\mathcal{A}| \leq 2^n n^{-\frac{1}{2}} \rho^{-1}.$$

Definition 3.3.13. Suppose $A = (T, E)$ is a simple directed graph (without multi-edges or self-loops) with vertex set T and edge set E. For each $e \in E$, we denote by $e^-(e^+)$ the starting (ending) vertex of e respectively. i.e. $e = (e^-, e^+)$. For two $e_1, e_2 \in E$, we say e_1 and e_2 *have no intersection* if $e_1^\pm, e_2^\pm$ are four different vertices; otherwise, we say e_1 and e_2 *have intersection.*

Definition 3.3.14. Given $k \in \mathbb{Z}_+$ and a simple directed graph $A = (T, E)$, A is called *k-colourable* if E can be written as a disjoint union $E = \bigcup_{j=1}^k E_j$ such that for each $j \in \{1, \cdots, k\}$ and $e_1 \neq e_2 \in E_j$, e_1 and e_2 have no intersection.

Lemma 3.3.15. *Suppose $A = (T, E)$ is a simple directed graph and $m \in \mathbb{Z}_+$. Assume for each $x \in T$,*

$$|\{e \in E : e^+ = x\} \cup \{e \in E : e^- = x\}| \leq m. \tag{3.3.9}$$

Then A is $2m - 1$-colourable.

Proof. By (3.3.9), each $e \in E$ has intersection with at most $2m - 2$ other edges. Thus we can color the edges of A by at most $2m - 1$ colors such that any two edges with the same color have no intersection. $\square$

The following lemma is a generalization of [DS20, Theorem 4.2] in the sense that [DS20, Theorem 4.2] is the special case when each graph A_i (see below) has two vertices and one directed edge.

Lemma 3.3.16. *Given $N, k, K_0 \in \mathbb{Z}_+$, suppose $A_i = (T_i, E_i)$ are simple directed graphs for $1 \leq i \leq N$. Assume A_i is k-colourable for each $1 \leq i \leq N$.*

Suppose $B \subset T_1 \times T_2 \times \cdots \times T_N$ satisfies the following:

1. *Each $\vec{x} = (x_1, x_2, \cdots, x_N) \in B$ is associated with K_0 indices $1 \leq I_1(\vec{x}) < I_2(\vec{x}) < \cdots < I_{K_0}(\vec{x}) \leq N$ and K_0 edges $e_j(\vec{x}) \in E_{I_j(\vec{x})}$ $(j = 1, \cdots, K_0)$ such that $e_j(\vec{x})^- = x_{I_j(\vec{x})}$ $(j = 1, \cdots, K_0)$.*

2. *$|B| > K_0^{-1} k^2 N^{\frac{1}{2}} |T_1||T_2| \cdots |T_N|$,*

then there exist $\vec{x}, \vec{y} \in B$ such that the following properties hold:

(a) *for each $i = 1, 2, \cdots, N$, either $x_i = y_i$ or $(x_i, y_i) \in E_i$,*

(b) *there exists $j \in \{1, 2, \cdots, K_0\}$ such that $(x_{I_j(\vec{x})}, y_{I_j(\vec{x})}) = e_j(\vec{x})$.*

Proof. Let us first consider an easier case when each A_i consists of two vertices and one directed edge (thus we can assume $k = 1$). Let e_i denote the single directed edge in A_i for $1 \leq i \leq N$. Then there is a bijection between $T_1 \times T_2 \times \cdots \times T_N$ and the power set of $\{1, \cdots, N\}$:

$$\vec{x} \longmapsto Y_{\vec{x}} = \{i : 1 \leq i \leq N, x_i = e_i^+\}. \tag{3.3.10}$$

We prove the lemma by contradiction. We assume there are no two elements in B satisfying both (a) and (b). Note that in our case, for any $\vec{x}, \vec{y} \in B$, condition (a) is equivalent to $Y_{\vec{x}} \subset Y_{\vec{y}}$ and condition (b) is equivalent to $Y_{\vec{y}} \cap \{I_j(\vec{x}) : 1 \leq j \leq K_0\} \neq \emptyset$. Hence, for any $\vec{x}, \vec{y} \in B$, $Y_{\vec{x}} \subset Y_{\vec{y}}$ implies $Y_{\vec{y}} \cap \{I_j(\vec{x}) : 1 \leq j \leq K_0\} = \emptyset$. By Definition 3.3.11, $\{Y_{\vec{z}} : \vec{z} \in B\}$ is K_0/N-Sperner. By Lemma 3.3.12, $|B| \leq 2^N N^{-\frac{1}{2}} K_0^{-1} = K_0^{-1} N^{\frac{1}{2}} |T_1||T_2| \cdots |T_N|$ which contradicts with assumption 2.

Now we consider the general case and we first prove that we can assume $k = 1$ without loss of generality. By assumption 2,

$$K_0^{-1} k^2 N^{\frac{1}{2}} |T_1||T_2| \cdots |T_N| < |B| \leq |T_1||T_2| \cdots |T_N|,$$

thus we have $k^2 N^{\frac{1}{2}} < K_0$. In particular, $k < K_0$.

Claim 3.3.17. *We can assume $k = 1$.*

Proof of the claim. For each i, since A_i is k-colourable, we can write E_i as a disjoint union $E_i = \bigcup_{m=1}^k E_i^{(m)}$ such that any two edges in $E_i^{(m)}$ have no intersection. For each $\vec{x} \in B$, by pigeonhole principle, there exists $m(\vec{x}) \in \{1, 2, \cdots, k\}$, such that

$$\left| \left\{ 1 \leq j \leq K_0 : e_j(\vec{x}) \in E_{I_j(\vec{x})}^{(m(\vec{x}))} \right\} \right| \geq \left\lceil \frac{K_0}{k} \right\rceil.$$

Since $B = \bigcup_{m=1}^k B_m$ with $B_m = \{\vec{x} \in B : m(\vec{x}) = m\}$, by pigeonhole principle again, there exists $m' \in \{1, \cdots, k\}$ with $|B_{m'}| \geq \left\lceil \frac{1}{k} |B| \right\rceil$ and thus

$$|B_{m'}| \geq \left\lceil \frac{K_0}{k} \right\rceil^{-1} N^{\frac{1}{2}} |T_1||T_2| \cdots |T_N|.$$

We substitute $A_i = (T_i, E_i)$ by $A_i' = (T_i, E_i^{(m')})$ for $i = 1, \cdots, N$, substitute B by $B_{m'}$, K_0 by $\lceil \frac{K_0}{k} \rceil$ and k by 1. $\square$

Now we assume $k = 1$. We prove the lemma by contradiction. We assume

$$\text{there are no two elements in } B \text{ satisfying both } (a) \text{ and } (b). \tag{3.3.11}$$

Given $i \in \{1, \cdots, N\}$, write $E_i = \{e_{is} : s = 1, \cdots, n_i\}$ and denote the set $T_i' = T_i \setminus \bigcup_{s=1}^{n_i} \{e_{is}^-, e_{is}^+\}$. For simplicity, denote

$$\bigtimes_{i=1}^{N} T_i = T_1 \times T_2 \times \cdots \times T_N.$$

Let $F_i = E_i \cup T_i'$ which consists of some edges and vertices. For each element in the Cartesian product $\vec{f} = (f_1, \cdots, f_N) \in F_1 \times F_2 \times \cdots \times F_N$, denote

$$C_{\vec{f}} = \left\{ \vec{x} \in \bigtimes_{i=1}^{N} T_i : \forall 1 \le i \le N,\, x_i = f_i \text{ if } f_i \in T_i';\, x_i \in \left\{ f_i^-, f_i^+ \right\} \text{ if } f_i \in E_i \right\}.$$

Then

$$T_1 \times T_2 \times \cdots \times T_N = \bigcup_{\vec{f} \in F_1 \times F_2 \times \cdots \times F_N} C_{\vec{f}}. \tag{3.3.12}$$

For each $1 \le i \le N$, since A_i is 1-colourable, any two edges in E_i have no intersection. Thus the union in (3.3.12) is a disjoint union. Since $|B|/(|T_1||T_2| \cdots |T_N|) > K_0^{-1} N^{\frac{1}{2}}$, by pigeonhole principle again, there exists $\vec{f}' = (f_1', \cdots, f_N') \in F_1 \times F_2 \times \cdots \times F_N$

such that

$$|B \cap C_{\vec{f}'}|/|C_{\vec{f}'}| > K_0^{-1} N^{\frac{1}{2}}. \tag{3.3.13}$$

Let $\mathcal{I} = \{1 \leq i \leq N : f_i' \in E_i\}$ be the set of coordinates such that f_i' is an edge. By assumption 1, for each $\vec{x} \in B \cap C_{\vec{f}'}$ and $j \in \{1, \cdots, K_0\}$, we have $e_j(\vec{x}) = f'_{I_j(\vec{x})}$, $e_j^-(\vec{x}) = x_{I_j(x)}$ and $I_j(\vec{x}) \in \mathcal{I}$. Denote

$$Y_{\vec{x}} = \{i \in \mathcal{I} : x_i = (f_i')^+\}$$

for each $\vec{x} \in B \cap C_{\vec{f}'}$. Then $\vec{z} \mapsto Y_{\vec{z}}$ is an injection from $B \cap C_{\vec{f}'}$ to the power set of $\mathcal{I}$. Note that, the definition of set $Y_{\vec{x}}$ is analog of (3.3.10) except that we are now restricting on the subset $\mathcal{I}$.

We claim that $\{Y_{\vec{x}} : \vec{x} \in B \cap C_{\vec{f}'}\}$ is $K_0/|\mathcal{I}|$-Sperner as a set of subsets of $\mathcal{I}$. To see this, suppose $Y_{\vec{x}} \subset Y_{\vec{y}}$ for some $\vec{x}, \vec{y} \in B \cap C_{\vec{f}'}$. Then $\vec{x}, \vec{y}$ satisfy property (a). By assumption (3.3.11), $\vec{x}, \vec{y}$ do not satisfy property (b), thus

$$Y_{\vec{y}} \cap \{I_j(\vec{x}) : j = 1, \cdots, K_0\} = \emptyset.$$

Since $\{I_j(\vec{x}) : j = 1, \cdots, K_0\} \subset \mathcal{I} \setminus Y_{\vec{x}}$, our claim follows from Definition 3.3.11.

Now Lemma 3.3.12 implies

$$|B \cap C_{\vec{f}'}| = |\{Y_{\vec{x}} : \vec{x} \in B \cap C_{\vec{f}'}\}| \leq 2^{|\mathcal{I}|} K_0^{-1} |\mathcal{I}|^{\frac{1}{2}} \leq |C_{\vec{f}'}| K_0^{-1} N^{\frac{1}{2}},$$

which contradicts with (3.3.13). $\square$

3.3.3 Proof of Wegner estimate

We now prove analogue of the Wegner estimate [DS20, Lemma 5.6].

Proposition 3.3.18 (Wegner estimate)**.** *Assume*

(1) $\varepsilon > \delta > 0$ are small and $c_2 > 0$ is a numerical constant

(2) integer $K \geq 1$, odd number $r > C_{\varepsilon,\delta,K}$ and real $\bar{V} > \exp(r^2)$

(3) $\lambda_0 \notin J_r^{\bar{V}}$ which is defined in Definition 3.2.12

(4) scales $R_0 \geq R_1 \geq \cdots \geq R_6 \geq \exp(c_2 r)$ with $R_k^{1-2\delta} \geq R_{k+1} \geq R_k^{1-\frac{1}{2}\varepsilon}$ and R_0, R_3 are r-dyadic

(5) $Q \subset \mathbb{Z}^2$ an r-dyadic box with $\ell(Q) = R_0$

(6) $Q'_1, \cdots, Q'_K \subset Q$ r-dyadic boxes, each with length R_3 (called "defects")

(7) $G \subset \cup_k Q'_k$ with $0 < |G| \leq R_0^\delta$

(8) $\Theta \subset Q$ and $Q \setminus \Theta = \cup_{b \in D} \Omega_r(b)$ for some $D \subset \dot{r}\mathbb{Z}^2 \cap Q$

(9) Θ is $\varepsilon_0^{\frac{1}{5}}$-regular in every box $Q' \subset Q \setminus \cup_k Q'_k$ with $\ell(Q') = R_6$, where ε_0 is defined by (3.2.2)

(10) potential $V' : \Theta \to \{0, \bar{V}\}$ satisfies the following: for any $V : Q \to \{0, \bar{V}\}$ with $V|_\Theta = V'$, any $\lambda \in \left[\lambda_0 - \bar{V}^{-R_5}, \lambda_0 + \bar{V}^{-R_5}\right]$, any $t \in [0,1]$ and any subset $\mathcal{R}$ of r-bits that do not affect $\Theta \cup \bigcup_k Q'_k$, each $Q_r(b) \in \mathcal{R}$ is admissible and $H_Q^{\mathcal{R},t} u = \lambda u$ implies

$$\bar{V}^{R_4} \|u\|_{\ell^2(Q \setminus \bigcup_k Q'_k)} \leq \|u\|_{\ell^2(Q)} \leq (1 + R_0^{-\delta}) \|u\|_{\ell^2(G)}. \tag{3.3.14}$$

Then we have

$$\mathbb{P}\left[\|(H_Q-\lambda_0)^{-1}\| \leq \bar{V}^{R_1}\middle| V|_\Theta = V'\right] \geq 1 - R_0^{10\varepsilon-\frac{1}{2}}. \tag{3.3.15}$$

As mentioned in Section 3.1.2, in order to prove the Wegner estimate, we need to prove the upper bound estimate and the lower bound estimate. In particular, the upper bound estimate is proved in Claim 3.3.21 and it provides a significantly smaller subset Λ_V of eigenvalues (depending on potential V) such that eigenvalues outside Λ_V have zero probability to be close to λ_0. Thus we only need to consider eigenvalues in Λ_V. The lower bound estimate is proved in Claim 3.3.23 and it implies that, any eigenvalue in Λ_V can be perturbed to move away from λ_0 by changing the potential function on any vertex in a significant portion of the box. By combining this fact and the Sperner lemma (Lemma 3.3.16), we prove a probability upper bound for the event that there is an eigenvalue in Λ_V which is close to λ_0 and thus prove the Wegner estimate.

Proof of Proposition 3.3.18. Throughout the proof, we allow constants $C > 1 > c > 0$ to depend on ε, δ, K.

Claim 3.3.19. *We can assume without loss of generality that $\cup_k Q'_k \subset \Theta$.*

Proof of the claim. Let $\Theta' = \cup_k Q'_k \setminus \Theta$ and observe that for any event $\mathcal{E}$,

$$\mathbb{P}\left[\mathcal{E}\middle|V|_\Theta = V'\right] = \mathbb{E}\left[\mathbb{P}\left[\mathcal{E}\middle|V|_{\Theta\cup\Theta'} = V' \cup V''\right]\right] \tag{3.3.16}$$

where the expectation is taking over all $V'' : \Theta' \to \{0, \bar{V}\}$. Thus, it suffices to estimate the term in the expectation. Now we replace Θ by $\Theta \cup \Theta'$ and check assumptions. Except for assumption (8), other assumptions are straightforward. As for assumption (8), note that $Q \setminus (\Theta \cup \Theta') = (Q \setminus \Theta) \setminus (\cup_k Q'_k)$. For any $a \in \dot{r}\mathbb{Z}^2$, by Lemma 3.2.15 and the assumption that Q'_k's are r-dyadic, either $\Omega_r(a) \subset (\cup_k Q'_k)$ or $\Omega_r(a) \cap (\cup_k Q'_k) = \emptyset$. Thus $Q \setminus (\Theta \cup \Theta') = \cup_{b \in D'} \Omega_r(b)$ where $D' = \{b \in D : \Omega_r(a) \cap (\cup_k Q'_k) = \emptyset\}$. The assumption follows. □

Now we assume $\cup_k Q'_k \subset \Theta$, then by Lemma 3.2.15, $\tilde{\Omega}_r(b) \cap (\cup_k Q'_k) = \emptyset$ for each $b \in D$.

We fix $\mathcal{R} = \{Q_r(b) : b \in D\}$. By assumption (10), when the square $Q_r(b) \in \mathcal{R}$, $(Q_r(b), V|_{F_r(b)})$ is admissible. For each $V : Q \to \{0, \bar{V}\}$ with $V|_\Theta = V'$ and $t \in [0, 1]$, denote all the eigenvalues of $H_Q^{\mathcal{R},t}$ by

$$\lambda_1^t(V) \leq \lambda_2^t(V) \leq \cdots \leq \lambda_{R_0^2}^t(V).$$

In particular, $\lambda_1^0(V) \leq \lambda_2^0(V) \leq \cdots \leq \lambda_{R_0^2}^0(V)$ are all the eigenvalues of H_Q. Let $u_{V,k} (k = 1, \cdots, R_0^2)$ be an orthonormal eigenbasis such that for each k,

$$H_Q u_{V,k} = \lambda_k^0(V) u_{V,k}.$$

Since $H_Q^{\mathcal{R},1}(x,y)=0$ whenever $\{x,y\}\in\bigcup_{b\in D}\partial S_r(b)$, we have

$$H_Q^{\mathcal{R},1}=\bigoplus_{b\in D}H_{S_r(b)}\bigoplus H_{Q\setminus(\cup_{b\in D}S_r(b))}. \tag{3.3.17}$$

Here, we also used the fact that $S_r(b)\cap S_r(b')=\emptyset$ whenever $b\neq b'\in D$ (see Remark 3.2.5).

Thus eigenvalues of $H_Q^{\mathcal{R},1}$ consist of eigenvalues of $H_{S_r(b)}(b\in D)$ and eigenvalues of $H_{Q\setminus(\cup_{b\in D}S_r(b))}$. By item 1 in Lemma 3.2.8, $Q\setminus(\cup_{b\in D}S_r(b))\subset\Theta$. Thus $H_{Q\setminus(\cup_{b\in D}S_r(b))}$ only depends on $V|_\Theta=V'$. Let $\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_m$ be all the eigenvalues of $H_{Q\setminus(\cup_{b\in D}S_r(b))}$. Let $\lambda_q\leq\lambda_{q+1}\leq\cdots\leq\lambda_{q+p}$ be all the eigenvalues of $H_{Q\setminus(\cup_{b\in D}S_r(b))}$ inside the closed interval $[\lambda_0-\bar{V}^{-R_4},\lambda_0+\bar{V}^{-R_4}]$. Then

$$\lambda_{q-1}<\lambda_0-\bar{V}^{-R_4}$$

if $q>1$. Denote

$$i(V)=|\{k:\lambda_k^1(V)<\lambda_0-\bar{V}^{-R_4}\}|+1=n(H_Q^{\mathcal{R},1};\lambda_0-\bar{V}^{-R_4})+1.$$

Because $\lambda_0\notin J_r^{\bar{V}}$, by item 2 in Proposition 3.2.13, any eigenvalue of $H_{S_r(b)}(b\in D)$ is outside the interval $[\lambda_0 - \frac{1}{2}\bar{V}^{-\frac{1}{4}}, \lambda_0 + \frac{1}{2}\bar{V}^{-\frac{1}{4}}]$. Thus by (3.3.17),

$$\begin{aligned}
&\sigma(H_Q^{\mathcal{R},1}) \cap [\lambda_0 - \bar{V}^{-R_4}, \lambda_0 + \bar{V}^{-R_4}] \\
&= \sigma(H_{Q\setminus(\cup_{b\in D}S_r(b))}) \cap [\lambda_0 - \bar{V}^{-R_4}, \lambda_0 + \bar{V}^{-R_4}] \\
&= \{\lambda_{q+j} : 0 \le j \le p\} \\
&= \{\lambda^1_{i(V)+j}(V) : 0 \le j \le p\}.
\end{aligned} \tag{3.3.18}$$

Claim 3.3.20. $p \le CR_0^{\delta}$.

Proof of the claim. Let

$$\left\{v_i \in \ell^2\big(Q \setminus (\cup_{b\in D} S_r(b))\big) : 0 \le i \le p\right\}$$

be an orthonormal set with $H_{Q\setminus(\cup_{b\in D}S_r(b))}v_i = \lambda_{q+i}v_i$ for each $0 \le i \le p$. Consider the function v_i' on Q defined by $v_i'|_{\cup_{b\in D}S_r(b)} = 0$ and $v_i'|_{Q\setminus(\cup_{b\in D}S_r(b))} = v_i$. By (3.3.17), v_i' is an eigenfunction of $H_Q^{\mathcal{R},1}$ with the eigenvalue λ_{q+i}. By assumption (10), $\|v_i'\|_{\ell^2(G)} \ge (1 + R_0^{-\delta})^{-1} \ge 1 - R_0^{-\delta}$. From $\langle v_i', v_j'\rangle_{\ell^2(Q)} = \delta_{ij}$ we deduce that

$$|\langle v_i', v_j'\rangle_{\ell^2(G)} - \delta_{ij}| \le R_0^{-\delta} \le (5|G|)^{-\frac{1}{2}}.$$

Thus, $\{v_i'|_G : 0 \le i \le p\}$ is a set of almost orthogonal vectors and Lemma 3.3.10 implies the claim. ∎

Claim 3.3.21. *Suppose* $\lambda_k^0(V) \in [\lambda_0 - \bar{V}^{-R_2}, \lambda_0 + \bar{V}^{-R_2}]$ *for some* $1 \le k \le R_0^2$. *Then* *there exists $j \in \{0, 1, \cdots, p\}$ such that $k = i(V) + j$.*

Proof of the claim. Fix such V and for simplicity, when $t \in [0,1]$ we denote $\lambda_k^t = \lambda_k^t(V)$ and choose u_k^t to be an ℓ^2-normalised eigenfunction of $H_Q^{\mathcal{R},t}$ with eigenvalue λ_k^t. Denote $X = \cup_{b\in D}\partial S_r(b)$. The first order variation implies (see [Kat13, Chapter 2, Section 6.5])

$$|\lambda_k^t - \lambda_k^0| = \left| \int_0^t \sum_{\substack{x\sim y \\ \{x,y\}\in X}} u_k^s(x) u_k^s(y) ds \right|. \tag{3.3.19}$$

By Lemma 3.2.15, $\bigcup_{b\in D} \tilde{\Omega}_r(b) \cap (\cup_k Q_k') = \emptyset$. Since $X \subset \bigcup_{b\in D} \tilde{\Omega}_r(b)$, assumption (10) and equation (3.3.14) imply

$$|\int_0^t \sum_{\substack{x\sim y \\ \{x,y\}\in X}} u_k^s(x) u_k^s(y) ds| \le 2t|X|\bar{V}^{-2R_4} \le 4tR_0^2\bar{V}^{-2R_4} < \frac{1}{2}\bar{V}^{-R_4}$$

as long as $|\lambda_k^t - \lambda_0| \le \bar{V}^{-R_5}$. Thus (3.3.19) implies

$$|\lambda_k^t - \lambda_0| \le \bar{V}^{-R_2} + \frac{1}{2}\bar{V}^{-R_4} + 4R_0^2 1_{\max_{0\le s\le t}|\lambda_k^s - \lambda_0| \ge \bar{V}^{-R_5}}. \tag{3.3.20}$$

Since λ_k^t is continuous with respect to t, by continuity, (3.3.20) implies $|\lambda_k^t - \lambda_0| \le \bar{V}^{-R_4}$ for each $t \in [0,1]$. In particular, $|\lambda_k^1 - \lambda_0| \le \bar{V}^{-R_4}$. Thus by (3.3.18), $k = i(V) + j$ for some $j \in \{0, 1, \cdots, p\}$. □

By Claim 3.3.21, we only need to consider eigenvalues in set $\{\lambda_{i(V)+k} : 0 \le k \le p\}$. We will prove that, with high probability, these $(p+1)$ eigenvalues are away from λ_0 with distance at least $\bar{V}^{-R_1}$. We first prove that each of these $(p+1)$ eigenvalues' eigenfunctions has a large support (Claim 3.3.22). Then we use these supports of eigenfunctions to do an eigenvalue perturbation argument which, combined with the Sperner lemma, proves that eigenvalues in $\{\lambda_{i(V)+k} : 0 \leq k \leq p\}$ are away from λ_0 with high probability (Claim 3.3.23).

Claim 3.3.22. $\mathbb{P}\left[\mathcal{E}_{uc} \middle| V|_\Theta = V'\right] \geq 1 - \exp(-R_0^\varepsilon)$, *where* $\mathcal{E}_{uc}$ *denotes the event that*

$$\left|\left\{a \in Q : |u(a)| \geq \bar{V}^{-\frac{1}{2}R_2}\|u\|_2\right\} \setminus \Theta\right| \geq R_4^{\frac{3}{2}}$$

holds whenever $|\lambda - \lambda_0| \leq \bar{V}^{-R_5}$ *and* $H_Q u = \lambda u$.

Proof of the claim. Our strategy here is that we first use Lemma 3.3.6 to find a vertex a_* with $|u(a_*)|$ being lower bounded, then use the unique continuation Lemma 3.3.5 to find $R_4^{\frac{3}{2}}$ vertices in $Q_{R_6}(a_*)$.

Recall the definition of $\mathcal{E}_{uc}^{\varepsilon;\alpha}(Q, \Theta)$ in Lemma 3.3.5 and that equation (3.2.2) implies $\varepsilon_0^{\frac{1}{5}} < \varepsilon_1$. By Lemma 3.3.5 and assumption (9), there exists $\alpha' > 1$ such that the event

$$\mathcal{E}'_{uc} = \bigcap_{\substack{Q' \subset Q \setminus \cup_k Q'_k \\ \ell(Q') = R_6}} \mathcal{E}_{uc}^{\varepsilon_0^{\frac{1}{5}},\alpha'}(Q', \Theta \cap Q')$$

satisfies

$$\mathbb{P}\left[\mathcal{E}'_{uc} \middle| V_\Theta = V'\right] \geq 1 - \exp(-\varepsilon_0^{\frac{1}{5}} R_6^{\frac{2}{3}} + C\log(R_0)) \geq 1 - \exp(-R_0^\varepsilon). \tag{3.3.21}$$

Thus it suffices to show $\mathcal{E}'_{uc} \subset \mathcal{E}_{uc}$. Suppose $\mathcal{E}'_{uc}$ holds, $|\lambda - \lambda_0| \leq \bar{V}^{-R_5}$, and $H_Q u = \lambda u$.

Lemma 3.3.6 provides an R_3-box Q_* with $Q_* \subset Q \setminus \cup_k Q'_k$ and $a_* \in \frac{1}{2}Q_*$ such that,

$$|u(a_*)| \geq \bar{V}^{-C_K R_3} \|u\|_{\ell^\infty(Q)} \geq \bar{V}^{-C'_K R_3} \|u\|_{\ell^2(Q)}. \tag{3.3.22}$$

Since $\mathcal{E}_{uc}^{\varepsilon_0^{\frac{1}{5}},\alpha'}(Q_{R_6}(a_*), \Theta \cap Q_{R_6}(a_*))$ holds and

$$|\lambda - \lambda_0| \leq \bar{V}^{-R_5} \leq (R_6 \bar{V})^{-\alpha' R_6},$$

we see that

$$|\{|u| \geq (R_6\bar{V})^{-\alpha' R_6} |u(a_*)|\} \cap Q_{R_6}(a_*) \setminus \Theta| \geq \frac{1}{2} \varepsilon_0^{\frac{3}{5}} R_6^2. \tag{3.3.23}$$

Thus by taking $r > C_{\varepsilon,\delta,K}$ large and observing $R_6 \geq \exp(c_2 r)$, we have

$$|\{|u| \geq \bar{V}^{-\frac{1}{2}R_2} \|u\|_{\ell^2(Q)}\} \cap Q \setminus \Theta| \geq \frac{1}{2} \varepsilon_0^{\frac{3}{5}} R_6^2 \geq R_4^{\frac{3}{2}}. \tag{3.3.24}$$

(3.3.24) provides the inclusion and the claim. ☐

Claim 3.3.23. *For $0 \leq k_1 \leq k_2 \leq p$ and $0 \leq \ell \leq C R_0^\delta$, we have*

$$\mathbb{P}\left[\mathcal{E}_{k_1,k_2,\ell} \cap \mathcal{E}_{uc} \middle| V|_\Theta = V'\right] \leq C r^6 R_0 R_4^{-\frac{3}{2}} \tag{3.3.25}$$

where $\mathcal{E}_{k_1,k_2,\ell}$ *denotes the event*

$$|\lambda^0_{i(V)+k_1}(V) - \lambda_0|, |\lambda^0_{i(V)+k_2}(V) - \lambda_0| < s_\ell \text{ and} \tag{3.3.26}$$

$$|\lambda^0_{i(V)+k_1-1}(V) - \lambda_0|, |\lambda^0_{i(V)+k_2+1}(V) - \lambda_0| \geq s_{\ell+1}, \tag{3.3.27}$$

where $s_i := \bar{V}^{-R_1+(R_2-\frac{1}{2}R_4+C)i}$ *for each* $i \in \mathbb{Z}$.

Proof. Conditioning on $V|_\Theta = V'$, we view events $\mathcal{E}_{uc}$ and $\mathcal{E}_{k_1,k_2,\ell}$ as subsets of $\{0, \bar{V}\}^{\cup_{b\in D}\Omega_r(b)}$. Given $\tau \in \{0,1\}$, denote by $\mathcal{E}_{k_1,k_2,\ell,\tau}$ the intersection of $\mathcal{E}_{k_1,k_2,\ell}$ and the event

$$|\{a' \in Q \setminus \Theta : |u_{V,i(V)+k_1}(a')| \geq \bar{V}^{-\frac{1}{2}R_2} \text{ and } V(a') = \tau\bar{V}\}| \geq \frac{1}{2}R_4^{\frac{3}{2}}. \tag{3.3.28}$$

Then

$$\mathcal{E}_{k_1,k_2,\ell} \cap \mathcal{E}_{uc} \subset \mathcal{E}_{k_1,k_2,\ell,0} \cup \mathcal{E}_{k_1,k_2,\ell,1}.$$

It suffices to prove that

$$\mathbb{P}\left[\mathcal{E}_{k_1,k_2,\ell,\tau}\middle|\, V|_\Theta = V'\right] \leq 200r^6 R_0 R_4^{-\frac{3}{2}} \tag{3.3.29}$$

for each $\tau \in \{0,1\}$.

We prove it for $\tau = 0$, the case where $\tau = 1$ is symmetric. We prove by contradiction, assume (3.3.29) does not hold for $\tau = 0$. That is,

$$\mathbb{P}\left[\mathcal{E}_{k_1,k_2,\ell,0}\middle|\, V|_\Theta = V'\right] > 200r^6 R_0 R_4^{-\frac{3}{2}}. \tag{3.3.30}$$

Given $V \in \mathcal{E}_{k_1,k_2,\ell,0}$ with $V|_\Theta = V'$ and $a \in \Omega_r(b)$ with some $b \in D$, we say a is a *"crossing" site* with respect to V (or *w.r.t.* V) if $V(a) = 0$ and

$$n\big((-\Delta + V + \bar{V}\delta_a)_{S_r(b)}; \lambda_0\big) = n\big((-\Delta + V)_{S_r(b)}; \lambda_0\big) - 1;$$

we say a is a *"non-crossing" site* with respect to V (or w.r.t. V) if $V(a) = 0$ and

$$n\big((-\Delta + V + \bar{V}\delta_a)_{S_r(b)}; \lambda_0\big) = n\big((-\Delta + V)_{S_r(b)}; \lambda_0\big).$$

Note that by rank one perturbation, for any $a \in Q \setminus \Theta$ with $V(a) = 0$, either a is a crossing site w.r.t. V or a is a non-crossing site w.r.t. V.

Denote by $\mathcal{E}_{k_1,k_2,\ell,0,cro}$ the intersection of $\mathcal{E}_{k_1,k_2,\ell,0}$ and the event

$$|\{|u_{V,i(V)+k_1}| \geq \bar{V}^{-\frac{1}{2}R_2}\} \cap \{a' \in Q \setminus \Theta : a' \text{ is a crossing site w.r.t. } V\}| \geq \frac{1}{4}R_4^{\frac{3}{2}}.$$

Denote by $\mathcal{E}_{k_1,k_2,\ell,0,ncr}$ the intersection of $\mathcal{E}_{k_1,k_2,\ell,0}$ and the event

$$|\{|u_{V,i(V)+k_1}| \geq \bar{V}^{-\frac{1}{2}R_2}\} \cap \{a' \in Q \setminus \Theta : a' \text{ is a non-crossing site w.r.t. } V\}| \geq \frac{1}{4}R_4^{\frac{3}{2}}.$$

Then by (3.3.28),

$$\mathcal{E}_{k_1,k_2,\ell,0} \subset \mathcal{E}_{k_1,k_2,\ell,0,cro} \cup \mathcal{E}_{k_1,k_2,\ell,0,ncr}.$$

By (3.3.30), one of the following holds:

$$\mathbb{P}\left[\mathcal{E}_{k_1,k_2,\ell,0,cro}\middle|\, V|_\Theta = V'\right] > 100r^6R_0R_4^{-\frac{3}{2}}, \tag{3.3.31}$$

or

$$\mathbb{P}\left[\mathcal{E}_{k_1,k_2,\ell,0,ncr}\middle|\, V|_\Theta = V'\right] > 100r^6R_0R_4^{-\frac{3}{2}}. \tag{3.3.32}$$

We will arrive at contradiction in each case.

Case 1. (3.3.31) holds.

For each $b \in D$, we define a directed graph $A_b = (T_b, E_b)$ with vertex set $T_b = \{0, \bar{V}\}^{\Omega_r(b)}$, and the edge set E_b is defined as follows. For each $w \in T_b$, let $\widetilde{w} \in \{0, \bar{V}\}^{S_r(b)}$ be $\widetilde{w} = w$ in $\Omega_r(b)$ and $\widetilde{w} = V'$ in $S_r(b) \setminus \Omega_r(b)$. Given $w_1, w_2 \in T_b$, there is an edge which starts from w_1 and ends at w_2 if $w_2 = w_1 + \bar{V}\delta_{b'}$ for some $b' \in \Omega_r(b)$ and $n\big((-\Delta)_{S_r(b)} + \widetilde{w_2}; \lambda_0\big) = n\big((-\Delta)_{S_r(b)} + \widetilde{w_1}; \lambda_0\big) - 1$.

For each $w \in T_b$, there are less than $2r^2$ edges which start from or end at w. By Lemma 3.3.15, A_b is $4r^2$-colourable.

For each $V \in \mathcal{E}_{k_1,k_2,\ell,0,cro} \cap \{V : V|_\Theta = V'\}$, by pigeonhole principle, we can find a subset $D_0(V) \subset D$ with $|D_0(V)| = \lceil \frac{1}{4} r^{-2} R_4^{\frac{3}{2}} \rceil$ such that for each $b \in D_0(V)$, there is a crossing site $b' \in \Omega_r(b)$ w.r.t. V with $|u_{V,i(V)+k_1}(b')| \geq \bar{V}^{-\frac{1}{2}R_2}$. This provides, for each $b \in D_0(V)$, an edge $e_b(V) \in E_b$ with $e_b(V)^- = V|_{\Omega_r(b)}$, $e_b(V)^+ = V|_{\Omega_r(b)} + \bar{V}\delta_{b'}$ and

$|u_{V,i(V)+k_1}(b')| \geq \bar{V}^{-\frac{1}{2}R_2}$. We use Lemma 3.3.16 with directed graphs $A_b = (T_b, E_b)$ ($b \in D$), subset $B = \mathcal{E}_{k_1,k_2,\ell,0,cro} \subset \bigtimes_{b\in D} T_b$, $N = |D| \leq R_0^2$, $K_0 = \lceil \frac{1}{4} r^{-2} R_4^{\frac{3}{2}} \rceil$, $k = 4r^2$, associated index set $D_0(V)$ and edge set $\{e_b(V) : b \in D_0(V)\}$ for each $V \in B$. Here, equation (3.3.32) serves as assumption 2 in Lemma 3.3.16. Lemma 3.3.16 provides $V_1, V_2 \in \mathcal{E}_{k_1,k_2,\ell}$ such that the following holds:

- $\forall b \in D$, either $V_1|_{Q_r(b)} = V_2|_{Q_r(b)}$ or $V_2|_{Q_r(b)} = V_1|_{Q_r(b)} + \bar{V}\delta_{b'}$ for some crossing site b' w.r.t. V_1.
- There exists a crossing site $a_0 \in Q \setminus \Theta$ w.r.t. V_1 such that $V_2(a_0) = \bar{V}$ and $|u_{V_1,i(V_1)+k_1}(a_0)| \geq \bar{V}^{-\frac{1}{2}R_2}$.

Denote $V_3 = V_1 + \bar{V}\delta_{a_0}$. Then by definition of crossing site and (3.3.17), $i(V_3) = i(V_1) - 1$ and

$$i(V_2) = i(V_1) - |\{a \in Q : V_1(a) \neq V_2(a)\}|.$$

By Cauchy interlacing theorem and the fact that $|\{a \in Q : V_2(a) \neq V_3(a)\}| = i(V_3) - i(V_2)$, we have

$$\lambda^0_{i(V_1)+k_1}(V_1) \geq \lambda^0_{i(V_3)+k_1}(V_3) \geq \lambda^0_{i(V_2)+k_1}(V_2). \tag{3.3.33}$$

By assumption (10), for each $1 \leq j \leq R_0^2$, we have $|u_{V_1,j}(a_0)| \leq \bar{V}^{-R_4}$ when $|\lambda^0_j(V_1) - \lambda_0| \leq \bar{V}^{-R_5}$. Since $|u_{V_1,i(V_1)+k_1}(a_0)| \geq \bar{V}^{-\frac{1}{2}R_2}$ and

$$\lambda_0 - s_\ell < \lambda^0_{i(V_1)+k_1}(V_1) < \lambda_0 + s_\ell,$$

we have

$$
\begin{aligned}
&\sum_{j=1}^{R_0^2}\frac{u_{V_1,j}(a_0)^2}{\lambda_j^0(V_1)-(\lambda_0-s_\ell)}\\
=&\sum_{j=i(V_1)+k_1}^{i(V_1)+k_2}\frac{u_{V_1,j}(a_0)^2}{\lambda_j^0(V_1)-(\lambda_0-s_\ell)}+\sum_{\substack{j\notin[i(V_1)+k_1,i(V_1)+k_2]\\|\lambda_j^0(V_1)-\lambda_0|\le\bar V^{-R_5}}}\frac{u_{V_1,j}(a_0)^2}{\lambda_j^0(V_1)-(\lambda_0-s_\ell)}\\
&+\sum_{|\lambda_j^0(V_1)-\lambda_0|>\bar V^{-R_5}}\frac{u_{V_1,j}(a_0)^2}{\lambda_j^0(V_1)-(\lambda_0-s_\ell)}\\
\ge&\frac{u_{V_1,i(V_1)+k_1}(a_0)^2}{\lambda_{i(V_1)+k_1}^0(V_1)-(\lambda_0-s_\ell)}-\sum_{\substack{j\notin[i(V_1)+k_1,i(V_1)+k_2]\\|\lambda_j^0(V_1)-\lambda_0|\le\bar V^{-R_5}}}\frac{\bar V^{-2R_4}}{s_{\ell+1}-s_\ell}\\
&-\sum_{|\lambda_j^0(V_1)-\lambda_0|>\bar V^{-R_5}}2\bar V^{R_5}\\
\ge&\frac{\bar V^{-R_2}}{2s_\ell}-R_0^2\frac{\bar V^{-2R_4}}{s_{\ell+1}-s_\ell}-2R_0^2\bar V^{R_5}\\
>&0.
\end{aligned}
\tag{3.3.34}
$$

Note that

$$\lambda^0_{i(V_3)+k_1}(V_1)=\lambda^0_{i(V_1)+k_1-1}(V_1)<\lambda_0-s_\ell<\lambda^0_{i(V_1)+k_1}(V_1).$$

By Lemma 3.3.9 and (3.3.34), $\lambda^0_{i(V_3)+k_1}(V_1+t\delta_{a_0})<\lambda_0-s_\ell$ for each $t>0$. Let $t=\bar V$, we have $\lambda^0_{i(V_3)+k_1}(V_3)<\lambda_0-s_\ell$. Thus by (3.3.33), $\lambda^0_{i(V_2)+k_1}(V_2)<\lambda_0-s_\ell$ and hence $V_2\notin\mathcal{E}_{k_1,k_2,\ell}$. We thus arrived at contradiction.

Case 2. (3.3.32) holds.

For each $b\in D$, we define a directed graph $A_b=(T_b,E_b)$ with vertex set $T_b=\{0,\bar V\}^{\Omega_r(b)}$, and edge set E_b is defined as follows. For each $w\in T_b$, let $\widetilde w\in\{0,\bar V\}^{S_r(b)}$

be $\widetilde{w} = w$ in $\Omega_r(b)$ and $\widetilde{w} = V'$ in $S_r(b) \setminus \Omega_r(b)$. Given $w_1, w_2 \in T_b$, there is an edge which starts from w_1 and ends at w_2 if $w_2 = w_1 + \bar{V}\delta_{b'}$ for some $b' \in \Omega_r(b)$ and $n\big((-\Delta)_{S_r(b)} + \widetilde{w_2}; \lambda_0\big) = n\big((-\Delta)_{S_r(b)} + \widetilde{w_1}; \lambda_0\big)$.

By the similar arguments used in Case 1, there exist $V_1, V_2 \in \mathcal{E}_{k_1,k_2,\ell}$ such that the following holds:

- $\forall b \in D$, either $V_1|_{Q_r(b)} = V_2|_{Q_r(b)}$ or $V_2|_{Q_r(b)} = V_1|_{Q_r(b)} + \bar{V}\delta_{b'}$ for some non-crossing site b' w.r.t. V_1.
- There exists a non-crossing site a_0 w.r.t. V_1 such that we have $V_2(a_0) = \bar{V}$ and $|u_{V_1,i(V_1)+k_1}(a_0)| \geq \bar{V}^{-\frac{1}{2}R_2}$.

Denote $V_3 = V_1 + \bar{V}\delta_{a_0}$. Then by (3.3.17) and definition of non-crossing site, $i(V_3) = i(V_1) = i(V_2)$. Since $V_1 \leq V_3 \leq V_2$, by monotonicity,

$$\lambda^0_{i(V_1)+k_2}(V_1) \leq \lambda^0_{i(V_3)+k_2}(V_3) \leq \lambda^0_{i(V_2)+k_2}(V_2). \tag{3.3.35}$$

Now we apply Lemma 3.3.8 to $H_Q - \lambda_0 + s_\ell$ with $r_1 = 2s_\ell$, $r_2 = s_{\ell+1}$, $r_3 = \bar{V}^{-R_2}$, $r_4 = \bar{V}^{-cR_4}$ and $r_5 = \bar{V}^{-R_5}$. Then $\lambda^0_{i(V_3)+k_2}(V_3) \geq \lambda_0 + s_\ell$. By (3.3.35), $\lambda^0_{i(V_2)+k_2}(V_2) \geq \lambda_0 + s_\ell$ and thus $V_2 \notin \mathcal{E}_{k_1,k_2,\ell}$. We thus arrived at contradiction. $\square$

Claim 3.3.24.

$$\{\|(H_Q - \lambda_0)^{-1}\| > \bar{V}^{R_1}\} \cap \{V|_\Theta = V'\} \subset \bigcup_{\substack{0 \leq k_1 \leq k_2 \leq p \\ 0 \leq \ell \leq CR_0^\delta}} \mathcal{E}_{k_1,k_2,\ell} \tag{3.3.36}$$

Proof of the claim. By Claim 3.3.20 and Claim 3.3.21, we can always find $0 \le \ell \le CR_0^\delta$ such that the annulus $(\lambda_0 - s_{\ell+1}, \lambda_0 + s_{\ell+1}) \setminus (\lambda_0 - s_\ell, \lambda_0 + s_\ell)$ contains no eigenvalue of H_Q. The claim follows. □

Finally by Claim 3.3.24,

$$\begin{aligned} &\mathbb{P}[\|(H_Q - \lambda_0)^{-1}\| > \bar{V}^{R_1} |\, V|_\Theta = V'] \\ &\qquad \le \sum_{0 \le k_1, k_2 \le p} \sum_{1 \le \ell \le CR_0^\delta} \mathbb{P}[\mathcal{E}_{k_1,k_2,\ell} \cap \mathcal{E}_{uc} |\, V|_\Theta = V'] + \mathbb{P}[\mathcal{E}_{uc}^c \big|\, V|_\Theta = V']. \end{aligned} \tag{3.3.37}$$

By Claim 3.3.22, 3.3.23 and let $C_{\varepsilon,\delta,K}$ be large enough,

$$\begin{aligned} &\mathbb{P}[\|(H_Q - \lambda_0)^{-1}\| > \bar{V}^{R_1} |\, V|_\Theta = V'] \\ &\le Cr^6 R_0^{1+3\delta} R_4^{-\frac{3}{2}} + \exp(-R_0^{-\varepsilon}) \\ &\le R_0^{10\varepsilon - \frac{1}{2}}. \end{aligned}$$

We used here $r \ge C_{\varepsilon,\delta,K}$ and $R_0 \ge \exp(c_2 r)$. □

3.4 Larger scales

We now prove Theorem 3.1.4 by a multi-scale analysis based on [DS20, Lemma 8.3] with Wegner estimate Proposition 3.3.18.

Definition 3.4.1. Suppose r is an odd number, $\mathcal{R}$ is a set of r-bits and $E \subset \mathbb{Z}^2$. We denote

$$\mathcal{R}_E = \{Q_r(b) \in \mathcal{R} : Q_r(b) \subset E\}. \tag{3.4.1}$$

We need the following gluing lemma in multi-scale analysis which is a direct modification of [DS20, Lemma 6.2] and it follows from the same proof as [DS20, Lemma 6.2].

Lemma 3.4.2 (Gluing lemma)**.** *If*

1. $\varepsilon > \delta > 0$ *small and* $c_3 > 0$ *a numerical constant*
2. $K \geq 1$ *an integer,* $r > C_{\varepsilon,\delta,K}$ *a large odd number and* $\bar{V} > \exp(r^2)$
3. $t \in [0,1]$ *and* $\lambda_0 \in [0,8]$
4. *scales* $R_0 \geq \cdots \geq R_6 \geq \exp(c_3 r)$ *with* $R_k^{1-\varepsilon} \geq R_{k+1}$
5. $1 \geq m \geq 2R_5^{-\delta}$ *represents the exponential decay rate*
6. $Q = Q_{R_0}(a) \subset \mathbb{Z}^2$ *an* r*-dyadic box*
7. $Q'_1, \cdots, Q'_K \subset Q$ *disjoint* r*-dyadic* R_2*-boxes with* $\|(H_{Q'_k} - \lambda_0)^{-1}\| \leq \bar{V}^{R_4}$ *(they are called "defects")*
8. $\mathcal{R}$ *a subset of admissible* r*-bits inside* Q *which do not affect* $\cup_k Q'_k$
9. *for all* $b \in Q$ *one of the following holds*
    - *there is* Q'_k *such that* $b \in Q'_k$ *and* $\operatorname{dist}(b, Q \setminus Q'_k) \geq \frac{1}{8}\ell(Q'_k)$

- *there is an r-dyadic R_5-box $Q'' \subset Q$ such that $b \in Q''$,* $\mathrm{dist}(b, Q \setminus Q'') \geq \frac{1}{8}\ell(Q'')$*, and* $|G_{Q''}^{\mathcal{R}_{Q''},t}(b', b''; \lambda_0)| \leq \bar{V}^{R_6 - m|b'-b''|}$ *for* $b', b'' \in Q''$*,*

then $|G_Q^{\mathcal{R},t}(b, b'; \lambda_0)| \leq \bar{V}^{R_1 - \bar{m}|b-b'|}$ *for* $b, b' \in Q$ *where* $\bar{m} = m - R_5^{-\delta}$*.*

Remark 3.4.3. As in [DS20, Remark 6.3], the scales $R_0, \cdots, R_6$ has the following interpretations:

1. R_0: large scale
2. $\exp(R_1)$: large scale resolvent bound
3. R_2: defect scale
4. $-R_3$: defect edge weight
5. $\exp(R_4)$: defect resolvent bound
6. R_5: small scale
7. $\exp(R_6)$: small scale resolvent bound

They are set up to be compatible with the multi-scale analysis (Theorem 3.4.7) below.

We also need a covering lemma which is a direct modification of [DS20, Lemma 8.1] and it follows from the same proof as [DS20, Lemma 8.1].

Lemma 3.4.4. *Suppose $K \geq 1$ is an integer, r is a large odd number, $\alpha \geq C^K$ is a power of 2, $R_0 \geq R_1 \geq R_2$ are r-dyadic scales with $R_i \geq \alpha R_{i+1} (i = 0, 1)$, $Q \subset \mathbb{Z}^2$ is an r-dyadic R_0-box and $Q_1'', \cdots, Q_K'' \subset Q$ are r-dyadic R_2-boxes. Then there is an*

r-dyadic scale $R_3 \in [R_1, \alpha R_1]$ and disjoint r-dyadic R_3-boxes $Q'_1, \cdots, Q'_K \subset Q$ such that,

$$\text{for each } Q''_k, \text{ there is } Q'_j \text{ with } Q''_k \subset Q'_j \text{ and } \operatorname{dist}(Q''_k, Q \setminus Q'_j) \geq \frac{1}{8} R_3. \tag{3.4.2}$$

The following lemma provides the continuity of resolvent bounds and its proof was given in [DS20].

Lemma 3.4.5 (Lemma 6.4 in [DS20])**.** *If square $Q \subset \mathbb{Z}^2$, $\lambda \in \mathbb{R}$, $\alpha > \beta > 0$, and*

$$|(H_Q - \lambda)^{-1}(x, y)| \leq \exp(\alpha - \beta|x - y|)$$

for any $x, y \in Q$, then for all $|\lambda' - \lambda| \leq c\beta|Q|^{-1} \exp(-\alpha)$, we have

$$|(H_Q - \lambda')^{-1}(x, y)| \leq 2 \exp(\alpha - \beta|x - y|)$$

for any $x, y \in Q$.

Definition 3.4.6. Suppose $\gamma, \varepsilon > 0$, large odd number r, real $\bar{V} > \exp(r^2)$, energy λ_0, r-dyadic box $Q_L(a)$, $\Theta \subset Q_L(a)$ and $V' : \Theta \to \{0, \bar{V}\}$. We say $(Q_L(a), \Theta, V')$ is (γ, ε)-good if the following holds:

Whenever we have

- $V : Q_L(a) \to \{0, \bar{V}\}$ with $V|_\Theta = V'$,
- $b, c \in Q_L(a)$,

- $t \in [0, 1]$,
- $\mathcal{R}$ a subset of r-bits inside $Q_L(a)$ such that each $Q_r(b) \in \mathcal{R}$ does not affect Θ,

then

- for each $Q_r(b) \in \mathcal{R}$, $(Q_r(b), V|_{F_r(b)})$ is admissible,
- the following inequality holds:

$$|G^{\mathcal{R},t}_{Q_L(a)}(b, c; \lambda_0)| \leq \bar{V}^{-\gamma|b-c|+L^{1-\varepsilon}}. \tag{3.4.3}$$

The following multi-scale analysis is a direct modification of [DS20, Lemma 8.3]. By using a standard argument (see, e.g. the proof of Theorem 2.3.1 in Chapter 2), it implies Theorem 3.1.4 with $Y_{\bar{V}} = J_r^{\bar{V}}$.

Recall that in Definition 3.2.17, we defined $\Theta^r = \cup_{a \in \dot{r}\mathbb{Z}^2} F_r(a)$ for any large odd number r.

Theorem 3.4.7 (Multi-scale Analysis)**.** *For each $\kappa < \frac{1}{2}$, we can pick $\varepsilon > \delta > 0$ such that, for each odd number $r > C_{\varepsilon,\delta}$, $\bar{V} > \exp(r^2)$ and $\lambda_0 \notin J_r^{\bar{V}}$, the following holds.*

There exist

1. *r-dyadic scales L_k for $k \geq 1$ with $L_{k+1} \in \left[\frac{1}{2} L_k^{\frac{1}{1-6\varepsilon}}, L_k^{\frac{1}{1-6\varepsilon}}\right]$ and the first scale satisfying $\frac{1}{2}\exp(\frac{1}{2}c_1\delta r) \leq L_1 \leq \exp(\frac{1}{2}c_1\delta r)$ where c_1 is the constant in Proposition 3.2.6,*
2. *decay rates $\gamma_k \geq \frac{1}{10r}$ with $\gamma_1 = \frac{1}{8r}$ and $\gamma_{k+1} = \gamma_k - L_k^{-\delta}$,*

3. densities $\eta_k < \varepsilon_0^{\frac{1}{5}}$ *with* $\eta_1 = \varepsilon_0^{\frac{1}{4}}$ *and* $\eta_{k+1} = \eta_k + L_k^{-\frac{1}{5}\varepsilon}$ *where* ε_0 *is defined by* (3.2.2),

4. random sets $\Theta_k \subset \Theta_{k+1} (k \geq 1)$ *where* $\Theta_1 = \Theta^r$,

such that the following statements are true for $k \geq 1$,

1. when $k \geq 2$, $\Theta_k \bigcap Q$ *is* $V|_{\Theta_{k-1} \bigcap 3Q}$*-measurable for any* r*-dyadic box* Q *with* $\ell(Q) \geq L_k$,

2. when $k \geq 2$, Θ_k *is a union of* Θ_{k-1} *and some* r*-bits,*

3. Θ_k *is* η_k*-regular in any* $Q_L(a) \subset \mathbb{Z}^2$ *with* $L \geq L_k^{1-\frac{5}{2}\varepsilon}$,

4. for any r*-dyadic box* Q *with* $\ell(Q) = L_k$,

$$\mathbb{P}\left[(Q, \Theta_k \cap Q, V|_{\Theta_k \cap Q}) \text{ is } (\gamma_k, \varepsilon)\text{-good}\right] \geq 1 - L_k^{-\kappa}. \tag{3.4.4}$$

Proof. Assume ε, δ are small and we impose further constraints on these objects during the proof. Set r-dyadic scale

$$L_1 \in \left[\frac{1}{2}\exp(\frac{1}{2}c_1\delta r), \exp(\frac{1}{2}c_1\delta r)\right]$$

where c_1 is the constant in Proposition 3.2.6. Set $\gamma_1 = \frac{1}{8r}$ and $\eta_1 = \varepsilon_0^{\frac{1}{4}}$. By letting $r > C_{\varepsilon,\delta}$, we can pick L_k, γ_k, η_k as in conditions 1, 2 and 3 for $k \geq 2$. Let M_0 be the largest integer such that $L_{M_0} \leq \exp(c_1 r)$. Then $M_0 \leq C'_{\varepsilon,\delta}$ for a constant

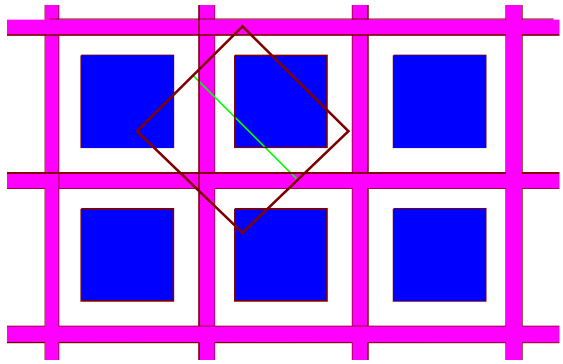

Figure 3.2: The figure illustrates the proof that Θ_1 (the pink region) is η_1-regular in $Q_L(a)$. The blue region indicates $\bigcup\{Q_{(1-100\sqrt{\varepsilon_0})r}(b) : b \in \dot{r}\mathbb{Z}^2\}$, the black tilted square indicates $\tilde{Q}$ and the green line indicates the diagonal $\mathcal{D}$.

$C'_{\varepsilon,\delta}$ depending on ε, δ and we have $L_{k-M_0} \le L_k^{\delta}$ for each $k > M_0$. Set $\Theta_k = \Theta_1$ for $k = 1, \cdots, M_0$. We prove by induction on k. We first prove the conclusion for $k \le M_0$. Statements 1 and 2 are true since $\Theta_k = \Theta_1$ when $k \le M_0$. To see Statement 3, let $Q_L(a) \subset \mathbb{Z}^2$ such that $L \ge L_1^{1-\frac{5}{2}\varepsilon}$. Suppose $\tilde{Q} \subset Q_L(a)$ is a tilted square. We claim that, if there exists $b_1 \in Q_L(a) \cap \dot{r}\mathbb{Z}^2$ such that $\tilde{Q} \cap Q_{(1-100\sqrt{\varepsilon_0})r}(b_1) \ne \emptyset$, then Θ_1 is $\varepsilon_0^{\frac{1}{4}}$-sparse in $\tilde{Q}$. We prove our claim by elementary geometry (see Figure 3.2). To see this, if $\tilde{Q} \cap \Theta_1 = \emptyset$ then our claim is obvious. Otherwise, note that we have

$$\begin{aligned}
&\operatorname{dist}(\Theta_1, Q_{(1-100\sqrt{\varepsilon_0})r}(b_1)) \\
=&\operatorname{dist}(F_r(b_1), Q_{(1-100\sqrt{\varepsilon_0})r}(b_1)) \\
=&\operatorname{dist}(Q_r(b_1) \setminus Q_{(1-2\varepsilon_0)r}(b_1), Q_{(1-100\sqrt{\varepsilon_0})r}(b_1)) \\
=&\left\lfloor \frac{(1-2\varepsilon_0)r-1}{2} \right\rfloor - \left\lfloor \frac{(1-100\sqrt{\varepsilon_0})r-1}{2} \right\rfloor + 1 \\
>&(50\sqrt{\varepsilon_0}-\varepsilon_0)r.
\end{aligned} \tag{3.4.5}$$

Thus $\tilde{Q} \cap \Theta_1 \ne \emptyset$ implies that the edge length of $\tilde{Q}$ is larger than $\frac{(50\sqrt{\varepsilon_0}-\varepsilon_0)r}{\sqrt{2}} > 25\sqrt{\varepsilon_0}r$.

Suppose $l \in \mathbb{Z}$ and $\varsigma \in \{+,-\}$ such that $\tilde{Q} \cap \mathcal{D}_l^\varsigma \neq \emptyset$ where $\mathcal{D}_l^\varsigma$ is a diagonal defined in Definition 3.3.2. Write $\mathcal{D} = \tilde{Q} \cap \mathcal{D}_l^\varsigma$ and then

$$|\mathcal{D}| > 25\sqrt{\varepsilon_0}r. \tag{3.4.6}$$

By elementary geometry,

$$|\{b \in \dot{r}\mathbb{Z}^2 : \mathcal{D} \cap Q_r(b) \neq \emptyset\}| \leq 10 + \frac{10|\mathcal{D}|}{r}. \tag{3.4.7}$$

Since $\mathcal{D}$ has at most one intersection with any horizontal or vertical line, we have

$$|\mathcal{D} \cap F_r(b)| \leq 10\varepsilon_0 r \tag{3.4.8}$$

for each $b \in \dot{r}\mathbb{Z}^2$. On the other hand, by Definition 3.2.17 we have

$$|\Theta_1 \cap \mathcal{D}| \leq \sum_{b \in \dot{r}\mathbb{Z}^2 : \mathcal{D} \cap Q_r(b) \neq \emptyset} |\mathcal{D} \cap F_r(b)|.$$

Thus by (3.4.8) and (3.4.7), we have $|\Theta_1 \cap \mathcal{D}| \leq 100\varepsilon_0 r + 100\varepsilon_0|\mathcal{D}| \leq \varepsilon_0^{\frac{1}{4}}|\mathcal{D}|$. The second inequality here is due to (3.4.6). Our claim follows.

Thus any tilted square in which Θ_1 is not $\varepsilon_0^{\frac{1}{4}}$-sparse is contained in

$$Q_L(a) \setminus \bigcup_{b \in Q_L(a) \cap \dot{r}\mathbb{Z}^2} Q_{(1-100\sqrt{\varepsilon_0})r}(b),$$

whose cardinality is less than $10^4\sqrt{\varepsilon_0}L^2 + 8rL \leq \varepsilon_0^{\frac{1}{4}}L^2$. Here, we used $L \geq \exp(c\delta r)$, $r > C_{\varepsilon,\delta}$ and ε_0 small enough (provided by (3.2.2)). Thus Θ_1 is $\varepsilon_0^{\frac{1}{4}}$-regular in $Q_L(a)$ and Statement 3 follows.

To see Statement 4, by Proposition 3.2.21, an r-dyadic Q is perfect implies $(Q, \Theta_1 \cap Q, V|_{\Theta_1 \cap Q})$ is $(\frac{1}{8r}, 1)$-good. Thus Proposition 3.2.19 implies Statement 4 when $k \leq M_0$.

Assume $k \geq M_0 + 1$ and our conclusions hold for any smaller k. We proceed to prove it for k. The general strategy is to apply Lemma 3.4.2.

For each $j < k$, we call an r-dyadic box $Q_{L_j}(a)$ "good" if

$$(Q_{L_j}(a), \Theta_j \cap Q_{L_j}(a), V|_{\Theta_j \cap Q_{L_j}(a)}) \text{ is } (\gamma_j, \varepsilon)\text{-good.}$$

Otherwise, we call it "bad". We must control the number of bad boxes in order to apply Lemma 3.4.2.

For any $0 < k' < k$, by Lemma 3.4.2, any bad r-dyadic $L_{k'}$-box Q must contain a bad $L_{k'-1}$-box. For any $0 < i \leq k$, and a bad L_{k-i}-box $Q' \subset Q$, we call Q' a *hereditary bad L_{k-i}-subbox of Q*, if there exists a sequence $Q' = \overline{Q}_i \subset \overline{Q}_{i-1} \subset \cdots \subset \overline{Q}_1 \subset Q$, where for each $j = 1, \cdots, i$, $\overline{Q}_j$ is a bad L_{k-j}-box. We also call such sequence $\{\overline{Q}_j\}_{1 \leq j \leq i}$ a hereditary bad chain of length i. Note that the set of hereditary bad chains of Q is $V_{\Theta_{k-1} \cap Q}$-measurable.

Claim 3.4.8. *If $\varepsilon < c_\kappa$ and $N > C_{M_0,\kappa}$, then for all $k > M_0$,*

$$\mathbb{P}[Q \text{ has no more than } N \text{ hereditary bad chains of length } M_0] \geq 1 - L_k^{-1}.$$

Proof of the claim. Let $N = (N')^{M_0}$ with N' to be determined. We can use the inductive hypothesis to estimate

$$\mathbb{P}\left[Q \text{ has more than } N \text{ hereditary bad chains of length } M_0\right] \tag{3.4.9}$$

$$\leq \sum_{\substack{r\text{-dyadic } Q' \subset Q \\ \ell(Q')=L_j \\ k-M_0<j\leq k}} \mathbb{P}\left[Q' \text{ has more than } N' \text{ bad } L_{j-1}\text{-subboxes}\right] \tag{3.4.10}$$

$$\leq \sum_{k-M_0<j\leq k} L_k^2 (L_j/L_{j-1})^{CN'} (L_{j-1}^{-\kappa})^{cN'} \tag{3.4.11}$$

$$\leq CM_0 L_k^2 (L_k^{(C\varepsilon - c\kappa)N'} + L_{k-M_0}^{(C\varepsilon - c\kappa)N'}) \tag{3.4.12}$$

$$\leq CM_0 L_k^2 (L_k^{(C\varepsilon - c\kappa)N'} + L_k^{(C\varepsilon - c\kappa)(1-6\varepsilon)^{M_0} N'}). \tag{3.4.13}$$

Here, c, C denote absolute constants. The claim follows by taking $c_\kappa = \frac{c\kappa}{2C}$ and $C_{M_0,\kappa} = (\frac{20+2M_0}{c\kappa(1-6c_\kappa)^{M_0}})^{M_0}$, and letting $\varepsilon < c_\kappa$ and $N' > C_{M_0,\kappa}^{\frac{1}{M_0}}$. $\square$

Now fix N as in the claim above. We call an L_k-box Q *ready* if Q is r-dyadic and Q contains no more than N' hereditary bad chains of length M_0. Note that the event that Q is ready is $V|_{\Theta_{k-1}\cap Q}$-measurable.

Suppose the L_k-box Q is ready. Let $Q_1''', \cdots, Q_N''' \subset Q$ be a complete list of L_{k-M_0}-boxes that includes every hereditary bad L_{k-M_0}-subboxes of Q. Let $Q_1'', \cdots, Q_N'' \subset Q$ be the corresponding bad L_{k-1}-subboxes of Q, such that $Q_i''' \subset Q_i''$ for each $i = 1, 2, \cdots, N$. These cubes are chosen in a way such that $\{Q_1'', \cdots, Q_N''\}$ contains all the bad L_{k-1}-subboxes in Q. Applying Lemma 3.4.4, we can choose an r-dyadic scale

$L' \in [c_N L_k^{1-3\varepsilon}, L_k^{1-3\varepsilon}]$ and disjoint r-dyadic L'-subboxes

$$Q'_1, \cdots, Q'_N \subset Q$$

such that, for each Q''_i, there is Q'_j such that $Q''_i \subset Q'_j$ and $\operatorname{dist}(Q''_i, Q \setminus Q'_j) \geq \frac{1}{8}L'$. Note that we can choose Q'_i, Q''_i, Q'''_i in a $V_{\Theta_{k-1}\cap Q}$-measurable way.

We define Θ_k to be the union of Θ_{k-1} and the subboxes $Q'_1, \cdots, Q'_N \subset Q$ of each ready L_k-box Q. We need to verify statements 1 to 4. Note that Statement 2 is true since each r-dyadic box is a union of r-bits (Lemma 3.2.15).

Claim 3.4.9. *Statements 1, 3 hold.*

Proof of the claim. For each L_k-box Q, the event that Q is ready, the scale L' and L'-boxes $Q'_i \subset Q$ are all $V|_{Q\cap\Theta_{k-1}}$-measurable. Thus $\Theta_k \cap Q$ is $V|_{\Theta_{k-1}\cap 3Q}$-measurable. Note that we have $3Q$ in place of Q because each r-dyadic L_k-box Q intersects 24 other r-dyadic L_k-boxes contained in $3Q$.

As for Statement 3, for each $L_k^{1-\frac{5}{2}\varepsilon}$-box $Q \subset \mathbb{Z}^2$, the set $Q \cap \Theta_k \setminus \Theta_{k-1}$ is covered by at most $25N$ boxes Q'_i with length at most $L_k^{1-3\varepsilon}$. Suppose $\tilde{Q}$ is a tilted square such that $Q \cap \Theta_{k-1}$ is η_{k-1}-sparse in $\tilde{Q}$ but $Q \cap \Theta_k$ is not η_k-sparse in $\tilde{Q}$, then $\tilde{Q}$ must intersect one of Q'_i's and have length at most $L_k^{1-\frac{11}{4}\varepsilon}$. This implies $\Theta_k \cap Q$ is η_k-regular in Q. □

Claim 3.4.10. *If the L_k-box Q is ready, $\mathcal{R}$ a subset of r-bits inside Q'_i that do not affect $\Theta_{k-1} \cup \bigcup_j Q''_j$, then each $Q_r(b) \in \mathcal{R}$ is admissible. Furthermore, if $|\lambda - \lambda_0| \leq \bar{V}^{-2L_{k-1}^{1-\varepsilon}}$,*

$t \in [0,1]$ and $H^{\mathcal{R},t}_{Q_i'} u = \lambda u$, then

$$\bar{V}^{cL_{k-1}^{1-\delta}} \|u\|_{\ell^\infty(E)} \leq \|u\|_{\ell^2(Q_i')} \leq (1 + \bar{V}^{-cL_{k-M_0}^{1-\delta}}) \|u\|_{\ell^2(G)},$$

where $E = Q_i' \setminus \cup_i Q_j''$ and $G = Q_i' \cap \cup_j Q_j'''$.

Proof of the claim. If r-bit $Q_r(b) \subset Q$ does not affect $\Theta_{k-1} \cup \bigcup_j Q_j''$, then it is contained in a good L_{k-1}-box $Q_{L_{k-1}}(a') \subset Q$. By Definition 3.4.6, since $Q_r(b)$ does not affect $\Theta_{k-1} \cap Q_{L_{k-1}}(a')$, it is admissible.

If $a \in Q_i' \setminus G$, then there is $j \in \{1, \cdots, M_0\}$ and a good L_{k-j}-box $Q'' \subset Q_i'$ with $a \in Q''$ and $\operatorname{dist}(a, Q_i' \setminus Q'') \geq \frac{1}{8} L_{k-j}$. Moreover, if $a \in E$, then $j = 1$. By Definition 3.4.6 and Lemma 3.4.5,

$$\begin{aligned} |u(a)| &= \left| \sum_{\substack{b \in Q'' \\ b' \in Q_i' \setminus Q'' \\ b \sim b'}} G^{\mathcal{R}_{Q''},t}_{Q''}(a,b;\lambda) u(b') \right| \\ &\leq 4 L_{k-j} \bar{V}^{L_{k-j}^{1-\varepsilon} - \frac{1}{8}\gamma_{k-j} L_{k-j}} \|u\|_{\ell^2(Q_i')} \\ &\leq \bar{V}^{-cL_{k-j}^{1-\delta}} \|u\|_{\ell^2(Q_i')}. \end{aligned}$$

Here we used $\gamma_{k-j} \geq \frac{1}{10r}$ and $L_{k-j} \geq \exp(c\delta r)$. In particular, we see that

$$\|u\|_{\ell^\infty(E)} \leq \bar{V}^{-cL_{k-1}^{1-\delta}} \|u\|_{\ell^2(Q_i')}$$

and

$$\|u\|_{\ell^\infty(Q_i'\setminus G)} \le \bar{V}^{-cL_{k-M_0}^{1-\delta}} \|u\|_{\ell^2(Q_i')}.$$

$\square$

Claim 3.4.11. *If Q is an r-dyadic L_k-box and $\mathcal{E}_i(Q)$ denotes the event that*

$$Q \text{ is ready and } \mathbb{P}[\|(H_{Q_i'} - \lambda_0)^{-1}\| \le \bar{V}^{L_k^{1-4\varepsilon}} | V|_{\Theta_k \cap Q}] = 1,$$

then $\mathbb{P}[\mathcal{E}_i(Q)] \ge 1 - L_k^{10\varepsilon - \frac{1}{2}}$.

Proof of the claim. Recall the event that Q ready and boxes $Q_i' \subset Q$ are $V|_{\Theta_{k-1}\cap Q}$-measurable. We may assume $i = 1$. We apply Proposition 3.3.18 to box Q_1' with $5\varepsilon > \delta > 0$, $K = N$, scales

$$L' \ge L_k^{1-4\varepsilon} \ge L_k^{1-5\varepsilon} \ge L_{k-1} \ge L_{k-1}^{1-2\delta} \ge 2L_{k-1}^{1-\varepsilon} \ge L_{k-1}^{1-\frac{5}{2}\varepsilon},$$

$\Theta = \Theta_{k-1} \cap Q_1'$, defects $\{Q_j'' : Q_j'' \subset Q_1'\}$, and $G = \cup\{Q_j''' : Q_j''' \subset Q_1'\}$. Assume $\varepsilon > 5\delta$ and note that $k \ge M_0 + 1$ and $L_{k-1} \ge L_{M_0} \ge \exp(\frac{1}{2}c_1 r)$. The previous claims provide the conditions to verify the hypothesis of Proposition 3.3.18. Since $Q_1' \subset \Theta_k$ when Q is ready, the claim follows. $\square$

Claim 3.4.12. *If Q is an r-dyadic L_k-box and $\mathcal{E}_1(Q), \cdots, \mathcal{E}_N(Q)$ hold, then Q is good.*

Proof of the claim. Suppose $\mathcal{R}$ is a subset of r-bits inside Q that do not affect Θ_k and $t \in [0,1]$. By Claim 3.4.10, each $Q_r(b') \in \mathcal{R}$ is admissible. We apply Lemma 3.4.2

to the box Q with small parameters $\frac{\varepsilon}{3} > \delta > 0$, decay rate γ_{k-1}, scales $L_k \geq L_k^{1-\varepsilon} \geq L' \geq L_k^{1-\frac{7}{2}\varepsilon} \geq L_k^{1-4\varepsilon} \geq L_{k-1} \geq L_{k-1}^{1-\varepsilon}$, and defects $Q_1', \cdots, Q_N'$. We conclude that

$$|G_Q^{\mathcal{R},t}(a,b)| \leq \bar{V}^{L_k^{1-\varepsilon} - \gamma_k |a-b|}$$

for each $a, b \in Q$. Since the events $\mathcal{E}_i(Q)$ are $V|_{\Theta_k \cap Q}$-measurable, we see that Q is good. $\square$

Finally we verify Statement 4. Combining the previous two claims, for any r-dyadic L_k-box Q, we have

$$\mathbb{P}[(Q, \Theta_k \cap Q, V|_{\Theta_k \cap Q}) \text{ is } (\gamma_k, \varepsilon)\text{-good}] \geq 1 - N L_k^{10\varepsilon - \frac{1}{2}} \geq 1 - L_k^{-\kappa},$$

provided $\kappa < \frac{1}{2} - 10\varepsilon$. $\square$

3.5 Proof of Lemma 3.3.5

Our approach follows the scheme in [DS20, Section 3] and [BLMS17]. The key for the proofs in [DS20], [BLMS17] and the current proof is the following observation for functions u satisfying $Hu = \lambda u$ on a tilted rectangle $R_{[1,a],[1,b]}$ defined in Definition 3.3.1.

Observation 3.5.1. Let $V : \mathbb{Z}^2 \to \mathbb{R}$ and $u : R_{[1,a],[1,b]} \to \mathbb{R}$. Suppose $a \geq 10b$ and

$-\Delta u + Vu = \lambda u$ in $R_{[2,a-1],[2,b-1]}$. If

$$\|u\|_{\ell^\infty(R_{[1,a],[1,2]})} \le 1$$

and $|u| \le 1$ on a $1-\varepsilon$ fraction of $R_{[1,a],[b-1,b]}$, then $\|u\|_{\ell^\infty(R_{[1,a],[1,b]})}$ is "suitably" bounded.

Observation 3.5.1 does not hold for arbitrary V and λ. It was proved in [BLMS17, Lemma 3.4] for the case when $V \equiv 0$ and $\lambda = 0$ (i.e. u is a harmonic function). It was also proved to hold with high probability for the case when $a \ge Cb^2 \log(a)$ and $\{V(x)\}_{x\in\mathbb{Z}^2}$ is a family of i.i.d. Bernoulli random variables taking values in $\{0, 1\}$ ([DS20, Lemma 3.13]).

In Lemma 3.5.20 below, we will prove that observation 3.5.1 holds with high probability only requiring $a \ge 10b$ and $\{V(x)\}_{x\in\mathbb{Z}^2}$ is a family of i.i.d. Bernoulli random variables taking values in $\{0, \bar{V}\}$.

Lemma 3.5.20 is the main new ingredient in the current proof. As long as Lemma 3.5.20 is proved, the rest of the proof of Lemma 3.3.5 follows the same scheme in [DS20] and [BLMS17] by proving a "growth lemma" (Lemma 3.5.22) and using a covering argument (Section 3.5.5) to conclude.

To prove the observation 3.5.1 (Lemma 3.5.20), we first consider the case when $u = 0$ on $R_{[1,a],[1,2]}$ (Lemma 3.5.8). We use the triangular matrix structure of the operator $M^{k,k'}_{[1,a]}$ defined in Definition 3.5.10. Then we use Lemma 3.1.6 to estimate the probability. We refer the reader to the beginning of Section 3.5.3 for an intuitive argument of the simple case when $u|_{R_{[1,a],[1,2]}\cup R_{[1,a],[b-1,b]}} = 0$.

3.5.1 Auxiliary lemmas

We first prove Lemma 3.1.6 by using Lemma 3.1.7.

Proof of Lemma 3.1.6. Write $\{e_j\}_{j=1}^n$ to be the standard normal basis in $\mathbb{R}^n$. Write $\Gamma = \Gamma_0 + a_0$ where Γ_0 is a k dimensional subspace and $a_0 \in \mathbb{R}^n$. Let Γ_1 be the orthogonal complement of Γ_0 and let $P : \mathbb{R}^n \to \Gamma_1$ be the orthogonal projection. Define $v_i = Pe_i$ for $i = 1, 2, \cdots, n$, then $\sum_{i=1}^n v_i v_i^\dagger = I_{n-k}$ (the identity operator on Γ_1).

Using Lemma 3.1.7 with $l = n$, $m = n - k$ and $m' = n - k - 1$, we can find $\mathcal{S} \subset \{1, 2, \cdots, n\}$ with $|\mathcal{S}| = n - k - 1$ such that the $n - k - 1$-th largest eigenvalue of

$$\sum_{i \in \mathcal{S}} v_i v_i^\dagger = \sum_{i \in \mathcal{S}} Pe_i e_i^\dagger P^\dagger \tag{3.5.1}$$

is at least $\frac{1}{4n(n-k)}$. Assume without loss of generality that $\mathcal{S} = \{1, 2, \cdots, n - k - 1\}$. Denote by Γ' the subspace generated by $\{e_i\}_{i=1}^{n-k-1}$ and let $Q : \mathbb{R}^n \to \Gamma'$ be the orthogonal projection onto Γ'. Then (3.5.1) is just $PQ^\dagger QP^\dagger$. Note that the dimension of the range of $QP^\dagger$ is at most $n-k-1$, thus the rank of the operator $PQ^\dagger QP^\dagger$ is at most $n - k - 1$. Hence the $n - k - 1$-th largest eigenvalue (which is also the smallest eigenvalue) of the positive semi-definite operator $QP^\dagger PQ^\dagger$ is at least $\frac{1}{4n(n-k)}$. This implies

$$\|PQ^\dagger a\|_2 \geq \sqrt{\frac{1}{4n(n-k)}}\|a\|_2 \tag{3.5.2}$$

for any $a \in \Gamma'$.

Consider the Boolean subcube $B' = \left\{\sum_{i=1}^{n-k-1} x_i e_i : x_i \in \{0,1\}\right\} \subset \Gamma'$. We claim that for any $v' \in \mathbb{R}^n$,

$$\#\{a \in B' + v' : \min_{b\in\Gamma} \|a - b\|_2 < \frac{1}{4} n^{-\frac{1}{2}} (n-k)^{-\frac{1}{2}}\} \leq 1. \tag{3.5.3}$$

To see this, assume the claim does not hold. Then for some $v'' \in \mathbb{R}^n$, there are two different $a_1, a_2 \in (B' + v'')$ with $\min_{b\in\Gamma} \|a_j - b\|_2 < \frac{1}{4} n^{-\frac{1}{2}} (n-k)^{-\frac{1}{2}}$ for $j = 1, 2$. Choose $b_1, b_2 \in \Gamma$ with $\|a_j - b_j\|_2 < \frac{1}{4} n^{-\frac{1}{2}} (n-k)^{-\frac{1}{2}}$ for $j = 1, 2$. Let $a' = a_1 - a_2$ and $b' = b_1 - b_2$. Then $\|a' - b'\|_2 < \frac{1}{2} n^{-\frac{1}{2}} (n-k)^{-\frac{1}{2}}$ and $a' \in \Gamma'$, $b' \in \Gamma_0$.

Since any two vectors in $B' + v''$ has ℓ^2 distance at least 1, $\|a'\|_2 \geq 1$. On the other hand, we have $\min_{b\in\Gamma_0} \|a' - b\|_2 < \frac{1}{2} n^{-\frac{1}{2}} (n-k)^{-\frac{1}{2}}$ which is equivalent to $\|PQ^\dagger a'\|_2 < \frac{1}{2} n^{-\frac{1}{2}} (n-k)^{-\frac{1}{2}}$. However, this contradicts with (3.5.2) and our claim (3.5.3) follows.

Finally, $B = \bigcup \left\{B' + \sum_{j=n-k}^{n} x_j e_j : x_j \in \{0,1\} \text{ for } n-k \leq j \leq n\right\}$. Thus by (3.5.3),

$$\begin{aligned}
&\# \left\{a \in B : \min_{b\in\Gamma} \|a - b\|_2 < \frac{1}{4} n^{-\frac{1}{2}} (n-k)^{-\frac{1}{2}}\right\} \\
&\leq \sum_{x_j \in \{0,1\} \text{ for } n-k \leq j \leq n} \# \left\{a \in B' + \sum_{j=n-k}^{n} x_j e_j : \min_{b\in\Gamma} \|a - b\|_2 < \frac{1}{4} n^{-\frac{1}{2}} (n-k)^{-\frac{1}{2}}\right\} \\
&\leq \sum_{x_j \in \{0,1\} \text{ for } n-k \leq j \leq n} 1 \\
&= 2^{k+1}.
\end{aligned}$$

$\square$

We will also need the following lemma to bound the inverse norm of principal submatrices of a triangular matrix.

Lemma 3.5.2. *Let $d > 0$ be an integer, $K > 1$ be a real number and $\{m_1 < m_2 < \cdots < m_d\}$ be a set of positive integers. Let $A = (a_{ij})_{1\leq i,j\leq d}$ be a lower (or upper) triangular matrix. Assume that $|a_{ii}| = 1$ for each $i = 1, \cdots, d$ and $|a_{ij}| \leq K^{|m_i - m_j|}$ for each $1 \leq i, j \leq d$. Then the Euclidean operator norm of the inverse A^{-1} satisfies $\|A^{-1}\| \leq d(2K)^{m_d}$.*

Proof. We assume A to be a lower triangular matrix, the case for upper triangular matrix follows the same argument. Denote $A^{-1} = \left(a'_{ij}\right)_{1\leq i,j\leq d}$.

We prove that $|a'_{ij}| \leq (2K)^{|m_i - m_j|}$ by induction on $k = i - j$. For $k = 0$, since A is lower triangular, $a'_{ii} = (a_{ii})^{-1}$ and thus $|a'_{ii}| = 1$. Assume our conclusion holds for $0 \leq k < k'$, we prove the case when $i - j = k'$. Note that

$$\sum_{l=1}^{d} a_{il} a'_{lj} = 0. \tag{3.5.4}$$

This implies

$$a_{ii} a'_{ij} = -\sum_{l=j}^{i-1} a_{il} a'_{lj}. \tag{3.5.5}$$

Since $|a_{ii}| = 1$, by inductive hypothesis and $|a_{ll'}| \leq K^{|m_l - m_{l'}|}$, we have

$$|a'_{ij}| \leq \sum_{l=j}^{i-1} K^{m_i - m_l} (2K)^{m_l - m_j} = K^{m_i - m_j} \sum_{l=j}^{i-1} 2^{m_l - m_j} \leq (2K)^{m_i - m_j}. \tag{3.5.6}$$

Thus the induction proves $|a'_{ij}| \le (2K)^{m_i - m_j}$ for $1 \le i, j \le d$. Finally,

$$\|A^{-1}\| \le \Big(\sum_{1 \le i,j \le d} |a'_{ij}|^2\Big)^{\frac{1}{2}} \le d(2K)^{m_d}$$

since $0 < m_1 < \cdots < m_d$. $\square$

3.5.2 Tilted rectangles

In this section, we collect basic lemmas on functions satisfying the equation $Hu = \lambda u$ on a tilted rectangle (see Definition 3.3.1). The following Lemma 3.5.4, Lemma 3.5.6 and Lemma 3.5.7 are rewrite of [DS20, Lemma 3.8], [DS20, Lemma 3.10] and [DS20, Lemma 3.11] respectively. They are modified to depend on $\bar{V}$ explicitly.

We will keep several notations from [DS20, Section 3]. In particular, we work in the tilted coordinates of

$$(s, t) = (x + y, x - y). \tag{3.5.7}$$

Under coordinate transformation (3.5.7), the transformed lattice is $\widetilde{\mathbb{Z}^2} = \{(s, t) \in \mathbb{Z}^2 : s - t \text{ is even}\}$. The equation

$$Hu = \lambda u \tag{3.5.8}$$

becomes

$$u(s,t) = (4 + V(s-1, t-1) - \lambda) u(s-1, t-1) - u(s-2, t) - u(s-2, t-2) - u(s, t-2). \tag{3.5.9}$$

Given two intervals $J_1, J_2 \subset \mathbb{Z}$, by Definition 3.3.1, under the coordinate transformation, the tilted rectangle $R_{J_1,J_2} \subset \mathbb{Z}^2$ is transformed to

$$\widetilde{R_{J_1,J_2}} = \{(s,t) \in J_1 \times J_2 : s - t \text{ is even}\}$$

in the new lattice $\widetilde{\mathbb{Z}^2}$. With a little abuse of notations, we also use R_{J_1,J_2} to denote $\widetilde{R_{J_1,J_2}}$ for the rest of this section.

Definition 3.5.3. Given integers $a_1 < a_2$ and $b_1 < b_2$, the west boundary of the tilted rectangle is

$$\partial^w R_{[a_1,a_2],[b_1,b_2]} = R_{[a_1,a_2],[b_1,b_1+1]} \cup R_{[a_1,a_1+1],[b_1,b_2]}.$$

The following lemma is a rewrite of [DS20, Lemma 3.8] and it follows from the same proof of [DS20, Lemma 3.8].

Lemma 3.5.4. *Suppose energy $\lambda \in \mathbb{R}$, real number $\bar{V} \in \mathbb{R}$ and integers $a_1 < a_2, b_1 < b_2$. Then every function $u : \partial^w R_{[a_1,a_2],[b_1,b_2]} \to \mathbb{R}$ has a unique extension*

$$u^0 = E^{(\lambda,\bar{V})}_{R_{[a_1,a_2],[b_1,b_2]}}(u) : R_{[a_1,a_2],[b_1,b_2]} \to \mathbb{R}$$

such that

$$Hu^0 = \lambda u^0 \tag{3.5.10}$$

in $R_{[a_1+1,a_2-1],[b_1+1,b_2-1]}$. Moreover, $E^{(\lambda,\bar{V})}_{R_{[a_1,a_2],[b_1,b_2]}}$ is a random linear operator and is $V|_{R_{[a_1+1,a_2-1],[b_1+1,b_2-1]}}$-measurable.

Remark 3.5.5. Given energy λ, real $\bar{V}$ and integers $a_1 < a_2$ and $b_1 < b_2$, we also denote $E^{(\lambda,\bar{V})}_{R_{[a_1,a_2],[b_1,b_2]}}$ by $E^{(\lambda,\bar{V})}_{[a_1,a_1],[b_1,b_2]}$ for simplicity. When energy λ and real number $\bar{V}$ are given in context, we also omit $\lambda, \bar{V}$ and denote $E^{(\lambda,\bar{V})}_{R_{[a_1,a_2],[b_1,b_2]}}$ by $E_{R_{[a_1,a_2],[b_1,b_2]}}$ and $E^{(\lambda,\bar{V})}_{[a_1,a_2],[b_1,b_2]}$ by $E_{[a_1,a_2],[b_1,b_2]}$.

Lemma 3.5.6. *Suppose we have real numbers $\lambda, \bar{V}$ and integers $a_1 \leq a_2$ and $b_1 \leq b_2$. Assume $\lambda \in [-2, 10]$ and $\bar{V} \geq 2$. If $Hu = \lambda u$ in $R_{[a_1+1,a_2-1],[b_1+1,b_2-1]}$ and $\|u\|_{\ell^\infty(\partial^w R_{[a_1,a_2],[b_1,b_2]})} = 1$, then*

$$\|u\|_{\ell^\infty(R_{[a_1,a_2],[b_1,b_2]})} \leq (\bar{V}(a_2 - a_1 + 1))^{C_1(b_2-b_1-1)\vee 0} \tag{3.5.11}$$

$$\|u\|_{\ell^\infty(R_{[a_1,a_2],[b_1,b_2]})} \leq (\bar{V}(b_2 - b_1 + 1))^{C_1(a_2-a_1-1)\vee 0} \tag{3.5.12}$$

for a numerical constant C_1.

Proof. We only prove (3.5.11), and (3.5.12) follows by symmetry.

Assume without loss of generality that $a_1 = b_1 = 1$. We prove

$$|u(s,t)| \leq (C\bar{V}s)^{(t-2)\vee 0} \tag{3.5.13}$$

by induction on $(s,t) \in R_{[1,a_2],[1,b_2]}$. Here, $C \geq 10$ is a universal constant to be determined. Firstly, if $(s,t) \in R_{[1,a_2],[1,2]}$, then $t \leq 2$ and

$$|u(s,t)| \leq 1 \leq (C\bar{V}s)^{(t-2)\vee 0}$$

by assumption. Secondly, if $(s,t) \in R_{[1,2],[3,b_2]}$, then $|u(s,t)| \le 1 \le (C\bar{V}s)^{(t-2)\vee 0}$ by assumption. Now suppose $(s,t) \in R_{[3,a_2],[3,b_2]}$ and assume (3.5.13) holds for $(s',t') \in R_{[1,s],[1,t]} \setminus \{(s,t)\}$. We use (3.5.9) to get

$$
\begin{aligned}
&|u(s,t)| \\
\le&(14+\bar{V})|u(s-1,t-1)| + |u(s-2,t)| + |u(s-2,t-2)| + |u(s,t-2)| \\
\le&(14+\bar{V})(C\bar{V}s)^{t-3} + (C\bar{V}(s-2))^{t-2} + (C\bar{V}(s-2))^{(t-4)\vee 0} + (C\bar{V}s)^{(t-4)\vee 0} \\
\le&(16+\bar{V})(C\bar{V}s)^{t-3} + (C\bar{V}(s-2))^{t-2} \\
\le&(C\bar{V}s)^{t-2}\left(\frac{16+\bar{V}}{C\bar{V}}s^{-1} + \left(\frac{s-2}{s}\right)^{t-2}\right) \\
\le&(C\bar{V}s)^{t-2}\left(\frac{16+\bar{V}}{C\bar{V}}s^{-1} + 1 - 2s^{-1}\right) \\
\le&(C\bar{V}s)^{t-2}.
\end{aligned}
$$

Here, we used $|\lambda| \le 10$, $\bar{V} \ge 2$ and $C \ge 10$. □

The following lemma follows the same proof of [DS20, Lemma 3.11].

Lemma 3.5.7. *Suppose real numbers $\lambda_1, \lambda_2, \bar{V}$ and positive integers $a, b > 2$. Assume $\lambda_1, \lambda_2 \in [-2, 10]$ and $\bar{V} \ge 2$. If $Hu_1 = \lambda_1 u_1$ and $Hu_2 = \lambda_2 u_2$ in $R_{[2,a-1],[2,b-1]}$ and $u_1 = u_2$ in $\partial^w R_{[1,a],[1,b]}$, then*

$$
\|u_1 - u_2\|_{\ell^\infty(R_{[1,a],[1,b]})} \le (a\bar{V})^{C_2 b}\|u_1\|_{\ell^\infty(\partial^w R_{[1,a],[1,b]})}|\lambda_1 - \lambda_2|, \tag{3.5.14}
$$

where C_2 is a numerical constant.

3.5.3 Key lemmas

The main task in this subsection is to prove the following Lemma 3.5.8 which will be used to prove the key estimate Lemma 3.5.20. See the context below observation 3.5.1 for a comparison between Lemma 3.5.20 and [DS20, Lemma 3.13].

Lemma 3.5.8. *There are constants $\alpha_1 > 1 > c_4 > 0$ such that, if*

1. *integers $a > b > \alpha_1$ with $10b \le a \le 60b$,*
2. *$\lambda_0 \in [0, 8]$ and $\bar{V} \ge 2$,*
3. *$\Theta \subset \mathbb{Z}^2$ is $(c_4, -)$-sparse in $R_{[1,a],[1,b]}$,*
4. *$V' : \Theta \to \{0, \bar{V}\}$,*
5. *$\mathcal{E}_{tr}(R_{[1,a],[1,b]})$ denotes the event that,*

$$\begin{cases} Hu = \lambda_0 u \ \text{ in } R_{[2,a-1],[2,b-1]} \\ \|u\|_{\ell^\infty(R_{[1,2],[1,b]})} = 1 \\ u \equiv 0 \ \text{ on } R_{[1,a],[1,2]} \end{cases} \tag{3.5.15}$$

implies $|u| \ge (a\bar{V})^{-\alpha_1 a}$ on a $\frac{1}{10^6}$ fraction of $R_{[1,a],[b-1,b]}$,

then $\mathbb{P}\left[\mathcal{E}_{tr}(R_{[1,a],[1,b]}) \middle| V|_\Theta = V'\right] \ge 1 - \exp(-c_4 a)$.

We give an intuitive argument here for the simple case when we have $Hu = \lambda u$ in $R_{[2,a-1],[2,b-1]}$ with $a \ge 10b$. We claim that, with high probability,

$$u|_{R_{[1,a],[1,2]} \cup R_{[1,a],[b-1,b]}} = 0$$

will force $u \equiv 0$ in $R_{[1,a],[1,b]}$ (which is implied by observation 3.5.1 and linearity).

To see this, by Lemma 3.5.4, we can regard function $u|_{R_{[1,a],[b-1,b]}}$ as the image of $u|_{R_{[1,a],[1,2]} \cup R_{[1,2],[3,b]}}$ under a linear mapping determined by the potential V. We assume $u|_{R_{[1,a],[1,2]}} = 0$ and $u(1,3) = 1$ (recall that we are working in the tilted coordinate (3.5.7)). It suffices to prove that, with high probability,

$$u|_{R_{[1,a],[b-1,b]}} \neq 0 \text{ for } \textit{any} \text{ choice of } u|_{R_{[1,2],[4,b]}}. \tag{3.5.16}$$

Once this is proved, $u|_{R_{[1,a],[1,2]} \cup R_{[1,a],[b-1,b]}} = 0$ will force $u(1,3) = 0$ and further $u|_{R_{[1,a],[1,3]}} = 0$. By repeating this argument, $u|_{R_{[1,a],[1,2]} \cup R_{[1,a],[b-1,b]}} = 0$ will force $u(s,t) = 0$ for each $(s,t) \in R_{[1,2],[3,b]}$ and then $u \equiv 0$ in $R_{[1,a],[1,b]}$ by Lemma 3.5.4.

To see (3.5.16), let us first calculate $u|_{R_{[1,a],\{3\}}}$. Using equation (3.5.9) for $t = 2$, we have $u(s,3) + u(s-2,3) = 0$ for any odd number $s \in [3,a]$. Since $u(1,3) = 1$, inductively we have

$$u(s,3) = (-1)^{\frac{s-1}{2}} \tag{3.5.17}$$

for odd $s \in [1,a]$. Let us calculate further $u|_{R_{[1,a],\{4\}}}$. Using equations (3.5.9) and (3.5.17) for $t = 3$, we have $u(s,4) + u(s-2,4) = (-1)^{\frac{s-2}{2}}(4 + V(s-1,3) - \lambda)$ for any even number $s \in [3,a]$. Inductively, for even $s \in [1,a]$,

$$u(s,4) = (-1)^{\frac{s-2}{2}} \left(u(2,4) + \sum_{2<s'<s,\ s' \text{ is odd}} (4 + V(s',3) - \lambda) \right). \tag{3.5.18}$$

By equations (3.5.17) and (3.5.18), we can write $u|_{R_{[1,a],[3,4]}} = u^{(1)} + u^{(2)} + u(2,4)u^{(3)}$

with $u^{(i)} \in \ell^2(R_{[1,a],[3,4]})$ for $i = 1, 2, 3$. Here, $u^{(1)}|_{R_{[1,a],\{3\}}} = 0$ and $u^{(1)}|_{R_{[1,a],\{4\}}} = A(\vec{V})$ in which A is a triangular matrix and the vector

$$\vec{V} = (V(3,3), V(5,3), \cdots, V(a - i_a, 3)) \tag{3.5.19}$$

satisfies $i_a \in \{1, 2\}$ and $a - i_a$ is an odd number. Moreover, $u^{(2)}(s, 3) = (-1)^{\frac{s-1}{2}}$ for odd $s \in [1, a]$ and $u^{(2)}(s, 4) = (-1)^{\frac{s-2}{2}} \frac{(s-2)(4-\lambda)}{2}$ for even $s \in [1, a]$; $u^{(3)}|_{R_{[1,a],\{3\}}} = 0$ and $u^{(3)}(s, 4) = (-1)^{\frac{s-2}{2}}$ for even $s \in [1, a]$. Note that, $u^{(2)}$ and $u^{(3)}$ are independent of potential V (in the sense of random variables). By Lemma 3.5.4, $u|_{R_{[1,a],[b-1,b]}}$ is determined linearly by $u|_{\partial^w R_{[1,a],[3,b]}}$. Hence, there are linear operators M_0, M_1 such that

$$u|_{R_{[1,a],[b-1,b]}} = M_0(u^{(1)} + u^{(2)} + u(2,4)u^{(3)}) + M_1(u|_{R_{[1,2],[5,b]}}). \tag{3.5.20}$$

Since $u^{(1)}$ is the zero extension of $A(\vec{V})$ and $u(2,4)u^{(3)}$ is determined linearly by $u(2,4)$, we have

$$u|_{R_{[1,a],[b-1,b]}} = M(A(\vec{V})) + M_0(u^{(2)}) + M_2(u|_{R_{[1,2],[4,b]}}) \tag{3.5.21}$$

with linear operators A, M, M_0 and M_2 all independent of $V|_{R_{[1,a],\{3\}}}$. Thus we have $u|_{R_{[1,a],[b-1,b]}} = 0$ implies

$$M(A(\vec{V})) + M_0(u^{(2)}) + M_2(u|_{R_{[1,2],[4,b]}}) = 0. \tag{3.5.22}$$

It will be proved later that M can be regarded as a triangular matrix and the operator MA is injective. Thus (3.5.22) implies

$$\vec{V} = -(MA)^{-1}(M_0(u^{(2)}) + M_2(u|_{R_{[1,2],[4,b]}})) \tag{3.5.23}$$

with $(MA)^{-1}$ defined on the range of MA.

However, the rank of operator M_2 is at most $|R_{[1,2],[4,b]}|$ which is bounded by $b \leq \frac{a}{10}$. Thus, conditioning on $V|_{R_{[1,a],[1,b]} \setminus R_{[1,a],\{3\}}}$,

$$\left\{-(MA)^{-1}(M_0(u^{(2)}) + M_2(v)) : v \in \ell^2(R_{[1,2],[4,b]})\right\}$$

is an affine subspace with dimension no larger than $\frac{a}{10}$. Recall (3.5.19), $\vec{V}$ is $V|_{R_{[1,a],\{3\}}}$-measurable and can be regarded as a random element in a Boolean cube with dimension larger than $\frac{a}{3}$. Thus by Lemma 3.1.6, with probability no less than $1-2^{\frac{a}{10}-\frac{a}{3}+1} > 1-\exp(-ca)$, (3.5.23) fails for *any* $u|_{R_{[1,2],[4,b]}}$. Our claim follows.

The proof of Lemma 3.5.8 below makes the above argument quantitative. Lemma 3.5.8 is also the key in proving Lemma 3.5.20. We start by defining the operator M in (3.5.21) and prove its triangular matrix structure.

Definition 3.5.9. Given $S_1 \subset S_2 \subset \widetilde{\mathbb{Z}^2}$, we use $P_{S_1}^{S_2} : \ell^2(S_2) \to \ell^2(S_1)$ to denote the restriction operator from S_2 to S_1. i.e. $P_{S_1}^{S_2}(u) = u|_{S_1}$ for $u \in \ell^2(S_2)$. We use $I_{S_1}^{S_2}$ to denote the adjoint operator $(P_{S_1}^{S_2})^\dagger$, i.e. $I_{S_1}^{S_2}(u) = u$ on S_1 and $I_{S_1}^{S_2}(u) = 0$ on $S_2 \setminus S_1$ for each $u \in \ell^2(S_1)$.

Definition 3.5.10. Given energy $\lambda \in [0,8]$, real number $\bar{V}$ and integers a,k,k' such that $a>1$ and $k<k'$, we define the linear operator

$$M_{[1,a]}^{k,k'} : \ell^2(R_{[1,a],\{k\}}) \to \ell^2(R_{[1,a],\{k'\}})$$

as follows:

$$M_{[1,a]}^{k,k'} = P_{R_{[1,a],\{k'\}}}^{R_{[1,a],[k-1,k']}} E_{[1,a],[k-1,k']} I_{R_{[1,a],\{k\}}}^{\partial^w R_{[1,a],[k-1,k']}}. \tag{3.5.24}$$

Lemma 3.5.11. *Given energy $\lambda \in [0,8]$, real number $\bar{V}$ and integers a,k,k' such that $a>1$ and $k<k'$, the linear operator $M_{[1,a]}^{k,k'}$ is $V|_{R_{[2,a-1],[k,k'-1]}}$-measurable.*

Proof. Lemma 3.5.4 implies that the extension operator $E_{[1,a],[k-1,k']}$ is $V|_{R_{[2,a-1],[k,k'-1]}}$-measurable, thus $M_{[1,a]}^{k,k'}$ is also $V|_{R_{[2,a-1],[k,k'-1]}}$-measurable. □

Given $(s,t) \in \widetilde{\mathbb{Z}^2}$, we use $\delta_{(s,t)}$ to denote the function that equals 1 on (s,t) and 0 elsewhere.

Proposition 3.5.12. *Suppose we have energy $\lambda \in [0,8]$, real number $\bar{V} > 2$ and integers a,k,k',s,s' such that $a \geq 4$, $k<k'$, $(s,k),(s',k') \in \widetilde{\mathbb{Z}^2}$ and $4 \leq s,s' \leq a$. Then*

$$|\langle \delta_{(s',k')}, M_{[1,a]}^{k,k'} \delta_{(s,k)} \rangle| = \begin{cases} 0 & \text{if } s' < s \\ 1 & \text{if } s' = s \end{cases} \tag{3.5.25}$$

and

$$|\langle \delta_{(s',k')}, M_{[1,a]}^{k,k'} \delta_{(s,k)} \rangle| \leq ((k'-k+2)\bar{V})^{C_1(s'-s)} \quad \text{if } s' > s. \tag{3.5.26}$$

Here, C_1 is the constant in Lemma 3.5.6.

Proof. Denote $R_1 = R_{[1,a],[k-1,k']}$. Assume the function $u : R_1 \to \mathbb{R}$ satisfies $u|_{\partial^w R_1} = \delta_{(s,k)}$ and $Hu = \lambda u$ in $R_{[2,a-1],[k,k'-1]}$. It suffices to show that

$$u(s',k') = 0 \qquad \text{if } s' < s \tag{3.5.27}$$

$$u(s',k') = (-1)^{\frac{k'-k}{2}} \qquad \text{if } s' = s \tag{3.5.28}$$

$$|u(s',k')| \le ((k'-k+2)\bar{V})^{C_1(s'-s)} \qquad \text{if } s' > s. \tag{3.5.29}$$

Firstly, since $u = 0$ on $\partial^w R_{[1,s-1],[k-1,k']}$, we have $u = 0$ on $R_{[1,s-1],[k-1,k']}$ by Lemma 3.5.4. Thus (3.5.27) holds. Secondly, we inductively prove $u(s,k+2i) = (-1)^i$ for $i = 0,1,\cdots,\left\lfloor \frac{k'-k}{2} \right\rfloor$. This is true for $i = 0$ since $u|_{\partial^w R_1} = \delta_{(s,k)}$. Suppose $u(s,k+2i) = (-1)^i$ for some $i < \left\lfloor \frac{k'-k}{2} \right\rfloor$. Since $s \ge 4$, we can use the equation $Hu = \lambda u$ at the point $(s-1,k+2i+1)$. By (3.5.27), we have $u(s,k+2i) + u(s,k+2i+2) = 0$ and thus $u(s,k+2i+2) = (-1)^{i+1}$. By induction we have $\left|u\left(s,k+2\left\lfloor \frac{k'-k}{2} \right\rfloor\right)\right| = 1$. Since $s = s'$ implies $k-k'$ is even, (3.5.28) follows.

Finally we suppose $s' > s$. By (3.5.27) and (3.5.28), $\|u\|_{\ell^\infty(\partial^w R_{[s-1,s'],[k-1,k']})} = 1$. Then by (3.5.12) in Lemma 3.5.6,

$$\|u\|_{\ell^\infty(R_{[s-1,s'],[k-1,k']})} \le (\bar{V}(k'-k+2))^{C_1(s'-s)}.$$

In particular, $|u(s',k')| \le (\bar{V}(k'-k+2))^{C_1(s'-s)}$ and (3.5.29) follows. $\square$

Corollary 3.5.13. *Suppose we have energy $\lambda \in [0,8]$, real number $\bar{V} > 2$ and integers*

a, k, k' such that $a \geq 6$ and $k < k'$. Assume k and k' have the same parity. Suppose

$$S_1 \subset R_{[4,a-1],\{k'\}}$$

and let $S_2 = \{(s,k) : (s,k') \in S_1\} \subset R_{[4,a-1],\{k\}}$. Then

$$\|(P_{S_1}^{R_{[1,a],\{k'\}}} M_{[1,a]}^{k,k'} I_{S_2}^{R_{[1,a],\{k\}}})^{-1}\| \leq a(2\bar{V}(k'-k+2))^{2C_1 a}. \tag{3.5.30}$$

Proof. By Proposition 3.5.12, $P_{R_{[4,a-1],\{k'\}}}^{R_{[1,a],\{k'\}}} M_{[1,a]}^{k,k'} I_{R_{[4,a-1],\{k\}}}^{R_{[1,a],\{k\}}}$ can be regarded as an upper triangular matrix $(a_{ij})_{1 \leq i,j \leq d}$ such that $|a_{ii}| = 1$ and

$$|a_{ij}| \leq ((k'-k+2)\bar{V})^{2C_1|i-j|}$$

for $1 \leq i, j \leq d$. Here, $d = |R_{[4,a-1],\{k\}}| \leq a$.

Since $P_{S_1}^{R_{[1,a],\{k'\}}} M_{[1,a]}^{k,k'} I_{S_2}^{R_{[1,a],\{k\}}}$ can be regarded as a principal submatrix which is also an upper triangular matrix, our conclusion follows from Lemma 3.5.2. □

Lemma 3.5.14. *Suppose we have real numbers $\lambda, \bar{V}$, integers $a > 1$ and $2 < b_* < b$. Denote $R_1 = R_{[1,a],[1,b]}$, $R_2 = R_{[1,a],[1,b_*+1]}$ and $R_3 = R_{[1,a],[1,b_*-1]}$. Then the following linear operator from $\ell^2(\partial^w R_3) \to \ell^2(R_{[1,a],\{b\}})$*

$$P_{R_{[1,a],\{b\}}}^{R_1} E_{R_1} I_{\partial^w R_3}^{\partial^w R_1} - M_{[1,a]}^{b_*+1,b} P_{R_{[1,a],\{b_*+1\}}}^{R_2} E_{R_2} I_{\partial^w R_3}^{\partial^w R_2} \tag{3.5.31}$$

is independent of $V|_{R_{[1,a],\{b_*\}}}$ (in the sense of random variables).

Lemma 3.5.14 allows us to write the operator $P^{R_1}_{R_{[1,a],\{b\}}} E_{R_1} I^{\partial^w R_1}_{\partial^w R_3}$ as the sum of two operators: a $V|_{R_{[1,a],\{b_*\}}}$-measurable operator and a $V|_{R_{[1,a],\{b_*\}}}$-independent operator. Here, the $V|_{R_{[1,a],\{b_*\}}}$-measurable operator can be written as the composition of a $V|_{R_{[1,a],\{b_*\}}}$-independent operator $M^{b_*+1,b}_{[1,a]}$ and the operator $P^{R_2}_{R_{[1,a],\{b_*+1\}}} E_{R_2} I^{\partial^w R_2}_{\partial^w R_3}$. Thus intuitively, Lemma 3.5.14 says that the $V|_{R_{[1,a],\{b_*\}}}$-measurable "part" of operator $P^{R_1}_{R_{[1,a],\{b\}}} E_{R_1} I^{\partial^w R_1}_{\partial^w R_3}$ is "contained" in $P^{R_2}_{R_{[1,a],\{b_*+1\}}} E_{R_2} I^{\partial^w R_2}_{\partial^w R_3}$. The proof is by direct calculation.

Proof of Lemma 3.5.14. Denote $R_4 = R_{[1,a],[b_*,b]}$ and let $u \in \ell^2(\partial^w R_3)$. Let $v = E_{R_1} I^{\partial^w R_1}_{\partial^w R_3}(u)$, then by uniqueness in Lemma 3.5.4,

$$v|_{R_{[1,a],\{b\}}} = P^{R_4}_{R_{[1,a],\{b\}}} E_{R_4}(v|_{\partial^w R_4}). \tag{3.5.32}$$

Let $v_1 = v|_{R_{[1,a],\{b_*\}}}$ and $v_2 = v|_{R_{[1,a],\{b_*+1\}}}$. Note that $v|_{R_{[1,2],[b_*,b]}} = 0$. By (3.5.32) and linearity of E_{R_4},

$$\begin{aligned} &v|_{R_{[1,a],\{b\}}} \\ &= P^{R_4}_{R_{[1,a],\{b\}}} E_{R_4} I^{\partial^w R_4}_{R_{[1,a],\{b_*\}}}(v_1) + P^{R_4}_{R_{[1,a],\{b\}}} E_{R_4} I^{\partial^w R_4}_{R_{[1,a],\{b_*+1\}}}(v_2) \\ &= P^{R_4}_{R_{[1,a],\{b\}}} E_{R_4} I^{\partial^w R_4}_{R_{[1,a],\{b_*\}}}(v_1) + M^{b_*+1,b}_{[1,a]}(v_2). \end{aligned} \tag{3.5.33}$$

Here, we used Definition 3.5.10.

By uniqueness in Lemma 3.5.4, $v_2 = P^{R_2}_{R_{[1,a],\{b_*+1\}}} E_{R_2} I^{\partial^w R_2}_{\partial^w R_3}(u)$. Thus the image of u under the operator (3.5.31) is $v|_{R_{[1,a],\{b\}}} - M^{b_*+1,b}_{[1,a]}(v_2)$. Thus by (3.5.33), in order to prove the conclusion, it suffices to prove that the linear operator

$$u \mapsto P^{R_4}_{R_{[1,a],\{b\}}} E_{R_4} I^{\partial^w R_4}_{R_{[1,a],\{b_*\}}}(v_1)$$

is independent of $V|_{R_{[1,a],\{b_*\}}}$.

To see this, note that E_{R_4} is independent of $V|_{R_{[1,a],\{b_*\}}}$ by Lemma 3.5.4. On the other hand, let $R_5 = R_{[1,a],[1,b_*]}$, then by uniqueness in Lemma 3.5.4 again, we have $v_1 = P^{R_5}_{R_{[1,a],\{b_*\}}} E_{R_5} I^{\partial^w R_5}_{\partial^w R_3}(u)$. Since E_{R_5} is also independent of $V|_{R_{[1,a],\{b_*\}}}$ by Lemma 3.5.4, the conclusion follows. $\square$

Proof of Lemma 3.5.8. For each $(s',t') \in R_{[1,2],[3,b]}$, let $\mathcal{E}^{(s',t')}_{tr}$ denote the following event:

$$\begin{cases} Hu = \lambda_0 u \text{ in } R_{[2,a-1],[2,b-1]} \\ u(s',t') = 1 \\ u(s,t) = 0 \text{ on } R_{[1,a],[1,2]} \\ |u(s,t)| \leq (a\bar{V})^{10C_1(t-t')} \text{ on } R_{[1,2],[1,t'-1]} \end{cases} \tag{3.5.34}$$

implies $|u| \geq (a\bar{V})^{-\frac{1}{2}\alpha_1 a}$ on a $\frac{1}{10^6}$ fraction of $R_{[1,a],[b-1,b]}$.

Claim 3.5.15. $\bigcap \left\{ \mathcal{E}^{(s',t')}_{tr} : (s',t') \in R_{[1,2],[3,b]} \right\} \subset \mathcal{E}_{tr}(R_{[1,a],[1,b]})$ *for* $\alpha_1 > 20C_1$.

Proof of the claim. Assume $\mathcal{E}^{(s,t)}_{tr}$ holds for each $(s,t) \in R_{[1,2],[3,b]}$, we prove that $\mathcal{E}_{tr}(R_{[1,a],[1,b]})$ also holds.

Given any $u : R_{[1,a],[1,b]} \to \mathbb{R}$ satisfying (3.5.15), let $(s',t') \in R_{[1,2],[3,b]}$ maximize

$(a\bar{V})^{-10C_1t'}|u(s',t')|$. Then $\|u\|_{\ell^\infty(R_{[1,2],[1,b]})}=1$ implies

$$|u(s',t')|\geq (a\bar{V})^{-10C_1b}.$$

Let $\tilde{u}=\frac{u}{u(s',t')}$, then $\tilde{u}$ satisfies (3.5.34) and thus $|\tilde{u}|\geq (a\bar{V})^{-\frac{1}{2}\alpha_1 a}$ on a $\frac{1}{10^6}$ fraction of $R_{[1,a],[b-1,b]}$. Hence $|u|\geq (a\bar{V})^{-(\frac{1}{2}\alpha_1+10C_1)a}$ on a $\frac{1}{10^6}$ fraction of $R_{[1,a],[b-1,b]}$. The claim follows from $\alpha_1>20C_1$. $\square$

Claim 3.5.16. *If $t'\in\{b-1,b\}$, then $\mathbb{P}\left[\mathcal{E}_{tr}^{(s',t')}\middle|\, V|_\Theta=V'\right]=1$.*

Proof. If $t'\in\{b-1,b\}$ and u satisfies (3.5.34), we claim that

$$\|u\|_{\ell^\infty(R_{[1,a],\{t\}})}\leq (a\bar{V})^{5C_1(t-t')} \tag{3.5.35}$$

for each $t=1,\cdots,t'-1$ and we prove (3.5.35) by induction. For $t=1,2$, this is true since $u=0$ on $R_{[1,a],[1,2]}$. Suppose our claim holds up to $t<t'-1$, using equation (3.5.9) on $R_{[1,a],\{t\}}$ and inductive hypothesis, we have

$$|u(s,t+1)+u(s+2,t+1)|\leq |16+\bar{V}|(a\bar{V})^{5C_1(t-t')} \tag{3.5.36}$$

for $s\in[1,a-2]$ with the same parity as $t+1$. By (3.5.34),

$$|u(s_0,t+1)|\leq (a\bar{V})^{10C_1(t+1-t')}$$

for $s_0 \in \{1,2\}$ with the same parity as $t+1$. Recursively using (3.5.36), we have

$$|u(s,t+1)| \leq s(16+\bar{V})(a\bar{V})^{5C_1(t-t')} \leq (a\bar{V})^{5C_1(t+1-t')}$$

for any $s \in [1,a]$ with the same parity as $t+1$. Thus induction proves (3.5.35) and

$$\|u\|_{\ell^\infty(R_{[1,a],\{t'-2,t'-1\}})} \leq (a\bar{V})^{-5C_1}.$$

Using equation (3.5.9) on $R_{[1,a],\{t'-1\}}$, we have

$$|u(s,t') + u(s+2,t')| \leq |16+\bar{V}|(a\bar{V})^{-5C_1} \leq (a\bar{V})^{-2} \tag{3.5.37}$$

for $s \in [1,a-2]$ with the same parity as t'. Since $u(s',t') = 1$, using (3.5.37) recursively, we have $|u(s,t')| \geq \frac{1}{2}$ for any $s \in [1,a]$ with the same parity as t'. Thus $\mathcal{E}_{tr}^{(s',t')}$ holds since $t' \in \{b-1,b\}$. □

Claim 3.5.17. *Suppose $(s',t') \in R_{[1,2],[3,b-2]}$. Let $s'' \in \{1,2\}$ and $b_0 \in \{b-1,b\}$ both have the same parity as $t'+1$. Then there exist*

1. *operators $A_1 : \ell^2(R_{[1,2],[1,t'-1]}) \to \ell^2(R_{[1,a],\{b_0\}})$ and $A_2 : \ell^2(R_{[1,2],[t'+1,b]}) \to \ell^2(R_{[1,a],\{b_0\}})$ which are independent of $V|_{R_{[1,a],\{t'\}}}$,*

2. *vector $v^* \in \ell^2(R_{[1,a],\{b_0\}})$ which is independent of $V|_{R_{[1,a],\{t'\}}}$,*

3. vector $\vec{V}_{t'} \in \mathbb{R}^{d_0}$ *with* $d_0 = |R_{[1,a],\{t'+1\}}| - 1$ *defined by*

$$\vec{V}_{t'} = \sum_{i=1}^{d_0} V(s'' + 2i - 1, t')e_i \tag{3.5.38}$$

where $\{e_i : 1 \leq i \leq d_0\}$ *is the standard basis of* $\mathbb{R}^{d_0}$*,*

4. $A_0 : \mathbb{R}^{d_0} \to \ell^2(R_{[1,a],\{t'+1\}})$ *defined as follows: for any* $(s, t'+1) \in R_{[1,a],\{t'+1\}}$ *and* $i \in \{1, \cdots, d_0\}$*,*

$$\langle \delta_{(s,t'+1)}, A_0 e_i \rangle = \begin{cases} (-1)^{\frac{s-s'-1}{2}} & \text{if } s > s'' \text{ and } 1 \leq i \leq \frac{s-s''}{2} \\ 0 & \text{otherwise,} \end{cases} \tag{3.5.39}$$

such that the following holds.

For any u *satisfying* (3.5.34)*, there exists* $u^* \in \ell^2(R_{[1,a],\{t'+1\}})$ *with*

$$\|u^*\| \leq (a\bar{V})^{-5}, \tag{3.5.40}$$

such that

$$u|_{R_{[1,a],\{b_0\}}} = M_{[1,a]}^{t'+1,b_0}(u^* + A_0(\vec{V}_{t'})) + A_1(u|_{R_{[1,2],[1,t'-1]}}) + A_2(u|_{R_{[1,2],[t'+1,b]}}) + v^*. \tag{3.5.41}$$

Proof. Assume u satisfies (3.5.34). Denote $R_1 = R_{[1,a],[1,b]}$. Let $u_0 = \delta_{(s',t')}$ on $\partial^w R_1$, $u_1 = u|_{R_{[1,2],[1,t'-1]}}$ and $u_2 = u|_{R_{[1,2],[t'+1,b]}}$. Then u is determined by u_1 and u_2 since we

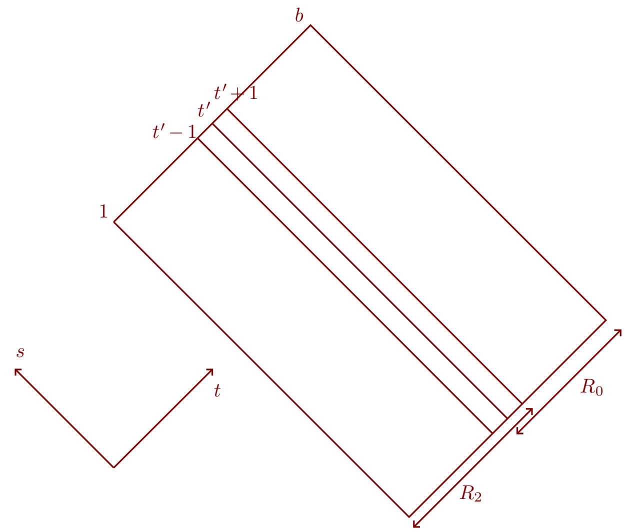

Figure 3.3: An illustration for tilted rectangles. Here, we have $R_0 = R_{[1,a],[t',b]}$ and $R_2 = R_{[1,a],[1,t'+1]}$ which are contained in $R_1 = R_{[1,a],[1,b]}$.

can decompose

$$u|_{\partial^w R_1} = u_1' + u_2' + u_0, \tag{3.5.42}$$

where $u_1' = I^{\partial^w R_1}_{R_{[1,2],[1,t'-1]}} u_1$ and $u_2' = I^{\partial^w R_1}_{R_{[1,2],[t'+1,b]}} u_2$. Thus

$$u = E_{R_1}(u_1') + E_{R_1}(u_2') + E_{R_1}(u_0)$$

and

$$u|_{R_{[1,a],\{b_0\}}} \tag{3.5.43}$$

$$= P^{R_1}_{R_{[1,a],\{b_0\}}} E_{R_1}(u_1') + P^{R_1}_{R_{[1,a],\{b_0\}}} E_{R_1}(u_2') + P^{R_1}_{R_{[1,a],\{b_0\}}} E_{R_1}(u_0). \tag{3.5.44}$$

We analyse each of three terms in (3.5.44) and will arrive at equation (3.5.41). More specifically, we will derive the correspondence between terms in (3.5.41) and (3.5.44) as follows:

1. $P^{R_1}_{R_{[1,a],\{b_0\}}} E_{R_1}(u_1') = A_1(u_1) + M^{t'+1,b_0}_{[1,a]}(u^*)$,
2. $P^{R_1}_{R_{[1,a],\{b_0\}}} E_{R_1}(u_2') = A_2(u_2)$,
3. $P^{R_1}_{R_{[1,a],\{b_0\}}} E_{R_1}(u_0) = v^* + M^{t'+1,b_0}_{[1,a]} A_0(\vec{V}_{t'})$.

Here, A_0, A_1, A_2, u^*, v^* and $\vec{V}_{t'}$ satisfy the properties in the conditions of this claim.

The first term in (3.5.44)**:** The strategy here is to apply Lemma 3.5.14. Note that

$$P^{R_1}_{R_{[1,a],\{b_0\}}} E_{R_1}(u_1') = P^{R_1}_{R_{[1,a],\{b_0\}}} E_{R_1} I^{\partial^w R_1}_{R_{[1,2],[1,t'-1]}}(u_1). \tag{3.5.45}$$

Denote $R_2 = R_{[1,a],[1,t'+1]}$ (see Figure 3.3), using Lemma 3.5.14 with $b_* = t'$, we can write

$$P^{R_1}_{R_{[1,a],\{b_0\}}} E_{R_1} I^{\partial^w R_1}_{R_{[1,2],[1,t'-1]}} = A_1 + M^{t'+1,b_0}_{[1,a]} P^{R_2}_{R_{[1,a],\{t'+1\}}} E_{R_2} I^{\partial^w R_2}_{R_{[1,2],[1,t'-1]}}. \tag{3.5.46}$$

Here, $A_1 : \ell^2(R_{[1,2],[1,t'-1]}) \to \ell^2(R_{[1,a],\{b_0\}})$ is a linear operator which is independent of $V|_{R_{[1,a],\{t'\}}}$. We claim that

$$\|P^{R_2}_{R_{[1,a],\{t'+1\}}} E_{R_2} I^{\partial^w R_2}_{R_{[1,2],[1,t'-1]}} u_1\|_2 \le (\bar{V}a)^{-5}. \tag{3.5.47}$$

To see this, let $v_1 = E_{R_2} I^{\partial^w R_2}_{R_{[1,2],[1,t'-1]}} u_1$. We inductively prove that

$$|v_1(s,t)| \le (a\bar{V})^{5C_1(t-t')} \tag{3.5.48}$$

for each $(s,t) \in R_{[1,a],[1,t'-1]}$. For $t = 1, 2$, this is true since $v_1 = 0$ on $R_{[1,a],[1,2]}$. Suppose (3.5.48) is true for t and $t+1$ and suppose $t+2 < t'$, using inductive hypothesis and (3.5.9) on $R_{[1,a],\{t+1\}}$, we have

$$|v_1(s,t+2) + v_1(s+2,t+2)| \le |16 + \bar{V}|(a\bar{V})^{5C_1(t+1-t')}$$

for each $s \in [1, a-2]$ with the same parity as t. Since by (3.5.34),

$$|v_1(s_1,t+2)| \le (a\bar{V})^{10C_1(t+2-t')}$$

for $s_1 \in \{1,2\}$ with the same parity as $t+2$. We recursively have

$$|v_1(s,t+2)| \le s(16+\bar{V})(a\bar{V})^{5C_1(t+1-t')} \le (a\bar{V})^{5C_1(t+2-t')} \tag{3.5.49}$$

for each $s \in [1,a]$ with the same parity as $t+2$. Thus by induction we have $|v_1(s,t)| \le (a\bar{V})^{-5C_1}$ for $(s,t) \in R_{[1,a],[1,t'-1]}$. Finally, since $v_1 = 0$ on $R_{[1,2],[t',t'+1]}$, we have

$$\|v_1\|_{\ell^\infty(\partial^w R_{[1,a],[t'-2,t'+1]})} \le (a\bar{V})^{-5C_1}.$$

Thus (3.5.47) follows from Lemma 3.5.6 and $C_1 \geq 10$.

In conclusion, by (3.5.45), (3.5.46) and (3.5.47),

$$P^{R_1}_{R_{[1,a],\{b_0\}}} E_{R_1}(u_1') = A_1(u_1) + M^{t'+1,b_0}_{[1,a]}(u^*), \tag{3.5.50}$$

where $u^* = P^{R_2}_{R_{[1,a],\{t'+1\}}} E_{R_2} I^{\partial^w R_2}_{R_{[1,2],[1,t'-1]}}(u_1)$ with

$$\|u^*\|_2 \leq (a\bar{V})^{-5}. \tag{3.5.51}$$

The second term in (3.5.44)**:** We have

$$P^{R_1}_{R_{[1,a],\{b_0\}}} E_{R_1}(u_2') = A_2(u_2) \tag{3.5.52}$$

where $A_2 = P^{R_1}_{R_{[1,a],\{b_0\}}} E_{R_1} I^{\partial^w R_1}_{R_{[1,2],[t'+1,b]}}$. We claim that A_2 is independent of $V|_{R_{[1,a],\{t'\}}}$. To see this, let $v_2 = E_{R_1} I^{\partial^w R_1}_{R_{[1,2],[t'+1,b]}}(u_2)$. Since $I^{\partial^w R_1}_{R_{[1,2],[t'+1,b]}}(u_2) = 0$ on $\partial^w R_{[1,a],[1,t']}$, we have $v_2 \equiv 0$ on $R_{[1,a],[1,t']}$ by Lemma 3.5.4. Using equation (3.5.9) for v_2 on $R_{[1,a],\{t'\}}$, we get

$$v_2(s, t'+1) = (-1)^{\frac{s-s''}{2}} u_2(s'', t'+1) \tag{3.5.53}$$

for $s \in [1,a]$ with the same parity as s''. By (3.5.53) and $v_2|_{R_{[1,a],\{t'\}}} = 0$ and Lemma 3.5.4, the linear transform

$$I^* : u_2 \mapsto v_2|_{\partial^w R_{[1,a],[t',b]}}$$

is independent of $V|_{R_{[1,a],\{t'\}}}$. Note that

$$\begin{aligned} v_2|_{R_{[1,a],\{b_0\}}} =& P^{R_{[1,a],[t',b]}}_{R_{[1,a],\{b_0\}}} E_{[1,a],[t',b]}(v_2|_{\partial^w R_{[1,a],[t',b]}}) \\ =& P^{R_{[1,a],[t',b]}}_{R_{[1,a],\{b_0\}}} E_{[1,a],[t',b]} I^*(u_2). \end{aligned}$$

By Lemma 3.5.4, $E_{[1,a],[t',b]}$ is independent of $V|_{R_{[1,a],\{t'\}}}$. Since

$$A_2 = P^{R_{[1,a],[t',b]}}_{R_{[1,a],\{b_0\}}} E_{[1,a],[t',b]} I^*,$$

thus A_2 is independent of $V|_{R_{[1,a],\{t'\}}}$.

The third term in (3.5.44)**:** Let $v_0 = E_{R_1}(u_0)$. The strategy here is to express $v_0|_{R_{[1,a],[t',t'+1]}}$ as a function of $\vec{V}_{t'}$. We have

$$v_0|_{R_{[1,a],[1,t'-1]}} = E_{[1,a],[1,t'-1]}(u_0|_{\partial^w R_{[1,a],[1,t'-1]}}) = 0. \tag{3.5.54}$$

Using equation (3.5.8) on the segment $R_{[2,a-1],\{t'-1\}}$, by (3.5.54), we have

$$v_0(s,t') + v_0(s+2,t') = 0$$

for each $(s,t') \in R_{[1,a-2],\{t'\}}$. Thus recursively from $v_0(s',t') = 1$, we have

$$v_0(s,t') = (-1)^{\frac{s-s'}{2}} \tag{3.5.55}$$

for $(s,t') \in R_{[1,a],\{t'\}}$. Using equation (3.5.8) on the segment $R_{[2,a-1],\{t'\}}$, we have

$$v_0(s+1,t'+1)+v_0(s-1,t'+1)=(V(s,t')-\lambda_0+4)v_0(s,t')$$

for each $(s,t') \in R_{[2,a-1],\{t'\}}$. Recall $s'' \in \{1,2\}$ has the same parity as $t'+1$. Then recursively from $v_0(s'',t'+1)=0$ and (3.5.55) we have

$$v_0(s_1,t'+1) \tag{3.5.56}$$

$$=(-1)^{\frac{s_1-s'-1}{2}} \sum_{\substack{s''<s<s_1 \\ s \not\equiv s'' \bmod 2}} (V(s,t')-\lambda_0+4) \tag{3.5.57}$$

$$=(-1)^{\frac{s_1-s'-1}{2}} \frac{(4-\lambda_0)(s_1-s'')}{2} + (-1)^{\frac{s_1-s'-1}{2}} \sum_{i=1}^{\frac{s_1-s''}{2}} V(s''+2i-1,t') \tag{3.5.58}$$

for any $s_1 \in (s'',a]$ with the same parity as s''. By (3.5.38) and (3.5.39), we can rewrite (3.5.58) as

$$v_0|_{R_{[1,a],\{t'+1\}}} = v_* + A_0(\vec{V}_{t'}), \tag{3.5.59}$$

where $v_* \in \ell^2(R_{[1,a],\{t'+1\}})$ satisfies

$$v_*(s,t'+1)=(-1)^{\frac{s-s'-1}{2}} \frac{(4-\lambda_0)(s-s'')}{2},$$

for $s \in [1,a]$ with the same parity as s''. Hence we have

$$v_0|_{R_{[1,a],[t',t'+1]}} = v_{**} + I_{R_{[1,a],\{t'+1\}}}^{R_{[1,a],[t',t'+1]}} A_0(\vec{V}_{t'}), \tag{3.5.60}$$

where $v_{**}|_{R_{[1,a],\{t'+1\}}} = v_*$ and $v_{**}(s,t') = (-1)^{\frac{s-s'}{2}}$ for $s \in [1,a]$ with the same parity as s'.

Denote $R_0 = R_{[1,a],[t',b_0]}$ (see Figure 3.3), then

$$v_0|_{R_{[1,a],\{b_0\}}} = P^{R_0}_{R_{[1,a],\{b_0\}}} E_{R_0} I^{\partial^w R_0}_{R_{[1,a],[t',t'+1]}} v_0|_{R_{[1,a],[t',t'+1]}}.$$

Together with (3.5.60), we have

$$\begin{aligned} &v_0|_{R_{[1,a],\{b_0\}}} \\ &= P^{R_0}_{R_{[1,a],\{b_0\}}} E_{R_0} I^{\partial^w R_0}_{R_{[1,a],[t',t'+1]}}(v_{**}) + P^{R_0}_{R_{[1,a],\{b_0\}}} E_{R_0} I^{\partial^w R_0}_{R_{[1,a],\{t'+1\}}} A_0(\vec{V}_{t'}) \\ &= P^{R_0}_{R_{[1,a],\{b_0\}}} E_{R_0} I^{\partial^w R_0}_{R_{[1,a],[t',t'+1]}}(v_{**}) + M^{t'+1,b_0}_{[1,a]} A_0(\vec{V}_{t'}) \\ &= v^* + M^{t'+1,b_0}_{[1,a]} A_0(\vec{V}_{t'}). \end{aligned} \tag{3.5.61}$$

Here, we used the Definition 3.5.10 of $M^{t'+1,b_0}_{[1,a]}$, and in the last equation we denoted

$$v^* = P^{R_0}_{R_{[1,a],\{b_0\}}} E_{R_0} I^{\partial^w R_0}_{R_{[1,a],[t',t'+1]}}(v_{**})$$

which is independent of $V|_{R_{[1,a],\{t'\}}}$ by Lemma 3.5.4. In conclusion,

$$P^{R_1}_{R_{[1,a],\{b_0\}}} E_{R_1}(u_0) = v_0|_{R_{[1,a],\{b_0\}}} = v^* + M^{t'+1,b_0}_{[1,a]} A_0(\vec{V}_{t'}). \tag{3.5.62}$$

Finally plug (3.5.50),(3.5.52) and (3.5.62) into equation (3.5.44), we have

$$u|_{R_{[1,a],\{b_0\}}} = M_{[1,a]}^{t'+1,b_0}(u^* + A_0(\vec{V}_{t'})) + A_1(u_1) + A_2(u_2) + v^* \tag{3.5.63}$$

which is equation (3.5.41) and our claim follows. $\square$

Now let $c_4 < \frac{1}{10^7}$. Fix $(s', t') \in R_{[1,2],[3,b-2]}$. Since Θ is $(c_4, -)$-sparse in $R_{[1,a],[1,b]}$,

$$|\Theta \cap R_{[1,a],\{t'\}}| \leq c_4 a \leq \frac{a}{10^7}. \tag{3.5.64}$$

Pick $b_0 \in \{b-1, b\}$ and $s'' \in \{1, 2\}$ with the same parity as $t'+1$. Denote

$$\Theta_* = \{(s, b_0) : (s-1, t') \in \Theta\} \cap R_{[4,a-1],\{b_0\}}. \tag{3.5.65}$$

For any $S \subset R_{[4,a-1],\{b_0\}}$, let $\mathcal{E}_S^{(s',t')}$ denote the event:

$$\text{(3.5.34) implies } \|u\|_{\ell^2(S)} \geq (a\bar{V})^{-\frac{1}{3}\alpha_1 a}. \tag{3.5.66}$$

Claim 3.5.18. *For any $a \geq 10^7$, we have*

$$\bigcap \left\{ \mathcal{E}_S^{(s',t')} : S \subset R_{[4,a-1],\{b_0\}} \setminus \Theta_*, |R_{[1,a],\{b_0\}} \setminus S| = \lfloor a/10^5 \rfloor \right\} \subset \mathcal{E}_{tr}^{(s',t')}.$$

Proof of the claim. Assume the event $\mathcal{E}_{tr}^{(s',t')}$ does not hold. Then we can find $u \in$

$\ell^2(R_{[1,a],[1,b]})$ satisfying (3.5.34) but

$$|\{(s,t) \in R_{[1,a],\{b-1,b\}} : |u(s,t)| \geq (a\bar{V})^{-\alpha_1 a}\}| \leq 10^{-6} a.$$

Hence by (3.5.64),

$$\begin{aligned}
&|\Theta_* \cup \{(s,t) \in R_{[1,a],\{b_0\}} : |u(s,t)| \geq (a\bar{V})^{-\alpha_1 a}\}| \\
&\leq 10^{-7} a + 10^{-6} a \\
&\leq 10^{-5} a - 5.
\end{aligned}$$

Thus there is $S \subset R_{[4,a-1],\{b_0\}} \setminus \Theta_*$ such that $|R_{[1,a],\{b_0\}} \setminus S| = \lfloor a/10^5 \rfloor$ and $\|u\|_{\ell^\infty(S)} \leq (a\bar{V})^{-\alpha_1 a}$. This implies

$$\|u\|_{\ell^2(S)} \leq a(a\bar{V})^{-\alpha_1 a} < (a\bar{V})^{-\frac{1}{3}\alpha_1 a}.$$

Hence $\mathcal{E}_S^{(s',t')}$ does not hold. $\square$

Claim 3.5.19. *For large enough a, $\mathbb{P}\left[\mathcal{E}_S^{(s',t')} \middle| V|_\Theta = V'\right] \geq 1 - \exp(-a/50)$ holds for any subset $S \subset R_{[4,a-1],\{b_0\}} \setminus \Theta_*$ such that $|R_{[1,a],\{b_0\}} \setminus S| = \lfloor a/10^5 \rfloor$.*

Proof of the claim. Denote $R_1 = R_{[1,a],[1,b]}$. It is sufficient to prove that

$$\mathbb{P}\left[\mathcal{E}_S^{(s',t')} \middle| V|_{\Theta \cup (R_1 \setminus R_{[1,a],\{t'\}})}\right] \geq 1 - \exp(-a/50) \tag{3.5.67}$$

for any $S \subset R_{[4,a-1],\{b_0\}} \setminus \Theta_*$ such that $|R_{[1,a],\{b_0\}} \setminus S| \leq a/50$. We pick an arbitrary $S_0 \subset R_{[4,a-1],\{b_0\}} \setminus \Theta_*$ with

$$|R_{[1,a],\{b_0\}} \setminus S_0| \leq a/50. \tag{3.5.68}$$

Let

$$S_0' = \{(s, t'+1) : (s, b_0) \in S_0\} \subset R_{[4,a-1],\{t'+1\}}$$

and

$$M_{S_0} = P_{S_0}^{R_{[1,a],\{b_0\}}} M_{[1,a]}^{t'+1,b_0} I_{S_0'}^{R_{[1,a],\{t'+1\}}}.$$

By Corollary 3.5.13,

$$\|M_{S_0}^{-1}\| \leq a(2(b_0 - t' + 1)\bar{V})^{2C_1 a} \leq (a\bar{V})^{3C_1 a}. \tag{3.5.69}$$

Let $d_0 = |R_{[1,a],\{t'+1\}}| - 1$ and $\{e_i\}_{i=1}^{d_0}$ be the standard basis in $\mathbb{R}^{d_0}$. For any $\mathcal{S} \subset \{1, \cdots, d_0\}$, let $P_{\mathcal{S}}$ be the orthogonal projection onto the span of $\{e_i : i \in \mathcal{S}\}$ and $P_{\mathcal{S}}^\dagger$ be its adjoint. Denote

$$\mathcal{S}_0 = \{(s - s'')/2 : (s, t'+1) \in S_0'\} \subset \{1, \cdots, d_0\}, \tag{3.5.70}$$

and let

$$A_{S_0} = P_{S_0'}^{R_{[1,a],\{t'+1\}}} A_0 P_{\mathcal{S}_0}^\dagger$$

where A_0 is defined in (3.5.39). By (3.5.39), A_{S_0} can be regarded as a triangular matrix and by simple calculations, we have

$$\|A_{S_0}^{-1}\| \leq a. \tag{3.5.71}$$

Denote $A' = I^{R_{[1,a],\{t'+1\}}}_{R_{[1,a],\{t'+1\}}\setminus S_0'} P^{R_{[1,a],\{t'+1\}}}_{R_{[1,a],\{t'+1\}}\setminus S_0'}$ and $\mathcal{S}_0^c = \{1, \cdots, d_0\} \setminus \mathcal{S}_0$. Then we can decompose the identity operator on $\ell^2(R_{[1,a],\{t'+1\}})$ by $I^{(1)} = A' + I^{R_{[1,a],\{t'+1\}}}_{S_0'} P^{R_{[1,a],\{t'+1\}}}_{S_0'}$, and the identity operator on $\mathbb{R}^{d_0}$ by $I^{(2)} = P^{\dagger}_{\mathcal{S}_0} P_{\mathcal{S}_0} + P^{\dagger}_{\mathcal{S}_0^c} P_{\mathcal{S}_0^c}$.

Suppose u satisfies (3.5.34). By Claim 3.5.17, there exists $u^* \in \ell^2(R_{[1,a],\{t'+1\}})$ with $\|u^*\| \leq (a\bar{V})^{-5}$ and we have

$$u|_{R_{[1,a],\{b_0\}}} = M_{[1,a]}^{t'+1,b_0}(u^* + A_0(\vec{V}_{t'})) + A_1(u|_{R_{[1,2],[1,t'-1]}}) + A_2(u|_{R_{[1,2],[t'+1,b]}}) + v^* \tag{3.5.72}$$

such that A_0, A_1, A_2, v^* are all independent of $V|_{R_{[1,a],\{t'\}}}$ and vector $\vec{V}_{t'} \in \mathbb{R}^{d_0}$ is $V|_{R_{[1,a],\{t'\}}}$-measurable. By the argument above, we can expand the first term in (3.5.72) (or (3.5.41)) as follows:

$$\begin{aligned}
&M_{[1,a]}^{t'+1,b_0}(u^* + A_0(\vec{V}_{t'}))\\
=&M_{[1,a]}^{t'+1,b_0} I^{(1)}(u^* + A_0(\vec{V}_{t'}))\\
=&M_{[1,a]}^{t'+1,b_0}\left(A' + I^{R_{[1,a],\{t'+1\}}}_{S_0'} P^{R_{[1,a],\{t'+1\}}}_{S_0'}\right)(u^* + A_0(\vec{V}_{t'}))\\
=&M_{[1,a]}^{t'+1,b_0} A'(u^* + A_0(\vec{V}_{t'})) + M_{[1,a]}^{t'+1,b_0} I^{R_{[1,a],\{t'+1\}}}_{S_0'} P^{R_{[1,a],\{t'+1\}}}_{S_0'}(u^*)\\
+&M_{[1,a]}^{t'+1,b_0} I^{R_{[1,a],\{t'+1\}}}_{S_0'} P^{R_{[1,a],\{t'+1\}}}_{S_0'} A_0(\vec{V}_{t'}),
\end{aligned} \tag{3.5.73}$$

and the last term in the last equation of (3.5.73) can be further expanded:

$$\begin{aligned}
&M_{[1,a]}^{t'+1,b_0} I_{S_0'}^{R_{[1,a],\{t'+1\}}} P_{S_0'}^{R_{[1,a],\{t'+1\}}} A_0(\vec{V}_{t'}) \\
=&M_{[1,a]}^{t'+1,b_0} I_{S_0'}^{R_{[1,a],\{t'+1\}}} P_{S_0'}^{R_{[1,a],\{t'+1\}}} A_0 I^{(2)}(\vec{V}_{t'}) \\
=&M_{[1,a]}^{t'+1,b_0} I_{S_0'}^{R_{[1,a],\{t'+1\}}} P_{S_0'}^{R_{[1,a],\{t'+1\}}} A_0 \left(P_{\mathcal{S}_0}^{\dagger} P_{\mathcal{S}_0} + P_{\mathcal{S}_0^c}^{\dagger} P_{\mathcal{S}_0^c}\right)(\vec{V}_{t'}) \\
=&M_{[1,a]}^{t'+1,b_0} I_{S_0'}^{R_{[1,a],\{t'+1\}}} A_{S_0} P_{\mathcal{S}_0}(\vec{V}_{t'}) \\
+&M_{[1,a]}^{t'+1,b_0} I_{S_0'}^{R_{[1,a],\{t'+1\}}} P_{S_0'}^{R_{[1,a],\{t'+1\}}} A_0 P_{\mathcal{S}_0^c}^{\dagger} P_{\mathcal{S}_0^c}(\vec{V}_{t'}).
\end{aligned} \tag{3.5.74}$$

Plug (3.5.73) and (3.5.74) into (3.5.41), after projecting onto S_0, we have

$$\begin{aligned}
&u|_{S_0} \\
=&M_{S_0} A_{S_0} (A_{S_0}^{-1} P_{S_0'}^{R_{[1,a],\{t'+1\}}}(u^*) + P_{\mathcal{S}_0}(\vec{V}_{t'})) \\
+&P_{S_0}^{R_{[1,a],\{b_0\}}} M_{[1,a]}^{t'+1,b_0} A'(u^* + A_0(\vec{V}_{t'})) + M_{S_0} P_{S_0'}^{R_{[1,a],\{t'+1\}}} A_0 P_{\mathcal{S}_0^c}^{\dagger} P_{\mathcal{S}_0^c}(\vec{V}_{t'}) \\
+&P_{S_0}^{R_{[1,a],\{b_0\}}} A_1(u|_{R_{[1,2],[1,t'-1]}}) + P_{S_0}^{R_{[1,a],\{b_0\}}} A_2(u|_{R_{[1,2],[t'+1,b]}}) \\
+&v^*|_{S_0}.
\end{aligned} \tag{3.5.75}$$

Let $\Gamma \subset \ell^2(S_0)$ be the direct sum of the ranges of the following four operators appeared in the third and fourth lines of (3.5.75):

$$P_{S_0}^{R_{[1,a],\{b_0\}}} M_{[1,a]}^{t'+1,b_0} A',\ M_{S_0} P_{S_0'}^{R_{[1,a],\{t'+1\}}} A_0 P_{\mathcal{S}_0^c}^{\dagger} P_{\mathcal{S}_0^c},\ P_{S_0}^{R_{[1,a],\{b_0\}}} A_1,\ P_{S_0}^{R_{[1,a],\{b_0\}}} A_2.$$

Let us denote

$$\Gamma - v^*|_{S_0} = \{-v^*|_{S_0} + x : x \in \Gamma\} \subset \ell^2(S_0)$$

and define the event $\mathcal{E}_{dist}$ as

$$\operatorname{dist}(P_{\mathcal{S}_0}(\vec{V}_{t'}), (M_{S_0}A_{S_0})^{-1}(\Gamma - v^*|_{S_0})) \geq \frac{\bar{V}}{4}a^{-1} \tag{3.5.76}$$

where dist is the Euclidean distance. We claim that $\mathcal{E}_{dist} \subset \mathcal{E}_{S_0}^{(s',t')}$ by choosing $\alpha_1 > 15C_1$ (recall definition (3.5.66) of $\mathcal{E}_S^{(s',t')}$). To see this, assume $\mathcal{E}_{dist}$ holds. (3.5.75) implies

$$\begin{aligned}
&\|u\|_{\ell^2(S_0)} \\
&\geq \operatorname{dist}\left(M_{S_0}A_{S_0}(A_{S_0}^{-1}P_{S_0'}^{R_{[1,a],\{t'+1\}}}(u^*) + P_{\mathcal{S}_0}(\vec{V}_{t'})), \Gamma - v^*|_{S_0}\right) \\
&\geq \|(M_{S_0}A_{S_0})^{-1}\|^{-1} \operatorname{dist}\left(A_{S_0}^{-1}P_{S_0'}^{R_{[1,a],\{t'+1\}}}(u^*) + P_{\mathcal{S}_0}(\vec{V}_{t'}), (M_{S_0}A_{S_0})^{-1}(\Gamma - v^*|_{S_0})\right) \\
&\geq (a\bar{V})^{-4C_1 a} \operatorname{dist}\left(A_{S_0}^{-1}P_{S_0'}^{R_{[1,a],\{t'+1\}}}(u^*) + P_{\mathcal{S}_0}(\vec{V}_{t'}), (M_{S_0}A_{S_0})^{-1}(\Gamma - v^*|_{S_0})\right).
\end{aligned} \tag{3.5.77}$$

Here, we used (3.5.69) and (3.5.71). By (3.5.40) and (3.5.71), we have

$$\|A_{S_0}^{-1}P_{S_0'}^{R_{[1,a],\{t'+1\}}}(u^*)\| \leq \|A_{S_0}^{-1}\|\|u^*\| \leq a(a\bar{V})^{-5} \leq a^{-4}.$$

Thus (3.5.77) further implies

$$\|u\|_{\ell^2(S_0)} \geq (a\bar{V})^{-4C_1 a}\Big(\operatorname{dist}\left(P_{\mathcal{S}_0}(\vec{V}_{t'}), (M_{S_0}A_{S_0})^{-1}(\Gamma - v^*|_{S_0})\right) - a^{-4}\Big). \tag{3.5.78}$$

By letting $\alpha_1 > 15C_1$, $\mathcal{E}_{dist}$ (or (3.5.76)) implies

$$\|u\|_{\ell^2(S_0)} \geq (a\bar{V})^{-4C_1 a}\left(\frac{\bar{V}}{4}a^{-1} - a^{-4}\right) \geq (a\bar{V})^{-5C_1 a} \geq (a\bar{V})^{-\frac{1}{3}\alpha_1 a}.$$

This proves our claim that $\mathcal{E}_{dist} \subset \mathcal{E}_{S_0}^{(s',t')}$. Thus in order to prove (3.5.67) with $S = S_0$, it suffices to prove

$$\mathbb{P}\left[\mathcal{E}_{dist}\middle|\, V|_{\Theta\cup(R_1\setminus R_{[1,a],\{t'\}})}\right] \geq 1 - \exp(-a/50). \tag{3.5.79}$$

To see this, we first prove an upper bound for the dimension of Γ. The ranks of operators $P_{S_0}^{R_{[1,a],\{b_0\}}}A_1$ and $P_{S_0}^{R_{[1,a],\{b_0\}}}A_2$ are less than b since the dimensions of their domains are less than b. On the other hand, the ranks of operators $M_{S_0}P_{S_0'}^{R_{[1,a],\{t'+1\}}}A_0P_{\mathcal{S}_0^c}^{\dagger}P_{\mathcal{S}_0^c}$ and $P_{S_0}^{R_{[1,a],\{b_0\}}}M_{[1,a]}^{t'+1,b_0}A'$ are at most $a/50$ since we have $|\mathcal{S}_0^c|, |R_{[1,a],\{t'+1\}} \setminus S_0'| \leq a/50$. Since $b \leq a/10$, the dimension of Γ is at most $b + b + a/50 + a/50 \leq \frac{2}{5}a$.

Together with Claim 3.5.17, these imply that $(M_{S_0}A_{S_0})^{-1}(\Gamma - v^*|_{S_0}) \subset \mathbb{R}^{\mathcal{S}_0}$ is an affine subspace with dimension at most $\frac{2}{5}a$ and is independent of $V|_{R_{[1,a],\{t'\}}}$. On the other hand, since $S_0 \subset R_{[4,a-1],\{b_0\}} \setminus \Theta_*$, by definition (3.5.65) of Θ_* and equations (3.5.38) and (3.5.70), $P_{\mathcal{S}_0}(\vec{V}_{t'})$ is independent of $V|_\Theta$. Moreover, by (3.5.68), we have

$$a \geq |\mathcal{S}_0| \geq d_0 - a/50 \geq \frac{2}{5}a + \frac{1}{20}a.$$

Thus by Lemma 3.1.6, conditioning on $V|_{\Theta\cup(R_1\setminus R_{[1,a],\{t'\}})}$, with probability no less than

$1 - 2^{-\frac{1}{20}a+1} \geq 1 - \exp(-a/50)$, we have

$$\mathrm{dist}(P_{\mathcal{S}_0}(\vec{V}_{t'}), (M_{S_0}A_{S_0})^{-1}(\Gamma - v^*|_{S_0})) \geq \frac{\bar{V}}{4}a^{-1}$$

which is (3.5.76). Hence (3.5.79) holds and Claim 3.5.19 follows. $\square$

Now, by Claim 3.5.18 and Claim 3.5.19, and letting c_4 be small enough,

$$\begin{aligned}
&\mathbb{P}\left[\left(\mathcal{E}_{tr}^{(s',t')}\right)^c \;\middle|\; V|_\Theta\right] \\
&\leq \sum_{\substack{S \subset R_{[4,a-1],\{b_0\}} \setminus \Theta_* \\ |R_{[1,a],\{b_0\}} \setminus S| = \lfloor a/10^5 \rfloor}} \mathbb{P}\left[\left(\mathcal{E}_{S}^{(s',t')}\right)^c \;\middle|\; V|_\Theta\right] \\
&\leq \sum_{\substack{S \subset R_{[4,a-1],\{b_0\}} \setminus \Theta_* \\ |R_{[1,a],\{b_0\}} \setminus S| = \lfloor a/10^5 \rfloor}} \exp(-a/50) \\
&\leq \binom{a}{\lfloor a/10^5 \rfloor} \exp(-a/50) \\
&\leq \exp(-2c_4 a)
\end{aligned}$$

for any large enough a.

Finally, by Claim 3.5.15 and Claim 3.5.16,

$$
\begin{aligned}
&\mathbb{P}\left[\mathcal{E}_{tr}(R_{[1,a],[1,b]}) \middle| V|_\Theta = V'\right] \\
&\geq 1 - \sum_{(s',t') \in R_{[1,2],[3,b]}} \mathbb{P}\left[\left(\mathcal{E}_{tr}^{(s',t')}\right)^c \middle| V|_\Theta = V'\right] \\
&\geq 1 - b\exp(-2c_4 a) \\
&\geq 1 - \exp(-c_4 a).
\end{aligned}
$$

Our conclusion follows. □

Lemma 3.5.20. *There are constants $\alpha_2 > 1 > c_5 > 0$ such that, if*

1. *integers $a > b > \alpha_2$ with $10b \leq a \leq 60b$,*
2. *$\lambda_0 \in [0, 8]$, $\bar{V} \geq 2$,*
3. *$\Theta \subset \mathbb{Z}^2$ is $(c_5, -)$-sparse in $R_{[1,a],[1,b]}$,*
4. *$V' : \Theta \to \{0, \bar{V}\}$,*
5. *$\mathcal{E}_{ni}(R_{[1,a],[1,b]})$ denotes the event that,*

$$
\begin{cases}
|\lambda - \lambda_0| \leq (a\bar{V})^{-\alpha_2 a} \\
Hu = \lambda u \text{ in } R_{[2,a-1],[2,b-1]} \\
|u| \leq 1 \text{ on } R_{[1,a],[1,2]} \\
|u| \leq 1 \text{ on a } 1 - 10^{-7} \text{ fraction of } R_{[1,a],[b-1,b]}
\end{cases}
\tag{3.5.80}
$$

implies $|u| \leq (a\bar{V})^{\alpha_2 a}$ in $R_{[1,a],[1,b]}$, *then* $\mathbb{P}\left[\mathcal{E}_{ni}(R_{[1,a],[1,b]})\middle|\, V|_{\Theta} = V'\right] \geq 1 - \exp(-c_5 a)$.

Proof. Denote $R_1 = R_{[1,a],[1,b]}$. Set $c_5 = c_4$ where c_4 is the constant in Lemma 3.5.8 and α_2 to be determined. We prove that $\mathcal{E}_{tr}(R_1) \subset \mathcal{E}_{ni}(R_1)$ where $\mathcal{E}_{tr}(R_1)$ is defined in Lemma 3.5.8. Suppose event $\mathcal{E}_{tr}(R_1)$ holds and u satisfies (3.5.80). By Lemma 3.5.4, there is $u_1 : R_1 \to \mathbb{R}$ such that

$$\begin{cases} Hu_1 = \lambda u_1 \text{ in } R_{[2,a-1],[2,b-1]} \\ u_1 = u \text{ on } R_{[1,a],[1,2]} \\ u_1 = 0 \text{ on } R_{[1,2],[3,b]}. \end{cases} \tag{3.5.81}$$

By Lemma 3.5.6, $\|u_1\|_{\ell^\infty(R_1)} \leq (a\bar{V})^{C_1 b}$ since $\|u_1\|_{\ell^\infty(\partial^w R_1)} \leq 1$. Let $u_2 = u - u_1$, then

$$|u_2| \leq 1 + (a\bar{V})^{C_1 b} \tag{3.5.82}$$

on a $1 - 10^{-7}$ fraction of $R_{[1,a],[b-1,b]}$. Define $u_3 : R_1 \to \mathbb{R}$ as follows:

$$\begin{cases} Hu_3 = \lambda_0 u_3 \text{ in } R_{[2,a-1],[2,b-1]} \\ u_3 = 0 \text{ on } R_{[1,a],[1,2]} \\ u_3 = u_2 \text{ on } R_{[1,2],[3,b]}. \end{cases} \tag{3.5.83}$$

By Lemma 3.5.7,

$$\begin{aligned}
&\|u_3 - u_2\|_{\ell^\infty(R_1)} \\
&\leq (a\bar{V})^{C_2 b}\|u_3\|_{\ell^\infty(\partial^w R_1)}|\lambda - \lambda_0| \\
&\leq (a\bar{V})^{C_2 b - \alpha_2 a}\|u_3\|_{\ell^\infty(\partial^w R_1)} \\
&\leq (a\bar{V})^{-2\alpha_1 a}\|u_3\|_{\ell^\infty(R_{[1,2],[3,b]})},
\end{aligned} \tag{3.5.84}$$

as long as $\alpha_2 > 2\alpha_1 + C_2$. By the definition of $\mathcal{E}_{tr}(R_1)$,

$$|u_3| \geq (a\bar{V})^{-\alpha_1 a}\|u_3\|_{\ell^\infty(R_{[1,2],[3,b]})}$$

on a 10^{-6} fraction of $R_{[1,a],[b-1,b]}$. Thus by (3.5.84),

$$|u_2| \geq \left((a\bar{V})^{-\alpha_1 a} - (a\bar{V})^{-2\alpha_1 a}\right)\|u_3\|_{\ell^\infty(R_{[1,2],[3,b]})} \geq (a\bar{V})^{-2\alpha_1 a}\|u_3\|_{\ell^\infty(R_{[1,2],[3,b]})}$$

on a 10^{-6} fraction of $R_{[1,a],[b-1,b]}$. By (3.5.82),

$$(a\bar{V})^{-2\alpha_1 a}\|u_3\|_{\ell^\infty(R_{[1,2],[3,b]})} \leq 1 + (a\bar{V})^{C_1 b}$$

and thus

$$\|u_2\|_{\ell^\infty(R_{[1,2],[3,b]})} = \|u_3\|_{\ell^\infty(R_{[1,2],[3,b]})} \leq (a\bar{V})^{2\alpha_1 a} + (a\bar{V})^{C_1 b + 2\alpha_1 a}.$$

Since $u_2 = 0$ on $R_{[1,a],[1,2]}$, by Lemma 3.5.6, we have $\|u_2\|_{\ell^\infty(R_1)} \leq 2(a\bar{V})^{2C_1 b + 2\alpha_1 a}$.

Finally,

$$\begin{aligned} \|u\|_{\ell^\infty(R_1)} &\leq \|u_1\|_{\ell^\infty(R_1)} + \|u_2\|_{\ell^\infty(R_1)} \\ &\leq (a\bar{V})^{C_1 b} + 2(a\bar{V})^{2C_1 b + 2\alpha_1 a} \\ &\leq (a\bar{V})^{\alpha_2 a} \end{aligned}$$

as long as $\alpha_2 > 2\alpha_1 + 3C_1$. Thus $\mathcal{E}_{tr}(R_1) \subset \mathcal{E}_{ni}(R_1)$ and our conclusion follows from Lemma 3.5.8. □

3.5.4 Growth lemma

Definition 3.5.21. Given a tilted square $R_{[a_1,a_2],[b_1,b_2]}$ and integer $k \in \mathbb{Z}^+$, we define $kR_{[a_1,a_2],[b_1,b_2]}$ to be $R_{[a_3,a_4],[b_3,b_4]}$ where $a_3 = \left\lceil \frac{(k+1)a_1-(k-1)a_2}{2} \right\rceil$, $a_4 = \left\lfloor \frac{(k+1)a_2-(k-1)a_1}{2} \right\rfloor$, $b_3 = \left\lceil \frac{(k+1)b_1-(k-1)b_2}{2} \right\rceil$ and $b_4 = \left\lfloor \frac{(k+1)b_2-(k-1)b_1}{2} \right\rfloor$.

For a tilted square $\tilde{Q}$, the following lemma allows us to estimate $\|u\|_{\ell^\infty(2\tilde{Q})}$ from an upper bound of $\|u\|_{\ell^\infty(\tilde{Q})}$, provided the portion of points with $|u| > 1$ is small enough in $4\tilde{Q}$. The proof is similar to that of [DS20, Lemma 3.18] and [BLMS17, Lemma 3.6].

Lemma 3.5.22. *For every small $\varepsilon > 0$, there is a large $\alpha > 1$ such that, if*

1. *$\tilde{Q}$ tilted square with $\ell(\tilde{Q}) > \alpha$,*

2. *$\Theta \subset \mathbb{Z}^2$ is ε-sparse in $2\tilde{Q}$,*

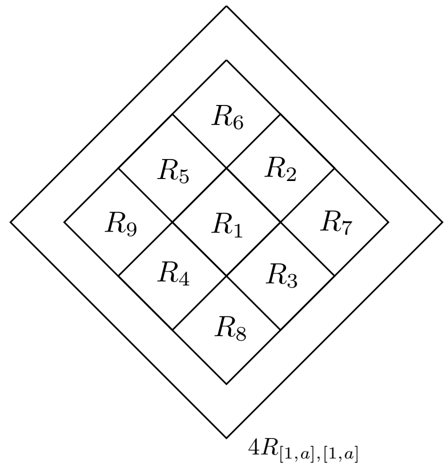

Figure 3.4: An illustration of covering argument.

3. $\lambda_0 \in [0, 8]$ *and* $\bar{V} \geq 2$,

4. $V' : \Theta \to \{0, \bar{V}\}$,

5. $\mathcal{E}_{ex}^{\varepsilon,\alpha}(\tilde{Q}, \Theta)$ *denotes the event that,*

$$\begin{cases} |\lambda - \lambda_0| \leq (\ell(\tilde{Q})\bar{V})^{-\alpha\ell(\tilde{Q})} \\ Hu = \lambda u \text{ in } 2\tilde{Q} \\ |u| \leq 1 \text{ in } \frac{1}{2}\tilde{Q} \\ |u| \leq 1 \text{ in a } 1-\varepsilon \text{ fraction of } 2\tilde{Q} \setminus \Theta \end{cases} \tag{3.5.85}$$

implies $|u| \leq (\ell(\tilde{Q})\bar{V})^{\alpha\ell(\tilde{Q})}$ *in* $\tilde{Q}$,

then $\mathbb{P}\left[\mathcal{E}_{ex}^{\varepsilon,\alpha}(\tilde{Q}, \Theta)\middle|\, V|_\Theta = V'\right] \geq 1 - \exp(-\varepsilon\ell(\tilde{Q}))$.

Proof. We identify $\tilde{Q}$ with $2R_{[1,a],[1,a]}$. Define $R_1 = R_{[1,a],[1,a]}$, $R_2 = R_{[1,a],[a+1,2a]}$, $R_3 = R_{[1-a,0],[1,a]}$, $R_4 = R_{[1,a],[1-a,0]}$, $R_5 = R_{[a+1,2a],[1,a]}$, $R_6 = R_{[a+1,2a],[a+1,2a]}$, $R_7 =$

$R_{[1-a,0],[a+1,2a]}$, $R_8 = R_{[1-a,0],[1-a,0]}$ and $R_9 = R_{[a+1,2a],[1-a,0]}$. See Figure 3.4 for an illustration. For some large a and $1 \leq i,j \leq 9$, let $\mathcal{E}'_{ex}(i,j)$ denote the event that

$$\begin{cases} |\lambda - \lambda_0| \leq (a\bar{V})^{-\alpha a} \\ Hu = \lambda u \text{ in } R_i \cup R_j \\ |u| \leq 1 \text{ in } R_i \\ |\{(s,t) \in R_j : |u(s,t)| > 1\}| \leq 100\varepsilon a^2 \end{cases} \tag{3.5.86}$$

implies $|u| \leq (a\bar{V})^{\frac{1}{2}\alpha a}$ in R_j.

Claim 3.5.23. *Let* $S = \{(1,2),(1,3),(1,4),(1,5),(2,6),(3,7),(4,8),(5,9)\}$. *Then we have* $\bigcap_{(i,j)\in S} \mathcal{E}'_{ex}(i,j) \subset \mathcal{E}^{\varepsilon,\alpha}_{ex}(2R_1,\Theta)$.

Proof of the claim. The strategy here is to use a covering argument from elementary geometry. Assume event $\bigcap_{(i,j)\in S} \mathcal{E}'_{ex}(i,j)$ holds and u satisfies (3.5.86). Our goal is to prove $|u| \leq (\ell(2R_1)\bar{V})^{\alpha\ell(2R_1)}$ in $2R_1$.

Since Θ is ε-spares in $4R_{[1,a],[1,a]}$ and $|u| \leq 1$ in a $1-\varepsilon$ fraction of $4R_{[1,a],[1,a]} \setminus \Theta$, we have $|\{(s,t) \in 4R_{[1,a],[1,a]} : |u(s,t)| > 1\}| \leq 100\varepsilon a^2$. Then the event $\mathcal{E}'_{ex}(1,2) \cap \mathcal{E}'_{ex}(1,3) \cap \mathcal{E}'_{ex}(1,4) \cap \mathcal{E}'_{ex}(1,5)$ implies $|u| \leq (a\bar{V})^{\frac{1}{2}\alpha a}$ in $\bigcup_{1\leq j\leq 5} R_j$. Finally, the event $\mathcal{E}'_{ex}(2,6) \cap \mathcal{E}'_{ex}(3,7) \cap \mathcal{E}'_{ex}(4,8) \cap \mathcal{E}'_{ex}(5,9)$ implies $|u| \leq (a\bar{V})^{\alpha a}$ in $\bigcup_{1\leq j\leq 9} R_j$. Since $2R_1 \subset \bigcup_{1\leq j\leq 9} R_j$, the claim follows. $\square$

Denote $\mathcal{E}'_{ex}(1,2)$ by $\mathcal{E}'_{ex}$ and let S be the set in the Claim 3.5.23. By Claim 3.5.23, it is sufficient to prove that $\mathbb{P}\left[\mathcal{E}'_{ex}(i,j) \middle| V|_\Theta = V'\right] \geq 1 - \exp(-\varepsilon a)$ for each

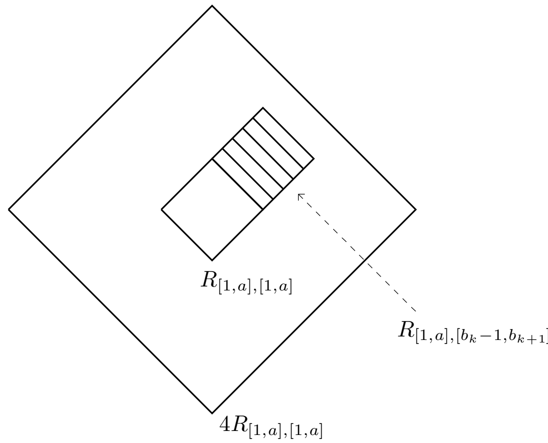

Figure 3.5: A schematic for the proof of Lemma 3.5.22

$(i,j) \in S$. By symmetry, we only need to prove for the case where $(i,j) = (1,2)$, i.e. $\mathbb{P}\left[\mathcal{E}'_{ex}(1,2) \middle| V|_\Theta = V'\right] \geq 1 - \exp(-\varepsilon a)$.

By Lemma 3.5.20 and a union bound, the event

$$\mathcal{E}_{ni} = \bigcap_{\substack{[c,d] \subset [1, \frac{5}{2}a] \\ \frac{a}{60} \leq d-c \leq \frac{a}{10}}} \mathcal{E}_{ni}(R_{[1,a],[c,d]})$$

satisfies $\mathbb{P}[\mathcal{E}_{ni} \big| V|_\Theta = V'] \geq 1 - \exp(-c_5 a + C\log(a))$ where c_5 is the constant in Lemma 3.5.20. It suffices to prove that, for every small $\varepsilon < \frac{1}{4}c_5$, there is a large α such that $\mathcal{E}_{ni} \subset \mathcal{E}'_{ex}(1,2)$. Assume $\mathcal{E}_{ni}$ and (3.5.86) hold, our goal is to prove $|u| \leq (a\bar{V})^{\frac{1}{2}\alpha a}$ in $R_2 = R_{[1,a],[a+1,2a]}$.

Claim 3.5.24. *Suppose $\varepsilon < 10^{-12}$. Then there is a sequence $b_0 \leq \cdots \leq b_{25} \in [a, \frac{5}{2}a - 2]$ such that*

1. $b_0 = a$

2. $b_{25} \geq 2a$

3. $\frac{1}{60}a \leq b_{k+1} - b_k \leq \frac{3}{40}a$ *for* $0 \leq k < 25$

4. $|u| \leq 1$ *on a* $1 - 10^{-7}$ *fraction of* $R_{[1,a],[b_{k+1}-1,b_{k+1}]}$ *for* $0 \leq k < 25$

Proof of the claim. Let $b_0 = a$. For each $k \in \{1, \cdots, 25\}$, let interval

$$J_k = \left(a + \frac{2k}{40}a, a + \frac{2k+1}{40}a\right].$$

Since $|u| > 1$ on at most $100\varepsilon a^2$ points in $R_{[1,a],[1,2a]}$, we have

$$\#\{(s,t) \in R_{[1,a],J_k} : |u(s,t)| > 1\} < 10^4\varepsilon \left|R_{[1,a],J_k}\right|$$

for each $k = 1, \cdots, 25$. The pigeonhole principle implies that, there is $b_k \in J_k \cap \mathbb{Z}$ such that

$$\#\{(s,t) \in R_{[1,a],[b_k-1,b_k]} : |u(s,t)| > 1\} < 10^5\varepsilon \left|R_{[1,a],[b_k-1,b_k]}\right|.$$

Since $\varepsilon < 10^{-12}$, we have

$$\#\{(s,t) \in R_{[1,a],[b_k-1,b_k]} : |u(s,t)| > 1\} < 10^{-7} \left|R_{[1,a],[b_k-1,b_k]}\right|$$

for each $k = 1, \cdots, 25$. On the other hand, $b_{k+1} - b_k \in \left[\frac{1}{40}a, \frac{3}{40}a\right] \subset \left[\frac{1}{60}a, \frac{3}{40}a\right]$ for $0 \leq k < 25$. Finally, $b_{25} > a + \frac{5}{4}a > 2a$ and our claim follows. $\square$

With the claim in hand, we apply $\mathcal{E}_{ni}(R_{[1,a],[b_k-1,b_{k+1}]})$ to conclude

$$\|u\|_{\ell^\infty(R_{[1,a],[b_k-1,b_{k+1}]})} \leq (a\bar{V})^{\alpha_2 a}(1 + \|u\|_{\ell^\infty(R_{[1,a],[b_k-1,b_k]})})$$

for $k = 0, \cdots, 24$. Since $\|u\|_{\ell^\infty(R_{[1,a],[1,a]})} \leq 1$, by induction, we obtain

$$\|u\|_{\ell^\infty(R_{[1,a],[1,2a]})} \leq 2^{25}(a\bar{V})^{25\alpha_2 a} < (a\bar{V})^{\frac{1}{2}\alpha a}$$

by setting $\alpha > 100\alpha_2$. $\square$

3.5.5 Covering argument

The proof of Lemma 3.3.5 below is a random version of [BLMS17, Proposition 3.9].

Definition 3.5.25. Given a tilted square $R_{[a_1,a_2],[b_1,b_2]}$ with $a_2 - a_1 = b_2 - b_1 > 0$, we call the point

$$\left(\left\lfloor\frac{a_1+a_2}{2}\right\rfloor, \left\lfloor\frac{b_1+b_2}{2}\right\rfloor + i\right) \in \widetilde{\mathbb{Z}^2}$$

the *center* of $R_{[a_1,a_2],[b_1,b_2]}$. Here, $i \in \{0,1\}$ such that $\lfloor\frac{a_1+a_2}{2}\rfloor - \lfloor\frac{b_1+b_2}{2}\rfloor - i$ is an even number.

Proof of Lemma 3.3.5. Let $\alpha' > 1 > \varepsilon' > 0$ be a pair of valid constants in Lemma 3.5.22. Let

$$\varepsilon_1 < 10^{-30}\varepsilon' \tag{3.5.87}$$

and suppose $\varepsilon < \varepsilon_1$. We impose further constraints on ε_1, α during the proof.

Assume without loss of generality that $Q = Q_n(\mathbf{0})$. Given integers $|s_1|, |t_1| \leq 10^{-10}\ell(Q)^{\frac{1}{3}}$ and $|s_2|, |t_2| \leq \varepsilon\ell(Q)^{\frac{2}{3}}$, let Q_{s_1,t_1,s_2,t_2} be the tilted square with center

$$\left(100\left\lceil \ell(Q)^{\frac{2}{3}}\right\rceil s_1 + 2\left\lceil \varepsilon^{-1}\right\rceil s_2, 100\left\lceil \ell(Q)^{\frac{2}{3}}\right\rceil t_1 + 2\left\lceil \varepsilon^{-1}\right\rceil t_2\right)$$

and length being any integer satisfying

$$(4\varepsilon)^{-1} \leq \ell(Q_{s_1,t_1,s_2,t_2}) \leq (2\varepsilon)^{-1}. \tag{3.5.88}$$

Then for different pairs (s_1, t_1, s_2, t_2) and (s_1', t_1', s_2', t_2'),

$$Q_{s_1,t_1,s_2,t_2} \cap Q_{s_1',t_1',s_2',t_2'} = \emptyset. \tag{3.5.89}$$

Meanwhile, for any $s_2, t_2 \in \left[-\varepsilon\ell(Q)^{\frac{2}{3}}, \varepsilon\ell(Q)^{\frac{2}{3}}\right]$,

$$\mathrm{dist}(Q_{s_1,t_1,s_2,t_2}, Q_{s_1',t_1',s_2,t_2}) > 50\ell(Q)^{\frac{2}{3}} \tag{3.5.90}$$

when $(s_1, t_1) \neq (s_1', t_1')$. Let

$$\mathcal{E}_{ex}^{s_1,t_1,s_2,t_2} = \bigcap\{\mathcal{E}_{ex}^{\alpha',\varepsilon'}(Q', \Theta) : Q' \supseteq Q_{s_1,t_1,s_2,t_2}, \ell(Q') \leq \ell(Q)^{\frac{2}{3}}\}.$$

By Lemma 3.5.22 and (3.5.88), for each s_1, t_1, s_2, t_2,

$$\mathbb{P}\left[\mathcal{E}_{ex}^{s_1,t_1,s_2,t_2} \middle| V|_\Theta = V'\right] \geq 1 - \sum_{l \geq (4\varepsilon)^{-1}} 10l^2 \exp(-\varepsilon' l) > \frac{9}{10} \tag{3.5.91}$$

by choosing $\varepsilon < \varepsilon_1$ small enough. Here, we used the fact that for any integer l, the number of tilted squares with length l that contain Q_{s_1,t_1,s_2,t_2} is less than $10l^2$. Note that, for each tilted Q', $\mathcal{E}_{ex}^{\alpha',\varepsilon'}(Q',\Theta)$ is $V|_{2Q'}$-measurable. Thus for any $s_2', t_2' \in \left[-\varepsilon\ell(Q)^{\frac{2}{3}}, \varepsilon\ell(Q)^{\frac{2}{3}}\right]$, by (3.5.90), we have

$$\left\{\mathcal{E}_{ex}^{s_1,t_1,s_2',t_2'} : |s_1|, |t_1| \leq 10^{-10}\ell(Q)^{\frac{1}{3}}\right\}$$

is a family of independent events. We denote by $\mathcal{E}_{ex}^{s_2',t_2'}$ the following event

$$\textit{at least half of events in } \left\{\mathcal{E}_{ex}^{s_1,t_1,s_2',t_2'} : |s_1|, |t_1| \leq 10^{-10}\ell(Q)^{\frac{1}{3}}\right\} \textit{ happen.} \tag{3.5.92}$$

Then by (3.5.91) and a large deviation estimate,

$$\mathbb{P}\left[\mathcal{E}_{ex}^{s_2',t_2'} \middle| V|_\Theta = V'\right] \geq 1 - \exp(-c\ell(Q)^{\frac{2}{3}}) \tag{3.5.93}$$

for a numerical constant c. Let

$$\begin{aligned}&\mathcal{E}_{ex}\\ =&\bigcap\{\mathcal{E}_{ex}^{s_2,t_2} : |s_2|, |t_2| \leq \varepsilon\ell(Q)^{\frac{2}{3}}\} \cap \bigcap\{\mathcal{E}_{ex}^{\alpha',\varepsilon'}(Q',\Theta) : \ell(Q') \geq \ell(Q)^{\frac{2}{3}}, Q' \subset Q\}.\end{aligned}$$

Then by Lemma 3.5.22 and (3.5.93),

$$\mathbb{P}[\mathcal{E}_{ex}\big|\, V|_\Theta = V'] \geq 1 - \exp(-c'\ell(Q)^{\frac{2}{3}} + C\log\ell(Q)) \geq 1 - \exp(-c''\ell(Q)^{\frac{2}{3}})$$

for constants c', c'' depending on ε'. Hence, it is sufficient to prove that

$$\mathcal{E}_{ex} \subset \mathcal{E}_{uc}^{\varepsilon,\alpha}(Q,\Theta).$$

Thus we assume $\mathcal{E}_{ex}$ holds and u satisfies (3.3.4). Our goal is to prove

$$\|u\|_{\ell^\infty(\frac{1}{100}Q)} \leq (\ell(Q)\bar{V})^{\alpha\ell(Q)}. \tag{3.5.94}$$

Let $\mathcal{Q}$ denote the subset of all Q_{s_1,t_1,s_2,t_2}'s such that $\mathcal{E}_{ex}^{s_1,t_1,s_2,t_2}$ happens. Then by definition of $\mathcal{E}_{ex}$ and (3.5.92), we have

$$|\mathcal{Q}| \geq 10^{-21}\varepsilon^2\ell(Q)^2. \tag{3.5.95}$$

Claim 3.5.26. *For any $Q_{s_1,t_1,s_2,t_2} \in \mathcal{Q}$ and $Q'' \subset Q$ with $Q'' \supseteq Q_{s_1,t_1,s_2,t_2}$, we have $\mathcal{E}_{ex}^{\alpha',\varepsilon'}(Q'',\Theta)$ holds.*

Proof. If $\ell(Q'') \leq \ell(Q)^{\frac{2}{3}}$, then $\mathcal{E}_{ex}^{s_1,t_1,s_2,t_2} \subset \mathcal{E}_{ex}^{\alpha',\varepsilon'}(Q'',\Theta)$. Otherwise,

$$\bigcap\{\mathcal{E}_{ex}^{\alpha',\varepsilon'}(Q',\Theta) : \ell(Q') \geq \ell(Q)^{\frac{2}{3}}, Q' \subset Q\} \subset \mathcal{E}_{ex}^{\alpha',\varepsilon'}(Q'',\Theta).$$

The claim follows from the definition of $\mathcal{E}_{ex}$. $\square$

Let

$$\mathcal{Q}_{sp} = \{Q' \in \mathcal{Q} : \exists Q'' \subset Q \text{ such that } Q'' \supseteq Q' \text{ and } \Theta \text{ is not } \varepsilon\text{-sparse in } Q''\}.$$

Write $\mathcal{Q}_{sp} = \{Q_{sp}^{(i)} : 1 \leq i \leq K_1\}$. For each $1 \leq i \leq K_1$, choose $Q_{spm}^{(i)} \subset Q$ to be a tilted square in which Θ is not ε-sparse and $Q_{sp}^{(i)} \subset Q_{spm}^{(i)}$. By Vitalli covering theorem, there exists $J' \subset \{1, \cdots, K_1\}$ such that

$$Q_{spm}^{(i_1)} \cap Q_{spm}^{(i_2)} = \emptyset$$

for each $i_1 \neq i_2 \in J'$ and

$$\begin{aligned} &|\bigcup\{Q_{spm}^{(i)} : i \in J'\}| \\ \geq &\frac{1}{100}|\bigcup\{Q_{spm}^{(i)} : 1 \leq i \leq K_1\}| \\ \geq &\frac{1}{100}|\bigcup\{Q_{sp}^{(i)} : 1 \leq i \leq K_1\}|. \end{aligned} \tag{3.5.96}$$

Since Θ is ε-regular in Q,

$$|\bigcup\{Q_{spm}^{(i)} : i \in J'\}| \leq \varepsilon \ell(Q)^2.$$

Thus by (3.5.96), $|\bigcup\{Q_{sp}^{(i)} : 1 \leq i \leq K_1\}| \leq 100\varepsilon\ell(Q)^2$. Note that by (3.5.89),

$\{Q_{sp}^{(i)} : 1 \leq i \leq K_1\}$ are pairwise disjoint. Thus by (3.5.88),

$$K_1 \leq 10^4 \varepsilon^3 \ell(Q)^2. \tag{3.5.97}$$

By choosing $\varepsilon < 10^{-26}$, (3.5.95) and (3.5.97) imply

$$|\mathcal{Q} \setminus \mathcal{Q}_{sp}| > 10^{-22} \varepsilon^2 \ell(Q)^2. \tag{3.5.98}$$

Now for any $Q' \in \mathcal{Q} \setminus \mathcal{Q}_{sp}$ and any $Q'' \subset Q$ with $Q'' \supseteq Q'$, Θ is ε-sparse in Q''. In particular, Θ is ε-sparse in Q' and by (3.5.88) and Definition 3.3.3, $\Theta \cap Q' = \emptyset$. Thus by (3.3.4),

$$\left| \{|u| > 1\} \cap \bigcup \{Q' : Q' \in \mathcal{Q} \setminus \mathcal{Q}_{sp}\} \right| < \varepsilon^3 \ell(Q)^2. \tag{3.5.99}$$

Equations (3.5.99), (3.5.89) and (3.5.98), together with $\varepsilon < 10^{-26}$, guarantee that there is $\mathcal{Q}_{good} \subset (\mathcal{Q} \setminus \mathcal{Q}_{sp})$ with

$$|\mathcal{Q}_{good}| > 10^{-23} \varepsilon^2 \ell(Q)^2 \tag{3.5.100}$$

such that

$$\|u\|_{\ell^\infty(Q')} \leq 1$$

for each $Q' \in \mathcal{Q}_{good}$.

We call a tilted square $Q' \subset Q$ "*tamed*" if the following holds:

1. the center of Q' is in $\frac{1}{50}Q$,

2. $Q' \supseteq Q''$ for some $Q'' \in \mathcal{Q}_{good}$,

3. $Q' \subset Q''' \subset Q$ implies Θ is ε-sparse in Q''',

4. $\|u\|_{\ell^\infty(Q')} \leq (\ell(Q')\bar{V})^{\alpha\ell(Q')}$.

Let $\mathcal{Q}_{ta}$ be the set of tamed squares. Then $\mathcal{Q}_{good} \subset \mathcal{Q}_{ta}$. We call $Q' \in \mathcal{Q}_{ta}$ maximal if any $Q'' \in \mathcal{Q}_{ta}$ with $Q'' \supseteq Q'$ implies $Q'' = Q'$.

Claim 3.5.27. *Suppose maximal $Q' \in \mathcal{Q}_{ta}$ with $\ell(Q') \leq \frac{1}{24}\ell(Q)$. Then $|u| > 1$ on at least a ε' fraction of $4Q' \setminus \Theta$.*

Proof. Since Q''s center is in $\frac{1}{50}Q$, $\ell(Q') \leq \frac{1}{24}\ell(Q)$ implies $4Q' \subset Q$. Assume $|u| \leq 1$ on a $1 - \varepsilon'$ fraction of $4Q' \setminus \Theta$. Since $Q' \supseteq Q''$ for some $Q'' \in \mathcal{Q}$, by Claim 3.5.26, $\mathcal{E}_{ex}^{\alpha',\varepsilon'}(2Q', \Theta)$ holds. Moreover, Q' containing some $Q'' \in \mathcal{Q}_{good}$ implies Θ is ε'-sparse in $4Q'$ and thus $\mathcal{E}_{ex}^{\alpha',\varepsilon'}(2Q', \Theta)$ implies

$$\|u\|_{\ell^\infty(2Q')} \leq (2\ell(Q')\bar{V})^{2\alpha'\ell(Q')}(1 + \|u\|_{\ell^\infty(Q')}) \leq (\ell(2Q')\bar{V})^{\alpha\ell(2Q')},$$

as long as $\alpha > 10\alpha'$. Thus $2Q'$ is also tamed and this contradicts with the maximality of Q'. $\square$

Write $\mathcal{Q}_{good} = \{Q^{(i)} : 1 \leq i \leq K_2\}$ and by (3.5.100),

$$K_2 > 10^{-23}\varepsilon^2\ell(Q)^2. \tag{3.5.101}$$

For each $1 \le i \le K_2$, pick a maximal $Q^{(i)}_{max} \in \mathcal{Q}_{ta}$ with $Q^{(i)} \subset Q^{(i)}_{max}$. Assume

$$\ell(Q^{(i_0)}_{max}) > \frac{1}{24}\ell(Q) \tag{3.5.102}$$

for some $1 \le i_0 \le K_2$. By definition of $\mathcal{Q}_{ta}$, the center of $Q^{(i_0)}_{max}$ is in $\frac{1}{50}Q$ and (3.5.102) implies $\frac{1}{100}Q \subset Q^{(i_0)}_{max}$. Hence

$$\|u\|_{\ell^\infty(\frac{1}{100}Q)} \le \|u\|_{\ell^\infty(Q^{(i_0)}_{max})} \le (\ell(Q^{(i_0)}_{max})\bar{V})^{\alpha\ell(Q^{(i_0)}_{max})} \le (\ell(Q)\bar{V})^{\alpha\ell(Q)}$$

and our conclusion (3.5.94) follows.

Now we assume $\ell(Q^{(i)}_{max}) \le \frac{1}{24}\ell(Q)$ for each $1 \le i \le K_2$ and we will arrive at contradiction. By Vitalli covering theorem, there is $J'' \subset \{1, \cdots, K_2\}$ such that

$$4Q^{(i_1)}_{max} \cap 4Q^{(i_2)}_{max} = \emptyset$$

for $i_1 \neq i_2 \in J''$ and

$$\begin{aligned} &\sum_{i \in J''} |4Q^{(i)}_{max}| \\ \ge &\frac{1}{100}|\bigcup\{4Q^{(i)}_{max} : 1 \le i \le K_2\}| \\ \ge &\frac{1}{100}|\bigcup\{Q^{(i)} : 1 \le i \le K_2\}|. \end{aligned} \tag{3.5.103}$$

By Claim 3.5.27, for each $1 \le i \le K_2$, $|u| > 1$ on a ε' fraction of $4Q_{max}^{(i)} \setminus \Theta$. Thus

$$
\begin{aligned}
|\{|u| > 1\} \setminus \Theta| \ge& \varepsilon' \sum_{i \in J''} |4Q_{max}^{(i)} \setminus \Theta| && (3.5.104)\\
\ge& \frac{1}{2}\varepsilon' \sum_{i \in J''} |4Q_{max}^{(i)}| && (3.5.105)\\
\ge& \frac{1}{200}\varepsilon' |\bigcup \{Q^{(i)} : 1 \le i \le K_2\}| && (3.5.106)\\
\ge& \frac{1}{200}\varepsilon' K_2 (4\varepsilon)^{-2} && (3.5.107)\\
\ge& 10^{-30}\varepsilon' \ell(Q)^2. && (3.5.108)
\end{aligned}
$$

Here, (3.5.105) is because Θ is ε-sparse in $4Q_{max}^{(i)}$; (3.5.106) is due to (3.5.103); (3.5.107) is due to (3.5.88) and (3.5.89); (3.5.108) is due to (3.5.101). However, by (3.5.87), (3.5.108) contradicts with $|\{|u| > 1\} \setminus \Theta| \le \varepsilon^3 \ell(Q)^2$ in (3.3.4). $\square$

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