TOPICS IN RANDOM CONFORMAL GEOMETRY: SLE BUBBLE MEASURES, CONFORMAL WELDINGS OF LIOUVILLE QUANTUM GRAVITY SURFACES, AND APPLICATIONS

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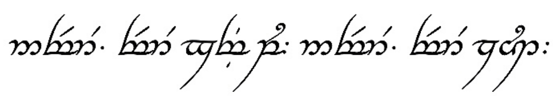
To my grandfather, Songnian Zou (1932-2023), for always asking me to not give up.

To Kobe Bryant (1978-2020), for all the inspirations and memories in purple and gold.



ABSTRACT

TOPICS IN RANDOM CONFORMAL GEOMETRY:

SLE BUBBLE MEASURES, CONFORMAL WELDINGS OF LIOUVILLE QUANTUM GRAVITY SURFACES, AND APPLICATIONS

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Xin Sun

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In this dissertation, we showed that the $\text{SLE}_\kappa(\rho)$ bubble measure recently constructed by Zhan arises naturally from the conformal welding of two Liouville quantum gravity (LQG) disks. The proof of the main results relies on (1) a "quantum version" of the limiting construction of the SLE bubble, (2) the conformal welding between quantum triangles and quantum disks due to Ang, Sun and Yu, and (3) the uniform embedding techniques of Ang, Holden and Sun. As a by-product of our proof, we obtained a decomposition formula of the $\text{SLE}_\kappa(\rho)$ bubble measure. Furthermore, we provided two applications of our conformal welding results. First, we computed the moments of the conformal radius of the $\text{SLE}_\kappa(\rho)$ bubble on $\mathbb{H}$ conditioning on surrounding i. The second application concerns the bulk-boundary correlation function in the Liouville Conformal Field Theory (LCFT). Within probabilistic frameworks, we derived a formula linking the bulk-boundary correlation function in the LCFT to the joint law of left & right quantum boundary lengths and the quantum area of the two-pointed quantum disk. This relation will be used by Ang, Remy, Sun and Zhu in a concurrent work to verify the formula of two-pointed bulk-boundary correlation function in physics predicted by Hosomichi (2001).

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Best of luck to all.

Da Wu

Philadelphia, PA

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CHAPTER 1

INTRODUCTION

The Schramm-Loewner evolution (SLE) and Liouville quantum gravity (LQG) are central objects in Random Conformal Geometry and it was shown in [She10] and [DMS20] that SLE curves arise naturally as the interfaces of LQG surfaces under conformal welding. Conformal welding results in [She10, DMS20] mainly focus on the infinite volume LQG surfaces.

Recently, Ang, Holden and Sun [AHS20] showed that conformal weldings of finite-volume quantum surfaces called two-pointed quantum disks can give rise to canonical variants of SLE curves with two marked points. Later, it was shown by Ang, Holden and Sun [AHS22] that another canonical variant of SLE called the SLE_κ Loop is the natural welding interface of two quantum disks. The resulting LQG surface is called the *quantum sphere*, which describes the scaling limit of classical planar map models with spherical topology.

As will be reviewed in Section 3.3, the rooted $\mathrm{SLE}_\kappa(\rho)$ bubble measure on $\mathbb{H}$ is an important one parameter family of random Jordan curves constructed by Zhan [Zha22] for all $\kappa > 0$ and $\rho > -2$. When $\kappa > 4$ and $\rho \in (-2, \frac{\kappa}{2} - 4]$, the law of the bubble is a probability measure on the space of rooted loops that satisfies conformal invariance property ([Zha22, Theorem 3.10]). When $\rho > (-2) \vee (\frac{\kappa}{2} - 4)$, the law of the bubble is a σ-finite infinite measure and satisfies conformal covariance property ([Zha22, Theorem 3.16]). In both cases, an instance η of $\mathrm{SLE}_\kappa(\rho)$ bubble is characterized by the following Domain Markov Property (DMP): suppose τ is a positive stopping time for η, then conditioning on the part of η before τ, i.e., $\eta[0,\tau]$, and the event that η is not completed at τ, the rest of η is a chordal $\mathrm{SLE}_\kappa(\rho)$ process on $\mathbb{H}\backslash\eta[0,\tau]$ ([Zha22, Theorem 3.16]). Moreover, it was shown that $\mathrm{SLE}_\kappa(\rho)$ bubble measures can be viewed as the weak limit of chordal $\mathrm{SLE}_\kappa(\rho)$ on $\mathbb{H}$ from 0 to ε as $\varepsilon \to 0^+$ (with force point at 0^-) after suitable rescaling ([Zha22, Theorem 3.20]).

On the other hand, it was shown in [ARS22, Section 4] that a particular $\mathrm{SLE}_\kappa(\rho)$ bubble curve can be obtained from conformally welding two Liouville quantum gravity surfaces of the disk topology. This was used to derive the *Fateev-Zamolodchikov-Zamolodchikov* (FZZ) formula in Liouville theory, which serves as a crucial input to the proof of the imaginary DOZZ formula for conformal loop ensemble (CLE) on the Riemann sphere [AS21]. This paper generalizes the conformal welding result in [ARS22] to all $\rho > -2$; see Remark 1.1.2 for the precise relation between our result and the one in [ARS22].

1.1. Statements of the main results

1.1.1. SLE bubble measures via conformal welding of quantum disks

Let $\mathrm{Bubble}_{\mathbb{H}}(p)$ be the space of rooted simple loops on $\mathbb{H}$ with root $p \in \mathbb{R}$. Precisely, an oriented simple closed loop η is in $\mathrm{Bubble}_{\mathbb{H}}(p)$ if and only if $p \in \eta$ and $(\eta \backslash \{p\}) \subseteq \overline{\mathbb{H}}$. Throughout this thesis, for an instance $\eta \in \mathrm{Bubble}_{\mathbb{H}}(p)$, let $D_\eta(p)$ be the connected component of $\mathbb{H}\backslash\eta$ which is encircled by η and let $D_\eta(\infty)$ be the domain $\mathbb{H}\backslash(\eta \cup D_\eta(p))$ containing ∞. The point p corresponds to two pseudo boundary marked points p^- and p^+ on $D_\eta(\infty)$. Let $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)$ denote the rooted $\mathrm{SLE}_\kappa(\rho)$ bubble measure with root 0 constructed in [Zha22] (see Definition 3.3.7) and note that this is a σ-finite infinite measure on $\mathrm{Bubble}_{\mathbb{H}}(0)$.

For each $\gamma \in (0, 2)$, there is a family of Liouville quantum gravity surfaces with disk topology called *quantum disks*. There is also a weight parameter $W > 0$ associated with the family of quantum disks. Let $\mathcal{M}_{0,2}^{\mathrm{disk}}(W)$ denote the two-pointed, weight-W quantum disk, i.e., both marked points are on the boundary, each with weight W; see Definition 2.3.2 and 2.3.4 for two regimes in terms of W. When $W = 2$, the two marked points on $\mathcal{M}_{0,2}^{\mathrm{disk}}(2)$ are *quantum typical* w.r.t. the quantum boundary length measure ([AHS20, Proposition A.8]) and we denote the $\mathcal{M}_{0,2}^{\mathrm{disk}}(2)$ by $\mathrm{QD}_{0,2}$. Let $\mathrm{QD}_{0,1}$ and $\mathrm{QD}_{1,1}$ denote the typical quantum disks with one boundary marked point and with one bulk & one boundary marked point respectively; see Definition 2.3.5 for the class of typical quantum disks and its variants.

Let $\mathrm{QD}_{0,1}(\ell)$ and $\mathrm{QD}_{1,1}(\ell)$ be the disintegration of $\mathrm{QD}_{0,1}$ and $\mathrm{QD}_{1,1}$ over its quantum boundary length respectively, i.e., $\mathrm{QD}_{0,1} = \int_0^\infty \mathrm{QD}_{0,1}(\ell)d\ell$ and $\mathrm{QD}_{1,1} = \int_0^\infty \mathrm{QD}_{1,1}(\ell)d\ell$, and both $\mathrm{QD}_{0,1}(\ell)$ and $\mathrm{QD}_{1,1}(\ell)$ should be understood as $\mathrm{QD}_{0,1}$ and $\mathrm{QD}_{1,1}$ restricted to having total boundary length ℓ respectively. Similarly, let $\mathcal{M}_{0,2}^{\text{disk}}(W;\cdot,\ell)$ be the disintegration of $\mathcal{M}_{0,2}^{\text{disk}}(W)$ over its right boundary, i.e., $\mathcal{M}_{0,2}^{\text{disk}}(W) = \int_0^\infty \mathcal{M}_{0,2}^{\text{disk}}(W;\cdot,\ell)d\ell$, and the $\mathcal{M}_{0,2}^{\text{disk}}(W;\cdot,\ell)$ again represents the $\mathcal{M}_{0,2}^{\text{disk}}(W)$ restricted to having the right boundary length ℓ. Let $\int_0^\infty \mathcal{M}_{0,2}^{\text{disk}}(W;\cdot,\ell) \times \text{QD}_{0,1}(\ell)d\ell$ be the curve-decorated quantum surface obtained by conformally welding the right boundary of $\mathcal{M}_{0,2}^{\text{disk}}(W)$ and total boundary of $\text{QD}_{0,1}$. Similarly, $\int_0^\infty \mathcal{M}_{0,2}^{\text{disk}}(W;\cdot,\ell) \times \text{QD}_{1,1}(\ell)d\ell$ is the quantum surface obtained by welding the right boundary of $\mathcal{M}_{0,2}^{\text{disk}}(W)$ and the total boundary of $\text{QD}_{1,1}$.

In theoretical physics, the Liouville quantum gravity originated in A. Polyakov's seminal work [Pol81], where he proposed a theory of summation over the space of Riemannian metrics on fixed two dimensional surface. The fundamental building block of his framework is the *Liouville Conformal Field Theory* (LCFT), which describes the law of the conformal factor of the metric tensor in a surface of fixed complex structure. The LCFT was made rigorous in probability theory in various different topologies; see [DKRV16] and [HRV18] for the cases of Riemann sphere and simply connected domains with boundary respectively, and [DRV15, Rem17, GRV19] for the cases of other topologies.

To be precise, let $P_{\mathbb{H}}$ be the probability measure corresponding to the law of the free-boundary Gaussian free field (GFF) on $\mathbb{H}$ normalized to having average zero on the unit circle in upper half plane unit circle $\partial\mathbb{D}\cap\mathbb{H}$. The infinite measure $\text{LF}_{\mathbb{H}}(d\phi)$ is defined by first sampling $(h, \mathbf{c})$ according to $P_{\mathbb{H}} \times \left[e^{-Qc}dc\right]$ and then letting $\phi(z) = h(z) - 2Q\log|z|_+ + \mathbf{c}$, where $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$ and $|z|_+ = \max\{|z|, 1\}$. We can further define the Liouville field with bulk or/and boundary insertion(s), e.g., $\text{LF}_{\mathbb{H}}^{(\beta,p)}$ and $\text{LF}_{\mathbb{H}}^{(\alpha,z),(\beta,p)}$, where $p \in \mathbb{R}$ and $z \in \mathbb{H}$. To make sense of $\text{LF}_{\mathbb{H}}^{(\beta,p)}$, where $p \in \partial\mathbb{H}$, let $\text{LF}_{\mathbb{H}}^{(\beta,p)} := \lim_{\varepsilon\to 0} \varepsilon^{\beta^2/4} e^{\frac{\beta}{2}\phi_\varepsilon(p)} \text{LF}_{\mathbb{H}}(d\phi)$, ϕ_ε being a suitable regularization at scale ε of ϕ. In terms of $\text{LF}_{\mathbb{H}}^{(\alpha,z),(\beta,p)}$ with $z \in \mathbb{H}$ and $p \in \partial\mathbb{H}$, we use the similar limiting procedure. Let $\text{LF}_{\mathbb{H}}^{(\beta,p),(\alpha,z)} := \lim_{\varepsilon\to 0} \varepsilon^{\alpha^2/2} e^{\alpha\phi_\varepsilon(z)} \text{LF}_{\mathbb{H}}^{(\beta,p)}(d\phi)$, $\phi_\varepsilon(z)$ being some suitable renormalization at scale ε. By Cameron-Martin shift (a.k.a. Girsanov's theorem), the $\text{LF}_{\mathbb{H}}^{(\beta,p)}$ represents a sample from $\text{LF}_{\mathbb{H}}$ plus a β-log singularity at boundary marked point p locally. Similarly, $\text{LF}_{\mathbb{H}}^{(\alpha,z),(\beta,p)}$ should be viewed as $\text{LF}_{\mathbb{H}}$ plus one boundary

β-log singularity at p and one bulk α-log singularity at z.

For $q \in \mathbb{H}$ and $p \in \partial\mathbb{H}$, let $\mathrm{Bubble}_{\mathbb{H}}(p,q)$ be the space of rooted simple loops on $\mathbb{H}$ rooted at p and surrounding q. Precisely, an oriented simple closed loop η is in $\mathrm{Bubble}_{\mathbb{H}}(p,q)$ if and only if $p \in \eta$, $(\eta\backslash\{p\}) \subseteq \overline{\mathbb{H}}$ and $q \in D_\eta(p)$. Let $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[d\eta|i \in D_\eta(0)]$ denote the conditional law of $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)$ on surrounding i and this is a probability measure on $\mathrm{Bubble}_{\mathbb{H}}(0,i)$.

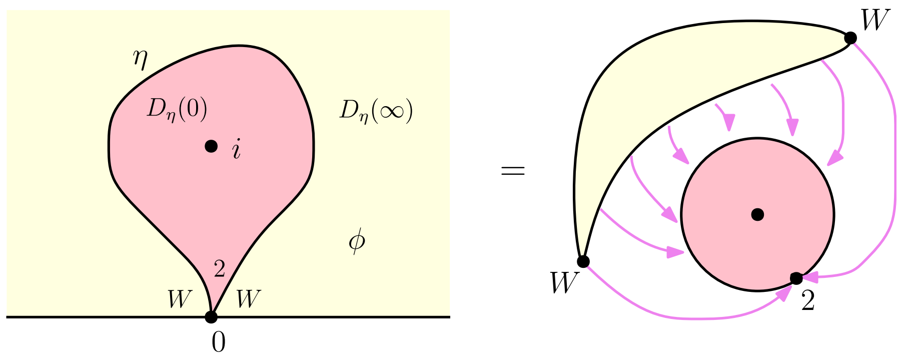


Figure 1.1: Illustration of Theorem 1.1.1 when $W \geq \frac{\gamma^2}{2}$: Suppose (ϕ,η) is sampled from $C \cdot \mathrm{LF}_{\mathbb{H}}^{(\gamma,i),(\beta_{2W+2},0)}(d\phi) \times \mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[d\eta|i \in D_\eta(0)]$ as shown on the left above, then the law of $(D_\eta(0),\phi,i,0)$ and $(D_\eta(\infty),\phi,0^-,0^+)$ viewed as a pair of marked quantum surfaces is equal to $\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W;\cdot,\ell) \times \mathrm{QD}_{1,1}(\ell)d\ell$, i.e., the quantum surface obtained by welding the total boundary of a sample from $\mathrm{QD}_{1,1}$ with the right boundary of a sample from $\mathcal{M}_{0,2}^{\mathrm{disk}}(W)$.

Theorem 1.1.1. *Fix $\gamma \in (0,2)$. For $W > 0$, let $\rho = W - 2$ and $\beta_{2W+2} = \gamma - \frac{2W}{\gamma}$. There exists some constant $C \in (0,\infty)$ such that suppose (ϕ,η) is sampled from*

$$C \cdot \mathrm{LF}_{\mathbb{H}}^{(\gamma,i),(\beta_{2W+2},0)}(d\phi) \times \mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[d\eta|i \in D_\eta(0)], \tag{1.1}$$

then the law of $(D_\eta(0),\phi,i,0)$ and $(D_\eta(\infty),\phi,0^-,0^+)$ viewed as a pair of marked quantum surfaces is equal to

$$\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W;\cdot,\ell) \times \mathrm{QD}_{1,1}(\ell)d\ell. \tag{1.2}$$

Remark 1.1.2. In [ARS22], the authors considered the same type of conformal welding with $W = \frac{\gamma^2}{2} - 2$ ([ARS22, Theorem 4.1]). The particular conformal welding result was used to obtained the so-called FZZ formula proposed in [FZZ00]. However in [ARS22, Theorem 4.1], the law of the welding interface was not explicitly specified. Here in the above Theorem 1.1.1, we generalized the [ARS22, Theorem 4.1] to all $W > 0$, and furthermore identified the law of the welding interface to be the $\mathrm{SLE}_\kappa(W-2)$ bubble.

The proof of Theorem 1.1.1 is separated into two parts. In Section 4.1, we show that the law of welding interface of curve-decorated quantum surface (1.2) is the $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)$ conditioning on surrounding i and it is independent of the underlying random field. To identify the law of the welding interface, we essentially use the "quantum version" of the limiting construction of the $\mathrm{SLE}_\kappa(\rho)$ bubble; see Corollary 3.3.8 for the statement on the Euclidean case. More precisely, we first consider the conformal welding of $\mathcal{M}_{0,2}^{\mathrm{disk}}(W)$ and $\mathrm{QD}_{1,2}$, i.e., the typical quantum disk with two boundary and one bulk marked points, whose welding interface is the chordal $\mathrm{SLE}_\kappa(\rho)$ conditioning on passing to the left of some fixed point in $\mathbb{H}$ (Lemma 4.1.5). Then conditioning on the quantum boundary length of $\mathrm{QD}_{1,2}$ between two boundary marked points shrinks to zero, we can construct a coupling with (1.2). Under such coupling, these two welding interfaces will match with high probability (Lemma 4.1.6). The independence of curve with the underlying random field follows from the coupling argument and Corollary 3.3.8 on the deterministic convergence of chordal $\mathrm{SLE}_\kappa(\rho)$.

The proof of the law of the underlying random field after conformal welding of two quantum disks, i.e., the quantum surface (1.2), is done in two steps. In Section 4.2, we first consider (1.2) when $0 < W < \frac{\gamma^2}{2}$, i.e., when the two-pointed quantum disk is thin. By Lemma 4.2.12, the thin quantum disk of weight W with one additional typical boundary marked point can be viewed as the concatenation of three independent disks: two thin disks of weight W and one thick disk of weight $\gamma^2 - W$ with one typical boundary marked point. Therefore, we can first sample one typical boundary marked point on $\mathcal{M}_{0,2}^{\mathrm{disk}}(W)$ and then sample two typical boundary marked points on $\mathrm{QD}_{1,1}(\gamma, \alpha)$, i.e., the quantum disk with one generic boundary insertion (Definition 4.2.9). The field law after conformally welding $\mathcal{M}_{2,\bullet}^{\mathrm{disk}}(W)$ and $\mathrm{QD}_{1,3}(\gamma, \alpha)$ can be derived from conformal welding results for quantum triangles in [ASY22]. After de-weighting all the additional marked points, we solve the case when $0 < W < \frac{\gamma^2}{2}$.

To extend to the full range $W > 0$, we inductively weld thin disks outside $\mathrm{QD}_{1,1}(\gamma, \alpha)$. By Theorem 3.2.2, a thick disk can be obtained by welding multiple thin disks. This concludes the outline of the proof of Theorem 1.1.1.

Next, we use the techniques of *uniform embedding* of quantum surfaces from [AHS21] to remove the bulk insertion in Theorem 1.1.1 so that the welding interface is the $\mathrm{SLE}_\kappa(\rho)$ bubble without conditioning. In order to introduce Theorem 1.1.3, we quickly recall the setups of the uniform embedding of upper half plane $\mathbb{H}$. Let $\mathrm{conf}(\mathbb{H})$ be the group of conformal automorphisms of $\mathbb{H}$ where the group multiplication $\cdot$ is the function composition $f \cdot g = f \circ g$. Let $\mathbf{m}_\mathbb{H}$ be a *Haar measure* on $\mathrm{conf}(\mathbb{H})$, which is both left and right invariant. Suppose $\mathfrak{f}$ is sampled from $\mathbf{m}_\mathbb{H}$ and $\phi \in C_0^\infty(\mathbb{H})'$, i.e., ϕ is a generalized function, then we call the random function

$$\mathfrak{f} \bullet_\gamma \phi = \phi \circ \mathfrak{f}^{-1} + Q|\log(\mathfrak{f}^{-1})'| \tag{1.3}$$

the *uniform embedding* of $(\mathbb{H}, \phi)$ via $\mathbf{m}_\mathbb{H}$. By invariance property of Haar measure, the law of $\mathfrak{f} \bullet_\gamma \phi$ only depends on $(\mathbb{H}, \phi)$ as quantum surface. We write

$$\mathbf{m}_\mathbb{H} \ltimes \left(\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W; \cdot, \ell) \times \mathrm{QD}_{0,1}(\ell) d\ell \right) \tag{1.4}$$

as the law of $(\mathfrak{f} \bullet_\gamma h, f(\eta), f(r))$, where $(\mathbb{H}, h, \eta, r)$ is an embedding of a sample from curve-decorated quantum surface $\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W; \cdot, \ell) \times \mathrm{QD}_{0,1}(\ell) d\ell$, and $\mathfrak{f}$ is sampled independently from $\mathbf{m}_\mathbb{H}$. Notice that here the $\mathbf{m}_\mathbb{H}$ does not fix our boundary marked point r, which initially is the root of η.

The equation (1.3) also provides a natural equivalence relation $\sim_\gamma$ over curve-decorated quantum surfaces; two curve-decorated quantum surfaces $(D_1, \phi_1, \eta_1, \omega_1, \ldots, \omega_n)$ with $\omega_i \in D_1 \cup \partial D_1$ and $(D_2, \phi_2, \eta_2, z_1 \ldots, z_n)$ with $z_i \in D_2 \cup \partial D_2$ are equivalent as quantum surfaces, denoted by $(D_1, \phi_1, \eta_1, \omega_1, \ldots, \omega_n) \sim_\gamma (D_2, \phi_2, \eta_2, z_1 \ldots, z_n)$, if there is a conformal map $\psi : D_1 \to D_2$ such that $\phi_2 = \psi \bullet_\gamma \phi_1$, $\eta_2 = \psi(\eta_1)$, and $\psi(\omega_i) = z_i, 1 \le i \le n$.

In addition, we also consider the case when the marked points are fixed under the action of Haar measure. For fixed $p \in \partial\mathbb{H}$, let $\mathrm{conf}(\mathbb{H}, p)$ be the subgroup of $\mathrm{conf}(\mathbb{H})$ fixing p and let $\mathbf{m}_{\mathbb{H},p}$ be a Haar measure on $\mathrm{conf}(\mathbb{H}, p)$. The curve-decorated quantum surface $\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W; \cdot, \ell) \times \mathrm{QD}_{0,1}(\ell) d\ell$ can be identified as a measure on the product space $(C_0^\infty(\mathbb{H})'/\mathrm{conf}(\mathbb{H}, p)) \times \mathrm{Bubble}_{\mathbb{H}}(p)$. Therefore, the measure

$$\mathbf{m}_{\mathbb{H},p} \ltimes \left(\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W; \cdot, \ell) \times \mathrm{QD}_{0,1}(\ell) d\ell \right) \tag{1.5}$$

can be defined in the exact same way as $\mathbf{m}_{\mathbb{H}} \ltimes \left(\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W; \cdot, \ell) \times \mathrm{QD}_{0,1}(\ell) d\ell \right)$.

For any fixed $p \in \mathbb{H}$, let $\mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}(\rho)$ denote the $\mathrm{SLE}_\kappa(\rho)$ bubble measure rooted at $p \in \mathbb{R}$. It is easily defined as the image of $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)$ under the shifting map $f_p : z \mapsto z + p$.

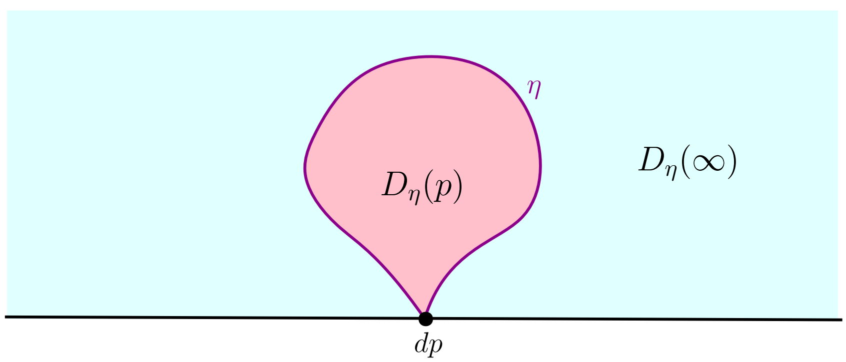


Figure 1.2: Illustration of the welding equation (1.6) in Theorem 1.1.3: first sample a root point p according to Lebesgue measure dp on $\mathbb{R}$, then sample (ϕ, η) according to the product measure $\mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2},p)}(d\phi) \times \mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}(W-2)(d\eta)$. The resulting quantum surface $(\mathbb{H}, \phi, \eta, p)/\sim_\gamma$ has the law of $C \int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W; \cdot, \ell) \times \mathrm{QD}_{0,1}(\ell) d\ell$ after uniform embedding.

Theorem 1.1.3. *Fix $\gamma \in (0, 2)$. For $W > 0$, let $\rho = W - 2$ and $\beta_{2W+2} = \gamma - \frac{2W}{\gamma}$. There exists some constant $C \in (0, \infty)$ such that*

$$\mathbf{m}_{\mathbb{H}} \ltimes \left(\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W; \cdot, \ell) \times \mathrm{QD}_{0,1}(\ell) d\ell \right) = C \cdot \mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2},p)}(d\phi) \times \mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}(\rho)(d\eta) dp, \tag{1.6}$$

where $\mathbf{m}_{\mathbb{H}}$ is a Haar measure on $\mathrm{conf}(\mathbb{H})$, i.e., the group of conformal automorphisms of $\mathbb{H}$.

Furthermore, there exists some constant $C \in (0,\infty)$ *such that*

$$\mathbf{m}_{\mathbb{H},0} \ltimes \left(\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W;\cdot,\ell) \times \mathrm{QD}_{0,1}(\ell) d\ell \right) = C \cdot \mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2},0)}(d\phi) \times \mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)(d\eta), \quad (1.7)$$

where $\mathbf{m}_{\mathbb{H},0}$ *is a Haar measure on* $\mathrm{conf}(\mathbb{H},0)$*, i.e., the group of conformal automorphisms of* $\mathbb{H}$ *fixing* 0*.*

The rigorous proof of Theorem 1.1.3 is presented in Section 4.3. The equation (1.7) should be viewed as the disintegration of equation (1.6) over its boundary root point. Unlike the case of Theorem 1.1.1, where there are two marked points:one in the bulk and one on the boundary, there is only one marked point in curve-decorated quantum surface $\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W;\cdot,\ell) \times \mathrm{QD}_{0,1}(\ell) d\ell$. Therefore, we do not have enough marked points to fix a conformal structure of $\mathbb{H}$. In this case, the LCFT describes the law of quantum surface $\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W;\cdot,\ell) \times \mathrm{QD}_{0,1}(\ell) d\ell$ after uniform embedding, whereas in Theorem 1.1.1, the LCFT describes the law of the quantum surface (1.2) under a fixed embedding.

Another way of stating Theorem 1.1.3 without using uniform embedding is to fix a particular embedding on the right hand side of equations (1.6) and (1.7). For instance, we can first sample (ϕ,η) from $\mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2},0)}(d\phi) \times \mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)(d\eta)$ and then fix the embedding by requiring $\nu_\phi(0,1) = \nu_\phi(1,\infty) = \nu_\phi(\infty,0)$, i.e., the quantum boundary lengths between $0,1$ and ∞ are all equal. By doing this, the law of $(D_\eta(0),\phi,0)$ and $(D_\eta(\infty),\phi,0^-,0^+)$ viewed as a pair of marked quantum surfaces is equal to $\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W;\cdot,\ell) \times \mathrm{QD}_{0,1}(\ell) d\ell$ up to some multiplicative constant.

As a by-product of the uniform embedding, we also obtained the following decomposition formula (Lemma 4.3.5 and Corollary 4.3.6) on the rooted SLE bubble measure $\mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}(\rho)$:

$$\begin{aligned} &\mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}(\rho)(d\eta) \\ &= C \cdot \frac{1}{|D_\eta(p)|} \int_{\mathbb{H}} |q-p|^{W - \frac{2W(W+2)}{\gamma^2}} (\Im q)^{\frac{W(W+2)}{\gamma^2} - \frac{W}{2}} \mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}(\rho)[d\eta | q \in D_\eta(p)] d^2 q, \end{aligned} \quad (1.8)$$

where $C \in (0,\infty)$, $|D_\eta(p)|$ is the euclidean area of $D_\eta(p)$, $\kappa = \gamma^2$, and $\rho = W - 2$. Equation

(1.8) also tells us that

$$\mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}(\rho)[q\in D_\eta(p)] \propto |q-p|^{W-\frac{2W(W+2)}{\kappa}}(\Im q)^{-\frac{W}{2}+\frac{W(W+2)}{\kappa}}. \tag{1.9}$$

In other words, for fixed $p\in\mathbb{R}$, the "probability" that $\mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}(\rho)$ surrounds q is proportional to $|q-p|^{W-\frac{2W(W+2)}{\kappa}}(\Im q)^{-\frac{W}{2}+\frac{W(W+2)}{\kappa}}$. As we will see in Section 4.3.1, it is the Haar measure together with "uniform symmetries" of the underlying Liouville field, or more concretely, the conformal covariance property of LCFT, that give us equation (1.9). The equation (1.8) provides a concrete relationship between the ordinary infinite bubble measure $\mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}(\rho)$ and the probability measure $\mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}(\rho)[d\eta|i\in D_\eta(p)]$ after conditioning and it builds the bridge between our two main theorems: Theorem 1.1.1 and Theorem 1.1.3.

Remark 1.1.4 (Scaling limits of random planar maps decorated by self-avoiding bubbles)**.** Motivated by [AHS22, Theorem 1.2], we conjecture that the scaling limit of the quadrangulated disk decorated by the self-avoiding discrete bubble converges in law to one-pointed quantum disk decorated by SLE bubble, i.e., the $\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(2;\cdot,\ell)\times \mathrm{QD}_{0,1}(\ell)d\ell$ in Theorem 1.1.3, for $\kappa=\gamma^2=\frac{8}{3}$ in the so-called *Gromov-Hausdorff-Prokhorov-Uniform topology* (GHPU topology). For the precise definition of GHPU topology, see [AHS22, Subsection 2.6]. The precise conjectures regarding the scaling limit of bubble-decorated quadrangulated disks will be stated in Subsection 6.3.

1.1.2. SLE bubble zippers with a generic insertion and applications

Next, we consider the generalization of Theorem 1.1.1 to the case when the bulk insertion of $\mathrm{QD}_{1,1}$ has generic weight.

Moments of the conformal radius of $\mathrm{SLE}_\kappa(\rho)$ bubbles

To generalize Theorem 1.1.1, we need to define the twisted $\mathrm{SLE}_\kappa(\rho)$ bubble measure on $\mathrm{Bubble}_{\mathbb{H}}(0,i)$ corresponding to weight-α bulk insertion of the quantum disk. Given $\eta\in\mathrm{Bubble}_{\mathbb{H}}(0,i)$, let $\psi_\eta:\mathbb{H}\to D_\eta(i)$ be the unique conformal map fixing i and 0. Let $\mathbf{m}$ denote the law of $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[d\eta|i\in D_\eta(0)]$ as in Theorem 1.1.1 and $\Delta_\alpha=\frac{\alpha}{2}(Q-\frac{\alpha}{2})$ is known as the *scaling dimension*. Define $\mathbf{m}_\alpha$ to be a *non-probability* measure on $\mathrm{Bubble}_{\mathbb{H}}(0,i)$ such that

$$\frac{d\mathbf{m}_\alpha}{d\mathbf{m}}(\eta) = |\psi'_\eta(i)|^{2\Delta_\alpha - 2}. \tag{1.10}$$

Fix $p \in \mathbb{R}, q \in \mathbb{H}$ and let $\mathrm{LF}_{\mathbb{H}}^{(\beta,p),(\alpha,q)}(\ell)$ be the disintegration of $\mathrm{LF}_{\mathbb{H}}^{(\beta,p),(\alpha,q)}$ over its total boundary length, i.e., $\mathrm{LF}_{\mathbb{H}}^{(\beta,p),(\alpha,q)} = \int_0^\infty \mathrm{LF}_{\mathbb{H}}^{(\beta,p),(\alpha,q)}(\ell)d\ell$. Like before, the measure $\mathrm{LF}_{\mathbb{H}}^{(\beta,p),(\alpha,q)}(\ell)$ represents the Liouville field $\mathrm{LF}_{\mathbb{H}}^{(\beta,p),(\alpha,q)}$ restricted to having total boundary length ℓ. The quantum surface $\mathrm{QD}_{1,1}(\alpha,\gamma)$ is the simple generalization of $\mathrm{QD}_{1,1}$ and has the LCFT description of $\mathrm{LF}_{\mathbb{H}}^{(\alpha,i),(\gamma,0)}$ under the particular embedding $(\mathbb{H},\phi,0,i)$; see Definition 4.2.7. Again, $\mathrm{QD}_{1,1}(\alpha,\gamma;\ell)$ is the disintegration of $\mathrm{QD}_{1,1}(\alpha,\gamma)$ over its total boundary length, i.e., $\mathrm{QD}_{1,1}(\alpha,\gamma) = \int_0^\infty \mathrm{QD}_{1,1}(\alpha,\gamma;\ell)d\ell$. We generalize Theorem 1.1.1 to Theorem 1.1.5 in order to compute the moments of conformal radius of the $\mathrm{SLE}_\kappa(\rho)$ bubble conditioning on surrounding i.

Theorem 1.1.5. *For $\alpha \in \mathbb{R}$ and $W > 0$, there exists some constant $C_W \in (0,\infty)$ such that the following holds: suppose (ϕ,η) is sampled from $\mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2},0),(\alpha,i)}(1) \times \mathbf{m}_\alpha$, then the law of $(D_\eta(0),\phi,i,0)$ and $(D_\eta(\infty),\phi,0^-,0^+)$ viewed as a pair of marked quantum surfaces is given by $C_W \cdot \int_0^\infty \mathrm{QD}_{1,1}(\alpha,\gamma;\ell) \times \mathcal{M}_{0,2}^{\mathrm{disk}}(W;1,\ell)d\ell$. In other words,*

$$\mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2},0),(\alpha,i)}(1) \times \mathbf{m}_\alpha = C_W \cdot \int_0^\infty \mathrm{QD}_{1,1}(\alpha,\gamma;\ell) \times \mathcal{M}_{0,2}^{\mathrm{disk}}(W;1,\ell)d\ell. \tag{1.11}$$

For technical convenience, we restrict the total boundary length of the curve-decorated quantum surface (1.11) to 1. For simply connected domain $D_\eta(0)$, ψ_η^{-1} is the conformal map from $D_\eta(0)$ to $\mathbb{H}$ that fixes 0 and 1. Let $g(z) = i\frac{z-i}{z+i}$ be the uniformizing map from $\mathbb{H}$ to $\mathbb{D}$ and let $\varphi_\eta : D_\eta(0) \mapsto \mathbb{D}$ be such that $\varphi_\eta := g \circ \psi_\eta^{-1}$. Notice that φ_η maps i to 0 and 0 to 1 respectively. Under our setups, the conformal radius of $D_\eta(0)$ viewed from i, denoted by $\mathrm{Rad}(D_\eta(0),i)$, is defined as $\frac{1}{|\varphi'_\eta(i)|}$, i.e.,

$$\mathrm{Rad}(D_\eta(0),i) := \frac{1}{|\varphi'_\eta(i)|}. \tag{1.12}$$

Notice that our definition of conformal radius (1.12) differs slightly with the classical literature of complex analysis, where the conformal map is chosen so that it maps i to 0 and its derivative at i is in $\mathbb{R}_+$. By simple computation,

$$\varphi'_\eta(i) = \left[g \circ \psi_\eta^{-1}\right]'(i) = g'(\psi_\eta^{-1}(i)) \cdot (\psi_\eta^{-1})'(i) = g'(i) \cdot \frac{1}{\psi'_\eta(i)}. \tag{1.13}$$

Therefore,

$$\mathrm{Rad}(D_\eta(0), i) = \frac{1}{|\varphi'_\eta(i)|} = \frac{|\psi'_\eta(i)|}{|g'(i)|} = 2|\psi'_\eta(i)|. \tag{1.14}$$

When η is sampled from $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[d\eta|i \in D_\eta(0)]$, we are interested in the moments of conformal radius $\mathrm{Rad}(D_\eta(0), i)$. Specifically, we want to compute $\mathbb{E}\left[\mathrm{Rad}(D_\eta(0), i)^{2\Delta_\alpha - 2}\right]$, which is the same as $2^{2\Delta_\alpha-2} \cdot \mathbb{E}\left[|\psi'_\eta(i)|^{2\Delta_\alpha-2}\right]$. To clear up the constant in our conformal welding equation (1.11), we further define the *renormalized moments of conformal radius* $\mathrm{CR}(\alpha, W)$ to be

$$\mathrm{CR}(\alpha, W) := \frac{\mathrm{Rad}(D_\eta(0), i)}{2^{2\Delta_\alpha-2} \cdot C_W} = \frac{\mathbb{E}\left[|\psi'_\eta(i)|^{2\Delta_\alpha-2}\right]}{C_W}. \tag{1.15}$$

Throughout this thesis, with a slight abuse of notation, when we talk about "the conformal radius of $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[d\eta|i \in D_\eta(0)]$", we really mean the conformal radius of the random simply connected domain $D_\eta(0)$ viewed from i when η is sampled from probability measure $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[d\eta|i \in D_\eta(0)]$.

Proposition 1.1.6 (Moments of conformal radius of SLE_κ bubbles)**.** *Fix $\kappa \in (0, 4)$, $W = 2, \rho = 0$ and $\frac{\gamma}{2} < \alpha < Q + \frac{2}{\gamma}$. Suppose η is sampled from $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}[d\eta|i \in D_\eta(0)]$, then we have*

$$\mathbb{E}\left[|\psi'_\eta(i)|^{2\Delta_\alpha-2}\right] = \frac{\Gamma(\frac{2\alpha}{\gamma})\Gamma(\frac{8}{\kappa} - \frac{2\alpha}{\gamma} + 1)}{\Gamma(\frac{8}{\kappa} - 1)}. \tag{1.16}$$

Consequently,

$$\mathbb{E}\left[\mathrm{Rad}(D_\eta(0), i)^{2\Delta_\alpha-2}\right] = 2^{2\Delta_\alpha-2} \cdot \frac{\Gamma(\frac{2\alpha}{\gamma})\Gamma(\frac{8}{\kappa} - \frac{2\alpha}{\gamma} + 1)}{\Gamma(\frac{8}{\kappa} - 1)}. \tag{1.17}$$

Moments of the conformal radius of the general $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[d\eta|i \in D_\eta(0)]$ bubbles are computed in Proposition 5.2.12. The key ingradients of the computation are the function $\overline{G}(\alpha,\beta)$ and the *Liouville reflection coefficient* $R(\beta,\mu_1,\mu_2)$ in [RZ22, AHS21], which describe the quantum boundary length laws of the two-pointed disk and the disk with one bulk and one boundary marked points, respectively.

The bulk-boundary correlation function in LCFT

As an another important application of Theorem 1.1.5, we derived a formula for the bulk-boundary correlation function in the LCFT within rigorous probabilistic frameworks. In theoretical physics, the LCFT is defined by the formal path integral. The most basic observable of Liouville theory is the correlation function with N bulk marked points $z_i \in \mathbb{H}$ with weights $\alpha_i \in \mathbb{R}$ and M boundary marked points $s_j \in \mathbb{R}$ with weights β_j. Precisely, for bulk insertions $(z_i)_{1\le i\le N}$ with weights $(\alpha_i)_{1\le i\le N}$ and boundary insertions $(s_j)_{1\le j\le M}$ with weights $(\beta_j)_{1\le j\le M}$, the correlation function in the LCFT at these points is defined using the following formal path integral:

$$\left\langle \prod_{i=1}^{N} e^{\alpha_i\phi(z_i)} \prod_{j=1}^{M} e^{\frac{\beta_j}{2}\phi(s_j)} \right\rangle_{\mu,\mu_\partial} = \int_{X:\mathbb{H}\to\mathbb{R}} DX \prod_{i=1}^{N} e^{\alpha_i X(z_i)} \prod_{j=1}^{M} e^{\frac{\beta_j}{2}X(s_j)} e^{-S^L_{\mu,\mu_\partial}(X)}, \tag{1.18}$$

where DX is the formal uniform measure on infinite dimensional function space and $S^L_{\mu,\mu_\partial}(X)$ is the *Liouville action functional* given by

$$S^L_{\mu,\mu_\partial}(X) := \frac{1}{4\pi}\int_{\mathbb{H}} \left(|\nabla_g X|^2 + QR_gX + 4\pi\mu e^{\gamma X}\right) d\lambda_g + \frac{1}{2\pi}\int_{\mathbb{R}} \left(QK_gX + 2\pi\mu_\partial e^{\frac{\gamma}{2}X}\right) d\lambda_{\partial g}. \tag{1.19}$$

For background Riemannian metric g on $\mathbb{H}$, $\nabla_g, R_g, K_g, d\lambda_g, d\lambda_{\partial g}$ stand for the gradient, Ricci curvature, Geodesic curvature, volume form and line segment respectively. The subscripts μ,μ_∂ emphasize the fact that both μ and μ_∂ are positive.

As a conformal field theory (CFT), the bulk correlation function $\langle e^{\alpha\phi(z)}\rangle_{\mu,\mu_\partial}$ in the LCFT takes the following form:

$$\left\langle e^{\alpha\phi(z)} \right\rangle_{\mu,\mu_\partial} = \frac{U(\alpha)}{|\Im z|^{2\Delta_\alpha}} \qquad \text{for } z\in\mathbb{H}, \tag{1.20}$$

where $U(\alpha)$ is known as the *structure constant* and $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$ is called the *scaling dimension* as mentioned before. In [FZZ00], the following formula for $U(\alpha)$ was proposed:

$$U_{\mathrm{FZZ}}(\alpha) := \frac{4}{\gamma} 2^{-\frac{\alpha^2}{2}} \left(\frac{\pi\mu}{2^{\gamma\alpha}} \frac{\Gamma\left(\frac{\gamma^2}{4}\right)}{\Gamma(1-\frac{\gamma^2}{4})} \right)^{\frac{Q-\alpha}{\gamma}} \Gamma\left(\frac{\gamma\alpha}{2} - \frac{\gamma^2}{4}\right) \Gamma\left(\frac{2\alpha}{\gamma} - \frac{4}{\gamma^2} - 1\right) \cos\left((\alpha - Q)\pi s\right), \tag{1.21}$$

where the parameter s is defined through the ratio of cosmological constants $\frac{\mu_\partial}{\sqrt{\mu}}$:

$$\cos \frac{\pi\gamma s}{2} = \frac{\mu_\partial}{\sqrt{\mu}} \sqrt{\sin \frac{\pi\gamma^2}{4}}, \qquad \text{with} \begin{cases} s \in [0, \frac{1}{\gamma}), & \text{when} \quad \frac{\mu_\partial^2}{\mu} \sin \frac{\pi\gamma^2}{4} \leq 1, \\ s \in i[0, +\infty), & \text{when} \quad \frac{\mu_\partial^2}{\mu} \sin \frac{\pi\gamma^2}{4} \geq 1. \end{cases}$$

In [ARS22], the (1.21) was proved within rigorous probability theory frameworks. From now on, for measure M on the space of distributions, let $M[f] := \int f(\phi) M(d\phi)$. For $\gamma \in (0, 2)$ and $\mu, \mu_\partial > 0$, let

$$\left\langle e^{\alpha\phi(z)} \right\rangle_{\mu,\mu_\partial} := \mathrm{LF}_{\mathbb{H}}^{(\alpha,z)} \left[e^{-\mu\mu_\phi(\mathbb{H}) - \mu_\partial \nu_\phi(\mathbb{R})} - 1 \right], \qquad \text{for } z \in \mathbb{H}, \tag{1.22}$$

where

$$\mu_\phi(\mathbb{H}) = \lim_{\varepsilon \to 0} \varepsilon^{\frac{\gamma^2}{2}} \int_{\mathbb{H}} e^{\gamma\phi_\varepsilon(z)} d^2z \qquad \text{and} \qquad \nu_\phi(\mathbb{R}) = \lim_{\varepsilon \to 0} \varepsilon^{\frac{\gamma^2}{4}} \int_{\mathbb{R}} e^{\frac{\gamma}{2}\phi_\varepsilon(z)} dz.$$

Since $|\Im z|^{2\Delta_\alpha} \langle e^{\alpha\phi(z)} \rangle_{\mu,\mu_\partial}$ does not depend on $z \in \mathbb{H}$, define $U(\alpha) := \langle e^{\alpha\phi(i)} \rangle_{\mu,\mu_\partial}$.

Theorem 1.1.7 ([ARS22, Theorem 1.1])**.** *For $\gamma \in (0, 2), \alpha \in (\frac{2}{\gamma}, Q)$ and $\mu, \mu_\partial > 0$, we have $U(\alpha) = U_{\mathrm{FZZ}}(\alpha)$.*

The above theorem is the first step towards rigorously solving the boundary LCFT. In this paper, we consider the bulk-boundary correlation in the LCFT. For $z \in \mathbb{H}$ and $s \in \mathbb{R}$, by the conformal invariance property, the bulk-boundary correlation function in the LCFT takes the following form:

$$\left\langle e^{\alpha\phi(z)} e^{\frac{\beta}{2}\phi(s)} \right\rangle_{\mu,\mu_\partial} = \frac{G_{\mu,\mu_\partial}(\alpha, \beta)}{|\Im z|^{2\Delta_\alpha - \Delta_\beta} |z - s|^{2\Delta_\beta}}. \tag{1.23}$$

Within probabilistic frameworks, define

$$\left\langle e^{\alpha\phi(z)} e^{\frac{\beta}{2}\phi(s)} \right\rangle_{\mu,\mu_\partial} := \mathrm{LF}_{\mathbb{H}}^{(\alpha,z),(\beta,s)} \left[e^{-\mu\mu_\phi(\mathbb{H})-\mu_\partial\nu_\phi(\mathbb{R})} \right] \tag{1.24}$$

and

$$G_{\mu,\mu_\partial}(\alpha,\beta) := \mathrm{LF}_{\mathbb{H}}^{(\alpha,i),(\beta,0)} \left[e^{-\mu\mu_\phi(\mathbb{H})-\mu_\partial\nu_\phi(\mathbb{R})} \right] \tag{1.25}$$

since $|\Im z|^{2\Delta_\alpha-\Delta_\beta}|z-s|^{2\Delta_\beta} \left\langle e^{\alpha\phi(z)} e^{\frac{\beta}{2}\phi(s)} \right\rangle_{\mu,\mu_\partial}$ does not depend on z and s. The function $G_{\mu,\mu_\partial}(\alpha,\beta)$ is also called the *structure constant* in boundary Liouville theory.

So far in the literature, all the exact formulas in LCFT except FZZ (1.21) have been derived by BPZ equations and the corresponding operator product expansion [BPZ84], including [KRV17] for the DOZZ formula and [Rem20, RZ20, RZ22] for different cases of boundary Liouville correlation functions with $\mu = 0$ and $\mu_\partial > 0$; see also discussions in [ARS22, Section 1.1]. In this thesis, from Theorem 1.1.5, we derive a formula linking the bulk-boundary correlation function to the joint law of left & right quantum boundary lengths and quantum area of $\mathcal{M}_{0,2}^{\mathrm{disk}}(W)$ when $0 < W < \frac{\gamma^2}{2}$.

Proposition 1.1.8 (Bulk-boundary correlation function in the LCFT)**.** *Fix $\gamma \in (0,2)$ and $\mu, \mu_\partial > 0$. When β_{2W+2} and α satisfy $0 < \beta_{2W+2} < \gamma < Q$ and $Q - \frac{\beta_{2W+2}}{2} < \alpha < Q$, we have*

$$\begin{aligned} G_{\mu,\mu_\partial}(\alpha,\beta_{2W+2}) = \mathrm{CR}(\alpha,W)^{-1}\frac{2}{\gamma}2^{-\frac{\alpha^2}{2}}\overline{U}_0(\alpha)\frac{2}{\Gamma(\frac{2}{\gamma}(Q-\alpha))}\left(\frac{1}{2}\sqrt{\frac{\mu}{\sin(\pi\gamma^2/4)}}\right)^{\frac{2}{\gamma}(Q-\alpha)} \times \\ \mathcal{M}_{0,2}^{\mathrm{disk}}(W)\left[e^{-\mu_\partial R_W-\mu A_W}\cdot K_{\frac{2}{\gamma}(Q-\alpha)}\left(L_W\sqrt{\frac{\mu}{\sin(\pi\gamma^2/4)}}\right)\right], \end{aligned} \tag{1.26}$$

where $\beta_{2W+2} = \gamma - \frac{2W}{\gamma}$, L_W, R_W and A_W denote the left, right quantum boundary length and the total quantum area of $\mathcal{M}_{0,2}^{\mathrm{disk}}(W)$ respectively. The $\mathrm{CR}(\alpha,W)$ is the renormalized moments of the conformal radius defined in (1.15) and takes an explicit formula (5.33). The

$\overline{U}_0(\alpha)$ *is defined in Theorem 5.2.17 and takes the following explicit formula:*

$$\overline{U}_0(\alpha) = \left(\frac{2^{-\frac{\gamma\alpha}{2}} 2\pi}{\Gamma(1-\frac{\gamma^2}{4})} \right)^{\frac{2}{\gamma}(Q-\alpha)} \Gamma\left(\frac{\gamma\alpha}{2} - \frac{\gamma^2}{4} \right) \qquad \textit{for all } \alpha > \frac{\gamma}{2}. \tag{1.27}$$

The $K_\nu(x)$ is the modified Bessel function of second kind. Precisely,

$$K_\nu(x) := \int_0^\infty e^{-x\cosh t} \cosh(\nu t) dt \qquad \textit{for } x > 0 \textit{ and } \nu \in \mathbb{R}.$$

The condition $0 < \beta_{2W+2} < \gamma$ in Proposition 1.1.8 is equivalent to $0 < W < \frac{\gamma^2}{2}$, i.e., the case when the two-pointed quantum disk is thin. By [HRV18, (3.5),(3.6),(3.7)], the *Seiberg bounds* correspond to

$$\alpha < Q, \quad \beta_{2W+2} < Q, \quad \text{and} \quad \alpha + \frac{1}{2}\beta_{2W+2} > Q, \tag{1.28}$$

which hold if and only if

$$0 < G_{\mu,\mu_\partial}(\alpha, \beta_{2W+2}) = \mathrm{LF}_{\mathbb{H}}^{(\alpha,i),(\beta_{2W+2},0)} \left[e^{-\mu\mu_\phi(\mathbb{H}) - \mu_\partial\nu_\phi(\mathbb{R})} \right] < \infty. \tag{1.29}$$

Notice that the range of α and β_{2W+2} in Proposition 1.1.8 are strictly contained in (1.28), and therefore the $G_{\mu,\mu_\partial}(\alpha, \beta_{2W+2})$ in (1.26) is nontrivial.

Remark 1.1.9. An explicit formula for the quantity

$$\mathcal{M}_{0,2}^{\mathrm{disk}}(W) \left[e^{-\mu_\partial R_W - \mu A_W} \cdot K_{\frac{2}{\gamma}(Q-\alpha)} \left(L_W \sqrt{\frac{\mu}{\sin(\pi\gamma^2/4)}} \right) \right] \tag{1.30}$$

in (1.26) is derived in the concurrent work of [ARSZ23, Lemma 4.4]. Combined with Proposition 1.1.8, this verifies the formula for $G_{\mu,\mu_\partial}(\alpha, \beta)$ proposed by Hosomichi [Hos01] in physics; see [ARSZ23, Theorem 1.2] for more details.

The contents of this thesis are essentially identical to my paper [Wu23], with only a few cosmetic changes.

1.2. Organization of the thesis

The rest of the thesis is organized as follows:

- In Chapter 2, we review all the necessary backgrounds on Liouville quantum gravities and the Liouville Conformal Field Theory.

- In Chapter 3, we review key concepts on Schramm-Loewner evolutions, conformal weldings of quantum surfaces, and constructions of $\mathrm{SLE}_\kappa(\rho)$ bubbles.

- In Chapter 4, we prove the main results of this thesis, including Theorem 1.1.1 and Theorem 1.1.3.

- In Chapter 5, we prove Theorem 1.1.5, which generalizes Theorem 1.1.1 to the case when the quantum bubble zipper has generic bulk insertions. As applications of Theorem 1.1.5, we compute the conformal radius of $\mathrm{SLE}_\kappa(\rho)$ bubbles on $\mathbb{H}$ conditioning on surrounding i and derive an analytic formula linking the bulk-boundary correlation function in LCFT to the joint law of left & right quantum boundary lengths and the total quantum area of the two-pointed quantum disk.

- In Chapter 6, we discuss several conjectures that arise naturally from the contexts of this thesis, including generalized SLE_κ bubbles and scaling limits of bubble-decorated quadrangulation disks.

CHAPTER 2

LIOUVILLE QUANTUM GRAVITIES AND THE LIOUVILLE CONFORMAL FIELD THEORY

The purpose of this chapter is to provide readers with all the necessary backgrounds on the Liouville Conformal Field Theory and Liouville quantum gravities. The discussion will be self-contained and centered around the main results of this thesis.

2.1. Notations and basic setups

Throughout this thesis, $\gamma \in (0,2)$ is the LQG coupling constant. Moreover,

$$Q = \frac{2}{\gamma} + \frac{\gamma}{2} \qquad \text{and} \qquad \kappa = \gamma^2.$$

For weight $W \in \mathbb{R}$, β_W is always a function of W with $\beta_W = Q + \frac{\gamma}{2} - \frac{W}{\gamma} = \gamma + \frac{2-W}{\gamma}$. We will work with planar domains in $\mathbb{C}$ including the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$, horizontal strip $\mathcal{S} = \mathbb{R} \times (0, \pi)$ and unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. For a domain $D \subset \mathbb{C}$, we denote its boundary by ∂D. For instance, $\partial \mathbb{H} = \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, $\partial \mathcal{S} = \{z \in \mathbb{C} : \Im(z) = 0 \text{ or } 1\} \cup \{\pm\infty\}$ and $\partial \mathbb{D} = \{z : |z| = 1\}$.

Fix a simply connected domain $D \subseteq \mathbb{C}$. Let $C_0^\infty(D)$ be the space of test functions equipped with the topology where a sequence (ϕ_k) satisfies $\phi_k \to 0$ in $C_0^\infty(D)$ if and only if there exists a compact set $K \subseteq D$ such that the support of ϕ_k is contained in K for every $k \in \mathbb{N}$ and ϕ_k as well as all of its derivatives converges uniformly to 0 as $k \to \infty$.

A *distribution* on D is a continuous linear functional from $C_0^\infty(D)$ to $\mathbb{R}$ with the aforementioned topology. Let $C_0^\infty(D)'$ denote the space of distributions on D.

We will frequently consider *non-probability* measures and extend the terminology of probability theory to this setting. More specifically, suppose M is a measure on a measurable space $(\Omega, \mathcal{F})$ with $M(\Omega)$ not necessarily 1 and X is a $\mathcal{F}$-measurable function, then we say $(\Omega, \mathcal{F})$ is a sample space and X is a random variable. We call the pushforward $M_X = X_* M$

the *law* of X and we say that X is sampled from M_X. We also write

$$M_X[f] := \int f(x) M_X(dx).$$

Weighting the law of X by $f(X)$ corresponds to working with measure $d\widetilde{M_X}$ with Randon-Nikodym derivative $\frac{d\widetilde{M_X}}{dM_X} = f$. For some event $E \in \mathcal{F}$ with $0 < M[E] < \infty$, let $M[\cdot|E]$ denote the probability measure $\frac{M[E\cap\cdot]}{M[E]}$ over the measure space $(E, \mathcal{F}_E)$ with $\mathcal{F}_E = \{A \cap E : A \in \mathcal{F}\}$. For a finite positive measure M, we denote its total mass by $|M|$ and let $M^{\#} = |M|^{-1}M$ denote the corresponding probability measure.

2.2. The Liouville Conformal Field Theory

2.2.1. Overview

In 1981, Polyakov introduced a path integral theory of summation over Riemannian metrics in the seminal paper "Quantum geometry of bosonic strings" [Pol81].

We start our discussion by recalling the *Feynman path integral formulation* of Brownian motions in $\mathbb{R}^d$. Let Σ be the space of simple continuous paths $\sigma : [0, T] \to \mathbb{R}^d$ with $\sigma(0) = \mathbf{0}$, and the d-dimensional Brownian motion may be regarded as a probability measure $\mathbb{P}$ (a.k.a. Wiener measure) on Σ. For $t_0 = 0 < t_1 < \ldots < t_{k-1} < t_k = T$ and $y_0 = 0$, we have

$$\mathbb{P}\left(\sigma(t_1) \in dy_1, \sigma(t_2) \in dy_2, \ldots, \sigma(t_k) \in dy_k\right) = \frac{1}{Z_{t_1,\ldots,t_k}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{k} \frac{|y_i - y_{i-1}|^2}{t_i - t_{i-1}}\right\}, \quad (2.1)$$

where

$$Z_{t_1,\ldots,t_k} = \prod_{i=1}^{k} \left(2\pi(t_i - t_{i-1})\right)^{d/2}. \quad (2.2)$$

Notice that if we choose a finer and finer partition $0 = t_0 < t_1 < \ldots < t_{k-1} < t_k = T$, then the energy function

$$\frac{1}{2}\sum_{i=1}^{k} \frac{|y_i - y_{i-1}|^2}{t_i - t_{i-1}} = \frac{1}{2}\sum_{i=1}^{k}(t_i - t_{i-1})\left(\frac{|y_i - y_{i-1}|}{t_i - t_{i-1}}\right)^2 \approx \frac{1}{2}\int_0^T |\dot{\sigma}(t)|^2 dt.$$

Therefore, we define the *Brownian action functional* S_{BM} (a.k.a. *Dirichlet energy functional*) by

$$S_{\text{BM}}(\sigma) := \frac{1}{2}\int_0^T |\dot{\sigma}(t)|^2 dt. \tag{2.3}$$

It is now well-known that the Brownian motion $(B_s)_{s\geq 0}$ can be understood via the following Feynman path integral representation:

$$\mathbb{E}\left[F((B_s)_{0\leq s\leq T})\right] = \frac{1}{Z}\int_\Sigma F(\sigma)e^{-S_{\text{BM}}(\sigma)}D\sigma, \tag{2.4}$$

where $D\sigma$ stands for the formal "uniform" measure on Σ and Z is the renormalization constant (a.k.a. partition function). The above path integral formulation came up frequently in the contexts of *Large deviation theory* in order to obtain the rate function.

Due to the fact the Brownian motion is the scaling limit of simple random walk, it is often be treated as the *canonical random path* in $\mathbb{R}^d$. Now that we have the *canonical random path*, what about the *canonical random surface*?

The answer is the *Liouville Conformal Field Theory.* In Polyakov's framework [Pol81], the *Liouville Conformal Field Theory* (LCFT hereafter) describes the conformal factor of the metric chosen "uniformly at random" and is a two-dimensional version of a *Feynman path integral* with an *exponential interaction* term. Mathematically, LCFT is an *infinite* measure on some infinite-dimensional function space and we call the underlying random field ϕ the *Liouville field.*

To be concrete, we consider the Riemann sphere $(\widehat{\mathbb{C}}, g)$ as our underlying Riemannian manifold, where $\widehat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$ and $g(z) = \frac{1}{|z|_+^4}$, $|z|_+ = \max\{1, |z|\}$. For $z, \alpha \in \mathbb{C}$, let

$$V_\alpha(z) := e^{\alpha\phi(z)} \tag{2.5}$$

be the *vertex operator* associated to ϕ.

The purpose of the LCFT (on Riemann sphere) is to study the vertex operator under some "uniform measure" $D\varphi$ twisted by $e^{-S_L(\varphi)}$, where $S_L(\cdot)$ is the so-called *Liouville action functional*. Let us define these terms more carefully. For distinct $(z_k)_{1\le k\le N} \in \mathbb{C}^N$ and $(\alpha_i)_{1\le i\le N} \in \mathbb{C}^N$, let

$$\left\langle \prod_{k=1}^{N} V_{\alpha_k}(z_k) \right\rangle_{\gamma,\mu} := \int_{\{\varphi:\widehat{\mathbb{C}}\to\mathbb{R}\}} \left(\prod_{k=1}^{N} e^{\alpha_k \phi(z_k)} \right) e^{-S_L(\varphi)} D\varphi, \tag{2.6}$$

where "$D\varphi$" is called the *Free field measure* in Physics literature and it should be understood as the "Lebesgue or uniform" measure on some infinite-dimensional function space $\{\varphi : \widehat{\mathbb{C}} \to \mathbb{R}\}$ and $S_L(\cdot)$ is the *Liouville action functional* taking the following form:

$$S_L(\varphi) := \frac{1}{4\pi} \int_{\widehat{\mathbb{C}}} \left(|\nabla_g \varphi(x)|^2 + Q \cdot R_g(x) \cdot \varphi(x) + 4\pi\mu e^{\gamma\varphi(x)} \right) g(d^2x), \tag{2.7}$$

where $R_g(x) = -\frac{1}{g(x)} \Delta \ln g(x)$ is the *Ricci curvature* of metric g, $\gamma \in (0,2)$, $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$ and $\mu > 0$ is the so-called the *cosmological constant.*

Since, in rigorous mathematical sense, the "Lebesgue measure" on infinite-dimensional function space does not exist, the (2.6) is an illegal definition. How to make it rigorous? We will represent the functions $\varphi : \widehat{\mathbb{C}} \to \mathbb{R}$ in terms of the eigenvector basis w.r.t. the Laplacian operator $-\frac{1}{g}\Delta$ and make sense of the measure "$e^{-S_L(\varphi)} D\varphi$" using the Gaussian free field (cf. Sheffield's *Proceedings of the ICM contribution for 2022* notes titled "What is a random surface?" [She22, Page 35]).

We will present the detailed computations regarding the above discussion in Section 2.2.3.

2.2.2. Gaussian Free Fields

Brownian motion can be viewed as a canonical random function from $\mathbb{R}$ to $\mathbb{R}$. One generalization of Brownian motion is called the *Gaussian free field* (GFF hereafter), which can be viewed as a random generalized function from $\mathbb{R}^d$ to $\mathbb{R}$. In this thesis, we will only consider the case when $d = 2$.

The zero-boundary GFF

Let $D \subset \mathbb{C}$ be a proper open domain with harmonically non-trivial boundary (i.e., Brownian motion starts from a point in D hits ∂D a.s.). If f and g are functions on D whose gradients are square integrable, then we can write

$$\langle f, g \rangle_\nabla = \frac{1}{2\pi} \int_D \nabla f(z) \cdot \nabla g(z) d^2 z \tag{2.8}$$

for the *Dirichlet inner product* between f and g. Let $H_0(D)$ be the *Hilbert space closure* of the space of compactly supported smooth functions on D w.r.t. (2.8). The *zero-boundary* GFF on D is the formal sum

$$\sum_{i \geq 1} \alpha_i f_i, \tag{2.9}$$

where $\{f_i\}_{i \geq 1}$ is an orthonormal basis of $H(D)$ and $\{\alpha_i\}_{i \geq 1}$ is an sequence of i.i.d. standard Gaussian variables. The formal sum (2.9) a.s. does not converge pointwise or in $H_0(D)$ but one can check that for each $f \in H_0(D)$, the formal inner product $\langle h, f \rangle_\nabla$ is a mean- zero Gaussian variable and these random variables satisfy the following covariance structure:

$$\mathbb{E}\left[\langle h, f \rangle_\nabla \langle h, g \rangle_\nabla\right] = \langle f, g \rangle_\nabla \tag{2.10}$$

By integration by parts, we can define the ordinary L^2 inner products

$$\langle h, f \rangle := -2\pi \langle h, \Delta_0^{-1} f \rangle_\nabla, \tag{2.11}$$

where Δ_0^{-1} is the inverse Laplacian with zero boundary conditions, whenever $\Delta_0^{-1} f \in H_0(D)$. Then the random variables $\langle h, f \rangle$ are jointly centered Gaussian with covariances

$$\mathrm{Cov}(\langle h, f \rangle, \langle h, g \rangle) = \frac{1}{2\pi} \int_D f(z) g(w) G_0^D(z, w) d^2 z d^2 w, \tag{2.12}$$

Therefore, we can check that zero-boundary GFF lies in the negative Sobolev space $H^{-\varepsilon}(D)$ for any $\varepsilon > 0$ [She07, Section 2.3] .

The free-boundary/whole-plane GFF

From now on, assume $D \subseteq \mathbb{C}$, i.e., we allow $D = \mathbb{C}$. Let $H(D)$ be the Hilbert space completion of

$$\left\{ f \in C^\infty(D) : \langle f, f \rangle_\nabla < \infty, \int_D f(z) d^2 z = 0 \right\} \tag{2.13}$$

with respect to (2.8). Note that when $\int_D f(z) d^2 z = 0$, the inner product (2.8) is positive definite.

The free-boundary GFF (if $D \neq \mathbb{C}$) or the whole-plane GFF (if $D = \mathbb{C}$) is again defined by the formal sum (2.9) but with the f_i's equal to orthonormal basis of $H(D)$ instead of $H_0(D)$. Same as the zero-boundary case, the formal inner products for $\langle h, f \rangle_\nabla$ for $f \in H(D)$ are well-defined and are jointly centered Gaussian variables with covariance structures given by

$$\mathbb{E}\left[\langle h, f \rangle_\nabla \langle h, g \rangle_\nabla\right] = \langle f, g \rangle_\nabla.$$

Next, let Δ^{-1} be the inverse of the Laplacian restricted to the space of functions such that $\int_D f(z) d^2 z = 0$, normalized so that $\int_D \Delta^{-1} f(z) d^2 z = 0$, with Neumann boundary conditions when $D \neq \mathbb{C}$. Again, whenever $\Delta^{-1} f \in H(D)$, we can define the L^2 inner product

$$\langle h, f \rangle := -2\pi \langle h, \Delta^{-1} f \rangle_\nabla$$

These L^2 inner products are jointly centered Gaussians with covariances

$$\mathrm{Cov}(\langle h, f \rangle, \langle h, g \rangle) = \frac{1}{2\pi} \int_D f(z) g(w) G^D(z, w) d^2 z d^2 w, \tag{2.14}$$

where G^D is the Green's function with Neumann boundary conditions if $D \neq \mathbb{C}$ and $G^D = -2\pi \log|z - w|$ if $D = \mathbb{C}$.

What about the case when $\langle \Delta^{-1} f, \Delta^{-1} f \rangle_\nabla < \infty$ but $\int_D f(z) d^2 z \neq 0$? We fix some f_0 such that $\langle \Delta^{-1} f_0, \Delta^{-1} f_0 \rangle_\nabla < \infty$ and $\int_D f_0(z) d^2 z = 1$. Let $\langle h, f_0 \rangle := c$ for some $c \in \mathbb{R}$. For any (whether generalized or not) function f, the function

$$\overline{f} := f - \left(\int_D f(z) d^2 z \right) f_0 \tag{2.15}$$

has total integral zero and we let

$$\langle h, f \rangle := \langle h, \overline{f} \rangle + c \cdot \int_D f(z) d^2 z. \tag{2.16}$$

Notice that the number c above is arbitrary. Therefore, free-boundary and whole-plane GFF are only defined *modulo a global additive constant.* Precisely, we view h as an equivalence class of distributions where two distributions are equivalent if and only if their difference is constant.

In the typical case when $D = \mathbb{C}$ (resp. $D = \mathbb{H}$), we fix the additive constant by requiring the circular average of h over $\partial\mathbb{D}$ (resp. $\partial\mathbb{D} \cap \mathbb{H}$) is zero, i.e., we consider the field $h - h_1(0)$ (well-defined random field). Note that $h_1(0)$ will be defined as below.

Circular averages of GFF

Let $D \subset \mathbb{C}$ and suppose h is an instance of GFF (zero-boundary, whole-plane, free-boundary) on D (with additive constants fixed in the two latter cases). Let $z \in D$ and $\varepsilon > 0$ be such that $\partial B(z, \varepsilon) \subset D$. Let $\rho_{z,\varepsilon}$ be the *uniform measure* on $\partial B(z, \varepsilon)$ and we define

$$h_\varepsilon(z) := \langle h, \rho_{z,\varepsilon} \rangle. \tag{2.17}$$

We should think $h_\varepsilon(z)$ as the random distribution h acting on the uniform measure on $\partial B(z, \varepsilon)$ or average of h on the $\partial B(z, \varepsilon)$.

2.2.3. The path integral construction of the LCFT on the Riemann sphere $\widehat{\mathbb{C}}$

The materials presented in this section follow closely from Vargas's lecture notes on DOZZ formula [Var17, Lecture 3].

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the *Riemann sphere* and we consider the metric $g(x) = \frac{1}{|x|_+^4}$ on $\widehat{\mathbb{C}}$. Notice that the Ricci curvature R_g of this metric g is a measure given (in generalized function notation) by

$$R_g(x)g(x)d^2x = -\Delta \ln g(x)d^2x = 4\mathfrak{u}(d^2x), \tag{2.18}$$

where $\mathfrak{u}$ is the uniform measure on the circle of center 0 and radius 1 with normalization such that $\int_{\widehat{\mathbb{C}}} \mathfrak{m}(d^2x) = 2\pi$. Let $H(x)$ be the GFF with covariance kernel given by (with an obvious abuse of notation since GFF is a generalized function)

$$\mathbb{E}[H(x)H(y)] = K_{\widehat{\mathbb{C}}}(x, y) := \log \frac{1}{|x-y|} + \log |x|_+ + \log |y|_+.$$

Notice that the GFF $H(x)$ has average 0 with respect to the curvature

$$\int_{\widehat{\mathbb{C}}} H(x)R_g(x)g(x)d^2x = 0. \tag{2.19}$$

Let

$$L^2(\widehat{\mathbb{C}}, g) := \left\{ \varphi : \int_{\widehat{\mathbb{C}}} |\varphi(x)|^2 g(d^2x) < \infty \right\} \tag{2.20}$$

be the space of square integrable functions from $\widehat{\mathbb{C}}$ to $\mathbb{R}$. The standard Sobolev space $H^1(\widehat{\mathbb{C}})$ is given by

$$H^1(\widehat{\mathbb{C}}, g) := \left\{ \varphi : \int_{\widehat{\mathbb{C}}} |\varphi(x)|^2 g(d^2x) + \int_{\widehat{\mathbb{C}}} |\nabla\varphi(x)|^2 d^2x < \infty \right\},$$

where ∇ is the standard gradient in $\mathbb{C}$ (with respect to the Euclidean metric d^2x). Let $(\varphi_j)_{j\geq 1}$ be the standard eigenvector basis for the operator $-\Delta_g := -\frac{1}{g}\Delta$, i.e.,

$$-\frac{1}{g(x)}\Delta\varphi_j(x) = \lambda_j\varphi_j(x). \tag{2.21}$$

We further renormalize them to have L^2 norm equal to 1, i.e.,

$$\int_{\widehat{\mathbb{C}}} \varphi_j(x)^2 g(d^2x) = 1.$$

Hence, every function φ in $L^2(\widehat{\mathbb{C}}, g)$ can be written uniquely in the orthonormal basis expansion in terms of $\{1, \varphi_1, \varphi_2, \ldots, \varphi_j, \ldots\}$. Precisely,

$$\varphi = c + \sum_{j \geq 1} c_j \varphi_j, \tag{2.22}$$

where

$$c_j = \int_{\widehat{\mathbb{C}}} \varphi(x)\varphi_j(x) g(d^2x)$$

for all $j \geq 1$. It is well-known that the "uniform measure "$D\varphi$ on the infinite-dimensional function space does not exist mathematically. However, at the very formal level, for any bounded continuous functional $F : L^2(\widehat{\mathbb{C}}, g) \to \mathbb{R}$, it is natural to write

$$\int_{L^2(\widehat{\mathbb{C}},g)} F(\varphi) D\varphi = \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}} F\left(c + \sum_{j \geq 1} c_j \varphi_j\right) dc \prod_{j=1}^{\infty} dc_j, \tag{2.23}$$

where dc and $dc_j, j \geq 1$ are standard Lebesgue measure on $\mathbb{R}$. If φ has eigenvector decomposition (2.22), then

$$\frac{1}{4\pi} \int_{\widehat{\mathbb{C}}} |\nabla_g \varphi(x)|^2 g(d^2x) = \frac{1}{4\pi} \sum_{j=1}^{\infty} c_j^2 \lambda_j. \tag{2.24}$$

Hence, formally we have

$$\int_{L^2(\widehat{\mathbb{C}},g)} F(\varphi) e^{-\frac{1}{4\pi}\int_{\widehat{\mathbb{C}}} |\nabla_g \varphi(x)|^2 g(d^2x)} D\varphi = \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}} F\left(c + \sum_{j \geq 1} c_j \varphi_j\right) dc \left(\prod_{j=1}^{\infty} e^{-\frac{c_j^2 \lambda_j}{4\pi}} dc_j\right) \tag{2.25}$$

By simple change of variables $u_j = \frac{c_j \sqrt{\lambda_j}}{\sqrt{2\pi}}$, we have that

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}} F\left(c + \sum_{j \geq 1} c_j \varphi_j\right) dc \left(\prod_{j=1}^{\infty} e^{-\frac{c_j^2 \lambda_j}{4\pi}} dc_j\right) \\ &= C \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}} F\left(c + \sqrt{2\pi} \sum_{j \geq 1} u_j \frac{\varphi_j}{\sqrt{\lambda_j}}\right) dc \left(\prod_{j=1}^{\infty} e^{-\frac{u_j^2}{2}} \frac{du_j}{\sqrt{2\pi}}\right), \end{aligned} \tag{2.26}$$

where the constant C is defined formally as $C := \prod_{j=1}^{\infty} \left(2\pi(\lambda_j)^{-1/2}\right)$. In probability theory, we know that for i.i.d. standard Gaussian $(\varepsilon_j)_{j\geq 1}$, the sum $\sqrt{2\pi}\sum_{j\geq 1}\varepsilon_j \frac{\varphi_j}{\sqrt{\lambda_j}}$ converges to the GFF H in $H^{-1}(\widehat{\mathbb{C}}, g)$, which is the dual space of $H^1(\widehat{\mathbb{C}}, g)$.

Therefore, following many steps of formal calculation, we can rigorously define

$$\int_{L^2(\widehat{\mathbb{C}},g)} F(\varphi)e^{-\frac{1}{4\pi}\int_{\widehat{\mathbb{C}}}|\nabla_g\varphi(x)|^2 g(d^2x)}D\varphi := \int_{\mathbb{R}} \mathbb{E}_H\left[F(H+c)\right]dc. \tag{2.27}$$

Here the notation $\mathbb{E}_H$ emphases the fact that we are taking expectations with respect to the law of the GFF. By construction,

$$\int_{\widehat{\mathbb{C}}} H(x)R_g(x)g(d^2x) = 0 \qquad \text{and} \qquad \int_{\widehat{\mathbb{C}}} R_g(x)g(d^2x) = 8\pi, \tag{2.28}$$

which leads to the following definition

$$\int F(\varphi)e^{-\frac{1}{4\pi}\int_{\widehat{\mathbb{C}}}|\nabla_g\varphi(x)|^2 g(d^2x)-\frac{1}{4\pi}\int_{\widehat{\mathbb{C}}}QR_g(x)\varphi(x)g(d^2x)}D\varphi := \int_{\mathbb{R}} e^{-2Qc}\mathbb{E}\left[F(H+c)\right]dc \tag{2.29}$$

Let us again emphasize the fact that the Gaussian free field is defined in the sense of *distributions* and should be viewed as *random generalized functions*.

Therefore, we have the following definition of the Liouville field on Riemann sphere $\widehat{\mathbb{C}}$.

Definition 2.2.1 (Liouville field on $\widehat{\mathbb{C}}$, [AHS21, Definition 2.6])**.** Let $(H, \mathbf{c})$ be sampled from $P_{\widehat{\mathbb{C}}} \times [e^{-2Qc}dc]$ and let $\phi(z) = H(z) - 2Q\log|z|_+ + \mathbf{c}$. Let $\mathrm{LF}_{\widehat{\mathbb{C}}}$ denote the law of ϕ and we call a sample from $\mathrm{LF}_{\widehat{\mathbb{C}}}$ a *Liouville field* on $\widehat{\mathbb{C}}$. Here $P_{\widehat{\mathbb{C}}}$ is the probability law of the Gaussian free field H.

Here we remark that the above definition is a rigorous way of making sense of the expression "$e^{-S_L(\varphi)}D\varphi$" when $\mu = 0$. When $\mu > 0$, we simply weight the zero-μ Liouville measure by $e^{-\mu A}$, where A is the quantum area of the LQG surface (will be discussed in later section).

2.2.4. The LCFT on the upper half plane $\mathbb{H}$

Let g be a smooth metric on $\mathbb{H}$ such that the metric completion of $(\mathbb{H}, g)$ is a compact Riemannian manifold. Let $H^1(\mathbb{H}, g)$ be the standard Sobolev space with norm defined by

$$|h|_{H^1(\mathbb{H},g)} := \left(\int_{\mathbb{H}} |\nabla h(z)|^2 + |h(z)|^2 g(d^2 z) \right)^{1/2}.$$

Let $H^{-1}(\mathbb{H}, g)$ be its dual space, which is defined as the completion of the set of smooth functions on $\mathbb{H}$ with respect to the following norm:

$$|f|_{H^{-1}(\mathbb{H},g)} := \sup_{h \in H^1(\mathbb{H},g), |h|_{H^1(\mathbb{H},g)} \leq 1} \left| \int_{\mathbb{H}} f(z)h(z)g(z)d^2 z \right|.$$

Here we remark that $H^{-1}(\mathbb{H})$ is a polish space and its topology does not depend on the choice of g. In this thesis, all the random functions considered are in $H^{-1}(\mathbb{H})$.

Let h be the centered Gaussian process on $\mathbb{H}$ with covariance kernel given by

$$\mathbb{E}[h(x)h(y)] = G_{\mathbb{H}}(x, y) := \log \frac{1}{|x-y||x-\overline{y}|} + 2\log |x|_+ + 2\log |y|_+,$$

where $|x|_+ = \max(|x|, 1)$. Notice that $h \in H^{-1}(\mathbb{H})$ and for test functions $f, g \in H^1(\mathbb{H})$, (h, f) and (h, g) are centred Gaussian variables with covariance given by

$$\mathbb{E}[(h, f), (h, g)] = \iint f(x)G_{\mathbb{H}}(x, y)g(y)d^2 x d^2 y.$$

Let $P_{\mathbb{H}}$ denote the law of h. For smooth test functions f and g with mean 0 on $\mathbb{H}$, i.e.,

$$\int_{\mathbb{H}} f(z)d^2 z = \int_{\mathbb{H}} g(z)d^2 z = 0,$$

we have that

$$\mathbb{E}[(h, f), (h, g)] = \frac{1}{2\pi} \int_{\mathbb{H}} \nabla f(z) \cdot \nabla g(z) d^2 z.$$

Notice that this characterizes the free boundary Gaussian free field, which is defined modulo an additive constant. We can fix a particular instance of field h by requiring the average around the upper half plane unit circle to be zero.

Definition 2.2.2 (Liouville field on $\mathbb{H}$, [AHS21, Definition 2.14])**.** Let $(h, \mathbf{c})$ be sampled from $P_{\mathbb{H}} \times [e^{-Qc}dc]$ on the product space $H^{-1}(\mathbb{H}) \times \mathbb{R}$. Let $\phi(z) = h(z) - 2Q \log |z|_+ + \mathbf{c}$ and let $\mathrm{LF}_{\mathbb{H}}$ denote the law of $\phi(z)$ on $H^{-1}(\mathbb{H})$. We call the sample from $\mathrm{LF}_{\mathbb{H}}$ the *Liouville field* on $\mathbb{H}$.

Lemma 2.2.3 ([ARS22, Lemma 2.2])**.** *For $\alpha \in \mathbb{R}$ and $z_0 \in \mathbb{H}$, the limit*

$$\mathrm{LF}_{\mathbb{H}}^{(\alpha,z_0)} := \lim_{\varepsilon \to 0} \varepsilon^{\alpha^2/2} e^{\alpha \phi_\varepsilon(z_0)} \mathrm{LF}_{\mathbb{H}}(d\phi)$$

exists in the vague topology. Moreover, sample $(h, \mathbf{c})$ from $(2\Im z_0)^{-\alpha^2/2} |z_0|_+^{-2\alpha(Q-\alpha)} P_{\mathbb{H}} \times \left[e^{(\alpha-Q)c}dc\right]$ and let

$$\phi(z) = h(z) - 2Q \log |z|_+ + \alpha G_{\mathbb{H}}(z, z_0) + \mathbf{c} \qquad \textit{for } z \in \mathbb{H},$$

then the law of ϕ is given by $\mathrm{LF}_{\mathbb{H}}^{(\alpha,z_0)}$. We call $\mathrm{LF}_{\mathbb{H}}^{(\alpha,z_0)}$ the Liouville field on $\mathbb{H}$ with α-insertion at z.

Next, we introduce the definition of Liouville field with multiple boundary insertions. The following definition is the combination of [AHS21, Definition 2.15] and [AHS21, Definition 2.17]:

Definition 2.2.4. Let $(\beta_i, s_i) \in \mathbb{R} \times \partial\mathbb{H}$ for $i = 1, \ldots, m$, where $m \geq 0$ and s_i are pairwise distinct. Let $(h, \mathbf{c})$ be sampled from $C_{\mathbb{H}}^{(\beta_i,s_i)_i} P_{\mathbb{H}} \times \left[e^{(\frac{1}{2}\sum_i \beta_i - Q)c}dc\right]$, where

$$C_{\mathbb{H}}^{(\beta_i,s_i)_i} = \begin{cases} \prod_{i=1}^m |s_i|_+^{-\beta_i(Q-\frac{\beta_i}{2})} e^{\sum_{j=i+1}^m \frac{\beta_i\beta_j}{4} G_{\mathbb{H}}(s_i,s_j)} & \text{if } s_1 \neq \infty, \\ \prod_{i=2}^m |s_i|_+^{-\beta_i(Q-\frac{\beta_i}{2}-\frac{\beta_1}{2})} e^{\sum_{j=i+1}^m \frac{\beta_i\beta_j}{4} G_{\mathbb{H}}(s_i,s_j)} & \text{if } s_1 = \infty. \end{cases}$$

Let

$$\phi(z)=\begin{cases} h(z)-2Q\log|z|_+ + \sum_{i=1}^m \frac{\beta_i}{2}G_{\mathbb{H}}(z,s_i)+\mathbf{c} & \text{if } s_1\neq\infty,\\ h(z)+(\beta_1-2Q)\log|z|_+ + \sum_{i=2}^m \frac{\beta_i}{2}G_{\mathbb{H}}(z,s_i)+\mathbf{c} & \text{if } s_1=\infty.\end{cases}$$

We write $\mathrm{LF}_{\mathbb{H}}^{(\beta_i,s_i)_i}$ for the law of ϕ and call a sample from $\mathrm{LF}_{\mathbb{H}}^{(\beta_i,s_i)_i}$ the Liouville field on $\mathbb{H}$ with boundary insertions $(\beta_i,s_i)_{1\le i\le m}$.

Lemma 2.2.5 ([AHS21, Lemma 2.18])**.** *We have the following convergence in the vague topology of measures on $H^{-1}(\mathbb{H})$:*

$$\lim_{r\to+\infty} r^{\beta(Q-\frac{\beta}{2})}\mathrm{LF}_{\mathbb{H}}^{(\beta,r),(\beta_i,s_i)_i}=\mathrm{LF}_{\mathbb{H}}^{(\beta,\infty),(\beta_i,s_i)_i}.$$

Definition 2.2.6. Let $(\alpha,q)\in\mathbb{R}\times\mathbb{H}$ and let $(\beta_i,p_i)\in\mathbb{R}\times\partial\mathbb{H}$ for $1\le i\le m$. Suppose $(h,\mathbf{c})$ is sampled from $C_{\mathbb{H}}^{(\beta_i,p_i)_i,(\alpha,q)}P_{\mathbb{H}}\times\left[e^{(\frac{1}{2}\sum_i\beta_i+\alpha-Q)c}dc\right]$, where

$$C_{\mathbb{H}}^{(\beta_i,p_i)_i,(\alpha,q)} = \begin{cases} \prod_{i=1}^m |p_i|_+^{-\beta_i(Q-\frac{\beta_i}{2})} e^{\sum_{j=i+1}^m \frac{\beta_i\beta_j}{4}G_{\mathbb{H}}(p_i,p_j)}(2\Im q)^{-\frac{\alpha^2}{2}}|q|_+^{-2\alpha(Q-\alpha)} & \text{if } p_1\neq\infty,\\ \prod_{i=2}^m |p_i|_+^{-\beta_i(Q-\frac{\beta_i}{2}-\frac{\beta_1}{2})} e^{\sum_{j=i+1}^m \frac{\beta_i\beta_j}{4}G_{\mathbb{H}}(p_i,p_j)}(2\Im q)^{-\frac{\alpha^2}{2}}|q|_+^{-2\alpha(Q-\alpha)} & \text{if } p_1=\infty.\end{cases}$$

Let

$$\phi(z)=\begin{cases} h(z)-2Q\log|z|_+ +\alpha G_{\mathbb{H}}(z,q)+ \sum_{i=1}^m \frac{\beta_i}{2}G_{\mathbb{H}}(z,p_i)+\mathbf{c} & \text{if } p_1\neq\infty,\\ h(z)+(\beta_1-2Q)\log|z|_+ +\alpha G_{\mathbb{H}}(z,q)+ \sum_{i=2}^m \frac{\beta_i}{2}G_{\mathbb{H}}(z,p_i)+\mathbf{c} & \text{if } p_1=\infty.\end{cases}$$

We denote the law of $\phi(z)$ on $H^{-1}(\mathbb{H})$ by $\mathrm{LF}_{\mathbb{H}}^{(\beta_i,p_i)_i,(\alpha,q)}$.

Finally, we recall the definition of the LCFT on horizontal strip $\mathcal{S}=\mathbb{R}\times(0,\pi)$. It is essentially the same procedure as defining LCFT on $\mathbb{H}$. Let

$$G_{\mathcal{S}}(z,w)=-\log|e^z-e^w|-\log|e^z-e^{\overline{w}}|+\max(2\Re z,0)+\max(2\Re w,0)$$

be the Green function on $\mathcal{S}$.

Definition 2.2.7 ([AHS21, Definition 2.19])**.** Let $(h, \mathbf{c})$ be sampled from $C_{\mathcal{S}}^{(\beta,\pm\infty),(\beta_3,s_3)} P_{\mathcal{S}} \times \left[e^{(\beta+\frac{\beta_3}{2}-Q)c} dc\right]$, where $\beta \in \mathbb{R}$ and $(\beta_3, s_3) \in \mathbb{R} \times \partial\mathcal{S}$, and

$$C_{\mathcal{S}}^{(\beta,\pm\infty),(\beta_3,s_3)} = e^{(-\frac{\beta_3}{2}(Q-\frac{\beta_3}{2})+\frac{\beta\beta_3}{2})|\Re s_3|}.$$

Let $\phi(z) = h(z) - (Q-\beta)|\Re z| + \frac{\beta_3}{2} G_{\mathcal{S}}(z, s_3) + \mathbf{c}$ and we denote the law of $\phi(z)$ on $H^{-1}(\mathbb{H})$ by $\mathrm{LF}_{\mathcal{S}}^{(\beta,\pm\infty),(\beta_3,s_3)}$.

Conformal symmetries of the Liouville Conformal Field Theory

Let $\mathrm{conf}(\mathbb{H})$ be the group of conformal automorphisms of $\mathbb{H}$ where group multiplication $\cdot$ is the function composition $f \cdot g = f \circ g$. The most results in this section can be directly adapted to the sphere case $\widehat{\mathbb{C}}$ (see [AHS21] for details).

Proposition 2.2.8 ([AHS21, Proposition 2.16])**.** *For $\beta \in \mathbb{R}$, let $\Delta_\beta = \frac{\beta}{2}(Q - \frac{\beta}{2})$. Let $f \in \mathrm{conf}(\mathbb{H})$ and $(\beta_i, s_i) \in \mathbb{R} \times \partial\mathbb{H}$ with $f(s_i) \neq \infty$ for all $1 \leq i \leq m$. Then $\mathrm{LF}_{\mathbb{H}} = f_*(\mathrm{LF}_{\mathbb{H}})$ and*

$$\mathrm{LF}_{\mathbb{H}}^{(\beta_i,f(s_i))_i} = \prod_{i=1}^{m} |f'(s_i)|^{-\Delta_{\beta_i}} f_* \left(\mathrm{LF}_{\mathbb{H}}^{(\beta_i,s_i)_i}\right).$$

Proposition 2.2.9. *For $1 \leq i \leq m$ and $1 \leq j \leq n$, let $(\alpha_i, z_i) \in \mathbb{R} \times \mathbb{H}$ and $(\beta_j, s_j) \in \mathbb{R} \times \partial\mathbb{H}$ with $f(s_j) \neq \infty$ for all $1 \leq j \leq n$. Let $f \in \mathrm{conf}(\mathbb{H})$ and we have*

$$\mathrm{LF}_{\mathbb{H}}^{(\alpha_i,f(z_i))_i,(\beta_j,f(s_j))_j} = \prod_{i=1}^{m}\prod_{j=1}^{n} |f'(z_i)|^{-2\Delta_{\alpha_i}} |f'(s_j)|^{-\Delta_{\beta_j}} \mathrm{LF}_{\mathbb{H}}^{(\alpha_i,z_i)_i,(\beta_j,s_j)_j}.$$

Proof. The proof is exactly the same as that of [AHS21, Proposition 2.9], which describes the case in $\widehat{\mathbb{C}}$ instead of $\mathbb{H}$. □

Lemma 2.2.10 ([ARS22, Lemma 3.14])**.** *Let $\alpha \in \mathbb{R}$ and $u \in \mathcal{S}$ with $\Re(u) = 0$, then we have*

$$\exp_* \mathrm{LF}_{\mathcal{S}}^{(\alpha,u)} = \mathrm{LF}_{\mathbb{H}}^{(\alpha,e^u)}.$$

Lemma 2.2.11 ([AHS21, Lemma 2.20]). *Let $\beta \in \mathbb{R}$ and $(\beta_3, s_3) \in \mathbb{R} \times \partial\mathcal{S}$, then we have*

$$\mathrm{LF}_{\mathbb{H}}^{(\beta,\infty),(\beta,0),(\beta_3,e^{s_3})} = e^{\frac{\beta_3^2}{4}\Re s_3} \exp_* \mathrm{LF}_{\mathcal{S}}^{(\beta,\pm\infty),(\beta_3,s_3)}.$$

Similarly, if $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ and $f \in \mathrm{conf}(\mathbb{H})$ satisfies $f(0) = 0, f(1) = 1$, and $f(-1) = \infty$, then

$$\mathrm{LF}_{\mathbb{H}}^{(\beta_1,\infty),(\beta_2,0),(\beta_3,1)} = 2^{\Delta_{\beta_1} - \Delta_{\beta_2} + \Delta_{\beta_3}} \cdot f_* \mathrm{LF}_{\mathbb{H}}^{(\beta_1,-1),(\beta_2,0),(\beta_3,1)}.$$

2.3. Liouville quantum gravities

The *Liouville quantum gravity* is a natural way to produce "random geometry" from the Gaussian free field. The study of natural probability laws on the space of two-dimensional Riemannian manifolds is called *two-dimensional quantum gravity.* By the Riemann uniformization theorem in complex analysis, every simply connected Riemann surface $\mathscr{S}$ is conformally equivalent to one of three Riemann surfaces: the open unit disk $\mathbb{D}$, the complex plane $\mathbb{C}$, or the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Therefore, $\mathscr{S}$ can be parametrized in coordinates $z = xi + y$ in one of such domains such that the metric takes the form

$$e^{\varphi(z)}(dx^2 + dy^2) \tag{2.30}$$

for some real-valued function φ. Therefore, the random Riemann surface $\mathscr{S}$ can be studied via the random function φ.

In Liouville quantum gravity, one views φ as the scalar multiple of the GFF and seeks to define the measure

$$\mu_h := e^{\gamma h(z)} d^2 z, \tag{2.31}$$

where h is an instance of GFF on some simply connected domain $D \subseteq \mathbb{C}$. Since h is a distribution, certain regularization procedure is needed in order to make (2.31) precise. The most common choice is to let $h_\varepsilon(z)$ be the circular average of h on $\partial B(z,\varepsilon)$ and set

$$\mu_h := \lim_{\varepsilon\to 0} \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)} d^2 z \tag{2.32}$$

It was in [DS11, Men17] that the above limit exists a.s. in the vague topology of Borel measures on D. The pair (D,μ_h) is described as a *random surface* $\mathscr{S}$ conformally parametrized by D with area measure μ_h. We can also parametrize $\mathscr{S}$ in a different domain $\widetilde{D}$. If $\psi : \widetilde{D} \to D$ is a conformal map, then we can write

$$\widetilde{h} = h \circ \psi + Q \log |\psi'|, \tag{2.33}$$

where $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$. The measure $\mu_{\widetilde{h}}$ on $\widetilde{D}$ is a.s. equivalent to the pullback via ψ^{-1} of the measure μ_h on D. It is shown in [Men17] that the (2.33) holds simultaneously for all possible ψ. The *quantum surface* is defined to be equivalence class of pairs (D,h) under relationship (2.33).

2.3.1. Quantum surfaces

Let $\gamma \in (0,2)$ and $\mathcal{DH} = \{(D,h) : D \subset \mathbb{C} \text{ open}, h \in C_0^\infty(D)'\}$. We define equivalence relation on $\mathcal{DH}$ by letting $(D,h) \sim_\gamma (\widetilde{D},\widetilde{h})$ if there is a conformal map $\psi : D \to \widetilde{D}$ such that $\widetilde{h} = \psi \bullet_\gamma h$, where

$$\psi \bullet_\gamma h := h \circ \psi^{-1} + Q \log |(\psi^{-1})'|. \tag{2.34}$$

A γ-*quantum surface* (a.k.a. γ-LQG surface) is an equivalence class of pairs $(D,h) \in \mathcal{DH}$ under the equivalence relation $\sim_\gamma$. An *embedding* of a quantum surface is a choice of representative (D,h). The transformation (2.34) is called the *coordinate change*. We can also consider quantum surfaces with marked points $(D,h,z_1,\dots,z_m,\omega_1,\dots,\omega_n)$ where $z_i \in D$ and $\omega_j \in \partial D$. We say

$$(D,h,z_1,\dots,z_m,\omega_1,\dots,\omega_n) \sim_\gamma (\widetilde{D},\widetilde{h},\widetilde{z_1},\dots,\widetilde{z_m},\widetilde{\omega_1},\dots,\widetilde{\omega_n})$$

if there is a conformal map $\psi : D \to \widetilde{D}$ such that $\widetilde{h} = \psi \bullet_\gamma h$ and $\psi(z_i) = \widetilde{z_i}, \psi(\omega_j) = \widetilde{\omega_j}$. Let $\mathscr{D}_{m,n}$ denote the set of equivalence class of such tuples under $\sim_\gamma$ and let $\mathscr{D} = \mathscr{D}_{0,0}$ for simplicity. We use (2.34) to define the equivalence relation because γ-LQG quantum area and γ-LQG quantum length measure is invariant under pushforward $\bullet_\gamma$. Since we will mainly work with $\mathbb{H}$, we view the set $\mathscr{D}_{m,n}$ as the quotient space

$$\{(\mathbb{H}, h, z_1, \ldots, z_m, \omega_1, \ldots, \omega_n) : h \text{ is a distribution on } \mathbb{H}, z_1 \ldots, z_m \in \mathbb{H}, \omega_1, \ldots, \omega_n \in \overline{\mathbb{R}}\}/\sim_\gamma .$$

The Borel σ-algebra of $\mathscr{D}_{m,n}$ is induced by the Borel sigma algebra on $H^{-1}(\mathbb{H})$.

In this thesis, we will work with quantum surfaces whose distribution h looks like GFF locally. The concrete definition is given as below.

Definition 2.3.1 ([GHS19, Definition 3.8])**.** Fix a simply connected domain $D \subseteq \mathbb{C}$ and let h be a random distribution on D. For $z \in D$, we say the distribution h is *GFF-like* near z if there exists a constant $r > 0$ such that the law of $h|_{B(z,r)}$ is absolutely continuous w.r.t. that of $(\widetilde{h} + g)|_{B(z,r)}$, where $(\widetilde{h}, g)$ is a coupling of zero-boundary GFF $\widetilde{h}$ on $\overline{B(z,r)}$ with a random continuous function g on $B(z,r)$. If $z \in \partial D$ and ∂D is analytic near z, we similarly call h *GFF-like* near z is h is locally absolutely continuous w.r.t. a free boundary GFF plus a continuous function in a similar manner.

If h is a random distribution which is GFF-like near z, then the measure

$$\mu_h := \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)} d^2 z$$

can be defined in $B(z,r)$ for some r as in Definition 2.3.1. Similarly, if the domain D has non-trivial boundary and the random distribution $\overline{h}$ is GFF-like near $z \in \partial D$, then the random measure

$$\nu_{\overline{h}} := \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/4} e^{\frac{\gamma}{2} \overline{h}_\varepsilon(z)} dz$$

exists almost surely, where for $z \in \partial D$, $\overline{h}_\varepsilon(z)$ is the circular average of $\overline{h}$ on $\partial B(z, \varepsilon) \cap D$

(see [DS11]).

For some random distribution h, we call random measures μ_h and ν_h *quantum area measure* and *quantum boundary length measure* respectively.

2.3.2. Quantum disks and quantum spheres

We recall the definitions of two-pointed quantum disk introduced in [AHS20]. It is a family of measures on $\mathscr{D}_{0,2}$. It is initially defined on the horizontal strip $\mathcal{S} = \mathbb{R} \times (0, \pi)$. Let $\exp : \mathcal{S} \to \mathbb{H}$ be the exponential map $z \mapsto e^z$ and let $h_{\mathcal{S}} = h_{\mathbb{H}} \circ \exp$ where $h_{\mathbb{H}}$ is sampled from $P_{\mathbb{H}}$. We call $h_{\mathcal{S}}$ the *free boundary GFF* on $\mathcal{S}$. It is known that $h_{\mathcal{S}}$ can be written as the sum of h^c and h^ℓ where h^c is constant on $u + [0, i\pi], u \in \mathbb{R}$ and h^ℓ has mean zero on all such vertical lines. We call h^ℓ the *lateral component* of free boundary GFF.

Definition 2.3.2 (Thick quantum disk)**.** Let $W \geq \frac{\gamma^2}{2}$, and let $\beta = Q + \frac{\gamma}{2} - \frac{W}{\gamma}$. Let

$$Y_t = \begin{cases} B_{2t} - (Q-\beta)t & \text{if } t \geq 0, \\ \widetilde{B_{-2t}} + (Q-\beta)t & \text{if } t < 0, \end{cases}$$

where $(B_s)_{s\geq 0}, (\widetilde{B}_s)_{s\geq 0}$ are independent standard Brownian motions conditional on $B_{2s} - (Q-\beta)s < 0$ and $\widetilde{B_{2s}} - (Q-\beta)s < 0$ for all $s > 0$. Let $h^1(z) = Y_t$ for all z with $\Re(z) = t$. Let $h^2(z)$ be the lateral component of free boundary GFF on $\mathcal{S}$ and let $\mathbf{c}$ be sampled from $\frac{\gamma}{2} e^{(\beta - Q)c} dc$ independent of h^1 and h^2. Let $\widehat{h}(z) = h^1(z) + h^2(z)$ and let $\phi(z) = \widehat{h}(z) + \mathbf{c}$. Let $\mathcal{M}_{0,2}^{\text{disk}}(W)$ denote the infinite measure on $\mathscr{D}_{0,2}$ describing the law of $(\mathcal{S}, \phi, -\infty, +\infty)$. We call a sample from $\mathcal{M}_{0,2}^{\text{disk}}(W)$ a weight-W quantum disk.

Definition 2.3.3 (Thick disk with one additional boundary marked point)**.** For $W \geq \frac{\gamma^2}{2}$, we first sample $(\mathcal{S}, \phi, +\infty, -\infty)$ from $\nu_\phi(\mathbb{R})\mathcal{M}_{0,2}^{\text{disk}}(W)[d\phi]$, then sample $s \in \mathbb{R}$ according to the probability measure proportional to $\nu_\phi|_{\mathbb{R}}$. We denote the law of the surface $(\mathcal{S}, \phi, +\infty, -\infty)/\sim_\gamma$ by $\mathcal{M}_{2,\bullet}^{\text{disk}}(W)$.

Definition 2.3.4 (Thin quantum disk)**.** Let $0 < W < \frac{\gamma^2}{2}$ and define the infinite measure $\mathcal{M}_{0,2}^{\text{disk}}(W)$ on two-pointed beaded surfaces as follows: first take T according to $(1-\frac{2}{\gamma^2}W)^{-2}\text{Leb}_{\mathbb{R}_+}$, then sample a Poisson point process $\{(u,\mathcal{D}_u)\}$ according to $\text{Leb}_{[0,T]}\times\mathcal{M}_{0,2}^{\text{disk}}(\gamma^2-W)$ and concatenate the $\mathcal{D}_u$ according to ordering induced by u.

When $W=2$, the two marked points of $\mathcal{M}_{0,2}^{\text{disk}}(2)$ are *typical* w.r.t. the quantum boundary length measure (see [AHS20, Proposition A.8]).

Definition 2.3.5. Let $(\mathcal{S},\phi,-\infty,+\infty)$ be an embedding of a sample from $\mathcal{M}_{0,2}^{\text{disk}}(2)$. Let $A=\mu_\phi(\mathcal{S})$ denote the total quantum area and $L=\nu_\phi(\partial\mathcal{S})$ denote the total quantum boundary length. Let QD denote the law of $(\mathcal{S},\phi)$ under reweighted measure $L^{-2}\mathcal{M}_{0,2}^{\text{disk}}(2)$, viewed as a measure on $\mathscr{D}$ by forgetting two marked points. For non-negative integers m and n, let $(\mathcal{S},\phi)$ be a sample from $A^mL^n\text{QD}$, then independently sample $z_1,\dots,z_m$ and $\omega_1,\dots,\omega_n$ according to $\mu_\phi^\#$ and $\nu_\phi^\#$, respectively. Let $\text{QD}_{m,n}$ denote the law of $(\mathcal{S},\phi,z_1,\dots,z_m,\omega_1,\dots,\omega_n)$ viewed as a measure on $\mathscr{D}_{m,n}$. We call a sample from $\text{QD}_{m,n}$ quantum disk with m bulk and n boundary marked points.

Let $\mathcal{C}$ denote the horizontal cylinder obtained by identifying two boundaries of $\mathbb{R}\times[0,2\pi]$ and let $h_{\mathcal{C}}$ be the centered Gaussian process with covariance kernel given by

$$G_{\mathcal{C}}(z,w):=-\log|e^z-e^\omega|+\max(\Re z,0)+\max(\Re\omega,0). \tag{2.35}$$

Let $H(\mathcal{C})$ be the Hilbert space closure of

$$\left\{f\in C^\infty(\mathcal{C}):\int_{\mathcal{C}}f(z)d^2z=0\right\}$$

with respect to the Dirichlet inner product $\langle\cdot,\cdot\rangle_\nabla$ (2.8). For notational convenience, we write the line segment $\{t\}\times[0,2\pi]$ on $\mathcal{C}$ as $[t,t+2\pi i]$ for each $t\in\mathbb{R}$. It is well-known that we have the *orthogonal decomposition* of Hilbert space (see, e.g. [She07])

$$H(\mathcal{C})=H_1(\mathcal{C})\oplus H_2(\mathcal{C}),$$

where $H_1(\mathcal{C}) \subset H(\mathcal{C})$ (resp. $H_2(\mathcal{C}) \subset H(\mathcal{C})$) denotes the subspace consisting of functions which are constant (resp. have mean zero) on $[t, t+2\pi i]$ for each $t \in \mathbb{R}$.

Definition 2.3.6 (Two-pointed quantum sphere, [AHS21, Definition 2.2])**.** For $W > 0$, let $\alpha = Q - \frac{W}{2\gamma} < Q$. Let

$$Y_t = \begin{cases} B_t - (Q-\alpha)t & \text{if } t \geq 0, \\ \widetilde{B_{-t}} + (Q-\alpha)t & \text{if } t < 0, \end{cases}$$

where $(B_s)_{s\geq 0}$ is a standard Brownian motion conditioned on $B_s - (Q-\alpha)s < 0$ for all $s > 0$. The $(\widetilde{B_s})_{s\geq 0}$ is an independent copy of $(B_s)_{s\geq 0}$. Let $h^1_{\mathcal{C}}(z) = Y_{\Re z}$ for $z \in \mathcal{C}$ and let $h^2_{\mathcal{C}}$ be independent of $h^1_{\mathcal{C}}$ and have the projection of $h_{\mathcal{C}}$ onto $H_2(\mathcal{C})$. Let $\widehat{h} = h^1_{\mathcal{C}} + h^2_{\mathcal{C}}$ and let $\mathbf{c}$ be a real number sampled from $\frac{\gamma}{2} e^{2(\alpha-Q)c} dc$ independent of $\widehat{h}$. Let $\phi = \widehat{\phi} + \mathbf{c}$. Let $\mathcal{M}^{\text{sph}}_2(W)$ be the infinite measure describing the law of $(\mathcal{C}, \phi, -\infty, +\infty)/\sim_\gamma$. We call a sample from $\mathcal{M}^{\text{sph}}_2(W)$ a two-pointed weight-W quantum sphere.

In the sphere case, the marked points are *typical* w.r.t. the quantum area measure when $W = 4 - \gamma^2$ (see [DMS20, Proposition A.13]).

Definition 2.3.7. Let $(\mathcal{C}, \phi, +\infty, -\infty)/\sim_\gamma$ be sampled from $\mathcal{M}^{\text{sph}}_2(4-\gamma^2)$. Let QS be the law of $(\mathcal{C}, \phi)/\sim_\gamma$ under the re-weighted measure $\mu_\phi(\mathcal{C})^{-2}\mathcal{M}^{\text{sph}}_2(4-\gamma^2)$. For $m \geq 0$, first sample $(\mathcal{C}, \phi)$ from $\mu_\phi(\mathcal{C})^m\text{QS}$, then sample m independent points $z_1, \ldots, z_m$ according to $\mu^{\#}_\phi$. Let QS_m be the law of $(\mathcal{C}, \phi, z_1, \ldots, z_m)/\sim_\gamma$. We call a sample from QS_m a quantum sphere with m marked points.

2.3.3. Relationships with the Liouville Conformal Field Theory

As reviewed in Section 2.2, the Liouville Conformal Field Theory is the Quantum Field Theory (QFT) corresponding the Liouville action functional which originates in [Pol81]. For each two dimensional Riemannian manifold, the LCFT associates it to a random field, which altogether form a conformal field theory. As mentioned in Section 1.1.1, LCFT was made rigorous in probability theory in [DKRV16] and [HRV18] for the case of Riemann sphere and simply connected domain with boundary respectively and in [DRV15, Rem17, GRV19] for the case of other topologies.

The LCFT and quantum surfaces provide two perspectives on LQG surfaces. Their close relationships has been demonstrated by Aru, Huang and Sun [AHS17] for the case of QS_3 and by Cerclé [Cer21] for the case of $\mathrm{QD}_{0,3}$. Precisely, [AHS17, Theorem 1.1] showed that, modulo some multiplicative constant, QS_3 has the law of $\mathrm{LF}_{\widehat{\mathbb{C}}}^{(\gamma,z_1),(\gamma,z_2),(\gamma,z_3)}$ under the particular embedding of $(\widehat{\mathbb{C}}, z_1, z_2, z_3)$. In the exact same spirit, [Cer21, Theorem 1.1] showed that $\mathrm{QD}_{0,3}$ has the law of $\mathrm{LF}_{\mathbb{H}}^{(\gamma,p_1),(\gamma,p_2),(\gamma,p_3)}, p_1, p_2, p_3 \in \partial\mathbb{H}$ when embedded into $(\mathbb{H}, p_1, p_2, p_3)$.

Notice that in all the cases above, we have enough marked points to fix the conformal structure. Traditionally, in the context of LCFT, we tend to assume that there are enough marked points since otherwise, we cannot properly define the Liouville correlation functions. What about the case when there are not enough marked points on the surface to fix the conformal structure?

The answer is given in [AHS21] and we should consider *uniform embeddings* of quantum surfaces. In words, when there are not enough marked points, the LCFT describes the law of quantum surfaces under uniform embeddings of marked points (maximal symmetries).

Now we set up the uniform embedding carefully. The discussion here is more general than that in subsection 1.1.1 before Theorem 1.1.3 in the sense that we also allow the domain $D = \widehat{\mathbb{C}}$ and more than one (bulk and/or boundary) marked points. Concretely, for simply connected domain D conformally equivalently to either $\widehat{\mathbb{C}}$ or $\mathbb{H}$, let $\mathrm{conf}(D)$ be the group of conformal automorphisms of D where group multiplication $\cdot$ is the function composition $f \cdot g = f \circ g$. Let $\mathbf{m}_D$ be a Haar measure on $\mathrm{conf}(D)$, which is both left and right invariant. Suppose $\mathfrak{f}$ is sampled from $\mathbf{m}_D$ and $\phi \in H^{-1}(D)$, then we call the random function $\mathfrak{f} \bullet_\gamma \phi = \phi \circ \mathfrak{f}^{-1} + Q|\log(\mathfrak{f}^{-1})'|$ the *uniform embedding* of (D, ϕ) via $\mathbf{m}_D$. By invariance property of Haar measure, the law of $\mathfrak{f} \bullet_\gamma \phi$ only depends on (D, ϕ) as quantum surface.

We write $\mathbf{m}_{\widehat{\mathbb{C}}} \ltimes \mathrm{QS}$ as the law of $\mathfrak{f} \bullet_\gamma \phi$, where $\mathfrak{f}$ is sampled from $\mathbf{m}_{\widehat{\mathbb{C}}}$ and $(\widehat{\mathbb{C}}, \phi)$ is an embedding of a sample from QS independent of $\mathbf{m}_{\widehat{\mathbb{C}}}$. We call $\mathbf{m}_{\widehat{\mathbb{C}}} \ltimes \mathrm{QS}$ the *uniform embedding of* QS via Haar measure $\mathbf{m}_{\widehat{\mathbb{C}}}$. The $\mathbf{m}_{\mathbb{D}} \ltimes \mathrm{QD}$ is defined in the exact same way.

Theorem 2.3.8 ([AHS21, Theorem 1.2]). *There exist constants C_1 and C_2 such that*

$$\mathbf{m}_{\widehat{\mathbb{C}}} \ltimes \mathrm{QS} = C_1 \cdot \mathrm{LF}_{\widehat{\mathbb{C}}} \qquad \textit{and} \qquad \mathbf{m}_{\mathbb{H}} \ltimes \mathrm{QD} = C_2 \cdot \mathrm{LF}_{\mathbb{H}}. \tag{2.36}$$

We can also consider the quantum surface with some (bulk and/or boundary) marked points but not enough to fix a conformal structure. Fix $a, b \in D \cup \partial D$ and let $\mathrm{conf}(D, a, b)$ be the subgroup of $\mathrm{conf}(D)$ fixing points a and b. Let $\mathbf{m}_{D,a,b}$ be a Haar measure on $\mathrm{conf}(D, a, b)$. The quantum surface with two (bulk and/or boundary) marked points can be identified as a measure on $C_0^\infty(D)'/\mathrm{conf}(D, a, b)$. Therefore, we can define, for instance, $\mathbf{m}_{D,a,b} \ltimes$ (quantum surface with two marked points) in the exact same way as $\mathbf{m}_{\widehat{\mathbb{C}}} \ltimes \mathrm{QS}$ and $\mathbf{m}_{\mathbb{D}} \ltimes \mathrm{QD}$.

Some other LCFT representations of the quantum surfaces without fixed conformal structures were also proved in [AHS21]. The below theorem describes the case the two-pointed quantum disk.

Theorem 2.3.9 ([AHS21, Theorem 2.22]). *Fix $W > \frac{\gamma^2}{2}$ and $\beta_W = \gamma + \frac{2-W}{\gamma}$. If we independently sample T from $\mathrm{Leb}_{\mathbb{R}}$ and $(\mathcal{S}, \phi, +\infty, -\infty)$ from $\mathcal{M}_{0,2}^{\mathrm{disk}}(W)$, then the law of $\widetilde{\phi} := \phi(\cdot + T)$ is $\frac{\gamma}{2(Q-\beta_W)^2}$ is $\frac{\gamma}{2(Q-\beta_W)^2}\mathrm{LF}_{\mathcal{S}}^{(\beta_W, \pm\infty)}$.*

Notice that two points on the boundary are not sufficient to fix a conformal structure of $\mathcal{S}$. Therefore, we have one degree of freedom described by horizontal shifting under Lebesgue measure. In the language of uniform embedding, Theorem 2.3.9 also tells us that the uniform embedding of $\mathcal{M}_{0,2}^{\mathrm{disk}}(W)$ in $(\mathcal{S}, +\infty, -\infty)$ has the law of LCFT with two boundary marked points module some multiplicative constant. Concretely speaking, we have

$$\mathbf{m}_{\mathcal{S},+\infty,-\infty} \ltimes \mathcal{M}_{0,2}^{\mathrm{disk}}(W) = \frac{\gamma}{2(Q-\beta_W)^2}\mathrm{LF}_{\mathcal{S}}^{(\beta_W, \pm\infty)}. \tag{2.37}$$

When $W = 2$, the equation (2.37) becomes

$$\mathbf{m}_{\mathcal{S},+\infty,-\infty} \ltimes \mathrm{QD}_{0,2} = \frac{\gamma}{2(Q-\gamma)^2} \mathrm{LF}_{\mathcal{S}}^{(\gamma,\pm\infty)}. \tag{2.38}$$

Similarly, in the sphere case, we have

$$\mathbf{m}_{\widehat{\mathbb{C}},p,q} \ltimes \mathrm{QS}_2 = C \cdot \mathrm{LF}_{\widehat{\mathbb{C}}}^{(\gamma,p),(\gamma,q)} \tag{2.39}$$

for some finite constant C. Note that equations (2.38) and (2.39) should be viewed as the disintegration of (2.36) over their marked points respectively.

CHAPTER 3

SCHRAMM-LOEWNER EVOLUTIONS, CONFORMAL WELDINGS OF QUANTUM SURFACES, AND $\mathrm{SLE}_\kappa(\rho)$ BUBBLE MEASURES

In this chapter, we review key results on Schramm-Loewner evolutions, conformal weldings of quantum surfaces, and the construction of $\mathrm{SLE}_\kappa(\rho)$ bubbles. Nothing substantial is proved in this chapter except some simple variations of known results that cannot be found in the literature.

3.1. Schramm-Loewner evolutions

3.1.1. Overview

In 1999, Schramm [Sch00] wanted to construct−for any simply connected domain $D \subseteq \mathbb{C}$ with boundary points a and b−a random non-self-crossing fractal curve η connecting a and b. He hoped that this η has the following two *nice* properties:

- The law of an SLE curve η is *conformally invariant* in the sense that if ψ is a conformal map (by definition, analytic and one-to-one) taking D to some other domain $\psi(D)$, then the image of η under ψ is again an SLE curve in $\psi(D)$ from $\psi(a)$ to $\psi(b)$ (up to different time parametrizations).

- The SLE path η is *Markovian* in the sense that conditioning on $\eta[0,\tau]$ for some positive stopping time τ and the event that η is not completed at τ, the conditional law of the rest of η is again an SLE in $D\backslash\eta[0,\tau]$.

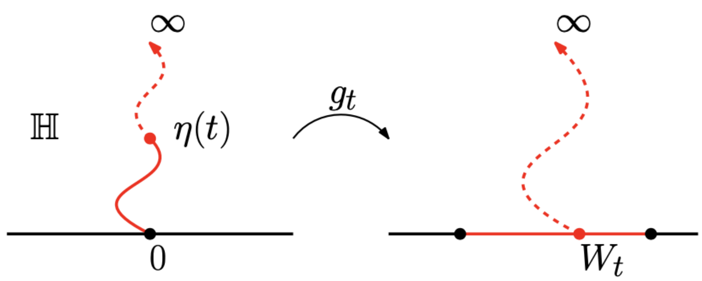


Figure 3.1: The $g_t(z) : \mathbb{H}\backslash\eta[0,t] \to \mathbb{H}$ is the Loewner evolution and $g_t(z) = z + \frac{2t}{z} + o(|z|^{-1})$ as $|z| \to \infty$ under capacity parametrizations.

Schramm [Sch00] showed that there is one and only one way of defining this family of random fractal curves if one insists on these properties. This family of curves is indexed by a parameter $\kappa \in [0, \infty)$. By conformal invariance property of SLE curves, it suffices to define the law of η on the upper half plane $\mathbb{H} \subseteq \mathbb{C}$ with $a = 0$ and $b = \infty$. Define the analytic functions $g_t : \mathbb{H}\backslash\eta[0,t] \to \mathbb{H}$ by requiring that $g_0(z) = z$ and for any fixed $z \in \mathbb{H}$, the ODE

$$\partial_t g_t(z) = \frac{2}{g_t - W_t} \tag{3.1}$$

is satisfied up to the stopping time τ when z is first hit by the curve $\eta[0,\tau]$, where $W_t := \sqrt{\kappa}B_t = B_{\kappa t}$ is the standard Brownian motion scaled by a factor of $\sqrt{k}$ (or sped up by a factor of κ). This requirement uniquely determines the functions g_t which in turn determine the curve η.

Intuitively, the bigger κ is (the faster Brownian motion moves up and down) the "more wildly "the curve becomes. It was shown by Rohde and Schramm [RS05] that η is a.s. a simple curve when $\kappa \in [0,4]$. The η a.s. hits (but does not cross) itself when $\kappa \in [4,8]$ and the η is a.s. *space-filling* when $\kappa \geq 8$.

3.1.2. Chordal $\mathrm{SLE}_\kappa(\underline{\rho})$ processes

In this subsection, we review the basic construction of chordal $\mathrm{SLE}_\kappa(\underline{\rho})$ process. The chordal $\mathrm{SLE}_\kappa(\underline{\rho})$ process, which was first studied in [LSW03, RS05], is a natural variant of chordal SLE_κ where one keeps track of extra marked *force points*.

First, we introduce some notations and terminologies that will be carried out throughout this thesis. Let (E, d_E) be a metric space and let $C([0,\widehat{T}],E)$ be the space of continuous functions from $[0,\widehat{T})$ to E. Let

$$\Sigma^E = \bigcup_{0<\widehat{T}\leq\infty} C([0,\widehat{T}],E).$$

For each $f \in \Sigma^E$, the lifetime $\widehat{T_f}$ of f is the extended number in $(0,\infty]$ such that $[0,\widehat{T_f})$ is the domain of f. Let $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ be the open upper half plane. A set $K \subset \mathbb{H}$ is called an $\mathbb{H}$-hull if K is bounded and $\mathbb{H}\backslash K$ is a simply connected domain. For each $\mathbb{H}$-hull K, there is a unique conformal map g_K from $\mathbb{H}\backslash K$ onto $\mathbb{H}$ such that $g_K(z) - z = O(1/z)$ as $z \to \infty$. The number $\text{hcap}(K) := \lim_{z\to\infty} z(g_K(z) - z)$ is called $\mathbb{H}$-capacity of K, which satisfies $\text{hcap}(\emptyset) = 0$ and $\text{hcap}(K) > 0$ if $K \neq \emptyset$. Let

$$\text{rad}_\omega(K) := \sup\{|z-\omega| : z \in K \cup \{\omega\}\} \tag{3.2}$$

for $\omega \in \mathbb{C}$ and $K \subset \mathbb{C}$. For $W \in \Sigma^{\mathbb{R}}$, the chordal Loewner equation driven by W is the following differential equation in $\mathbb{C}$:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}$$

with $0 \leq t < \widehat{T_W}$ and $g_0(z) = z$. For each $z \in \mathbb{C}$, let τ_z^* be the biggest extended number in $[0,\widehat{T_W}]$ such that the solution $t \mapsto g_t(z)$ exists on $[0,\tau_z^*)$. For $0 \leq t < \widehat{T_W}$, let $K_t = \{z \in \mathbb{H} : \tau_z^* \leq t\}$ and $H_t = \mathbb{H}\backslash K_t$. It turns out that each K_t is an $\mathbb{H}$-hull with $\text{hcap}(K_t) = 2t$ and $g_t = g_{K_t}$. We call g_t and K_t the chordal Loewner maps and hulls respectively.

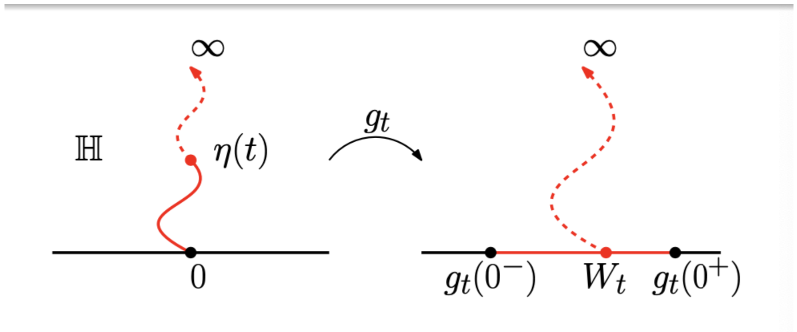


Figure 3.2: The chordal $\mathrm{SLE}_\kappa(\rho_-,\rho_+)$ process: initially, we have two force points at 0^- and 0^+ with weights ρ_- and ρ_+ respectively. Correspondingly, we have two additional force point processes V_t^- and V_t^+.

We now review the definition of multi-force-point $\mathrm{SLE}_\kappa(\underline{\rho})$ process. Here, all the force points lie on the boundary. Let $\kappa > 0$ and $\underline{\rho} = (\rho_1, \dots, \rho_m) \in \mathbb{R}^m$. Let $\omega \in \mathbb{R}$ and $v_1, \dots, v_m$ be such that

$$\sum_{j:v_j=\omega^+} \rho_j > -2 \qquad \text{and} \qquad \sum_{j:v_j=\omega^-} \rho_j > -2. \tag{3.3}$$

Consider the following system of SDE:

$$\begin{aligned} dW_t &= \sum_{j=1}^m \mathbb{1}_{\{W_t \neq V_t^j\}} \frac{\rho_j}{W_t - V_t^j} dt + \sqrt{\kappa} dB_t, \qquad W_0 = \omega; \\ dV_t^j &= \mathbb{1}_{\{W_t \neq V_t^j\}} \frac{2}{V_t^j - W_t} dt, \qquad V_0^j = v_j, \qquad 1 \le j \le m. \end{aligned} \tag{3.4}$$

If some $v_j = \infty$, then V_t^j is ∞, and $\frac{1}{V_t^j - W_t^j}$ is 0. It is known that a weak solution of the system (3.4), in the integral sense, exists and is unique in law, and the W_t in the solution a.s. generates a Loewner curve η, which we call $\mathrm{SLE}_\kappa(\underline{\rho})$ curve starts from ω with force points $\underline{v} = (v_1, \dots, v_m)$. The V_t^j is called the *force point process* started from v_j.

3.2. Conformal weldings of quantum surfaces

In this section, we review the key results in [AHS20] and [AHS22] regarding the conformal welding of quantum disks and quantum spheres.

First we define for $W > 0$ and $\ell, \ell' > 0$, the family of measures $\{\mathcal{M}_{0,2}^{\mathrm{disk}}(W; \ell, \ell')\}_{\ell,\ell'>0}$ such that $\mathcal{M}_{0,2}^{\text{disk}}(W;\ell,\ell')$ is supported on quantum surfaces with left and right boundary arc lengths ℓ and ℓ' respectively. This family of measures satisfies the following disintegration relation:

$$\mathcal{M}_{0,2}^{\text{disk}}(W) = \int_0^\infty \int_0^\infty \mathcal{M}_{0,2}^{\text{disk}}(W;\ell,\ell')d\ell d\ell'. \tag{3.5}$$

The disintegration (3.5) characterizes $\mathcal{M}_{0,2}^{\text{disk}}(W;\ell,\ell')$ modulo a Lebesgue measure zero set of (ℓ,ℓ'). This ambiguity was removed by some suitable topology introduced in [AHS20, Section 2.6 and 4] for which the measure $\mathcal{M}_{0,2}^{\text{disk}}(W;\ell,\ell')$ is continuous in (ℓ,ℓ').

Theorem 3.2.1 (Conformal welding of two quantum disks, [AHS20, Theorem 2.2])**.** *Let $W_1, W_2 > 0$ and there exists some constant $c = c_{W_1,W_2}$ such that*

$$\mathcal{M}_{0,2}^{\text{disk}}(W_1+W_2;\ell,\ell') \otimes \text{SLE}_\kappa(W_1-2,W_2-2) = c\int_0^\infty \mathcal{M}_{0,2}^{\text{disk}}(W;\ell,r) \times \mathcal{M}_{0,2}^{\text{disk}}(W;r,\ell')dr. \tag{3.6}$$

When $W_1+W_2 \geq \frac{\gamma^2}{2}$, the $\mathcal{M}_{0,2}^{\text{disk}}(W_1+W_2;\ell,\ell') \otimes \text{SLE}_\kappa(W_1-2,W_2-2)$ in (3.6) denote the measure on the curve-decorated quantum surfaces obtained by first sampling a quantum disk $(\mathcal{S},\psi,+\infty,-\infty)$ according to $\mathcal{M}_{0,2}^{\text{disk}}(W;\ell,\ell')$ with an arbitrary embedding then independently sampling η according to $\text{SLE}_\kappa(W_1-2,W_2-2)$ on $(\mathcal{S},+\infty,-\infty)$.

When $W \in (0,\frac{\gamma^2}{2})$, the measure $\mathcal{M}_{0,2}^{\text{disk}}(W_1+W_2;\ell,\ell') \otimes \text{SLE}_\kappa(W_1-2,W_2-2)$ corresponds to sampling independent $\text{SLE}_\kappa(W_1-2,W_2-2)$ in each component of the thin quantum disk.

Here we emphasize that for all $W > 0$, the $\mathcal{M}_{0,2}^{\text{disk}}(W_1+W_2;\ell,\ell') \otimes \text{SLE}_\kappa(W_1-2,W_2-2)$ is a measure on the curve-decorated quantum surface (equivalence class of surfaces), which means it does not depend on the particular embedding. In this thesis, we will frequently encounter this kind of measures (on different curve-decorated quantum surface) and they are defined in the exact same manner.

The following theorem describes the conformal welding of n quantum disks, which is a natural generalization of Theorem 3.2.1.

Theorem 3.2.2 (Conformal welding of n disks, [AHS20, Theorem 2.2]). *Fix $W_1, \dots, W_n > 0$ and $W = W_1 + \dots + W_n$. There exists a constant $C = C_{W_1,\dots,W_n} \in (0, \infty)$ such that for all $\ell, r > 0$, the identity*

$$\begin{aligned} &\mathcal{M}_{0,2}^{\text{disk}}(W; \ell, r) \otimes \mathcal{P}^{\text{disk}}(W_1, \dots, W_n) \\ &= C \iiint_0^\infty \mathcal{M}_{0,2}^{\text{disk}}(W_1; \ell, \ell_1) \times \mathcal{M}_{0,2}^{\text{disk}}(W_2; \ell_1, \ell_2) \times \dots \times \mathcal{M}_{0,2}^{\text{disk}}(W_n; \ell_{n-1}, r) d\ell_1 \dots d\ell_{n-1} \end{aligned} \tag{3.7}$$

holds as measures on the space of curve-decorated quantum surfaces.

The measure $\mathcal{P}^{\text{disk}}(W_1, \dots, W_n)$ is defined in [AHS20, Definition 2.25] on tuple of curves $(\eta_1, \dots, \eta_{n-1})$ in a domain (D, x, y). It was defined by the following induction procedure: first sample η_{n-1} from $\text{SLE}_\kappa(W_1 + \dots + W_{n-1} - 2; W_n - 2)$ then $(\eta_1, \dots, \eta_{n-2})$ from

$$\mathcal{P}^{\text{disk}}(W_1, \dots, W_{n-1})$$

on connected component (D', x', y') on the left of $D \backslash \eta_{n-1}$ where x' and y' are the first and the last point hit by η_{n-1}.

As reviewed in Section 3.1.1, for simply connected domain (D, p, q) $(D \neq \mathbb{C})$ with two marked boundary points. The chordal SLE_κ is a family of conformally invariant random curves from p to q. When $0 < \kappa < 4$, SLE_κ is a.s. simple and only intersects ∂D at $\{p, q\}$.

One can also construct a whole-plane variant of SLE_κ: for $(\mathbb{C}, p, q)$ with $p \neq q$ and $\rho > -2$, there is a SLE-like random curve connecting p and q called the *whole-plane* $\text{SLE}_\kappa(\rho)$. The definition is not important for our presentation here so we simply omit it (check [MS17, Section 2.1.3]).

For $\kappa \in (0, 8)$, on $(\mathbb{C}, p, q)$, the *two-sided whole-plane* SLE_κ, which is denoted by $\text{SLE}_\kappa^{p \rightleftharpoons q}$, is defined by first running a whole-plane $\text{SLE}_\kappa(2)$ curve η_1 from p to q, then running a chordal

SLE_κ curve η_2 on $\mathbb{C}\backslash\eta_1$ from q to p. That being said, the $\mathrm{SLE}_\kappa^{p\rightleftharpoons q}$ is a probability measure on pairs of curves on $\mathbb{C}$ connecting p and q (form an oriented loop) and satisfying the *bi-chordal resampling property*: conditioning on one arm, the other arm is a chordal SLE_κ in the complement.

It was shown in [Bef08] that $\mathrm{SLE}_\kappa^{p\rightleftharpoons q}$ almost surely has Hausdorff dimension $d = 1 + \frac{\kappa}{8}$. Its d-dimensional *Minkowski content measure* $\mathrm{Cont}_d(\eta)$ exists [LR15]. The following *unrooted SLE Loop measure* was constructed by Zhan in [Zha21]:

$$\mathrm{SLE}_\kappa^{\mathrm{loop}}(d\eta) := \frac{1}{\mathrm{Cont}_d(\eta)^2}\int_{\mathbb{C}}\int_{\mathbb{C}} |p-q|^{-2(2-d)}\mathrm{SLE}_\kappa^{p\rightleftharpoons q}(d\eta)d^2pd^2q. \tag{3.8}$$

The rooted version can be easily obtained by the disintegration on the outside integral $(\int_{\mathbb{C}} \cdots dq^2)$ in (3.8). The (3.8) is a rather constructive definition in the sense that it tells us that the unrooted SLE loop measure $\mathrm{SLE}_\kappa^{\mathrm{loop}}(d\eta)$ can be sampled by the following three steps:

1. Sample a pair of points (p,q) according to the measure

$$|p-q|^{-2(2-d)}d^2pd^2q \qquad \text{on } \mathbb{C}\times\mathbb{C}. \tag{3.9}$$

2. Sample $\eta = (\eta_1, \eta_2)$ according to the two-sided whole-plane $\mathrm{SLE}_\kappa^{p\rightleftharpoons q}(d\eta)$.

3. Re-weight the η by the square of its Minkowski content measure $\mathrm{Cont}_d(\eta)^2$.

For $\gamma \in (0,2)$, recall that the *unmarked quantum disk* QD is defined in Definition 2.3.5. Let $\mathrm{QD}(\ell)$ be the disintegration of QD over its total boundary length, i.e.,

$$\mathrm{QD} = \int_0^\infty \mathrm{QD}(\ell)d\ell.$$

For $\ell > 0$, let $(\mathcal{D}_1, \mathcal{D}_2)$ be sampled from $\mathrm{QD}(\ell)\times\mathrm{QD}(\ell)$ and let $\mathrm{Weld}(\mathcal{D}_1,\mathcal{D}_2)$ be the curve-decorated quantum surface obtained by first uniformly sampling points a and b on the boundaries of $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively and then conformally welding $\mathcal{D}_1$ and $\mathcal{D}_2$ by identifying a and b. Let $\mathrm{Weld}(\mathrm{QD}(\ell), \mathrm{QD}(\ell))$ denote the distribution of $\mathrm{Weld}(\mathcal{D}_1, \mathcal{D}_2)$ and define

$$\mathrm{Weld}(\mathrm{QD}, \mathrm{QD}) := \int_0^\infty \ell \cdot \mathrm{Weld}(\mathrm{QD}(\ell), \mathrm{QD}(\ell)) d\ell.$$

Theorem 3.2.3 ([AHS22, Theorem 1.1])**.** *For $\gamma \in (0, 2)$, we have*

$$\mathrm{QS} \otimes \mathrm{SLE}_\kappa^{\mathrm{loop}} = C \cdot \mathrm{Weld}(\mathrm{QD}, \mathrm{QD}) \tag{3.10}$$

for some finite constant C.

The proof of above theorem relies on the *uniform embedding* of the three-pointed curve-decorated sphere $\mathrm{QS}_3 \otimes \mathrm{SLE}_\kappa^{p \rightleftharpoons q}$ (see [AHS22, Figure 1] for a nice summary of the proof pipeline).

3.3. $\mathrm{SLE}_\kappa(\rho)$ bubble measures

In this section, we review the basic terminologies and limiting constructions of rooted $\mathrm{SLE}_\kappa(\rho)$ bubble measure in [Zha22].

3.3.1. Basic notations and terminologies

First, we introduce some basic notations and terminologies. Let $f \in \Sigma^E$. For a continuous and strictly increasing function θ on $[0, \widehat{T_f})$ with $\theta(0) = 0$, the function $g := f \circ \theta^{-1} \in \Sigma^E$ is called the time-change of f via θ, and we write $f \sim g$. Let $\widetilde{\Sigma^E} := \Sigma^E / \sim$ and an element of $\widetilde{\Sigma^E}$, denoted by $[f]$, where $f \in \Sigma^E$, is called an MTC (module time-changes) function or curve. Throughout this thesis, all the curves considered are MTC curve. Therefore, we will simply write f instead of $[f]$ without confusion. The $\widetilde{\Sigma^E}$ is a metric space with the distance defined by

$$d_{\widetilde{\Sigma^E}} := \inf \left\{ \sup\{ d_E(f'(t), g'(t)) : 0 \le t < \widehat{T_{f'}} \} : f' \in [f], g' \in [g], \widehat{T_{f'}} = \widehat{T_{g'}} \right\}. \tag{3.11}$$

An element $f \in \Sigma^E$ is called a rooted loop if

$$\lim_{t \to \widehat{T}_f} f(t) = f(0)$$

and $f(0)$ is called its root. If $f \in \Sigma^E$ is called a rooted loop, then $[f] \in \widetilde{\Sigma}^E$ is called a rooted MTC loop. Notice that all the elements in $\text{Bubble}_{\mathbb{H}}(p)$ are MTC loops.

3.3.2. Constructions of $\text{SLE}_\kappa(\rho)$ bubble measures via radial Bessel processes

We now review Zhan's constructions on $\text{SLE}_\kappa(\rho)$ bubble measures via the *Radial Bessel Processes.*

Let $\delta_-, \delta_+ > 0$ and B_t be a standard Brownian motion. A stochastic process $(Z_t)_{t\geq 0}$ with $Z_0 = x \in [-1, 1]$ satisfying the following stochastic differential equation (SDE)

$$dZ_t = \sqrt{1 - Z_t^2} dB_t - \frac{\delta_+}{4}(Z_t + 1)dt - \frac{\delta_-}{4}(Z_t - 1)dt \tag{3.12}$$

is called the *Radial Bessel Process* with dimension (δ_+, δ_-) starting from x. It has a unique strong solution with infinite lifetime and we let $\nu_x^{\delta_+,\delta_-}$ denote its (probability) law on the path space $C([0, \infty), \mathbb{R})$. It stays in the interval $[-1, 1]$ and behaves like a squared Bessel process of dimension $\delta_\pm$ near ± 1. This process satisfies the Markov property in the sense that for any stopping time τ w.r.t. the filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$, conditioning on $(Z_t)_{0\leq t\leq \tau}$, the rest of the process is a radial Bessel process with dimension (δ_+, δ_-) starting from Z_τ.

Proposition 3.3.1 (Transition density of radial Bessel processes, [Zha22, Proposition 2.14])**.** *When $\delta_-, \delta_+ > 0$ and $x \in [-1, 1]$, the solution of (3.12) has the following transition kernel:*

$$p_t(x, y) = \omega_{\alpha_+,\alpha_-}(y) \sum_{n=0}^{\infty} \frac{P_n^{(\alpha_+,\alpha_-)}(x) P_n^{(\alpha_+,\alpha_-)}(y) e^{-\beta_n t}}{\int_{-1}^{1} \omega_{\alpha_+,\alpha_-}(s) P_n^{(\alpha_+,\alpha_-)}(s)^2 ds}, \tag{3.13}$$

where $\alpha_\pm = \frac{\delta_\pm}{2} - 1$, $\omega_{\alpha_+,\alpha_-}(s) = (1-s)^{\alpha_+}(1+s)^{\alpha_-}$, $\beta_n = \frac{1}{2}n(n+1+\alpha_+ + \alpha_-)$, and $P_n^{(\alpha_+,\alpha_-)}$ are Jacobi polynomials with parameters (α_+, α_-), which is a class of orthogonal polynomials w.r.t. $\mathbb{1}_{(-1,1)}\omega_{\alpha_+,\alpha_-}$ (cf. [KWKS]).

Proposition 3.3.2 (Invariant probability density of radial Bessel processes, [Zha22, Proposition 2.15])**.** *Under the above settings, let*

$$p_\infty(x) = \frac{\omega_{\alpha_+,\alpha_-}(x)}{\int_{-1}^{1} \omega_{\alpha_+,\alpha_-}(s)ds} \tag{3.14}$$

and we have that for $y \in [-1, 1]$ *and* $t > 0$*,*

$$\int_{-1}^{1} p_\infty(x)p_t(x, y)dx = p_\infty(y). \tag{3.15}$$

Moreover, there exist constants $C, L \in (0, \infty)$ *such that for any* $x, y \in [-1, 1]$*,*

$$|p_t(x, y) - p_\infty(y)| < C \cdot p_\infty(y) \cdot e^{-\frac{\delta_+ + \delta_-}{4} t} \tag{3.16}$$

for $t > L$*.*

By standard arguments in SDE, there exists a stochastic process $(Z_t)_{t\in\mathbb{R}}$ such that for any fixed $\tau \in \mathbb{R}$, Z_τ follows the law of $\mathbb{1}_{(-1,1)} p_\infty(y)dy$. Moreover, conditioning on $(Z_t)_{t\leq\tau}$, the random process $(Z_{t+t_0})_{t\geq 0}$ is again a radial Bessel process with dimension (δ_+, δ_-) starting from Z_{t_0}. We call such process Z_t the *stationary radial Bessel process with dimension* (δ_+, δ_-). Let $\nu_{\mathbb{R}}^{\delta_+,\delta_-}$ denote its unique probability law on $C(\mathbb{R}, \mathbb{R})$.

In (3.12), when $\delta_+ > 0$, $\delta_- < 2$, and the initial value $x \in (-1, 1]$. The process Z_t will never visit $(1, \infty)$ but will visit -1 at some finite time. Let $\mu_x^{\delta_+,\delta_-}$ denote the law of Z_t killed once it hits -1.

Lemma 3.3.3 ([Zha22, Lemma 3.6])**.** *Let* $\delta_-^* = 4 - \delta_- > 2$ *and we have that*

$$\left.\frac{d\mu_x^{\delta_+,\delta_-}}{d\nu_x^{\delta_+,\delta_-^*}}(Z)\right|_{\mathcal{F}_\tau} = \frac{M_\tau^Z}{M_0^Z} \tag{3.17}$$

for any stopping time τ and

$$M_t^Z := e^{-\frac{1}{8}\delta_+(2-\delta_-)t}\left(\frac{1+Z_t}{2}\right)^{\frac{\delta_-}{2}-1}. \tag{3.18}$$

The relationship (3.17) allows us to show that there exists a σ-finite measure $\mu_{\mathbb{R}}^{\delta_+,\delta_-}$ [Zha22, Lemma 3.7] on the space

$$\Sigma^{\mathbb{R}} := \bigcup_{\widehat{T}\in(-\infty,\infty]} C((-\infty,\widehat{T}),\mathbb{R})$$

such that

$$\left.\frac{d\mu_{\mathbb{R}}^{\delta_+,\delta_-}}{d\nu_{\mathbb{R}}^{\delta_+,\delta_-^*}}(Z)\right|_{\mathcal{F}_\tau} = M_\tau^Z. \tag{3.19}$$

In fact, one can derive the radial Bessel process from the chordal $\mathrm{SLE}_\kappa(\rho_-,\rho_+)$ process. Let η be an instance of $\mathrm{SLE}_\kappa(\rho_-,\rho_+)$ curve starting from 0 with force points $v^+ > 0 > v^-$. Let W_t be the driving function and let V_t^- and V_t^+ be the other two force point processes.

Define

$$Z_t := \frac{2W_t - V_t^+ - V_t^-}{V_t^+ - V_t^-} \tag{3.20}$$

and let

$$p(t) = \frac{\kappa}{2}\log\left(\frac{V_t^+ - V_t^-}{v^+ - v^-}\right). \tag{3.21}$$

Let $\widehat{Z_s} := Z_{p^{-1}(s)}$ and $\widehat{Z_s}$ is a radial Bessel process with dimension (δ_+,δ_-), where $\delta_\pm = \frac{4}{\kappa}(\rho_\pm + 2)$.

By SLE coordinate change [SW05], we are interested in the case when

$$\rho_+ = \rho \qquad \text{and} \qquad \rho_- = \kappa - 6 - \rho.$$

The process is stopped when the curve swallows v^-. This corresponds to the radial Bessel process is stopped once hits -1. Recall that the truncated radial Bessel at -1 has the law of μ^{δ_+,δ_-}.

Let $\rho_-^* = \kappa - 6 - \rho_- \geq \frac{\kappa}{2} - 2$ and $\delta_-^* = \frac{4}{\kappa}(\rho_-^* + 2) = 4 - \delta_- > 2$. Next, we consider the $\mathrm{SLE}_\kappa(\rho_+, \rho_-^*)$ process starting from 0 with force points at 0^+ and 0^-. Let Z_t and $p(t)$ be as in (3.20) and (3.21) respectively and let $\widehat{Z_s} = Z_{p^{-1}(s)}$. Then $(\widehat{Z_s})_{s\in\mathbb{R}}$ has the law of stationary Bessel with law $\nu_{\mathbb{R}}^{\delta_+,\delta_-^*}$.

Moreover, we can recover W from $\widehat{Z_t}$ from the following [Zha22, Lemma 3.14] :

$$\begin{cases} \widehat{V_s^{\pm}} = \int_{-\infty}^{s} \frac{1}{\kappa}\left(1 \pm \widehat{Z_r}\right) e^{\frac{4}{\kappa}r} dr \\ \widehat{W}_s = \frac{1+\widehat{Z_s}}{2}\widehat{V_s^+} + \frac{1-\widehat{Z_s}}{2}\widehat{V_s^-} \\ q(t) = \int_{-\infty}^{s} \frac{1}{4\kappa}\left(1 - \left(\widehat{Z_r^2}\right)\right) e^{\frac{4}{\kappa}r} dr \\ W_t = \widehat{W_{q^{-1}(t)}} \end{cases} \tag{3.22}$$

Recall the measure $\mu_{\mathbb{R}}^{\delta_+,\delta_-}$ from the Randon-Nikodym derivative relationship (3.19). The push-forward of $\mu_{\mathbb{R}}^{\delta_+,\delta_-}$ under the composition map $\widehat{Z_t} \mapsto W \mapsto \eta$ is an infinite measure on $\mathrm{Bubble}_{\mathbb{H}}(0)$, which we denote by μ_0. When $\kappa \in (0,4)$ and $\rho > -2$, under suitable capacity parametrization, it is the measure that we want.

Theorem 3.3.4 (Existence of $\mathrm{SLE}_\kappa(\rho)$ bubbles, [Zha22, Theorem 3.16]). *Let $0 < \kappa < 4$ and $\rho > -2$. Then there exists a non-zero σ-finite measure $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)$ on $\mathrm{Bubble}_{\mathbb{H}}(0)$ such that the followings hold:*

1. *It satisfies the domain Markov property in the sense that conditioning on the initial segment and the event that the curve is not completed, the rest of the curve has the law of the chordal $\mathrm{SLE}_\kappa(\rho)$ in the remaining domain.*

2. *The $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)$ has the law of μ_0 under capacity parametrizaton map.*

3. *For any $r > 0$, the restriction of $\mathrm{SLE}_{k,0}^{\mathrm{bubble}}(\rho)$ to the event that $\{\eta : \mathrm{rad}_0(\eta) > r\}$ is a finite measure.*

Moreover, $\mathrm{SLE}_{k,0}^{\mathrm{bubble}}(\rho)$ satisfies the conformal covariance with exponent $\alpha := \frac{(\rho+2)(2\rho+8)}{2\kappa}$, *i.e., for any* $\psi \in \mathrm{conf}(\mathbb{H})$ *fixing* 0,

$$\psi(\mathrm{SLE}_{k,0}^{\mathrm{bubble}}(\rho)) = \psi'(0)^{\alpha}\mathrm{SLE}_{k,0}^{\mathrm{bubble}}(\rho). \tag{3.23}$$

The 1. and 3. characterize $\mathrm{SLE}_{k,0}^{\mathrm{bubble}}(\rho)$ *up to a multiplicative constant, i.e., any measure on* $\mathrm{Bubble}_{\mathbb{H}}(0)$ *satisfies 1. and 3. equals some constant times* $\mathrm{SLE}_{k,0}^{\mathrm{bubble}}(\rho)$*. Moreover,*

- *If* $\rho \geq \frac{\kappa}{2} - 2$, $\mathrm{SLE}_{k,0}^{\mathrm{bubble}}(\rho)$ *is supported on the loops that intersect* $\overline{\mathbb{R}}$ *only at* 0.
- *If* $\rho < \frac{\kappa}{2} - 2$, $\mathrm{SLE}_{k,0}^{\mathrm{bubble}}(\rho)$ *is supported on the loops whose intersection with* $\overline{\mathbb{R}}$ *is a compact subset of* $\overline{\mathbb{R}}$*, of which* 0 *is an accumulation point.*

The explicit construction of the rooted $\mathrm{SLE}_\kappa(\rho)$ bubble is carried out by taking the weak limit of chordal $\mathrm{SLE}_\kappa(\rho)$ measures under suitable rescaling. We use $\xrightarrow{w}$ to denote the *weak convergence*. Recall that for bounded measures $\mu_n, n \in \mathbb{N}$, and μ defined on some metric space E, $\mu_n \xrightarrow{w} \mu$ if and only if for any $f \in C_b(E, \mathbb{R})$, $\mu_n(f) \xrightarrow{w} \mu(f)$.

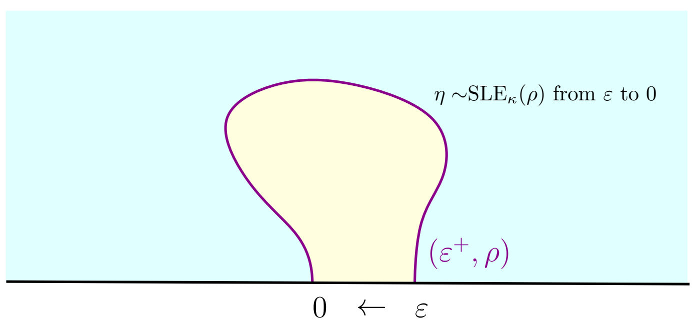


Figure 3.3: Illustration of Theorem 3.3.5: $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)$ as the weak limit of chordal $\mathrm{SLE}_\kappa(\rho)$ with suitable rescaling.

Theorem 3.3.5 ([Zha22, Theorem 3.20])**.** *Let* $0 < \kappa < 4$ *and* $\rho > -2$*. There exists a non-zero* σ*-finite measure* $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)$ *on* $\mathrm{Bubble}_{\mathbb{H}}(0)$ *such that the following holds: For any fixed* $S > 0$*, let* $E_S = \{\eta : \mathrm{rad}_0(\eta) > S\}$*. Then as* $\varepsilon \to 0^+$*,*

$$\varepsilon^{\frac{(\rho+2)(\kappa-8-2\rho)}{2\kappa}} \mathbb{1}_{E_S}\mathrm{SLE}_{\kappa,(\varepsilon;\varepsilon^+)\to 0}^{\mathbb{H}}(\rho) \xrightarrow{w} \mathbb{1}_{E_S}\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho) \tag{3.24}$$

in the space $\widetilde{\Sigma}^{\mathbb{C}}$ *with distance defined by (3.11), where* $\mathrm{SLE}^{\mathbb{H}}_{\kappa,(\varepsilon;\varepsilon^+)\to 0}(\rho)$ *denotes the law of a single-force-point* $\mathrm{SLE}_\kappa(\rho)$ *on* $\mathbb{H} : (\varepsilon;\varepsilon^+) \to 0$.

For general simply connected domain (D, a, b), let $\mathrm{SLE}^{D}_{\kappa,(a,c)\to b}(\rho)$ denote the chordal $\mathrm{SLE}_\kappa(\rho)$ process on D from a to b with force point c. Throughout this thesis, $c \in \{a^+, a^-\}$ in most cases.

Remark 3.3.6. Notice that in [AHS21, Theorem 3.20], the author considered

$$\mathrm{SLE}^{\mathbb{H}}_{\kappa,(r,r^+)\to -r}(\rho)$$

for $r > 0$ as the limiting sequence of measures. To get (3.24), first apply the shift map $f_r : \mathbb{H} \to \mathbb{H}$ such that $f_r(z) = z + r$ then let $\varepsilon = 2r$.

Definition 3.3.7 (Rooted $\mathrm{SLE}_\kappa(\rho)$ bubble measure)**.** For $0 < \kappa < 4$ and $\rho > -2$, we define the weak limit $\mathrm{SLE}^{\mathrm{bubble}}_{\kappa,0}(\rho)$ in Theorem 3.3.5 as the rooted $\mathrm{SLE}_\kappa(\rho)$ bubble measure with root 0. More generally, for any $p \in \partial\mathbb{H}$, let $f_p : \mathbb{H} \to \mathbb{H}$ be such that $f_p(z) = z + p$ and define

$$\mathrm{SLE}^{\mathrm{bubble}}_{\kappa,p}(\rho) := f_p(\mathrm{SLE}^{\mathrm{bubble}}_{\kappa,0}(\rho)).$$

If $\rho = 0$, then we omit the existence of ρ and write $\mathrm{SLE}^{\mathrm{bubble}}_{\kappa,p}$ for fixed $p \in \partial\mathbb{R}$.

Corollary 3.3.8. *Let* $\widetilde{E_{i,0}}$ *be the set of curves on* $\overline{\mathbb{H}}$ *starting from some point on* $[0,\infty]$*, ending at* 0*, and surrounding* i*. Under the same settings as Theorem 3.3.5, we have*

$$\mathrm{SLE}^{\mathbb{H}}_{\kappa,(\varepsilon;\varepsilon^+)\to 0}(\rho)[d\eta|\widetilde{E_{i,0}}] \xrightarrow{w} \mathrm{SLE}^{\mathrm{bubble}}_{\kappa,0}(\rho)[d\eta|\widetilde{E_{i,0}}] \qquad \text{as } \varepsilon \to 0^+ \tag{3.25}$$

in the metric space $\widetilde{\Sigma}^{\mathbb{C}}$ *with distance defined by (3.11).*

Proof. Let $E_1 = \{\eta : \mathrm{rad}_0(\eta) > 1\}$. It is clear that $\widetilde{E_{i,0}} \subset E_1$. Moreover, $\widetilde{E_{i,0}}$ is open in $\widetilde{\Sigma}^{\mathbb{C}}$ and $\partial\widetilde{E_{i,0}}$ contains the curves that end at 0 and pass through i. For $0 < \delta < 1$, let $E_\delta = \{\eta : \mathrm{rad}_0(\eta) \geq \delta\}$ and $\tau_\delta = \inf_{t>0}\{t : \mathrm{rad}_0(\eta[0,t]) = \delta\}$ be the first time that $\mathrm{SLE}^{\mathrm{bubble}}_{\kappa,0}(\rho)$

curve has radius δ under capacity parametrization. For any $\eta \in E_\delta$, let $\eta_\delta = \eta[0, \tau_\delta]$. For any fixed instance of η_δ, let $\widetilde{\partial E_{i,\eta_\delta}}$ be the set of curves from $\eta(\tau_\delta)$ to 0 on $\mathbb{H}\backslash\eta_\delta$ that pass through i. By Domain Markov Property of $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)$ stated in [Zha22, Theorem 3.16], we have that

$$\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[\widetilde{\partial E_{i,0}}] = \int_{E_\delta} \mathrm{SLE}_{\kappa,(\eta_\delta,v(\eta_\delta))\to 0}^{\mathbb{H}\backslash\eta_\delta}(\rho)[\widetilde{\partial E_{i,\eta_\delta}}]\mathrm{SLE}_{k,0}^{\mathrm{bubble}}(\rho)(d\eta_\delta). \tag{3.26}$$

By [Zha22, Theorem 3.20], $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[E_\delta] > 0$. Moreover, it is well-known that when $0 < \kappa < 4$, the probability that chordal $\mathrm{SLE}_\kappa(\rho)$ passes through a fixed interior point is zero (see, for instance, [Zha19]). Therefore, $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[\widetilde{\partial E_{i,0}}] = 0$. By (3.24) and [Zha22, (F3)],

$$\varepsilon^{\frac{(\rho+2)(\kappa-8-2\rho)}{2\kappa}} \mathbb{1}_{E_1}\mathbb{1}_{\widetilde{E_{i,0}}}\mathrm{SLE}_{\kappa,(\varepsilon;\varepsilon^+)\to 0}^{\mathbb{H}}(\rho) \xrightarrow{w} \mathbb{1}_{E_1}\mathbb{1}_{\widetilde{E_{i,0}}}\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho). \tag{3.27}$$

Equivalently,

$$\varepsilon^{\frac{(\rho+2)(\kappa-8-2\rho)}{2\kappa}} \mathbb{1}_{\widetilde{E_{i,0}}}\mathrm{SLE}_{\kappa,(\varepsilon;\varepsilon^+)\to 0}^{\mathbb{H}}(\rho) \xrightarrow{w} \mathbb{1}_{\widetilde{E_{i,0}}}\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho). \tag{3.28}$$

In order to prove (3.25), it remains to show that $0 < \mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[\widetilde{E_{i,0}}] < \infty$. By [Zha22, Theorem 3.16],

$$\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[\widetilde{E_{i,0}}] \le \mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[E_1] < \infty. \tag{3.29}$$

For any $\eta \in E_\delta$, let $\eta_\delta = \eta[0, \tau_\delta]$. For any fixed instance of η_δ, let $\widetilde{E_{i,\eta_\delta}}$ denote the set of curves on $\mathbb{H}\backslash\eta_\delta$ from $\eta(\tau_\delta)$ to 0 that surround i. Again, by Domain Markov Property of $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)$ ([Zha22, Theorem 3.16]),

$$\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[\widetilde{E_{i,0}}] = \int_{E_\delta} \mathrm{SLE}_{\kappa,(\eta_\delta,v(\eta_\delta))\to 0}^{\mathbb{H}\backslash\eta_\delta}(\rho)[\widetilde{E_{i,\eta_\delta}}]\mathrm{SLE}_{k,0}^{\mathrm{bubble}}(\rho)(d\eta_\delta), \tag{3.30}$$

where the force point $v(\eta_\delta)$ is defined in [Zha22, (3.17)]. For each instance of η_δ, we claim that

$$\mathrm{SLE}_{\kappa,(\eta_\delta,v(\eta_\delta))\to 0}^{\mathbb{H}\backslash\eta_\delta}(\rho)[\widetilde{E_{i,\eta_\delta}}] > 0. \tag{3.31}$$

Assume otherwise, i.e., $\mathrm{SLE}_{\kappa,(\eta_\delta,v(\eta_\delta))\to 0}^{\mathbb{H}\backslash\eta_\delta}(\rho)[\widetilde{E_{i,\eta_\delta}}] = 0$. By conformal invariance property of chordal $\mathrm{SLE}_\kappa(\rho)$, we only need to consider the $\mathrm{SLE}_\kappa(\rho)$ on $\mathbb{H}$ from 0 to ∞ conditional on passing to the left of i. By scaling property of chordal $\mathrm{SLE}_\kappa(\rho)$, the probability that $\mathrm{SLE}_\kappa(\rho)$ conditional on passing to the left of $ai, a > 0$ is zero, i.e., $\mathrm{SLE}_\kappa(\rho)$ will almost surely stay to the right of positive imaginary axis. This is impossible and leads to a contradiction. Therefore, $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[\widetilde{E_{i,0}}] > 0$ and this completes the proof. $\square$

CHAPTER 4

$\mathrm{SLE}_\kappa(\rho)$ BUBBLES VIA CONFORMAL WELDING OF QUANTUM SURFACES

4.1. Law of welding interface via the limiting procedure

In this section, we prove Proposition 4.1.1, i.e., we show that under the same setups as Theorem 1.1.1, the law of the welding interface is $\mathrm{SLE}_\kappa(\rho)$ bubble measure conditioning on surrounding i.

Proposition 4.1.1. *Fix $\gamma \in (0,2)$. For $W > 0$, let $\rho = W - 2$. Let $(\mathbb{H}, \phi, \eta, 0, i)$ be an embedding of the quantum surface*

$$\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W; \cdot, \ell) \times \mathrm{QD}_{1,1}(\ell) d\ell. \tag{4.1}$$

Let M_ϕ denote the marginal law of ϕ in $(\mathbb{H}, \phi, \eta, 0, i)$, then (ϕ, η) has the law of $M_\phi \times \mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[\cdot | i \in D_\eta(0)]$.

4.1.1. The LCFT description of three-pointed quantum disks

We start with the definition of two-pointed quantum disk with one additional typical bulk insertion.

Definition 4.1.2 ([ARS22, Definition 3.10])**.** For $W \geq \frac{\gamma^2}{2}$, recall the definition of thick quantum disk $\mathcal{M}_{0,2}^{\mathrm{disk}}(W)$ from Definition 2.3.2. Sample ϕ on $H^{-1}(\mathbb{H})$ such that $(\mathbb{H}, \phi, 0, \infty)$ is an embedding of $\mathcal{M}_{0,2}^{\mathrm{disk}}(W)$. Let L denote the law of ϕ and let (ϕ, z) be sampled from $L(d\phi)\mu_\phi(dz^2)$. We write $\mathcal{M}_{1,2}^{\mathrm{disk}}(W)$ for the law $(\mathbb{H}, \phi, z, 0, \infty)$ viewed as a marked quantum surface.

Lemma 4.1.3. *For $\gamma \in (0,2)$ and $W \in \mathbb{R}$, let $\beta_W = \gamma + \frac{2-W}{\gamma}$. Suppose $(\phi, \mathbf{x})$ is sampled from $\mathrm{LF}_{\mathbb{H}}^{(\gamma,i),(\beta_W,\infty),(\beta_W,\mathbf{x})} \times dx$, then the law of $(\mathbb{H}, \phi, \eta, i, \infty, \mathbf{x})$ as a marked quantum surface is equal to $\frac{2(Q-\beta_W)^2}{\gamma} \mathcal{M}_{1,2}^{\mathrm{disk}}(W)$.*

Proof. By [ARS22, Lemma 3.13], if $\mathcal{M}_{1,2}^{\text{disk}}(W)$ is embedded as $(\mathcal{S}, \phi, i\theta, +\infty, -\infty)$, then (ϕ, θ) has the law of

$$\frac{\gamma}{2(Q-\beta_W)^2}\mathrm{LF}_{\mathcal{S}}^{(\beta_W,\pm\infty),(\gamma,i\theta)}(d\phi)\mathbb{1}_{\widetilde{\theta}\in(0,\pi)}d\widetilde{\theta}. \tag{4.2}$$

Fix $\theta \in (0,\pi)$ and let $\exp : \mathcal{S} \to \mathbb{H}$ be the map $z \mapsto e^z$. By [ARS22, Lemma 3.14] and [AHS21, Lemma 2.20], we have

$$\exp_*\left(\mathrm{LF}_{\mathcal{S}}^{(\beta_W,\pm\infty),(\gamma,i\theta)}\right) = \mathrm{LF}_{\mathbb{H}}^{(\gamma,e^{i\theta}),(\beta_W,\infty),(\beta_W,0)}.$$

Let $f_\theta(z) = \frac{z}{\sin\theta} - \cot\theta$, which sends $e^{i\theta} \mapsto i$, $\infty \mapsto \infty$, and $0 \mapsto \mathbf{x} = -\cot\theta$. By [AHS21, Proposition 2.16], for any $r \in \mathbb{R}$, we have

$$\mathrm{LF}_{\mathbb{H}}^{(\gamma,i),(\beta_W,-\cot\theta),(\beta_W,\frac{r}{\sin\theta}-\cot\theta)} = (\sin\theta)^{2\Delta_\gamma+2\Delta_\beta}(f_\theta)_*\mathrm{LF}_{\mathbb{H}}^{(\gamma,e^{i\theta}),(\beta_W,0),(\beta_W,r)},$$

where $\Delta_\alpha = \frac{\alpha}{2}(Q-\frac{\alpha}{2})$. After multiplying both sides by $\left(\frac{r}{\sin\theta}-\cot\theta\right)^{2\Delta_{\beta_W}}$, we have

$$\begin{aligned}&\left(\frac{r}{\sin\theta}-\cot\theta\right)^{2\Delta_{\beta_W}}\mathrm{LF}_{\mathbb{H}}^{(\gamma,i),(\beta_W,-\cot\theta),(\beta_W,\frac{r}{\sin\theta}-\cot\theta)}\\ &=(\sin\theta)^{2\Delta_\gamma+2\Delta_{\beta_W}}\left(\frac{1}{\sin\theta}-\frac{\cot\theta}{r}\right)^{2\Delta_{\beta_W}}(f_\theta)_*\left(r^{2\Delta_{\beta_W}}\mathrm{LF}_{\mathbb{H}}^{(\gamma,e^{\theta i}),(\beta_W,0),(\beta_W,r)}\right).\end{aligned}$$

By [AHS21, Lemma 2.18], taking limit as $r \to \infty$ yields

$$\frac{1}{(\sin\theta)^2}\mathrm{LF}_{\mathbb{H}}^{(\gamma,i),(\beta_W,-\cot\theta),(\beta_W,\infty)} = (f_\theta)_*\,\mathrm{LF}_{\mathbb{H}}^{(\gamma,e^{\theta i}),(\beta_W,0),(\beta_W,\infty)}.$$

Here the convergence is in the vague topology. When θ is sampled from $\mathbb{1}_{(0,\pi)}(\widetilde{\theta})d\widetilde{\theta}$, we have

$$\frac{1}{(\sin\theta)^2}\mathrm{LF}_{\mathbb{H}}^{(\gamma,i),(\beta_W,-\cot\theta),(\beta_W,\infty)} = \mathrm{LF}_{\mathbb{H}}^{(\gamma,i),(\beta_W,\infty),(\beta_W,\mathbf{x})}\times dx$$

by change of variables $\mathbf{x} = -\cot\theta$. This completes the proof. □

A direct consequence of [AHS20, Theorem 2.2] is the following.

Theorem 4.1.4. *Let* $(\mathbb{H}, \phi, 0, \infty)$ *be the embedding of a sample from* $\mathcal{M}_{0,2}^{\text{disk}}(W+2)$*. Let* η *be sampled from* $\text{SLE}_\kappa(W-2, 0)$ *on* $(\mathbb{H}, 0, \infty)$ *independent of* ϕ*, then*

$$\mathcal{M}_{0,2}^{\text{disk}}(W+2) \otimes \text{SLE}_\kappa(W-2, 0) = C_{W,2} \int_0^\infty \mathcal{M}_{0,2}^{\text{disk}}(W, \cdot, \ell) \times \mathcal{M}_{0,2}^{\text{disk}}(2, \ell, \cdot) d\ell \qquad (4.3)$$

for some constant $C_{W,2} \in (0, \infty)$*.*

For $W > 0$, let $\beta_{W+2} = \gamma - \frac{W}{\gamma}$. Let $(\phi, \mathbf{x})$ be sampled from $\text{LF}_{\mathbb{H}}^{(\gamma,i),(\beta_{W+2},\infty),(\beta_{W+2},\mathbf{x})} \times dx$ and let η be sampled from the chordal $\text{SLE}^{\mathbb{H}}_{\kappa,(\mathbf{x};\mathbf{x}^-)\to\infty}(W-2)$. Denote $\nu_\phi(a, b)$ the quantum boundary length of (a, b) with respect to the random field ϕ. Fix $\delta \in (0, \frac{1}{2})$ and let M_δ denote the law of $(\phi, \mathbf{x}, \eta)$ restricted to the event that $\nu_\phi(\mathbf{x}, \infty) \in (\delta, 2\delta)$, $\nu_\phi(\mathbb{R}) \in (1, 2)$ and i is to the right of η. Let $M_\delta^\# = \frac{1}{|M_\delta|} M_\delta$ be the corresponding probability measure.

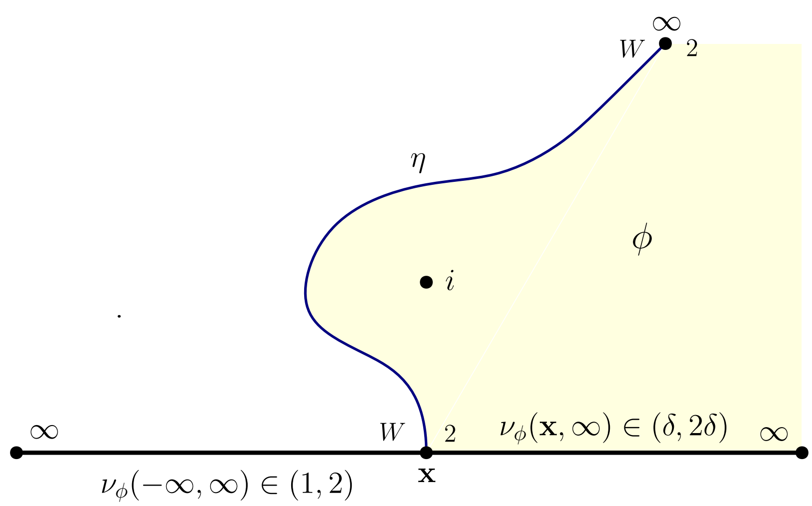


Figure 4.1: Illustration of M_δ: first sample $(\phi, \mathbf{x})$ from $\text{LF}_{\mathbb{H}}^{(\gamma,i),(\beta_{W+2},\infty),(\beta_{W+2},\mathbf{x})} \times dx$ and then sample η according to $\text{SLE}^{\mathbb{H}}_{\kappa,(\mathbf{x};\mathbf{x}^-)\to\infty}(W-2)$. The M_δ is the restriction $(\phi, \mathbf{x}, \eta)$ to the event that $\nu_\phi(\mathbf{x}, \infty) \in (\delta, 2\delta)$, $\nu_\phi(-\infty, \infty) \in (1, 2)$ and i is to the right of η.

Lemma 4.1.5. *Fix* $W > 0$*. There exists some constant* $C \in (0, \infty)$ *such that for each* $\delta \in (0, \frac{1}{2})$*, if* $(\phi, \mathbf{x}, \eta)$ *is sampled from* M_δ*, then the law of marked quantum surface* $(\mathbb{H}, \phi, \eta, i, \mathbf{x}, \infty)$

is
$$C\cdot\int_{\delta}^{2\delta}\int_{1-\ell'}^{2-\ell'}\int_0^\infty \mathcal{M}_{0,2}^{\text{disk}}(W;\ell,\ell_1)\times\mathcal{M}_{1,2}^{\text{disk}}(2,\ell_1,\ell')d\ell_1 d\ell d\ell'. \tag{4.4}$$

Proof. By Lemma 4.1.3, if we sample $(\phi,\mathbf{x})$ from

$$\mathrm{LF}_{\mathbb{H}}^{(\gamma,i),(\beta_{W+2},\infty),(\beta_{W+2},\mathbf{x})}\times dx,$$

then $(\mathbb{H},\phi,i,\mathbf{x},\infty)$ viewed as a marked quantum surface has the law of $C\cdot\mathcal{M}_{1,2}^{\text{disk}}(W+2)$ for some constant $C\in(0,\infty)$. Furthermore, if we sample η from $\mathrm{SLE}_{\kappa,(\mathbf{x};\mathbf{x}^-)\to\infty}^{\mathbb{H}}(W-2)$ conditioning on i is to the right of η, then by Theorem 4.1.4, the quantum surface $(\mathbb{H},\phi,\eta,i,\mathbf{x},\infty)$ has the law of

$$C\cdot\int_0^\infty\int_0^\infty\int_0^\infty \mathcal{M}_{0,2}^{\text{disk}}(W;\ell,\ell_1)\times\mathcal{M}_{1,2}^{\text{disk}}(2;\ell_1,\ell')d\ell d\ell_1 d\ell'. \tag{4.5}$$

Conditioning on $\nu_\phi(\mathbf{x},\infty)\in(\delta,2\delta)$ and $\nu_\phi(\mathbb{R})\in(1,2)$ gives the desired result. □

4.1.2. Proof of Proposition 4.1.1 via coupling

Fix $W>0$. Sample a pair of quantum surfaces $(\mathcal{D}_1,\mathcal{D}_2)$ from

$$\int_1^2\int_0^\infty \mathcal{M}_{0,2}^{\text{disk}}(W;a,p)\times \mathrm{QD}_{1,1}(p)dpda \tag{4.6}$$

and let $\mathcal{D}_1\oplus\mathcal{D}_2$ be the curve-decorated quantum surface obtained by conformally welding the right boundary of $\mathcal{D}_1$ and total boundary of $\mathcal{D}_2$. Notice that $\mathcal{D}_1\oplus\mathcal{D}_2$ has a interior marked point and a boundary marked point. Let $(\mathbb{D},\phi_{\mathbb{D}},\eta_{\mathbb{D}},0,i)$ be the unique embedding of $\mathcal{D}_1\oplus\mathcal{D}_2$ on $(\mathbb{D},0,i)$ and let $f:\mathbb{H}\to\mathbb{D}$ be the conformal map with $f(i)=0$ and $f(\infty)=i$. Denote $M_{\mathbb{D}}$ the joint law of $(\mathbb{D},\phi_{\mathbb{D}},\eta_{\mathbb{D}},0,i)$ and let $M_{\mathbb{D}}^{\#}=\frac{1}{|M_{\mathbb{D}}|}M_{\mathbb{D}}$ be the probability measure obtained from $M_{\mathbb{D}}$.

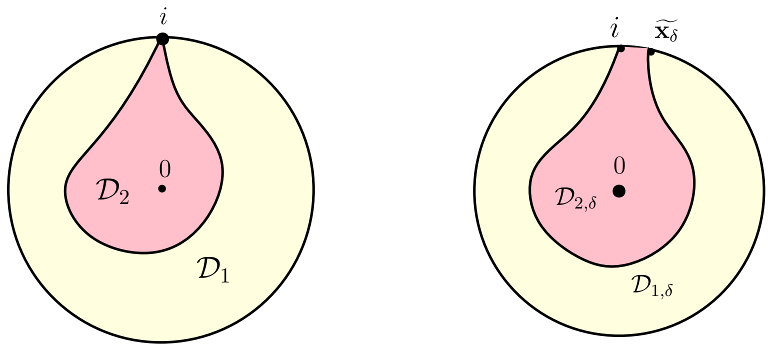


Figure 4.2: **Left:** $(\phi_{\mathbb{D}}, \eta_{\mathbb{D}})$ from $M_{\mathbb{D}}^{\#}$ is obtained by embedding $\mathcal{D}_1 \oplus \mathcal{D}_2$ into $(\mathbb{D}, 0, i)$. **Right:** $(\phi^\delta, \eta^\delta)$ from $M_\delta^{\#}$ is obtained by embedding $\mathcal{D}_{1,\delta} \oplus \mathcal{D}_{2,\delta}$ into $(\mathbb{D}, 0, i)$.

Next, we recall the definition of $M_\delta^{\#}$. For $0 < \gamma < 2$ and $W > 0$, let $\beta_{W+2} = \gamma - \frac{W}{\gamma}$. Sample $(\phi, \mathbf{x})$ from $\mathrm{LF}_{\mathbb{H}}^{(\gamma,i),(\beta_{W+2},\infty),(\beta_{W+2},\mathbf{x})} \times dx$ and let η be sampled from $\mathrm{SLE}^{\mathbb{H}}_{\kappa,(\mathbf{x};\mathbf{x}^-)\to\infty}(W-2)$. Fix $\delta \in (0, \frac{1}{2})$ and let M_δ be the law of $(\phi, \mathbf{x}, \eta)$ restricted to the event that $\nu_\phi(\mathbf{x}, \infty) \in (\delta, 2\delta)$, $\nu_\phi(\mathbb{R}) \in (1, 2)$ and i is to the right of η. Let $M_\delta^{\#} = \frac{1}{|M_\delta|} M_\delta$ be the corresponding probability measure.

Sample $(\phi, \mathbf{x}, \eta)$ from $M_\delta^{\#}$ and let $\mathcal{D}_{1,\delta}$ and $\mathcal{D}_{2,\delta}$ be the two components such that $(\mathbb{H}, \phi, \eta, i, \mathbf{x})$ is the embedding of the surface $\mathcal{D}_{1\,\delta} \oplus \mathcal{D}_{2,\delta}$ after conformal welding. Let $\phi^\delta = \phi \circ f^{-1} + \log |(f^{-1})'|$ and $\eta^\delta = f \circ \eta$ be such that $(\mathbb{D}, \phi^\delta, \eta^\delta, 0, i)$ is the embedding of $\mathcal{D}_{1,\delta} \oplus \mathcal{D}_{2,\delta}$. Here η^δ is the welding interface between $\mathcal{D}_{1,\delta}$ and $\mathcal{D}_{2,\delta}$. Let $\widetilde{\mathbf{x}_\delta} = f(\mathbf{x})$ be the image of $\mathbf{x}$ under f.

Lemma 4.1.6. *There exists a coupling between $M_{\mathbb{D}}^{\#}$ and $M_\delta^{\#}$ such that the followings hold: There exist random simply connected domains U_δ and $\widetilde{U_\delta} \subset \mathbb{D}$ and a conformal map $g_\delta : \widetilde{U_\delta} \to U_\delta$ satisfying the following properties: With probability $1 - o_\delta(1)$, we have*

1. $\phi_{\mathbb{D}}(z) = \phi^\delta \circ g_\delta(z) + Q \log |g_\delta'(z)|, \qquad$ *for* $z \in \widetilde{U_\delta}$.

2. $\mathrm{diam}(\mathbb{D}\backslash U_\delta) = o_\delta(1)$ *and* $\mathrm{diam}(\mathbb{D}\backslash \widetilde{U_\delta}) = o_\delta(1)$.

3. $|\widetilde{\mathbf{x}_\delta} - i| = o_\delta(1)$.

4. $\sup_{z\in K} |g_\delta(z) - z| = o_\delta(1)$, *for any compact set* $K \subset \mathbb{D}$.

In order to prove Lemma 4.1.6, we need the following two basic coupling results on the quantum disk. The first one is on $\mathrm{QD}_{1,1}$. Suppose $\mathcal{D}$ as a quantum surface has the law of $\mathrm{QD}_{1,1}$ and it has emebdding $(\mathbb{H},\phi,i,-1)$. Let $\mathcal{D}^\varepsilon := (\mathbb{H}_\varepsilon,\phi,i,-1,-1-2\varepsilon)$, where $\mathbb{H}_\varepsilon = \mathbb{H}\backslash B_\varepsilon(-1-\varepsilon)$ with $B_\varepsilon(-1-\varepsilon) = \{z\in\mathbb{C} : |z+1+\varepsilon|\le\varepsilon\}$.

Lemma 4.1.7 ([ARS22, Lemma 5.17])**.** *For $\varepsilon>0$ and $\ell>0$, suppose $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ are sampled from $\mathrm{QD}_{1,1}(\ell)^\#$ and $\mathrm{QD}_{1,1}(\widetilde{\ell})^\#$ respectively, then the law of $\widetilde{\mathcal{D}^\varepsilon}$ converges in total variation distance to $\mathcal{D}^\varepsilon$ as $\widetilde{\ell}\to\ell$.*

The second coupling result is on $\mathcal{M}_{0,2}^{\mathrm{disk}}(W)$. Suppose $\mathcal{D}$ is sampled from $\mathcal{M}_{0,2}^{\mathrm{disk}}(W)$ and it has embedding $(\mathbb{D},\phi,-i,i)$. With a slight abuse of notation, let $\mathcal{D}^\varepsilon := (\mathbb{D}_\varepsilon,\phi,\alpha_\varepsilon,\alpha'_\varepsilon,i,-i)$, where $B_\varepsilon(i)=\{z\in\mathbb{C} : |z-i|\le\varepsilon\}$, $\mathbb{D}_\varepsilon=\mathbb{D}\backslash B_\varepsilon(i)$, and $\{\alpha_\varepsilon,\alpha'_\varepsilon\} = \partial\mathbb{D}\cap\partial B_\varepsilon(i)$.

Lemma 4.1.8. *Fix $W>0$. For $\varepsilon,\ell,r,\widetilde{\ell},\widetilde{r}>0$, suppose $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ are sampled from $\mathcal{M}_{0,2}^{\mathrm{disk}}(W;\ell,r)^\#$ and $\mathcal{M}_{0,2}^{\mathrm{disk}}(W;\widetilde{\ell},\widetilde{r})^\#$ respectively, then $\widetilde{\mathcal{D}^\varepsilon}$ converges in total variation distance to $\mathcal{D}$ as $(\widetilde{\ell},\widetilde{r})\to(\ell,r)$.*

Proof. The proof follows directly from [AHS20, Proposition 2.23]. □

Lemma 4.1.9. *Suppose $(\phi,\mathbf{x},\eta)$ is sampled from $M_\delta^\#$ and let $A=\nu_\phi(-\infty,\mathbf{x})$, $B=\nu_\phi(\mathbf{x},\infty)$ and $P=\nu_\phi(\eta)$, then as $\delta\to0$, B converges to 0 in probability and the $M_\delta^\#$-law of (A,P) converges in total variation distance to a probability measure on $(1,2)\times(0,\infty)$ whose density function is proportional to*

$$f_W(a,p)p^{-\frac{4}{\gamma^2}+1}dadp, \tag{4.7}$$

where $f_W(a,p)=|\mathcal{M}_{0,2}^{\mathrm{disk}}(W;a,p)|$.

Proof. By Proposition 5.1 and [AHS21, Lemma 3.3], we have

$$|\mathcal{M}_{0,2}^{\mathrm{disk}}(W;\ell,r)| = f_W(\ell,r) \qquad \text{and} \qquad |\mathcal{M}_{1,2}^{\mathrm{disk}}(2;\ell,r)| = C(\ell+r)^{-\frac{4}{\gamma^2}+1}. \tag{4.8}$$

By (4.4), the $M_\delta^\#$-law of (A, P, B) is a probability measure on the space

$$S_\delta = \left\{(a,p,b) \in (0,\infty)^3 : b \in (\delta, 2\delta), a+b \in (1,2)\right\},$$

whose density function is proportional to

$$m(a,p,b) = f_W(a,p)(p+b)^{-\frac{4}{\gamma^2}+1}.$$

Therefore, we have

$$|M_\delta| = \int_{S_\delta} m(a,p,b)\, da\, dp\, db.$$

By definition of $M_\delta^\#$, for any $\varepsilon > 0$, we have $\lim_{\delta\to 0} M_\delta^\#[B > \varepsilon] = 0$. As $\delta \to 0$, the limiting $M_\delta^\#$-law of (A, P) is a probability measure on $(1,2)\times(0,\infty)$ whose density function is proportional to $f_W(a,p)p^{-\frac{4}{\gamma^2}+1}$. This completes the proof. □

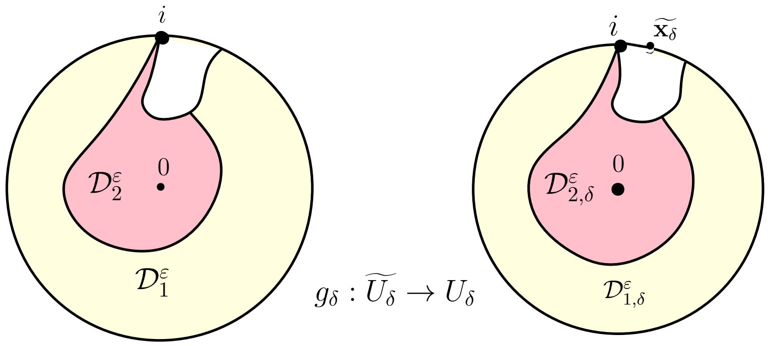


Figure 4.3: We can couple $M_\mathbb{D}^\#$ and $M_\delta^\#$ so that the light green and pink quantum surfaces agree with high probability. The domain $\widetilde{U_\delta}$ is the interior of $\overline{\mathcal{D}_1^\varepsilon \cup \mathcal{D}_2^\varepsilon}$ in the embedding of $\mathcal{D}_1 \oplus \mathcal{D}_2$ and U_δ is the interior of $\overline{\mathcal{D}_{1,\delta}^\varepsilon \cup \mathcal{D}_{2,\delta}^\varepsilon}$ in the embedding of $\mathcal{D}_{1,\delta} \oplus \mathcal{D}_{2,\delta}$.

Proof of Lemma 4.1.6. Recall the definition of marked quantum surfaces $\mathcal{D}_1$ and $\mathcal{D}_2$ embedded as $(\mathbb{D}, \phi_\mathbb{D}, \eta_\mathbb{D}, 0, i)$. Let $\widetilde{A}$ and $\widetilde{P}$ be the left and right boundary length of $\mathcal{D}_1$ respectively. The law of $(\widetilde{A}, \widetilde{P})$ is the probability measure on $[1,2]\times(0,\infty)$ proportional to

$$\left|\mathcal{M}_{0,2}^{\text{disk}}(W; a, p)\right| |\text{QD}_{1,1}(p)| \propto f_W(a,p)p^{-\frac{4}{\gamma^2}+1}. \tag{4.9}$$

Conditioning on $(\widetilde{A}, \widetilde{P})$, the joint law of $(\mathcal{D}_1, \mathcal{D}_2)$ is $\mathcal{M}_{0,2}^{\text{disk}}(W; \widetilde{A}, \widetilde{P})^{\#} \times \mathrm{QD}_{1,1}(\widetilde{P})^{\#}$.

Next, let A_δ and P_δ be the left and right boundary of $D_{1,\delta}$ respectively and let B_δ be the right boundary of $D_{2,\delta}$. By Lemma 4.1.9, as $\delta \to 0$, $M_\delta^{\#}$-law of (A_δ, P_δ) converges in law to $(\widetilde{A}, \widetilde{P})$ and $B_\delta \to 0$ in probability. Therefore, we can couple $M_\delta^{\#}$ and $M_{\mathbb{D}}^{\#}$ so that $(A_\delta, P_\delta) = (\widetilde{A}, \widetilde{P})$ with probability $1 - o_\delta(1)$. By Lemma 4.1.7 and 4.1.8, there exists a coupling between $(\mathcal{D}_1^\varepsilon, \mathcal{D}_2^\varepsilon)$ and $(\mathcal{D}_{1,\delta}^\varepsilon, \mathcal{D}_{2,\delta}^\varepsilon)$ such that

$$\lim_{\delta \to 0} \mathbb{P}\left[(\mathcal{D}_1^\varepsilon, \mathcal{D}_2^\varepsilon) = (\mathcal{D}_{1,\delta}^\varepsilon, \mathcal{D}_{2,\delta}^\varepsilon)\right] = 1 \tag{4.10}$$

for some $\varepsilon = o_\delta(1)$ with sufficiently slow decay. Let $\widetilde{U_\delta}$ denote the interior of $\overline{\mathcal{D}_1^\varepsilon \cup \mathcal{D}_2^\varepsilon}$ in the embedding of $\mathcal{D}_1 \oplus \mathcal{D}_2$ and U_δ denote the interior of $\overline{\mathcal{D}_{1,\delta}^\varepsilon \cup \mathcal{D}_{2,\delta}^\varepsilon}$ in the embedding of $\mathcal{D}_{1,\delta} \oplus \mathcal{D}_{2,\delta}$. By conformal welding, the marked quantum surfaces $(\widetilde{U_\delta}, \phi_{\mathbb{D}}, 0, i^-)$ and $(U_\delta, \phi^\delta, 0, i^-)$ agree with probability $1 - o_\delta(1)$. On this high probability event, there exists a unique conformal map $g_\delta : \widetilde{U_\delta} \to U_\delta$ such that $\phi_{\mathbb{D}} = \phi^\delta \circ g_\delta + Q \log |g_\delta'|$ with $g_\delta(0) = 0$ and $g_\delta(i^-) = i^-$.

Notice that the random simply connected domain $\widetilde{U_\delta}$ is completely determined by $M_{\mathbb{D}}^{\#}$. Almost surely under $M_{\mathbb{D}}^{\#}$, the $\{\overline{\mathbb{D}} \backslash \widetilde{U_\delta}\}_\delta$ is a sequence of shrinking compact sets in the euclidean sense, i.e., $\mathrm{diam}(\overline{\mathbb{D}} \backslash \widetilde{U_\delta}) = o_\delta(1)$ and $\bigcap_{\delta > 0} \overline{\mathbb{D}} \backslash \widetilde{U_\delta} = \{i\}$. By the coupling between $M_{\mathbb{D}}^{\#}$ and $M_\delta^{\#}$, we know that $\mathrm{diam}(\overline{\mathbb{D}} \backslash \widetilde{U_\delta}) = o_\delta(1)$ with probability $1 - o_\delta(1)$. Notice that $\mathrm{diam}(\overline{\mathbb{D}} \backslash \widetilde{U_\delta}) = 0$ if and only if the harmonic measure of $\mathbb{D} \backslash \widetilde{U_\delta}$ viewed from 0 in $\widetilde{U_\delta}$ tends to 0 as $\delta \to 0$. Therefore, in our coupling, with probability $1 - o_\delta(1)$, the harmonic measure of $\mathbb{D} \backslash \widetilde{U_\delta}$ viewed from 0 in $\widetilde{U_\delta}$ is $o_\delta(1)$. Since the harmonic measure is conformally invariant and by (4.10), with probability $1 - o_\delta(1)$, harmonic measure of $\mathbb{D} \backslash U_\delta$ viewed from 0 in U_δ is also $o_\delta(1)$. Hence, we have $\mathrm{diam}(\mathbb{D} \backslash U_\delta) = o_\delta(1)$ with probability $1 - o_\delta(1)$. This proves (2) in Lemma 4.1.6.

By construction, we know that $\widetilde{\mathbf{x}_\delta} \in \mathbb{D} \backslash U_\delta$ and $|\widetilde{\mathbf{x}_\delta} - i| \leq \mathrm{diam}\,(\mathbb{D} \backslash U_\delta)$. The above argument directly implies that $|\widetilde{\mathbf{x}_\delta} - i| = o_\delta(1)$ with probability $1 - o_\delta(1)$. Therefore (3) is also proved.

Finally, by (4.10), we have $g_\delta(0)=0$, $g_\delta(i^-)=i^-$, $\mathrm{diam}(\mathbb{D}\backslash U_\delta)=o_\delta(1)$, and $\mathrm{diam}(\mathbb{D}\backslash\widetilde{U_\delta})=o_\delta(1)$ with probability $1-o_\delta(1)$, the standard conformal distortion estimates imply (4). $\square$

Proof of Proposition 4.1.1. For the convenience of readers, we first recall the definition and basic setup regarding $M_\delta^\#$ on $\mathbb{H}$: For $W>0$, let $\beta_{W+2}=\gamma-\frac{W}{\gamma}$. Sample $(\phi,\mathbf{x})$ from $\mathrm{LF}_{\mathbb{H}}^{(\gamma,i),(\beta_{W+2},\infty),(\beta_{W+2},\mathbf{x})}\times dx$ and let η be sampled from $\mathrm{SLE}^{\mathbb{H}}_{\kappa,(\mathbf{x};\mathbf{x}^-)\to\infty}(W-2)$. Fix $\delta\in(0,\frac{1}{2})$ and let $M_\delta^\#$ be the probability law of $(\phi,\mathbf{x},\eta)$ restricted to the event that $\nu_\phi(\mathbf{x},\infty)\in(\delta,2\delta)$, $\nu_\phi(\mathbb{R})\in(1,2)$ and i is to the right of η. Sample $(\phi,\mathbf{x},\eta)$ from $M_\delta^\#$ and let $\mathcal{D}_{1,\delta}$ and $\mathcal{D}_{2,\delta}$ be the two components such that $(\mathbb{H},\phi,\eta,i,\mathbf{x})$ is the embedding of the conformally welded surface $\mathcal{D}_{1\,\delta}\oplus\mathcal{D}_{2,\delta}$.

We first prove the results on $(\mathbb{D},0,i)$ instead of $(\mathbb{H},i,\infty)$. Let $f:\mathbb{H}\to\mathbb{D}$ be the conformal map such that $f(i)=0$ and $f(\infty)=i$. In the end, since both $M_\delta^\#$ and $M_{\mathbb{D}}^\#$ are probability laws, we can pull back all the results via f^{-1}. Let $\phi^\delta=\phi\circ f^{-1}+\log|(f^{-1})'|$ and $\eta^\delta=f\circ\eta$ be such that $(\mathbb{D},\phi^\delta,\eta^\delta,0,i)$ is an embedding of $\mathcal{D}_{1,\delta}\oplus\mathcal{D}_{2,\delta}$. Let $\widetilde{\mathbf{x}_\delta}=f(\mathbf{x})$ be the image of $\mathbf{x}$ under f. Here η^δ represents the welding interface between $\mathcal{D}_{1,\delta}$ and $\mathcal{D}_{2,\delta}$.

By Lemma 4.1.6, there exists a coupling between $M_{\mathbb{D}}^\#$ and $M_\delta^\#$ such that

$$\lim_{\delta\to0}\mathbb{P}\left[(\mathcal{D}_1^\varepsilon,\mathcal{D}_2^\varepsilon)=(\mathcal{D}_{1,\delta}^\varepsilon,\mathcal{D}_{2,\delta}^\varepsilon)\right]=1 \tag{4.11}$$

for some $\varepsilon=o_\delta(1)$ with sufficiently slow decay (this is (4.10)). Moreover, let U_δ be the interior of $\overline{\mathcal{D}_{1,\delta}^\varepsilon\cup\mathcal{D}_{2,\delta}^\varepsilon}\subset\mathbb{D}$ and let $\widetilde{U_\delta}$ be the interior of $\overline{\mathcal{D}_1^\varepsilon\cup\mathcal{D}_2^\varepsilon}\subset\mathbb{D}$. Then there exists a unique conformal map $g_\delta:\widetilde{U_\delta}\to U_\delta$ such that with probability $1-o_\delta(1)$, $|\widetilde{\mathbf{x}_\delta}-i|=o_\delta(1)$ and $\sup_{z\in K}|g_\delta(z)-z|=o_\delta(1)$ for any compact set $K\subset\mathbb{D}$. Take $K=\overline{\mathcal{D}_1}\subseteq\mathbb{D}$ and by definition of $M_{\mathbb{D}}^\#$, $\eta_{\mathbb{D}}\subseteq\partial\overline{\mathcal{D}_1}$. The image of $\eta_{\mathbb{D}}$ under g_δ is $\eta^\delta\subset\partial\overline{\mathcal{D}_{1,\delta}}$. Since $\sup_z|g_\delta(z)-z|=o_\delta(1)$, there exist parametrizations $p_\delta:[0,1]\to\eta^\delta$ and $p_{\mathbb{D}}:[0,1]\to\eta_{\mathbb{D}}$ such that $|g_\delta(p_{\mathbb{D}}(t))-p_{\mathbb{D}}(t)|=|p_\delta(t)-p_{\mathbb{D}}(t)|=o_\delta(1)$ for all $t\in[0,1]$. Hence, under such coupling between $M_{\mathbb{D}}^\#$ and $M_\delta^\#$, with probability $1-o_\delta(1)$, there exist parametrizations p_δ and $p_{\mathbb{D}}$ of η^δ and $\eta_{\mathbb{D}}$ respectively, such that $\sup_{t\in[0,1]}|p_\delta(t)-p_{\mathbb{D}}(t)|=o_\delta(1)$, which implies the topology of convergence under coupling is the same as (3.11).

Next, by Lemma 4.1.6, $|\widetilde{\mathbf{x}_\delta} - i| = o_\delta(1)$ with probability $1 - o_\delta(1)$, and for any instance of $\widetilde{\mathbf{x}_\delta}$, η^δ has the law of $\mathrm{SLE}^{\mathbb{D}}_{\kappa,(\widetilde{\mathbf{x}_\delta};\widetilde{\mathbf{x}_\delta}^+)\to i}(W-2)[\cdot|0 \in \mathcal{D}_{2,\delta}]$. By Corollary 3.3.8, for any deterministic sequence $\mathbf{x}_\delta$ on $\partial\mathbb{D}$ that converges to i in euclidean distance as $\delta \to 0$,

$$\mathrm{SLE}^{\mathbb{D}}_{\kappa,(\mathbf{x}_\delta;\mathbf{x}_\delta^+)\to i}(W-2)[\cdot|0 \in \mathcal{D}_{2,\delta}] \xrightarrow{w} \mathrm{SLE}^{\mathrm{bubble}}_{\kappa,i}(W-2)[\cdot|0 \in \mathcal{D}_2] \tag{4.12}$$

in the distance (3.11). Hence , under $M^{\#}_{\mathbb{D}}$, $\eta_{\mathbb{D}}$ is independent of $\phi_{\mathbb{D}}$ and has the law of $\mathrm{SLE}^{\mathrm{bubble}}_{\kappa,i}(W-2)[\cdot|0 \in \mathcal{D}_2]$. By pulling back all the results above on $\mathbb{D}$ to $\mathbb{H}$ via f^{-1}, we have that

$$\int_1^2 \phi(\ell)d\ell \times \mathrm{SLE}^{\mathrm{bubble}}_{\kappa,0}(W-2)[\cdot|i \in D_\eta(0)] = \int_1^2\int_0^\infty \mathcal{M}^{\mathrm{disk}}_{0,2}(W;\ell,r)\times \mathrm{QD}_{1,1}(r)drd\ell \tag{4.13}$$

for some unknown Liouville field ϕ. Finally, by the identical scaling argument in the proof of [ARS22, Theorem 4.1], the integration on $[1,2]$ in (4.13) can be replaced by $(0,\infty)$. This completes the proof. □

4.2. Law of field via quantum triangles

4.2.1. Preliminaries on quantum triangles

Our derivation of field law relies on the conformal welding of quantum triangles with quantum disks. In this section, we recall the definition of quantum triangles and review the conformal welding theorem between quantum triangle and quantum disk ([ASY22]).

Definition 4.2.1 (Thick quantum triangle, [ASY22, Definition 2.17])**.** For $W_1, W_2, W_3 > \frac{\gamma^2}{2}$, set $\beta_i = \gamma + \frac{2-W_i}{\gamma} < Q$ for $i = 1,2,3$, and let $\mathrm{LF}_{\mathcal{S}}^{(\beta_1,+\infty),(\beta_2,-\infty),(\beta_3,0)}$ be the Liouville field on $\mathcal{S}$ with insertion $\beta_1, \beta_2, \beta_3$ at $+\infty, -\infty$ and 0, respectively. Let ϕ be sampled from

$$\frac{1}{(Q-\beta_1)(Q-\beta_2)(Q-\beta_3)}\mathrm{LF}_{\mathcal{S}}^{(\beta_1,+\infty),(\beta_2,-\infty),(\beta_3,0)}.$$

Define $\mathrm{QT}(W_1, W_2, W_3)$ to be the law of the three-pointed quantum surface

$$(\mathcal{S}, \phi, +\infty, -\infty, 0)/\sim_\gamma$$

and we call a sample from $\mathrm{QT}(W_1, W_2, W_3)$ a quantum triangle of weight (W_1, W_2, W_3).

One can also define the conditional law of quantum disks/triangles on fixed boundary length. This is again done by disintegration.

Definition 4.2.2 ([ASY22, Definition 2.26])**.** Fix $W_1, W_2, W_3 > \frac{\gamma^2}{2}$. Let $\beta_i = \gamma + \frac{2-W_i}{\gamma}$ and $\overline{\beta} = \beta_1 + \beta_2 + \beta_3$. Sample h from $P_{\mathbb{H}}$ and set

$$\widetilde{h(z)} = h(z) + (\overline{\beta} - 2Q)\log|z|_+ - \beta_1 \log|z| - \beta_2 \log|z-1|.$$

Fix $\ell > 0$ and let $L_{12} = \nu_{\widetilde{h}}([0,1])$. We define $\mathrm{QT}(W_1, W_2, W_3; \ell)$, the quantum triangles of weights W_1, W_2, W_3 with left boundary length ℓ, to be the law of $\widetilde{h} + \frac{2}{\gamma}\log\frac{\ell}{L_{12}}$ under the reweighted measure $\frac{2}{\gamma}\frac{\ell^{\frac{1}{\gamma}(\overline{\beta}-2Q)-1}}{L_{12}^{\frac{1}{\gamma}(\overline{\beta}-2Q)}} P_{\mathbb{H}}(dh)$. The same thing holds if we replace $L_{12} = \nu_{\widetilde{h}}([0,1])$ by $L_{13} = \nu_{\widetilde{h}}((-\infty, 0])$ or $L_{23} = \nu_{\widetilde{h}}([1, +\infty))$.

Lemma 4.2.3 ([ASY22, Lemma 2.27])**.** *In the same settings of Definition 4.2.2, the sample from* $\mathrm{QT}(W_1, W_2, W_3; \ell)$ *has left boundary length* ℓ*, and we have*

$$\mathrm{QT}(W_1, W_2, W_3) = \int_0^\infty \mathrm{QT}(W_1, W_2, W_3; \ell) d\ell. \tag{4.14}$$

Let $\mathrm{SLE}_\kappa(\rho_-; \rho_+, \rho_1)$ be the law of a chordal SLE_κ on $\mathbb{H}$ from 0 to ∞ with force points $0^-, 0^+, 1$, with corresponding weights ρ_-, ρ_+, ρ_1 respectively. Moreover, suppose η is a curve from 0 to ∞ on $\mathbb{H}$ that does not touch 1. Let D_η be the connected component of $\mathbb{H}\backslash\eta$ containing 1 and ψ_η is the unique conformal map from the component D_η to $\mathbb{H}$ fixing 1 and sending the first (resp. last) point on ∂D_η hit by η to 0 (resp. ∞). Define the measure

$\widetilde{\mathrm{SLE}}_\kappa(\rho_-;\rho_+,\rho_1;\alpha)$ on curves from 0 to ∞ on $\mathbb{H}$ as follows:

$$\frac{d\widetilde{\mathrm{SLE}}_\kappa(\rho_-;\rho_+,\rho_1;\alpha)}{d\mathrm{SLE}_\kappa(\rho_-;\rho_+,\rho_1)}(\eta)=\psi_\eta'(1)^\alpha. \tag{4.15}$$

Theorem 4.2.4 ([ASY22, Theorem 1.2])**.** *Suppose $W,W_1,W_2,W_3>0$ and*

$$\frac{\gamma^2}{2}\notin\{W_1,W_2,W_3,W+W_1,W+W_2\}.$$

Let

$$\alpha=\frac{W_3+W_2-W_1-2}{4\kappa}\left(W_3+W_1+2-W_2-\kappa\right). \tag{4.16}$$

Then there exist some constant $C=C_{W,W_1,W_2}\in(0,\infty)$ such that

$$\begin{aligned}\mathrm{QT}(W+W_1,W+W_2,W_3)&\otimes\widetilde{\mathrm{SLE}}_\kappa(W-2;W_2-2,W_1-W_2;\alpha)\\&=C\cdot\int_0^\infty\mathcal{M}_{0,2}^{\mathrm{disk}}(W;\ell)\times\mathrm{QT}(W_1,W_2,W_3;\ell)d\ell.\end{aligned} \tag{4.17}$$

4.2.2. Quantum disks with generic bulk and boundary insertions

Definition 4.2.5 (Special case of Definition 2.2.6)**.** Let $\alpha,\beta\in\mathbb{R}$. Fix $p\in\mathbb{R}$ and $q\in\mathbb{H}$. Suppose $(h,\mathbf{c})$ is sampled from $C_{\mathbb{H}}^{(\beta,p),(\alpha,q)}P_{\mathbb{H}}\times\left[e^{\left(\frac{1}{2}\beta+\alpha-Q\right)c}dc\right]$, where

$$C_{\mathbb{H}}^{(\beta,p),(\alpha,q)}=|p|_+^{-\beta(Q-\frac{\beta}{2})}(2\Im q)^{-\frac{\alpha^2}{2}}|q|_+^{-2\alpha(Q-\alpha)}.$$

Then the field $\phi(z)=h(z)-2Q\log|z|_++\alpha G_{\mathbb{H}}(z,q)+\frac{\beta}{2}G_{\mathbb{H}}(z,p)+\mathbf{c}$ has the law of $\mathrm{LF}_{\mathbb{H}}^{(\beta,p),(\alpha,q)}$. Moreover, If $p=\infty$, let $(h,\mathbf{c})$ be sampled from $C_{\mathbb{H}}^{(\beta,\infty),(\alpha,q)}P_{\mathbb{H}}\times\left[e^{\left(\frac{1}{2}\beta+\alpha-Q\right)c}dc\right]$, where

$$C_{\mathbb{H}}^{(\beta,\infty),(\alpha,q)}=(2\Im q)^{-\frac{\alpha^2}{2}}|q|_+^{-2\alpha(Q-\alpha)}.$$

Let $\phi_\infty(z)=h(z)+(\beta-2Q)\log|z|_++\alpha G_{\mathbb{H}}(z,q)+\mathbf{c}$ and ϕ_∞ has the law of $\mathrm{LF}_{\mathbb{H}}^{(\beta,\infty),(\alpha,p)}$.

Proposition 4.2.6 ([ARS22, Proposition 3.9])**.** *Suppose $(\mathbb{H},\phi,i,0)$ is an embedding of*

$\mathrm{QD}_{1,1}$, *then* ϕ *has the law of* $C_0 \cdot \mathrm{LF}_{\mathbb{H}}^{(\gamma,i),(\gamma,0)}$ *for some fixed finite constant* C_0.

Definition 4.2.7. Fix $\alpha, \beta \in \mathbb{R}$. Define the quantum surface $\mathrm{QD}_{1,1}(\alpha,\beta)$ as follows: suppose $(\mathbb{H}, \phi, i, 0)$ is an embedding of $\mathrm{QD}_{1,1}(\alpha,\beta)$, then the law of ϕ is $\mathrm{LF}_{\mathbb{H}}^{(\alpha,i),(\beta,0)}$. Notice that $\mathrm{QD}_{1,1}(\gamma,\gamma) = C \cdot \mathrm{QD}_{1,1}$ for some finite constant C.

Lemma 4.2.8. *Fix* $\alpha, \beta \in \mathbb{R}$ *and let* h *be sampled from* $P_{\mathbb{H}}$. *Let* $\widetilde{h}(z) = h(z) + \alpha G_{\mathbb{H}}(z,i) + \frac{\beta}{2} G_{\mathbb{H}}(z,0) - 2Q \log |z|_+$ *and* $L = \nu_{\widetilde{h}}(\mathbb{R})$. *Let* $\mathrm{LF}_{\mathbb{H}}^{(\alpha,i),(\beta,0)}(\ell)$ *be the law of* $\widetilde{h} + \frac{2}{\gamma}\log\frac{\ell}{L}$ *under the reweighted measure* $2^{-\alpha^2/2}\frac{2}{\gamma}\frac{\ell^{\frac{2}{\gamma}(\alpha+\frac{\beta}{2}-Q)-1}}{L^{\frac{2}{\gamma}(\alpha+\frac{\beta}{2}-Q)}} P_{\mathbb{H}}$, *and let* $\mathrm{QD}_{1,1}(\alpha,\beta;\ell)$ *be the measure on quantum surfaces* $(\mathbb{H}, \phi, 0, i)$ *with* ϕ *being sampled from* $\mathrm{LF}_{\mathbb{H}}^{(\alpha,i),(\beta,0)}(\ell)$. *Then* $\mathrm{QD}_{1,1}(\alpha,\beta;\ell)$ *is a measure on quantum surfaces with (quantum) boundary length* ℓ, *and*

$$\mathrm{LF}_{\mathbb{H}}^{(\alpha,i),(\beta,0)} = \int_0^\infty \mathrm{LF}_{\mathbb{H}}^{(\alpha,i),(\beta,0)}(\ell) d\ell \qquad \text{and} \qquad \mathrm{QD}_{1,1}(\alpha,\beta) = \int_0^\infty \mathrm{QD}_{1,1}(\alpha,\beta;\ell) d\ell. \tag{4.18}$$

Proof. Suppose ϕ has the law of $\widetilde{h} + \frac{2}{\gamma}\log\frac{\ell}{L}$, then we have

$$\nu_\phi(\mathbb{R}) = \int_{\mathbb{R}} e^{\frac{\gamma}{2}\phi(x)} dx = \frac{\ell}{L}\int_{\mathbb{R}} e^{\frac{\gamma}{2}\widetilde{h(x)}} dx = \ell. \tag{4.19}$$

Therefore, we have $\nu_\phi(\mathbb{R}) = \ell$ almost surely under $\mathrm{LF}_{\mathbb{H}}^{(\alpha,i),(\beta,0)}(\ell)$. Moreover, for any non-negative measurable function F on $H^{-1}(\mathbb{H})$, we have

$$\begin{aligned}\int_0^\infty \int F\left(\widetilde{h} + \frac{2}{\gamma}\log\frac{\ell}{L}\right) 2^{-\alpha^2/2}\frac{2}{\gamma}\frac{\ell^{\frac{2}{\gamma}(\alpha+\frac{\beta}{2}-Q)-1}}{L^{\frac{2}{\gamma}(\alpha+\frac{\beta}{2}-Q)}} P_{\mathbb{H}}(dh) d\ell \\ = \int\int_{-\infty}^\infty F(\widetilde{h}+c) 2^{-\alpha^2/2} e^{(\alpha+\frac{\beta}{2}-Q)c} dc P_{\mathbb{H}}(dh)\end{aligned} \tag{4.20}$$

by Fubini's theorem and change of variable $c = \frac{2}{\gamma}\log\frac{\ell}{L}$. This matches the field law in Definition 2.2.6. Hence (4.18) is proved. □

Definition 4.2.9 (QD with one general boundary insertion)**.** Fix $\alpha \in \mathbb{R}$ and let $(\mathbb{H}, \phi, i, 0)$ be an embedding of $\mathrm{QD}_{1,1}(\gamma,\alpha)$. Let $L = \nu_\phi(\mathbb{R})$ denote the total quantum boundary length and $A = \mu_\phi(\mathbb{H})$ denote the total quantum area. Let $\mathrm{QD}_{0,1}(\gamma,\alpha)$ be the law of $(\mathbb{H},\phi,0)$ under the reweighted measure $A^{-1}\mathrm{QD}_{1,1}(\gamma,\alpha)$. For integers $n \geq 0$ and $m \geq 1$, let $(\mathbb{H},\phi)$ be sampled from the re-weighted measure $A^n L^{m-1}\mathrm{QD}_{0,1}(\gamma,\alpha)$, then independently sample $\omega_1,\ldots,\omega_{m-1}$ and $z_1,\ldots,z_n$ according to $\nu_\phi^\#$ and $\mu_\phi^\#$ respectively. Let $\mathrm{QD}_{n,m}(\gamma,\alpha)$ denote the law of $(\mathbb{H},\phi,0,\omega_1,\ldots,\omega_{m-1},z_1,\ldots,z_n)$ viewed as a measure on equivalence class $\mathscr{D}_{n,m}$.

Notice that in the above definition, we only have one general boundary insertion (with weight α). All the other insertions (both bulk and boundary) are quantum typical.

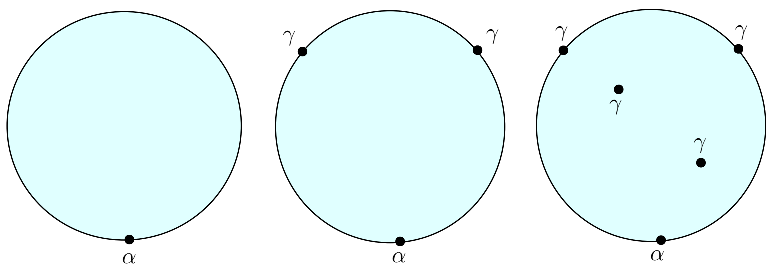


Figure 4.4: Left: $\mathrm{QD}_{0,1}(\gamma,\alpha)$ Middle: $\mathrm{QD}_{0,3}(\gamma,\alpha)$ Right: $\mathrm{QD}_{2,3}(\gamma,\alpha)$

More generally, for fixed $\ell_1,\ldots,\ldots,\ell_m$, like in [AHS20, Section 2.6], we can define the measure $\mathrm{QD}_{1,m}(\gamma,\alpha)(\ell_1,\ell_2,\ldots,\ell_m)$ using disintegration and it satisfies

$$\mathrm{QD}_{1,m}(\gamma,\alpha) = \int_0^\infty \ldots \int_0^\infty \mathrm{QD}_{1,m}(\gamma,\alpha;\ell_1,\ldots,\ell_m)d\ell_1\ldots d\ell_m. \tag{4.21}$$

4.2.3. Conformal weldings of thin and thick disks

Lemma 4.2.10. *For $W > \frac{\gamma^2}{2}$, let $\beta_W = \gamma + \frac{2-W}{\gamma} < Q$. Then we have*

$$\mathrm{QD}_{0,3}(\gamma,\beta_W) = C \cdot \mathrm{QT}(2,2,W) \tag{4.22}$$

for some finite constant C.

Proof. After applying [AHS21, Lemma 2.31] twice, we have

$$\mathrm{LF}_{\mathbb{H}}^{(\beta_W,0)}(d\phi)\nu_\phi(dx)\nu_\phi(dy) = \mathrm{LF}_{\mathbb{H}}^{(\beta_W,0),(\gamma,x),(\gamma,y)}(d\phi)dxdy. \tag{4.23}$$

By disintegration, we can fix an embedding of $\mathrm{QD}_{0,3}(\gamma,\alpha)$ to be $(\mathbb{H},\phi,-1,0,1)$ so that ϕ has the law of $C\cdot\mathrm{LF}_{\mathbb{H}}^{(\beta_W,0),(\gamma,-1),(\gamma,1)}$ for some finite constant C. Let $f:\mathbb{H}\to\mathcal{S}$ be the conformal map such that $f(-1)=-\infty, f(1)=\infty$ and $f(0)=0$. Therefore, by Definition 4.2.1, it has the law of $\mathrm{QT}(2,2,W)$ under push-forward of f. This completes the proof. □

Lemma 4.2.11. *Recall* $\mathrm{LF}_{\mathbb{H}}^{(\beta_i,z_i)_i}$ *from Definition 2.2.4. We have*

$$\mathrm{LF}_{\mathbb{H}}^{(\beta_i,z_i)_i}\left[f(\phi)\int_{\mathbb{H}}g(u)\mu_\phi(du)\right] = \mathrm{LF}_{\mathbb{H}}^{(\beta_i,z_i)_i,(\gamma,u)}[f(\phi)]g(u)d^2u \tag{4.24}$$

for non-negative measurable functions f and g.

Proof. The proof is identical to that of [AHS21, Lemma 2.33] with $\widehat{\mathbb{C}}$ replaced by $\mathbb{H}$. □

Next we recall the decomposition theorem of thin quantum disk with one additional typical boundary marked point that is crucial to our derivation of the field law.

Lemma 4.2.12 ([AHS20, Proposition 4.4])**.** *For $W\in(0,\frac{\gamma^2}{2})$, we have*

$$\mathcal{M}_{2,\bullet}^{\mathrm{disk}}(W) = \left(1-\frac{2}{\gamma^2}W\right)^2\mathcal{M}_{0,2}^{\mathrm{disk}}(W)\times\mathcal{M}_{2,\bullet}^{\mathrm{disk}}(\gamma^2-W)\times\mathcal{M}_{0,2}^{\mathrm{disk}}(W). \tag{4.25}$$

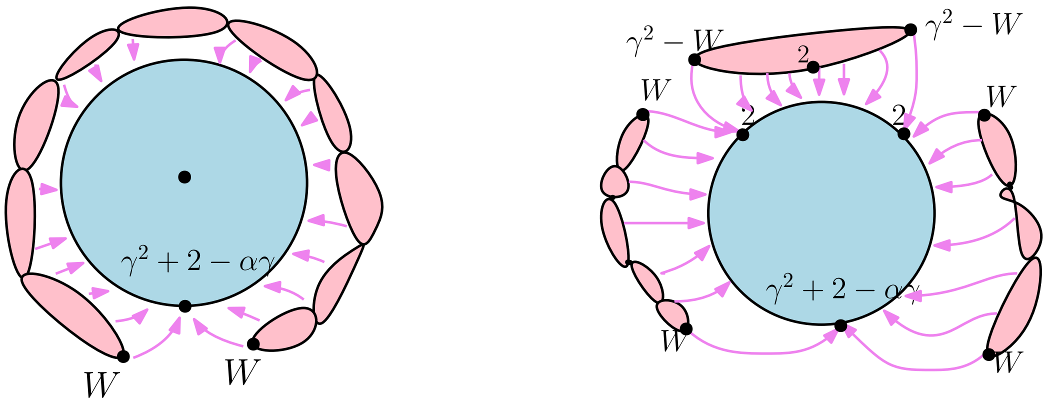


Figure 4.5: When $0 < W < \frac{\gamma^2}{2}$, welding $\mathrm{QD}_{1,1}(\gamma,\alpha)$ with $\mathcal{M}_{0,2}^{\mathrm{disk}}(W)$ is equivalent to first welding $\mathrm{QD}_{0,3}(\gamma,\alpha)$ with three independent quantum disks $\mathcal{M}_{0,2}^{\mathrm{disk}}(W)$, $\mathcal{M}_{2,\bullet}^{\mathrm{disk}}(\gamma^2-W)$ and $\mathcal{M}_{0,2}^{\mathrm{disk}}(W)$ separately then de-weighting all the three additional boundary marked points and sampling an bulk marked point in the blue region according to quantum area measure.

Proposition 4.2.13. *Fix $0<\gamma<2$ and $0<W<\frac{\gamma^2}{2}$. For $\alpha\le\gamma<Q$, let $W_\alpha = 2-(\alpha-\gamma)\gamma\ge 2>\frac{\gamma^2}{2}$. Let $(\mathbb{H},\phi,\eta,0,i)$ be an embedding of*

$$\int_0^\infty \mathrm{QD}_{1,1}(\gamma,\alpha;\ell)\times\mathcal{M}_{0,2}^{\mathrm{disk}}(W;\ell)d\ell. \tag{4.26}$$

Then ϕ has the law of $C\cdot \mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+W_\alpha},0),(\gamma,i)}$ for some finite constant C. Notice that $\alpha = \beta_{W_\alpha} = Q+\frac{\gamma}{2}-\frac{W_\alpha}{\gamma}$.

Proof. Fix $0<W<\frac{\gamma^2}{2}$ and $\alpha\le\gamma$. Start with the following four quantum surfaces:

$$\mathrm{QD}_{0,3}(\gamma,\alpha), \mathcal{M}_{0,2}^{\mathrm{disk}}(W), \mathcal{M}_{0,2}^{\mathrm{disk}}(\gamma^2-W) \text{ and } \mathcal{M}_{0,2}^{\mathrm{disk}}(W). \tag{4.27}$$

Notice that $\mathrm{QD}_{0,3}(\gamma,\alpha)$ has one α insertion and two γ insertions along its boundary. First, weld two $\mathcal{M}_{0,2}^{\mathrm{disk}}(W)$ disks along the boundaries of $\mathrm{QD}_{0,3}(\gamma,\alpha)$ with γ and α insertions, then weld $\mathcal{M}_{0,2}^{\mathrm{disk}}(\gamma^2-W)$ along the boundary of $\mathrm{QD}_{0,3}(\gamma,\alpha)$ with two γ insertions. Precisely, we consider

$$\begin{aligned}
&\text{Weld}\left(\text{QD}_{0,3}(\gamma,\alpha), \mathcal{M}^{\text{disk}}_{0,2}(W) \times \mathcal{M}^{\text{disk}}_{0,2}(\gamma^2 - W) \times \mathcal{M}^{\text{disk}}_{0,2}(W)\right) \\
&:= \int_0^\infty \left(\iint_0^\infty \mathcal{M}^{\text{disk}}_{0,2}(W;\ell_1)\text{QD}_{0,3}(\gamma,\alpha;\ell_1,\ell_2,\ell_3)\mathcal{M}^{\text{disk}}_{0,2}(W;\ell_3)d\ell_1 d\ell_3 \right) \cdot \\
&\mathcal{M}^{\text{disk}}_{0,2}(\gamma^2 - W;\ell_2)d\ell_2 \\
&= \int_0^\infty \left(\iint_0^\infty \mathcal{M}^{\text{disk}}_{0,2}(W;\ell_1)\text{QD}_{0,3}(\gamma,\alpha;\ell_1,\ell_2,\ell_3)\mathcal{M}^{\text{disk}}_{0,2}(W;\ell_3)d\ell_1 d\ell_3 \right) \cdot \\
&\ell_2^{-1}\mathcal{M}^{\text{disk}}_{2,\bullet}(\gamma^2 - W;\ell_2)d\ell_2 \\
&= L_2^{-1} \cdot \text{Weld}(\text{QD}_{0,3}(\gamma,\alpha), \mathcal{M}^{\text{disk}}_{0,2}(W) \times \mathcal{M}^{\text{disk}}_{2,\bullet}(\gamma^2 - W) \times \mathcal{M}^{\text{disk}}_{0,2}(W)) \\
&= \left(1 - \frac{2}{\gamma^2}W\right)^{-2} \cdot L_2^{-1} \cdot \text{Weld}\left(\text{QD}_{0,3}(\gamma,\alpha), \mathcal{M}^{\text{disk}}_{2,\bullet}(W)\right),
\end{aligned} \tag{4.28}$$

where L_2 denotes the quantum length of welding interface between $\text{QD}_{0,3}(\gamma,\alpha)$, $\mathcal{M}^{\text{disk}}_{0,2}(\gamma^2 - W)$ and

$$\text{Weld}\left(\text{QD}_{0,3}(\gamma,\alpha), \mathcal{M}^{\text{disk}}_{2,\bullet}(W)\right) := \int_0^\infty \text{QD}_{0,3}(\gamma,\alpha;\ell) \times \mathcal{M}^{\text{disk}}_{2,\bullet}(W;\cdot,\ell)d\ell. \tag{4.29}$$

In (4.29), $\text{QD}_{0,3}(\gamma,\alpha;\ell)$ represents the $\text{QD}_{0,3}(\gamma,\alpha)$ conditioning on having total boundary length ℓ and $\mathcal{M}^{\text{disk}}_{2,\bullet}(W;\cdot,\ell)$ represents the $\mathcal{M}^{\text{disk}}_{2,\bullet}(W)$ conditioning on having left boundary length ℓ. By de-weighting all the three marked points on the welding interface and sampling an additional bulk marked points in the inner region of (4.29), we have

$$\begin{aligned}
\text{Weld}\left(\text{QD}_{0,3}(\gamma,\alpha), \mathcal{M}^{\text{disk}}_{2,\bullet}(W)\right) &= \int_0^\infty \text{QD}_{0,3}(\gamma,\alpha;\ell) \times \mathcal{M}^{\text{disk}}_{2,\bullet}(W;\ell)d\ell \\
&= \int_0^\infty \ell^2 \cdot \text{QD}_{0,1}(\gamma,\alpha;\ell) \times \mathcal{M}^{\text{disk}}_{2,\bullet}(W;\ell)d\ell \\
&= \int_0^\infty \ell^3 \cdot \text{QD}_{0,1}(\gamma,\alpha;\ell) \times \mathcal{M}^{\text{disk}}_{0,2}(W;\ell)d\ell \\
&= L_T^3 \cdot \text{Weld}(\text{QD}_{0,1}(\gamma,\alpha), \mathcal{M}^{\text{disk}}_{0,2}(W)) \\
&= L_T^3 \cdot A_I^{-1} \cdot \text{Weld}(\text{QD}_{1,1}(\gamma,\alpha), \mathcal{M}^{\text{disk}}_{0,2}(W)),
\end{aligned} \tag{4.30}$$

where L_T denotes the quantum length of the total welding interface and A_I denotes the quantum area of $\mathrm{QD}_{0,1}(\gamma,\alpha)$. Hence, by (4.28), (4.30), we have

$$\begin{aligned}&\mathrm{Weld}\left(\mathrm{QD}_{0,3}(\gamma,\alpha),\mathcal{M}_{0,2}^{\mathrm{disk}}(W)\times\mathcal{M}_{0,2}^{\mathrm{disk}}(\gamma^2-W)\times\mathcal{M}_{0,2}^{\mathrm{disk}}(W)\right)\\&=\left(1-\frac{2}{\gamma^2}W\right)^{-2}\cdot L_2^{-1}\cdot L_T^3\cdot A_I^{-1}\cdot\mathrm{Weld}(\mathrm{QD}_{1,1}(\gamma,\alpha),\mathcal{M}_{0,2}^{\mathrm{disk}}(W)).\end{aligned}\tag{4.31}$$

By applying Theorem 4.2.4 three times, we know that suppose $(\mathbb{H},\phi,\eta_1,\eta_2,\eta_3,0,1,-1)$ is an embedding of

$$\mathrm{Weld}\left(\mathrm{QD}_{0,3}(\gamma,\alpha),\mathcal{M}_{0,2}^{\mathrm{disk}}(W)\times\mathcal{M}_{0,2}^{\mathrm{disk}}(\gamma^2-W)\times\mathcal{M}_{0,2}^{\mathrm{disk}}(W)\right),$$

then ϕ is independent of (η_1,η_2,η_3) and has the law of $C\cdot\mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+W_\alpha},0),(0,-1),(0,1)}$ for some finite constant C. Here we emphasize the fact that weights of insertions -1 and 1 are both zero due to the computation

$$\beta_{2+W+(\gamma^2-W)}=\beta_{2+\gamma^2}=0,$$

where the 2 comes from the insertion γ on $\mathrm{QD}_{0,3}(\gamma,\alpha)$, the W comes from $\mathcal{M}_{0,2}^{\mathrm{disk}}(W)$ and the γ^2-W comes from $\mathcal{M}_{0,2}^{\mathrm{disk}}(\gamma^2-W)$. Finally, by quantum surface relationship (4.31) and Lemma 4.2.11, we know that suppose $(\mathbb{H},\phi,0,i,\eta)$ is an embedding of

$$\mathrm{Weld}\left(\mathrm{QD}_{1,1}(\gamma,\alpha),\mathcal{M}_{0,2}^{\mathrm{disk}}(W)\right),$$

then ϕ has the law of $C\cdot\mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+W_\alpha},0),(\gamma,i)}$ for some finite constant C. □

4.2.4. Proof of Theorem 1.1.1

In this section, we prove Theorem 1.1.1 by inductively welding thin disks along the $\mathrm{QD}_{1,1}$.

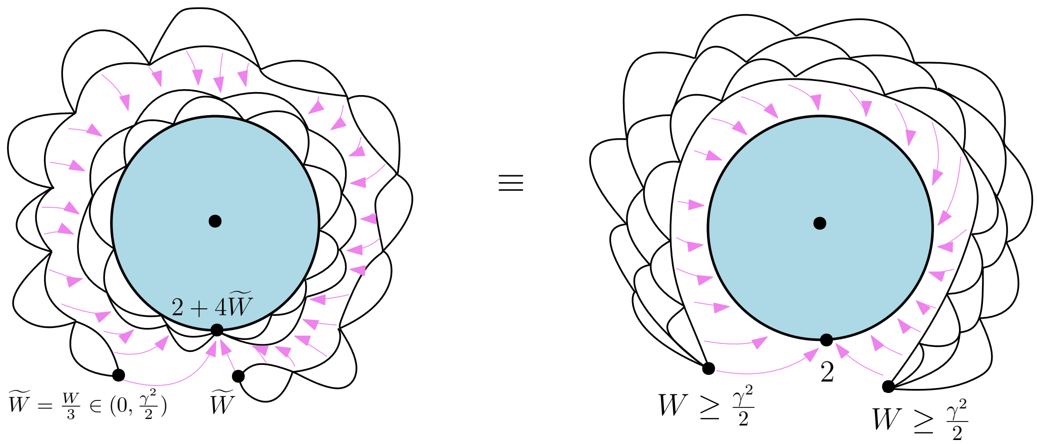


Figure 4.6: Illustration of the induction procedure in the proof of Theorem 1.1.1: suppose $W \geq \frac{\gamma^2}{2}$ and $\widetilde{W} = \frac{W}{3} \in (0, \frac{\gamma^2}{2})$, then welding a thick quantum disk $\mathcal{M}_{0,2}^{\text{disk}}(W)$ is equivalent to welding three thin quantum disks $\mathcal{M}_{0,2}^{\text{disk}}(\widetilde{W})$. Notice that here we only care about the law of the underlying random field.

Proof of Theorem 1.1.1. By Proposition 4.1.1, we have the correct curve law and know that the curve law is independent of the underlying random field. Therefore, it remains to derive the field law. Fix $0 < \gamma < 2$ and $0 < W < \frac{\gamma^2}{2}$. For $\alpha \leq \gamma$, let $W_\alpha = 2 - (\alpha - \gamma)\gamma \geq 2$. Let $(\mathbb{H}, \phi, \eta, 0, i)$ be an embedding of quantum surface

$$\int_0^\infty \text{QD}_{1,1}(\gamma, \alpha; \ell) \times \mathcal{M}_{0,2}^{\text{disk}}(W; \ell) d\ell. \tag{4.32}$$

By Proposition 4.2.13, ϕ has the law of $C \cdot \text{LF}_{\mathbb{H}}^{(\beta_{2W+W_\alpha}, 0), (\gamma, i)}$ for some finite constant C. Therefore, in order to prove the Theorem 1.1.1, we only need to extend the range of W from $(0, \frac{\gamma^2}{2})$ to $(0, \infty)$. For any $W \geq \frac{\gamma^2}{2}$, there exists some integer $n \geq 2$ such that $\widetilde{W} = \frac{W}{n} \in$

$(0, \frac{\gamma^2}{2})$. Moreover, by Theorem 3.2.2, we have

$$\begin{aligned}
&\int_0^\infty \mathrm{QD}_{1,1}(\ell) \times \mathcal{M}_{0,2}^{\mathrm{disk}}(W;\ell)d\ell \\
&= \int_0^\infty \mathrm{QD}_{1,1}(\ell)\cdot \\
&\left(\underbrace{\mathcal{M}_{0,2}^{\mathrm{disk}}(\widetilde{W};\ell,\ell_1)\mathcal{M}_{0,2}^{\mathrm{disk}}(\widetilde{W};\ell_1,\ell_2)\dots\mathcal{M}_{0,2}^{\mathrm{disk}}(\widetilde{W};\ell_{n-2},\ell_{n-1})\mathcal{M}_{0,2}^{\mathrm{disk}}(\widetilde{W};\ell_{n-1},\ell_n)}_{n \text{ thin disks}}\right) d\vec{\ell} \\
&= \int_0^\infty \left(\left(\left(\left(\mathrm{QD}_{1,1}(\ell)\mathcal{M}_{0,2}^{\mathrm{disk}}(\widetilde{W};\ell,\ell_1)\right)\mathcal{M}_{0,2}^{\mathrm{disk}}(\widetilde{W};\ell_1,\ell_2)\right)\dots\right)\mathcal{M}_{0,2}^{\mathrm{disk}}(\widetilde{W};\ell_{n-1},\ell_n)\right) d\vec{\ell},
\end{aligned} \tag{4.33}$$

where $d\vec{\ell} = d\ell d\ell_1, \dots, d\ell_n$. Notice that $\mathrm{QD}_{1,1} = C \cdot \mathrm{QD}_{1,1}(\gamma,\gamma)$ by definition and $W_\gamma = 2$. By applying Proposition 4.2.13 n times from the inner bracket to outer bracket, we have that suppose $(\mathbb{H}, \phi, \eta, 0, i)$ is an embedding of (4.32), then ϕ has the law of $C \cdot \mathrm{LF}_{\mathbb{H}}^{(\gamma,i),(\beta_{2+2n\widetilde{W}},0)}$, which is the same as $C \cdot \mathrm{LF}_{\mathbb{H}}^{(\gamma,i),(\beta_{2+2W},0)}$ for some finite constant C. This completes the proof. □

4.3. Proof of Theorem 1.1.3 via uniform embeddings of quantum surfaces

4.3.1. Uniform embeddings of quantum surfaces

To start, let us recall the setups of the *uniform embedding of quantum surfaces* described in Section 2.3.3. Let $\mathrm{conf}(\mathbb{H})$ be the group of conformal automorphisms of $\mathbb{H}$ where group multiplication $\cdot$ is the function composition $f \cdot g = f \circ g$. Let $\mathbf{m}_{\mathbb{H}}$ be a Haar measure on $\mathrm{conf}(\mathbb{H})$, which is both left and right invariant. Suppose $\mathfrak{f}$ is sampled from $\mathbf{m}_{\mathbb{H}}$ and $\phi \in H^{-1}(\mathbb{H})$, then we call the random function $\mathfrak{f} \bullet_\gamma \phi = \phi \circ \mathfrak{f}^{-1} + Q|\log(\mathfrak{f}^{-1})'|$ the *uniform embedding* of $(\mathbb{H}, \phi)$ via $\mathbf{m}_{\mathbb{H}}$. By invariance property of Haar measure, the law of $\mathfrak{f} \bullet_\gamma \phi$ only depends on $(\mathbb{H}, \phi)$ as quantum surface. Let $(z_i)_{1\le i\le n} \in \mathbb{H}$, $(s_j)_{1\le j\le m} \in \partial\mathbb{H}$ be groups of bulk and boundary marked points respectively. Suppose $(\mathbb{H}, h, z_1 \dots, z_n, s_1, \dots, s_m)$ is a marked quantum surface, then we call $\mathbf{m}_{\mathbb{H}} \ltimes (\mathbb{H}, h, z_1 \dots, z_n, s_1, \dots, s_m)$ the *uniform embedding* of $(\mathbb{H}, h, z_1 \dots, z_n, s_1, \dots, s_m)$ via $\mathbf{m}_{\mathbb{H}}$.

Lemma 4.3.1 ([ARS22, Lemma 3.7])**.** *Define three measures A, N, K on the conformal automorphism group* $\mathrm{conf}(\mathbb{H})$ *on* $\mathbb{H}$ *as follows. Sample* $\mathbf{t}$ *from* $1_{t>0}\frac{1}{t}dt$ *and let* $a : z \mapsto \mathbf{t}z$*. Sample* $\mathbf{s}$ *from Lebesgue measure on* $\mathbb{R}$ *and let* $n : z \mapsto z + \mathbf{s}$*. Sample* $\mathbf{u}$ *from* $\mathbb{1}_{-\frac{\pi}{2}<u<\frac{\pi}{2}}du$ *and let* $k : z \mapsto \frac{z\cos\mathbf{u}-\sin\mathbf{u}}{z\sin\mathbf{u}+\cos\mathbf{u}}$*. Let A, N, K be the law of a, n, k respectively, then the law of $a \circ n \circ k$ under $A \times N \times K$ is equal to* $\mathbf{m}_{\mathbb{H}}$*.*

Lemma 4.3.2. *Suppose* $\mathfrak{f}$ *is sampled from* $\mathbf{m}_{\mathbb{H}}$*, then the joint law of* $(\mathfrak{f}(0), \mathfrak{f}(i))$ *is*

$$\frac{1}{\Im q \cdot |p-q|^2} dpdq^2. \tag{4.34}$$

Proof. By the definition of A, N and K in Lemma 4.3.1, the $\mathfrak{f}(i)$ and $\mathfrak{f}(0)$ have the marginal law of $\mathbf{ts} + \mathbf{t}i$ and $\mathbf{t}\tan\mathbf{u} + \mathbf{ts}$ respectively, where $\mathbf{t}$ is sampled from $\mathbf{1}_{\{t>0\}}\frac{1}{t}dt, \mathbf{s}$ is sampled from ds, and $\mathbf{u}$ is sampled from $\mathbb{1}_{\{-\frac{\pi}{2}<u<\frac{\pi}{2}\}}du$. Let $x = ts, y = t$ and $z = t\tan u + st$, then we have

$$\frac{1}{t}dsdtdu = \left(\frac{1}{y^2}dxdy\right)\left(\frac{y}{y^2+(z-x)^2}dz\right) = \frac{1}{\Im q \cdot |p-q|^2}dpd^2q.$$

Therefore the joint law of $(\mathfrak{f}(0), \mathfrak{f}(i))$ is $\frac{1}{\Im q \cdot |p-q|^2}dpdq^2$. ☐

Lemma 4.3.3. *Let* $f \in \mathrm{conf}(\mathbb{H})$ *be such that $f(0) = p \in \mathbb{R}$ and $f(i) = q \in \mathbb{H}$, then we have that*

$$|f'(i)| = \Im q \qquad \text{and} \qquad f'(0) = \frac{|q-p|^2}{\Im q}. \tag{4.35}$$

Proof. Write $f(z) = \frac{az+b}{cz+d}$ with $ad - bc = 1$. Since $f(0) = p$ and $f(i) = q$, we have that

$$\begin{cases} \Re(q) & = \frac{ac+bd}{c^2+d^2}, \\ \Im(q) & = \frac{1}{c^2+d^2}, \\ p & = \frac{b}{d}, \\ ad-bc & = 1. \end{cases}$$

Furthermore, we have $|f'(i)| = \frac{1}{c^2+d^2}$ and $f'(0) = \frac{1}{d^2}$. Since $\frac{c^2}{d^2} = \frac{|\Re(q-p)|^2}{|\Im q|^2}$ and $c^2 + d^2 = \frac{1}{\Im q}$,

$f'(0) = \frac{|\Re(q-p)|^2}{\Im q} + \Im q = \frac{|q-p|^2}{\Im q}$ and $|f'(i)| = \Im q$. This completes the proof. □

4.3.2. Proof of Theorem 1.1.3

Fix $p \in \mathbb{R}$ and $\gamma \in (0,2)$. Recall that for any $\eta \in \mathrm{Bubble}_{\mathbb{H}}(p)$, the $D_\eta(p)$ denotes the component of $\mathbb{H} \backslash \eta$ which is encircled by η. Let $|D_\eta(p)|$ denote the euclidean area of $D_\eta(p)$. For $W > 0$, let $\rho = W - 2$. Define

$$\widetilde{\mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}}(\rho) := \frac{1}{|D_\eta(p)|} \int_{\mathbb{H}} |q-p|^{W - \frac{2W(W+2)}{\gamma^2}} (\Im q)^{\frac{W(W+2)}{\gamma^2} - \frac{W}{2}} \mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}(\rho)[d\eta | q \in D_\eta(p)] d^2 q. \tag{4.36}$$

Lemma 4.3.4. *For $W > 0$, let $\beta_{2W+2} = \gamma - \frac{2W}{\gamma}$. There exists some constant $C \in (0,\infty)$ such that*

$$\mathbf{m}_{\mathbb{H}} \ltimes \left(\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W; \cdot, \ell) \times \mathrm{QD}_{0,1}(\ell) d\ell \right) = C \cdot \mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2}, p)}(d\phi) \times \widetilde{\mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}}(\rho)(d\eta) dp. \tag{4.37}$$

Furthermore, we have

$$\mathbf{m}_{\mathbb{H},0} \ltimes \left(\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W; \cdot, \ell) \times \mathrm{QD}_{0,1}(\ell) d\ell \right) = C \cdot \mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2}, 0)}(d\phi) \times \widetilde{\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}}(\rho)(d\eta), \tag{4.38}$$

where recall that $\mathbf{m}_{\mathbb{H},0}$ is a Haar measure on $\mathrm{conf}(\mathbb{H}, 0)$, i.e., the group of conformal automorphisms of $\mathbb{H}$ fixing 0.

Proof. By Theorem 1.1.1, suppose $(\mathbb{H}, \phi, \eta, 0, i)$ is an embedding of the quantum surface

$$\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W; \cdot, \ell) \times \mathrm{QD}_{1,1}(\ell) d\ell,$$

then (ϕ, η) has the law of

$$C \cdot \mathrm{LF}_{\mathbb{H}}^{(\gamma, i), (\beta_{2W+2}, 0)}(d\phi) \times \mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[d\eta | i \in D_\eta(0)] \tag{4.39}$$

for some constant $C \in (0,\infty)$. By Proposition 2.2.9 and Lemma 4.3.3, for any $f \in \mathrm{conf}(\mathbb{H})$ with $f(0) = p \in \mathbb{R}$ and $f(i) = q \in \mathbb{H}$, we have

$$\begin{aligned} f_*\mathrm{LF}_{\mathbb{H}}^{(\gamma,i),(\beta_{2W+2},0)} &= |f'(0)|^{\Delta_{\beta_{2W+2}}}|f'(i)|^{2\Delta_\gamma}\mathrm{LF}_{\mathbb{H}}^{(\gamma,q),(\beta_{2W+2},p)} \\ &= \left(\frac{|q-p|^2}{\Im q}\right)^{\Delta_{\beta_{2W+2}}} \cdot (\Im q)^{2\Delta_\gamma} \cdot \mathrm{LF}_{\mathbb{H}}^{(\gamma,q),(\beta_{2W+2},p)}. \end{aligned} \tag{4.40}$$

Recall that for $\alpha \in \mathbb{R}$, $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$. By Lemma 4.3.2, if f is sampled from a $\mathbf{m}_{\mathbb{H}}$, then the joint law of $(f(0), f(i))$ is $\frac{1}{\Im q \cdot |p-q|^2} dpdq^2$. Therefore, suppose f is sampled from a $\mathbf{m}_{\mathbb{H}}$, then $f_*\mathrm{LF}_{\mathbb{H}}^{(\gamma,i),(\beta_{2W+2},0)}$ has the law of

$$\begin{aligned} &\frac{1}{\Im q \cdot |p-q|^2} \cdot \left(\frac{|q-p|^2}{\Im q}\right)^{\Delta_{\beta_{2W+2}}} \cdot (\Im q)^{2\Delta_\gamma} \cdot \mathrm{LF}_{\mathbb{H}}^{(\gamma,q),(\beta_{2W+2},p)} dpdq^2 \\ &= |q-p|^{W - \frac{2W(W+2)}{\gamma^2}} (\Im q)^{\frac{W(W+2)}{\gamma^2} - \frac{W}{2}} \mathrm{LF}_{\mathbb{H}}^{(\gamma,q),(\beta_{2W+2},p)} dpd^2q. \end{aligned} \tag{4.41}$$

Moreover, since $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[d\eta | i \in D_\eta(0)]$ is a probability measure, for fixed $f \in \mathrm{conf}(\mathbb{H})$ with $f(0) = p$ and $f(i) = q$, we have

$$f_*\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[d\eta | i \in D_\eta(0)] = \mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[d\eta | q \in D_\eta(p)]. \tag{4.42}$$

Combining (4.39), (4.41) and (4.42), we have

$$\begin{aligned} &\mathbf{m}_{\mathbb{H}} \ltimes \left(\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W;\cdot,\ell) \times \mathrm{QD}_{1,1}(\ell) d\ell\right) \\ &= C \cdot |q-p|^{W - \frac{2W(W+2)}{\gamma^2}} (\Im q)^{\frac{W(W+2)}{\gamma^2} - \frac{W}{2}} \mathrm{LF}_{\mathbb{H}}^{(\gamma,q),(\beta_{2W+2},p)} \times \mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[d\eta | q \in D_\eta(p)] dpd^2q. \end{aligned} \tag{4.43}$$

On the other hand, by [AHS21, Lemma 2.32] (the proof is identical with the domain replaced by $\mathbb{H}$), we have that

$$\begin{aligned}
&\mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2},p)}(d\phi) \times \widetilde{\mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}(\rho)}(d\eta)\mathbb{1}_{D_\eta(p)}(\mu_\phi(d^2q))dp \\
&= \mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2},p),(\gamma,q)}(d\phi) \times \widetilde{\mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}(\rho)}(d\eta)\mathbb{1}_{D_\eta(p)}(d^2q)dp \\
&= \mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2},p),(\gamma,q)}(d\phi) \cdot |q-p|^{W-\frac{2W(W+2)}{\gamma^2}} (\Im q)^{\frac{W(W+2)}{\gamma^2}-\frac{W}{2}} \mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}(\rho)[d\eta | q \in D_\eta(p)] d^2q dp.
\end{aligned} \tag{4.44}$$

Hence, by (4.43) and (4.44), we have

$$\begin{aligned}
&\mathbf{m}_{\mathbb{H}} \ltimes \left(\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W;\cdot,\ell) \times \mathrm{QD}_{1,1}(\ell) d\ell \right) \\
&\qquad = C \cdot \mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2},p)}(d\phi) \times \widetilde{\mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}(\rho)}(d\eta)\mu_\phi(\mathbb{1}_{D_\eta(p)} d^2q)dp
\end{aligned} \tag{4.45}$$

for some constant $C \in (0,\infty)$. After de-weighting both sides of (4.45) by the quantum area of $D_\eta(p)$ and forgetting the bulk marked point, we have

$$\mathbf{m}_{\mathbb{H}} \ltimes \left(\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W;\cdot,\ell) \times \mathrm{QD}_{0,1}(\ell) d\ell \right) = C \cdot \mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2},p)}(d\phi) \times \widetilde{\mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}(\rho)}(d\eta)dp \tag{4.46}$$

since quantum area is invariant under the Haar measure $\mathbf{m}_{\mathbb{H}}$. Furthermore, if we consider the $\mathbf{m}_{\mathbb{H},0}$, which is a Haar measure on the subgroup of $\mathrm{conf}(\mathbb{H})$ fixing 0, i.e., $\mathrm{conf}(\mathbb{H},0)$, then we have

$$\mathbf{m}_{\mathbb{H},0} \ltimes \left(\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W;\cdot,\ell) \times \mathrm{QD}_{0,1}(\ell) d\ell \right) = C \cdot \mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2},0)}(d\phi) \times \widetilde{\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)}(d\eta). \tag{4.47}$$

Note that equation (4.47) should be viewed as the disintegration of equation (4.46) over its boundary marked point. □

Lemma 4.3.5. *Fix $\rho > -2$. Then there exists some constant $C \in (0,\infty)$ such that*

$$\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho) = C \cdot \widetilde{\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)}, \tag{4.48}$$

where the constant C equals to $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[i \in D_\eta(0)]$.

Proof. Notice that

$$\begin{aligned}|D_\eta(0)| \cdot \mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)(d\eta) &= \int_{\mathbb{H}} \mathbb{1}_{q \in D_\eta(0)} \mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)(d\eta) d^2q \\ &= \int_{\mathbb{H}} \mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[q \in D_\eta(0)] \cdot \mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[d\eta | q \in D_\eta(0)] d^2q.\end{aligned}$$

Let $\psi \in \mathrm{conf}(\mathbb{H})$ be such that $\psi(i) = q$ and $\psi(0) = 0$ and it is easy to show that $\psi'(0) = \frac{|q|^2}{\Im q}$. By [Zha22, Theorem 3.16], we have

$$\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[q \in D_\eta(0)] = \psi'(0)^{-\alpha} \cdot \mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[i \in D_\eta(0)], \tag{4.49}$$

where $\alpha = \frac{(\rho+2)(2\rho+8-\kappa)}{2\kappa}$. Since $W = \rho + 2$, we have

$$\psi'(0)^{-\alpha} = |q|^{W - \frac{2W(W+2)}{\kappa}} (\Im q)^{-\frac{W}{2} + \frac{W(W+2)}{\kappa}}. \tag{4.50}$$

Hence,

$$\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[q \in D_\eta(0)] = C \cdot |q|^{W - \frac{2W(W+2)}{\kappa}} (\Im q)^{-\frac{W}{2} + \frac{W(W+2)}{\kappa}}, \tag{4.51}$$

where $C = \mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[i \in D_\eta(0)] \in (0,\infty)$ by Corollary 3.3.8. Therefore, by (4.36),

$$\begin{aligned}\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho) &= C \cdot \frac{1}{|D_\eta(0)|} \int_{\mathbb{H}} |q|^{W - \frac{2W(W+2)}{\kappa}} (\Im q)^{-\frac{W}{2} + \frac{W(W+2)}{\kappa}} \mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[d\eta | q \in D_\eta(0)] d^2q \\ &= C \cdot \widetilde{\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}}(\rho).\end{aligned} \tag{4.52}$$

This completes the proof. □

Corollary 4.3.6. *Fix $\rho > -2$ and $p \in \mathbb{R}$. Then there exists some constant $C \in (0, \infty)$ such that*

$$\mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}(\rho) = C \cdot \widetilde{\mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}}(\rho). \tag{4.53}$$

Proof. Fix $p \in \mathbb{R}$. Let $f_p \in \mathrm{conf}(\mathbb{H})$ be such that $f_p(z) = z + p$. Since $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}[d\eta | q \in D_\eta(0)]$ is a probability measure on $\mathrm{Bubble}_{\mathbb{H}}(0,q)$ for all q,

$$f_p\left(\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}[d\eta|q \in D_\eta(0)]\right) = \mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}[d\eta|q \in D_\eta(p)]. \tag{4.54}$$

Hence,

$$\begin{aligned}
&f_p\left(\widetilde{\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}}(\rho)\right)\\
&= f_p\left(\frac{1}{|D_\eta(0)|}\int_{\mathbb{H}} |q|^{W-\frac{2W(W+2)}{\kappa}}(\Im q)^{-\frac{W}{2}+\frac{W(W+2)}{\kappa}}\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho)[d\eta|q \in D_\eta(0)]d^2q\right)\\
&= \frac{1}{|D_\eta(p)|}\int_{\mathbb{H}} |q-p|^{W-\frac{2W(W+2)}{\kappa}}(\Im(q-p))^{-\frac{W}{2}+\frac{W(W+2)}{\kappa}}\mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}(\rho)[d\eta|q \in D_\eta(p)]d^2q\\
&= \frac{1}{|D_\eta(p)|}\int_{\mathbb{H}} |q-p|^{W-\frac{2W(W+2)}{\kappa}}(\Im q)^{-\frac{W}{2}+\frac{W(W+2)}{\kappa}}\mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}(\rho)[d\eta|q \in D_\eta(p)]d^2q\\
&= \widetilde{\mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}}(\rho).
\end{aligned} \tag{4.55}$$

By Lemma 4.3.5, we have

$$\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(\rho) = C \cdot \widetilde{\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}}.(\rho) \tag{4.56}$$

The (4.53) follows from applying f_p on both sides of (4.56). $\square$

Proof of Theorem 1.1.3. Theorem 1.1.3 follows immediately from Lemma 4.3.4, Lemma 4.3.5. $\square$

CHAPTER 5

SLE BUBBLE ZIPPERS WITH A GENERIC INSERTION AND APPLICATIONS

5.1. SLE bubble zippers with a generic bulk insertion

5.1.1. Quantum disks with a generic bulk insertion

Definition 5.1.1 (same as Definition 2.2.6)**.** For $\alpha, \beta \in \mathbb{R}$, let ϕ be sampled from $\mathrm{LF}_{\mathbb{H}}^{(\alpha,i),(\beta,0)}$. We denote $\mathrm{QD}_{1,1}(\alpha,\beta)$ the infinite measure describing the law of quantum surface $(\mathbb{H},\phi,0,i)$.

Lemma 5.1.2. *Fix $\alpha, \beta \in \mathbb{R}$ and $q \in \mathbb{H}$, and we have*

$$(\Im q)^{2\Delta_\alpha - \Delta_\beta} |q|^{2\Delta_\beta} \mathrm{LF}_{\mathbb{H}}^{(\alpha,q)(\beta,0)} = (f_\infty)_* \mathrm{LF}_{\mathbb{H}}^{(\beta,\infty),(\alpha,i)}, \tag{5.1}$$

where $f_\infty \in \mathrm{conf}(\mathbb{H})$ is the conformal map with $f_\infty(\infty) = 0$ and $f_\infty(i) = q$.

Proof. For each $r > 0$, let $f_r \in \mathrm{conf}(\mathbb{H})$ be a conformal map such that $f_r(r) = 0$ and $f_r(i) = q$. By Proposition 2.2.9, we have

$$\mathrm{LF}_{\mathbb{H}}^{(\beta,0),(\alpha,q)} = |f_r'(i)|^{-2\Delta_\alpha} |f_r'(r)|^{-\Delta_\beta} (f_r)_* \mathrm{LF}_{\mathbb{H}}^{(\beta,r),(\alpha,i)}. \tag{5.2}$$

Assume $f_r(z) = \frac{a_r z + b_r}{c_r z + d_r}$, where $a_r d_r - b_r c_r = 1$. Trivially, we have $|f_r'(z)| = \frac{1}{(c_r z + d_r)^2}$. Since $f_r(r) = 0$ and $f_r(i) = q$, we have

$$\begin{cases} \Im q = \frac{1}{c_r^2 + d_r^2}, \\ \Re q = \frac{a_r c_r + b_r d_r}{c_r^2 + d_r^2}, \\ a_r r + b_r = 0. \end{cases}$$

After solving the above equations, we have

$$\begin{cases} |f_r'(r)| = a_r^2 = \frac{|q|^2}{(r^2+1)\Im q}, \\ |f_r'(i)| = \Im q, \end{cases} \quad \text{and} \quad \begin{cases} a_r = \frac{|q|}{\sqrt{r^2+1}\sqrt{\Im q}}, \\ b_r = -\frac{r|q|}{\sqrt{r^2+1}\sqrt{\Im q}}, \\ c_r = \frac{\sqrt{|q|^2(r^2+1)-(\Im q - r\Re q)^2}}{\sqrt{r^2+1}\sqrt{\Im q}|q|}, \\ d_r = \frac{\sqrt{\Im q}(1-r\frac{\Re q}{\Im q})}{\sqrt{r^2+1}|q|}. \end{cases}$$

After multiplying $r^{\beta(Q-\frac{\beta}{2})}$ on both sides of (5.2), we have

$$\left(\frac{r^{\beta(Q-\frac{\beta}{2})}}{(r^2+1)^{\frac{\beta}{2}(Q-\frac{\beta}{2})}}\right)(\Im q)^{2\Delta_\alpha - \Delta_\beta}|q|^{2\Delta_\beta}\mathrm{LF}_{\mathbb{H}}^{(\alpha,q),(\beta,0)} = r^{\beta(Q-\frac{\beta}{2})}(f_r)_*\left[\mathrm{LF}_{\mathbb{H}}^{(\beta,r),(\alpha,i)}\right].$$

As $r \to \infty$, the left hand side becomes $(\Im q)^{2\Delta_\alpha - \Delta_\beta}|q|^{2\Delta_\beta}\mathrm{LF}_{\mathbb{H}}^{(\alpha,q),(\beta,0)}$. The right hand side converges in vague topology to $(f_\infty)_*\mathrm{LF}_{\mathbb{H}}^{(\beta,\infty),(\alpha,i)}$ follows from the facts that $f_r \to f_\infty$ in the topology of uniform convergence of analytic function and its derivatives on all compact sets and [AHS21, Lemma 2.18]. This completes the proof. □

Lemma 5.1.3. *Let $\alpha_1, \alpha_2, \beta \in \mathbb{R}$ and $\ell > 0$. For $\varepsilon > 0$, we define the measure $\mathrm{LF}_{\mathbb{H},\varepsilon}^{(\alpha_2,i),(\beta,0)}$ through the Radon-Nikodym derivative as follows:*

$$\frac{\mathrm{LF}_{\mathbb{H},\varepsilon}^{(\alpha_2,i),(\beta,0)}(\ell)}{\mathrm{LF}_{\mathbb{H}}^{(\alpha_1,i),(\beta,0)}(\ell)}(\phi) := \varepsilon^{\frac{1}{2}(\alpha_2^2-\alpha_1^2)}e^{(\alpha_2-\alpha_1)\phi_\varepsilon(i)}.$$

Furthermore, we have the weak convergence of measures

$$\lim_{\varepsilon\to 0}\mathrm{LF}_{\mathbb{H},\varepsilon}^{(\alpha_2,i),(\beta,0)}(\ell) = \mathrm{LF}_{\mathbb{H}}^{(\alpha_2,i),(\beta,0)}(\ell).$$

Proof. We know that if ϕ is sampled from $\mathrm{LF}_{\mathbb{H}}^{(\alpha,i),(\beta,0)}(1)^\#$, then $\phi + \frac{2}{\gamma}\log\ell$ has the law of $\mathrm{LF}_{\mathbb{H}}^{(\alpha,i),(\beta,0)}(\ell)^\#$. Moreover, we have

$$\frac{|\mathrm{LF}_{\mathbb{H},\varepsilon}^{(\alpha,i),(\beta,0)}(\ell)|}{|\mathrm{LF}_{\mathbb{H},\varepsilon}^{(\alpha,i),(\beta,0)}(1)|} = \frac{|\mathrm{LF}_{\mathbb{H}}^{(\alpha,i),(\beta,0)}(\ell)|}{|\mathrm{LF}_{\mathbb{H}}^{(\alpha,i),(\beta,0)}(1)|} = \ell^{\frac{2}{\gamma}(\frac{\beta}{2}+\alpha-Q)-1}.$$

Let

$$\widetilde{h^j} = h - 2Q\log|\cdot|_+ + \alpha_j G_{\mathbb{H}}(\cdot,i) + \frac{\beta}{2}G_{\mathbb{H}}(\cdot,0), \qquad j=1,2$$

and $\widetilde{h^{2,\varepsilon}} = \widetilde{h^1} + (\alpha_2-\alpha_1)G_{\mathbb{H},\varepsilon}(\cdot,i)$, where $G_{\mathbb{H},\varepsilon}(z,i)$ is the average of Green function $G_{\mathbb{H}}(z,\cdot)$ over $\partial B(i,\varepsilon)$. Notice that $\mathrm{Var}(h_\varepsilon(i)) = -\log\varepsilon - \log 2 + o_\varepsilon(1)$ and $\mathbb{E}\left[e^{(\alpha_2-\alpha_1)h_\varepsilon(i)}\right] = (1+o_\varepsilon(1))(2\varepsilon)^{-\frac{1}{2}(\alpha_2-\alpha_1)^2}$. Furthermore, the average of $-2Q\log|\cdot| + \alpha G_{\mathbb{H}}(\cdot,i) + \frac{\beta}{2}G_{\mathbb{H}}(\cdot,0)$ over $\partial B(i,\varepsilon)$ is $-\alpha\log(2\varepsilon)+o_\varepsilon(1)$. Let $L_1 = \nu_{\widetilde{h^1}}(\mathbb{R})$, $L_2 = \nu_{\widetilde{h^2}}(\mathbb{R})$ and $L_{2,\varepsilon} = \nu_{\widetilde{h^{2,\varepsilon}}}(\mathbb{R})$. For any bounded continuous function F on $H^{-1}(\mathbb{H})$, we have

$$\begin{aligned}
&\int \varepsilon^{\frac{1}{2}(\alpha_2^2-\alpha_1^2)} e^{(\alpha_2-\alpha_1)(\widetilde{h^1_\varepsilon}(i)-\frac{2}{\gamma}\log L_1)} F(\widetilde{h^1}-\frac{2}{\gamma}\log L_1)\cdot 2^{-\frac{\alpha_1^2}{2}}\cdot\frac{2}{\gamma}L_1^{-\frac{2}{\gamma}(\frac{\beta}{2}+\alpha_1-Q)}dh\\
&= \int \frac{(1+o_\varepsilon(1))}{\mathbb{E}[e^{(\alpha_2-\alpha_1)h_\varepsilon(i)}]} e^{(\alpha_2-\alpha_1)h_\varepsilon(i)} F(\widetilde{h^1}-\frac{2}{\gamma}\log L_1)\cdot 2^{-\frac{\alpha_2^2}{2}}\cdot\frac{2}{\gamma}L_1^{-\frac{2}{\gamma}(\frac{\beta}{2}+\alpha_2-Q)}dh\\
&= \int (1+o_\varepsilon(1)) F(\widetilde{h^{2,\varepsilon}}-\frac{2}{\gamma}\log L_{2,\varepsilon})\cdot 2^{-\frac{\alpha_2^2}{2}}\cdot\frac{2}{\gamma}L_{2,\varepsilon}^{-\frac{2}{\gamma}(\frac{\beta}{2}+\alpha_2-Q)}dh\\
&\xrightarrow{\varepsilon\to 0} \int F(\widetilde{h^2}-\frac{2}{\gamma}\log L_2)\cdot 2^{-\frac{\alpha_2^2}{2}}\cdot\frac{2}{\gamma}L_2^{-\frac{2}{\gamma}(\frac{\beta}{2}+\alpha_2-Q)}dh.
\end{aligned}$$

The second equality follows from the Girsanov's Theorem. Since $L_2 = (1+o_\varepsilon(1))L_{2,\varepsilon}$ and $\sup_{x\in\mathbb{R}}|G_{\mathbb{H}}(x,i) - G_{\mathbb{H},\varepsilon}(x,i)| = o_\varepsilon(1)$, the final ε limit follows from the the Dominated Convergence Theorem. □

5.1.2. Proof of Theorem 1.1.5

Proof of Theorem 1.1.5. By Theorem 1.1.1, we have

$$\mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2},0),(\gamma,i)}(1)\times\mathbf{m} = C_W\cdot\int_0^\infty \mathrm{QD}_{1,1}(\ell)\times\mathcal{M}_{0,2}^{\mathrm{disk}}(W;1,\ell)d\ell.$$

Let (Y,η) be sampled from the left hand side. Let $\psi_\eta:\mathbb{H}\to D_\eta(i)$ be the conformal map fixing 0 and i and $\xi_\eta:\mathbb{H}\to D_\eta(\infty)$ be such that $\xi_\eta(0)=0^-$, $\xi_\eta(1)=0^+$ and $\xi_\eta(\infty)=\infty$. Let $X,Z\in H^{-1}(\mathbb{H})$ be such that

$$X = Y\circ\psi_\eta + Q\log|\psi_\eta'| \qquad\text{and}\qquad Z = Y\circ\xi_\eta + Q\log|\xi_\eta'|.$$

Notice that $\mathrm{QD}_{1,1}(\ell)$ embedded in $(\mathbb{H},0,i)$ has the law of $C\cdot \mathrm{LF}_{\mathbb{H}}^{(\gamma,i),(\gamma,0)}(r)$. Therefore, the X has the law of

$$C_W\int_0^\infty |\mathcal{M}_{0,2}^{\mathrm{disk}}(W;1,\ell)|\cdot \mathrm{LF}_{\mathbb{H}}^{(\gamma,i),(\gamma,0)}(\ell)d\ell.$$

The conditional law of marked quantum surface $(\mathbb{H},Z,0,1)$ given X is $\mathcal{M}_{0,2}^{\mathrm{disk}}(\beta_W;1,\nu_X(\mathbb{R}))^{\#}$. Next, if we re-weight X by $\varepsilon^{\frac{1}{2}(\alpha^2-\gamma^2)}e^{(\alpha-\gamma)X_\varepsilon(i)}$ and send ε to 0, the law of X converges weakly to

$$C_W\int_0^\infty |\mathcal{M}_{0,2}^{\mathrm{disk}}(W;1,\ell)|\cdot \mathrm{LF}_{\mathbb{H}}^{(\alpha,i),(\gamma,0)}(\ell)d\ell.$$

Consequently, the law of Z conditioned on re-weighted X is $\mathcal{M}_{0,2}^{\mathrm{disk}}(W;1,\nu_X(\mathbb{R}))^{\#}$.

Next, let $\theta_{i,\varepsilon}$ be the uniform probability measure on $\partial B(i,\varepsilon)$ for sufficiently small ε. Let $\theta_{i,\varepsilon}^{\eta}=(\psi_\eta)_*(\theta_{i,\varepsilon})$ be the push-forward of $\theta_{i,\varepsilon}$ under ψ_η. Since ψ_η' is holomorphic and $\log|\psi_\eta'|$ is harmonic,

$$X_\varepsilon(i)=(X,\theta_{i,\varepsilon})=(Y\circ\psi_\eta+Q\log|\psi_\eta'|,\theta_{i,\varepsilon}^\eta)=(Y,\theta^\eta_{i,|\psi_\eta'(i)|\varepsilon})+Q\log|\psi_\eta'(i)|.$$

Therefore, re-weighting by $\varepsilon^{\frac{1}{2}(\alpha^2-\gamma^2)}e^{(\alpha-\gamma)X_\varepsilon(i)}$ is equivalent to re-weighting by

$$\begin{aligned}&\varepsilon^{\frac{1}{2}(\alpha^2-\gamma^2)}e^{(\alpha-\gamma)[(Y,\theta^\eta_{i,|\psi_\eta'(i)|\varepsilon})+Q\log|\psi_\eta'(i)|]}\\&=\left(\varepsilon|\psi_\eta'(i)|\right)^{\frac{1}{2}(\alpha^2-\gamma^2)}e^{(\alpha-\gamma)(Y,\theta^\eta_{i,|\psi_\eta'(i)|\varepsilon})}|\psi_\eta'(i)|^{-\frac{1}{2}\alpha^2+Q\alpha-2}.\end{aligned}$$

Hence, we conclude that for any bounded continuous F on $H^{-1}(\mathbb{H})^3$ and bounded continuous function g on $\mathrm{Bubble}_{\mathbb{H}}(0,i)$ equipped with Hausdorff topology,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\iint C_W\left(\varepsilon|\psi_\eta'(i)|\right)^{\frac{1}{2}(\alpha^2-\gamma^2)}e^{(\alpha-\gamma)(Y,\theta^\eta_{i,|\psi_\eta'(i)|\varepsilon})}F(X,Y,Z)\mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2},0),(\gamma,i)}(1)(dY)g(\eta)\mathbf{m}_\alpha(d\eta)\\&=\iint C_W F(\widetilde{X},\widetilde{Y},\widetilde{Z})\mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2},0),(\gamma,i)}(1)(d\widetilde{Y})g(\widetilde{\eta})\mathbf{m}_\alpha(d\widetilde{\eta}).\end{aligned}$$

By conformal welding, (X,Z) is uniquely determined by (Y,η). Similarly, $(\widetilde{Y},\widetilde{\eta})$ is uniquely determined by $(\widetilde{X},\widetilde{Z})$. Therefore, when $(\widetilde{Y},\widetilde{\eta})$ is sampled from $\mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2},0),(\gamma,i)}\times\mathbf{m}_\alpha$, $\widetilde{X}$ has the law of

$$C_W\cdot\int_0^\infty|\mathcal{M}_{0,2}^{\mathrm{disk}}(W;1,\ell)|\cdot\mathrm{LF}_{\mathbb{H}}^{(\alpha,i),(\gamma,0)}(\ell)d\ell$$

and the conditional law of marked quantum surface $(\mathbb{H},\widetilde{Z},0,i)$ given $\widetilde{X}$ is $\mathcal{M}_{0,2}^{\mathrm{disk}}(W;1,\nu_{\widetilde{X}}(\mathbb{R}))$. This finishes the proof. □

5.2. Applications

5.2.1. Preliminary results on integrabilities of LCFT

First, we recall the double gamma function $\Gamma_b(z)$. For b such that $\Re(b)>0$, $\Gamma_b(z)$ is the meromorphic on $\mathbb{C}$ such that

$$\ln\Gamma_b(z)=\int_0^\infty\frac{1}{t}\left(\frac{e^{-zt}-e^{-(b+\frac{1}{b})t/2}}{(1-e^{-bt})(1-e^{-\frac{1}{b}t})}-\frac{\left(\frac{1}{2}(b+\frac{1}{b})-z\right)^2}{2}e^{-t}+\frac{z-\frac{1}{2}(b+\frac{1}{b})}{t}\right)dt$$

for $\Re(z)>0$ and it satisfies the following two shift equations:

$$\frac{\Gamma_b(z)}{\Gamma_b(z+b)}=\frac{1}{\sqrt{2\pi}}\Gamma(bz)b^{-bz+\frac{1}{2}}\qquad\text{and}\qquad\frac{\Gamma_b(z)}{\Gamma_b(z+\frac{1}{b})}=\frac{1}{\sqrt{2\pi}}\Gamma\left(\frac{1}{b}z\right)\left(\frac{1}{b}\right)^{-\frac{1}{b}z+\frac{1}{2}}.\quad(5.3)$$

The above two shift equations allow us to extend $\Gamma_b(z)$ meromorphically from $\Re(z)>0$ to the entire complex plane $\mathbb{C}$. It has simple poles at $-nb-m\frac{1}{b}$ for nonnegative integers m,n. The double sine function is defined as

$$S_b(z):=\frac{\Gamma_b(z)}{\Gamma_b(b+\frac{1}{b}-z)}.\qquad(5.4)$$

We can now define the Liouville reflection coefficient R. For fixed $\mu_1,\mu_2>0$, let $\sigma_j\in\mathbb{C}$ satisfy $\mu_j=e^{i\pi\gamma(\sigma_j-\frac{Q}{2})}$ and $\Re\sigma_j=\frac{Q}{2}$ for $j=1,2$ and define the following two meromorphic functions for $\beta \in \mathbb{C}$ as belows::

$$\overline{R}(\beta,\mu_1,\mu_2) = \frac{(2\pi)^{\frac{2}{\gamma}(Q-\beta)-\frac{1}{2}}(\frac{2}{\gamma})^{\frac{\gamma}{2}(Q-\beta)-\frac{1}{2}}}{(Q-\beta)\Gamma(1-\frac{\gamma^2}{4})^{\frac{2}{\gamma}(Q-\beta)}} \frac{\Gamma_{\frac{\gamma}{2}}(\beta-\frac{\gamma}{2})e^{i\pi(\sigma_1+\sigma_2-Q)(Q-\beta)}}{\Gamma_{\frac{\gamma}{2}}(Q-\beta)S_{\frac{\gamma}{2}}(\frac{\beta}{2}+\sigma_2-\sigma_1)S_{\frac{\gamma}{2}}(\frac{\beta}{2}+\sigma_1-\sigma_2)}, \tag{5.5}$$

$$R(\beta,\mu_1,\mu_2) = -\Gamma\left(1-\frac{2}{\gamma}(Q-\beta)\right)\overline{R}(\beta,\mu_1,\mu_2). \tag{5.6}$$

Proposition 5.2.1 ([RZ22, Theorem 1.7])**.** *Let $\beta_W = Q = \frac{\gamma}{2} - \frac{W}{\gamma} \in (\frac{\gamma}{2}, Q)$. Let $\mu_1, \mu_2 \geq 0$ not both be zero. Recall random field $\widehat{h}$ defined in Definition* 2.3.2 *of $\mathcal{M}_{0,2}^{\text{disk}}(W)$. We have that*

$$\mathbb{E}\left[\left(\mu_1\nu_{\widehat{h}}(\mathbb{R}) + \mu_2\nu_{\widehat{h}}(\mathbb{R}+\pi i)\right)^{\frac{2}{\gamma}(Q-\beta_W)}\right] = \overline{R}(\beta_W,\mu_1,\mu_2). \tag{5.7}$$

Lemma 5.2.2 ([AHS21, Lemma 3.3])**.** *For $W \in [\frac{\gamma^2}{2}, \gamma Q)$ and $\beta_W = Q + \frac{\gamma}{2} - \frac{W}{\gamma}$, let L_1, L_2 denote the left and right boundary length of weight W quantum disk $\mathcal{M}_{0,2}^{\text{disk}}(W)$, then the law of $\mu_1 L_1 + \mu_2 L_2$ is*

$$\mathbb{1}_{\ell>0}\overline{R}(\beta_W,\mu_1,\mu_2)\ell^{-\frac{2}{\gamma^2}W}d\ell.$$

Let $W = 2, \mu_1 = \mu_2 = 1$ and by independent sampling property of $\mathcal{M}_{0,2}^{\text{disk}}(2)$, we have the following results on the joint law of left and right boundary length.

Proposition 5.2.3 ([DMS20], Proposition 5.1)**.** *For $\ell, \gamma > 0$, we have*

$$|\mathcal{M}_{0,2}^{\text{disk}}(2;\ell,r)| = \frac{(2\pi)^{\frac{4}{\gamma^2}-1}}{(1-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma^2}{4})^{\frac{4}{\gamma^2}}}(\ell+r)^{-\frac{4}{\gamma^2}-1}. \tag{5.8}$$

Proposition 5.2.4 ([AHS21, Proposition 3.4])**.** *For $W \in (\frac{\gamma^2}{2}, \gamma^2)$ and $\beta_W = Q + \frac{\gamma}{2} - \frac{W}{\gamma}$. Let L_1 and L_2 be the left and right quantum boundary lengths of weight-W quantum disk*

$\mathcal{M}_{0,2}^{\text{disk}}(W)$, *and we have*

$$\mathcal{M}_{0,2}^{\text{disk}}(W)\left[1-e^{-\mu_1 L_1-\mu_2 L_2}\right]=-\frac{\gamma}{2(Q-\beta)}R(\beta_W;\mu_1,\mu_2). \tag{5.9}$$

Next, we recall the two-pointed correlation function of the Liouville theory on $\mathbb{H}$ that was introduced in Section 1.1.2 when $\mu=0,\mu_\partial>0$. For bulk insertions z_i with weights α_i and boundary insertions s_j with weights β_j, the correlation function of LCFT at these points is defined using the following formal path integral:

$$\left\langle \prod_{i=1}^{N} e^{\alpha_i\phi(z_i)}\prod_{j=1}^{M} e^{\frac{\beta_j}{2}\phi(s_j)}\right\rangle_{\mu_\partial}=\int_{X:\mathbb{H}\to\mathbb{R}} DX\prod_{i=1}^{N} e^{\alpha_i X(z_i)}\prod_{j=1}^{M} e^{\frac{\beta_j}{2}X(s_j)}e^{-S^L_{\mu_\partial}(X)}. \tag{5.10}$$

In the above formula, DX is the formal uniform measure on infinite dimensional function space and $S_L^{\mu_\partial}(X)$ is the *Liouville action functional* given by

$$S^L_{\mu_\partial}(X):=\frac{1}{4\pi}\int_{\mathbb{H}}\left(|\nabla_g X|^2+QR_gX\right)d\lambda_g+\frac{1}{2\pi}\int_{\mathbb{R}}\left(QK_gX+2\pi\mu_\partial e^{\frac{\gamma}{2}X}\right)d\lambda_{\partial g}. \tag{5.11}$$

For background Riemannian metric g on $\mathbb{H}$, $\nabla_g, R_g, K_g, d\lambda_g, d\lambda_{\partial g}$ stand for the gradient, Ricci curvature, Geodesic curvature, volume form and line segment respectively. The subscript μ_∂ emphases the fact that we are considering the case when $\mu=0,\mu_\partial>0$. For $z\in\mathbb{H}$ and $s\in\mathbb{R}$, the *bulk-boundary correlator* is

$$\left\langle e^{\alpha\phi(z)}e^{\frac{\beta}{2}\phi(s)}\right\rangle_{\mu_\partial}=\frac{G(\alpha,\beta)}{|z-\overline{z}|^{2\Delta_\alpha-\Delta_\beta}|z-s|^{2\Delta_\beta}}. \tag{5.12}$$

Next, we introduce the rigorous mathematical definition of $G(\alpha,\beta)$.

Definition 5.2.5 ([RZ22, Definition 1.5])**.** The function $G(\alpha,\beta)$ is defined as

$$G(\alpha,\beta)=\frac{2}{\gamma}\Gamma\left(\frac{2\alpha+\beta-2Q}{\gamma}\right)\left(\mu_B^{\frac{2Q-2\alpha-\beta}{\gamma}}\right)\overline{G}(\alpha,\beta),$$

where for $\beta < Q$ and $\frac{\gamma}{2} - \alpha < \frac{\beta}{2} < \alpha$:

$$\overline{G}(\alpha,\beta) = \mathbb{E}\left[\left(\int_{\mathbb{R}} \frac{g(x)^{\frac{\gamma}{4}(\frac{2}{\gamma}-\alpha-\frac{\beta}{2})}}{|x-i|^{\gamma\alpha}} e^{\frac{\gamma}{2}h(x)}dx\right)^{\frac{2}{\gamma}(Q-\alpha-\frac{\beta}{2})}\right]. \tag{5.13}$$

In the above formula, $g(x) = \frac{1}{|x|_+^4}$, $|x|_+ = \max(|x|,1)$ and $h(x)$ is sampled from $P_{\mathbb{H}}$.

Theorem 5.2.6 ([RZ22, Theorem 1.7])**.** *For $\gamma \in (0,2), \beta < Q$ and $\frac{\gamma}{2} - \alpha < \frac{\beta}{2} < \alpha$,*

$$\overline{G}(\alpha,\beta) = \left(\frac{2^{\frac{\gamma}{2}(\frac{\beta}{2}-\alpha)}2\pi}{\Gamma(1-\frac{\gamma^2}{4})}\right)^{\frac{2}{\gamma}(Q-\alpha-\frac{\beta}{2})} \frac{\Gamma(\frac{\gamma\alpha}{2}+\frac{\gamma\beta}{4}-\frac{\gamma^2}{4})\Gamma_{\frac{\gamma}{2}}(\alpha-\frac{\beta}{2})\Gamma_{\frac{\gamma}{2}}(\alpha+\frac{\beta}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\beta}{2})^2}{\Gamma_{\frac{\gamma}{2}}(Q-\frac{\beta}{2})\Gamma_{\frac{\gamma}{2}}(\alpha)^2\Gamma_{\frac{\gamma}{2}}(Q)}. \tag{5.14}$$

Lemma 5.2.7. *Fix $\ell > 0$. Let γ,β,α be such that $\gamma \in (0,2), \beta < Q, \frac{\gamma}{2} - \alpha < \frac{\beta}{2} < \alpha$. Let h be sampled from $P_{\mathbb{H}}$ and let $h_\infty(z) = h(z) + (\beta - 2Q)\log|z|_+ + \alpha G_{\mathbb{H}}(z,i)$. Let ϕ be sampled from $\mathrm{LF}_{\mathbb{H}}^{(\beta,\infty),(\alpha,i)}(d\phi)$ and for each bounded non-negative measurable function f on $(0,\infty)$, we have*

$$\mathrm{LF}_{\mathbb{H}}^{(\beta,\infty),(\alpha,i)}[f(\nu_\phi(\mathbb{R}))] = \int_0^\infty f(\ell)2^{-\frac{\alpha^2}{2}}\ell^{\frac{2}{\gamma}(\frac{1}{2}\beta+\alpha-Q)-1}\cdot\frac{2}{\gamma}\cdot\overline{G}(\alpha,\beta)d\ell,$$

where $\overline{G}(\alpha,\beta)$ is the two point (one bulk, one boundary) correlation function of Liouville theory on $\mathbb{H}$.

Proof. It suffices to consider the case when $f(\ell) = \mathbb{1}_{a<\ell<b}(\ell)$. By direct computation,

$$\begin{aligned}
\mathrm{LF}_{\mathbb{H}}^{(\beta,\infty),(\alpha,i)}[f(\nu_\phi(\mathbb{R}))] &= \mathbb{E}\left[\int_{\mathbb{R}} \mathbb{1}_{\{e^{\frac{\gamma}{2}c}\nu_{h_\infty(\mathbb{R})}\in(a,b)\}}2^{-\frac{\alpha^2}{2}}e^{(\frac{1}{2}\beta+\alpha-Q)c}dc\right] \\
&= \mathbb{E}\left[\int_a^b \nu_{h_\infty}(\mathbb{R})^{\frac{2}{\gamma}(Q-\alpha-\frac{1}{2}\beta)}2^{-\frac{\alpha^2}{2}}\ell^{\frac{2}{\gamma}(\frac{1}{2}\beta+\alpha-Q)-1}\cdot\frac{2}{\gamma}d\ell\right] \\
&= \int_a^b 2^{-\frac{\alpha^2}{2}}\ell^{\frac{2}{\gamma}(\frac{1}{2}\beta+\alpha-Q)-1}\cdot\frac{2}{\gamma}\cdot\mathbb{E}\left[\nu_{h_\infty(\mathbb{R})}^{\frac{2}{\gamma}(Q-\alpha-\frac{1}{2}\beta)}\right]d\ell.
\end{aligned}$$

The second line follows from the change of variable $\ell = e^{\frac{\gamma}{2}c}\nu_{h_\infty}(\mathbb{R})$. The third line follows from the finiteness of $\mathbb{E}\left[\nu_{h_\infty(\mathbb{R})}^{\frac{2}{\gamma}(Q-\alpha-\frac{1}{2}\beta)}\right]$ and Fubini's theorem. The finiteness of

$$\mathbb{E}\left[\nu_{h_\infty(\mathbb{R})}^{\frac{2}{\gamma}(Q-\alpha-\frac{1}{2}\beta)}\right]$$

is proved in [RZ22, Proposition 5.1]. Furthermore,

$$\begin{aligned}
\mathbb{E}\left[\nu_{h_\infty(\mathbb{R})}^{\frac{2}{\gamma}(Q-\alpha-\frac{1}{2}\beta)}\right] &= \lim_{\varepsilon\downarrow 0}\mathbb{E}\left[\left(\int_{\mathbb{R}}\varepsilon^{\frac{\gamma^2}{4}}e^{\frac{\gamma}{2}h_\varepsilon(x)}e^{\frac{\gamma}{2}[(\beta-\frac{4}{\gamma})\log|x|_+ + \alpha\widetilde{G}_{\mathbb{H}}(x,i)]}dx\right)^{\frac{2}{\gamma}(Q-\alpha-\frac{1}{2}\beta)}\right] \\
&= \lim_{\varepsilon\downarrow 0}\mathbb{E}\left[\left(\int_{\mathbb{R}}e^{\frac{\gamma}{2}h_\varepsilon(x)-\frac{\gamma^2}{2}\mathbb{E}[h_\varepsilon(x)^2]}\frac{|x|_+^{\gamma\alpha+\frac{\gamma\beta}{2}-2}}{|x-i|^{\gamma\alpha}}dx\right)^{\frac{2}{\gamma}(Q-\alpha-\frac{1}{2}\beta)}\right] \\
&= \mathbb{E}\left[\left(\int_{\mathbb{R}}e^{\frac{\gamma}{2}h_\varepsilon(x)-\frac{\gamma^2}{2}\mathbb{E}[h_\varepsilon(x)^2]}\frac{|x|_+^{\gamma\alpha+\frac{\gamma\beta}{2}-2}}{|x-i|^{\gamma\alpha}}dx\right)^{\frac{2}{\gamma}(Q-\alpha-\frac{1}{2}\beta)}\right] \\
&= \mathbb{E}\left[\left(\int_{\mathbb{R}}e^{\frac{\gamma}{2}h(x)}\frac{|x|_+^{\gamma\alpha+\frac{\gamma\beta}{2}-2}}{|x-i|^{\gamma\alpha}}dx\right)^{\frac{2}{\gamma}(Q-\alpha-\frac{1}{2}\beta)}\right] \\
&= \overline{G}(\alpha,\beta).
\end{aligned}$$

This completes the proof. □

5.2.2. Moments of the conformal radius of $\mathrm{SLE}_\kappa(\rho)$ bubbles

By (1.11) in Theorem 1.1.5,

$$\mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2},0),(\alpha,i)}(1)\times\mathbf{m}_\alpha = C_W\cdot\int_0^\infty \mathrm{QD}_{1,1}(\alpha,\gamma;\ell)\times\mathcal{M}_{0,2}^{\mathrm{disk}}(W;1,\ell)d\ell \tag{5.15}$$

for $W>0$ and $\alpha\in\mathbb{R}$. By definition of $\mathbf{m}_\alpha$ (1.10),

$$|\mathbf{m}_\alpha| = \mathbb{E}\left[|\psi_\eta'(i)|^{2\Delta_\alpha-2}\right] \tag{5.16}$$

since $\mathbf{m}$ is a probability measure. Therefore, taking mass on both sides of (5.15) yields

$$\left|\mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2},0),(\alpha,i)}(1)\right| \cdot \mathbb{E}\left[|\psi_\eta'(i)|^{2\Delta_\alpha-2}\right] = C_W \cdot \int_0^\infty \left|\mathrm{QD}_{1,1}(\alpha,\gamma;\ell)\right| \left|\mathcal{M}_{0,2}^{\mathrm{disk}}(W;1,\ell)\right| d\ell. \tag{5.17}$$

Lemma 5.2.8. *Fix $\ell > 0$ and $q \in \mathbb{H}$. Let γ, β, α be such that $\gamma \in (0,2), \beta < Q$ and $\frac{\gamma}{2} - \alpha < \frac{\beta}{2} < \alpha$. Then we have*

$$|\mathrm{LF}_{\mathbb{H}}^{(\beta,0),(\alpha,q)}(\ell)| = |q|^{-2\Delta_\beta} (\Im q)^{\Delta_\beta - 2\Delta_\alpha} \cdot 2^{-\frac{\alpha^2}{2}} \ell^{\frac{2}{\gamma}(\frac{1}{2}\beta+\alpha-Q)-1} \cdot \frac{2}{\gamma} \cdot \overline{G}(\alpha,\beta). \tag{5.18}$$

Moreover, for $\mu > 0$, $\beta < Q$ and $Q - \alpha < \frac{\beta}{2} < \alpha$, we have

$$\begin{aligned}&\mathrm{LF}_{\mathbb{H}}^{(\beta,0),(\alpha,q)}\left[e^{-\mu\nu_\phi(\mathbb{R})}\right] \\ &\quad = |q|^{-2\Delta_\beta}(\Im q)^{\Delta_\beta-2\Delta_\alpha} 2^{-\frac{\alpha^2}{2}} \frac{2}{\gamma} \cdot \overline{G}(\alpha,\beta) \mu^{\frac{2}{\gamma}(Q-\alpha-\frac{1}{2}\beta)} \Gamma\left(\frac{2}{\gamma}\left(\frac{1}{2}\beta+\alpha-Q\right)\right).\end{aligned} \tag{5.19}$$

Proof. By Lemma 5.2.7 and Lemma 5.1.2, for bounded continuous function f on $(0,\infty)$, $\beta < Q$ and $\frac{\gamma}{2} - \alpha < \frac{\beta}{2} < \alpha$,

$$\begin{aligned}\mathrm{LF}_{\mathbb{H}}^{(\beta,0),(\alpha,q)}[f(\nu_\phi(\mathbb{R}))] &= |q|^{-2\Delta_\beta}(\Im q)^{\Delta_\beta-2\Delta_\alpha} \mathrm{LF}_{\mathbb{H}}^{(\beta,\infty),(\alpha,i)}[f(\nu_\phi(\mathbb{R}))] \\ &= |q|^{-2\Delta_\beta}(\Im q)^{\Delta_\beta-2\Delta_\alpha} \cdot 2^{-\frac{\alpha^2}{2}} \int_0^\infty f(\ell) \ell^{\frac{2}{\gamma}(\frac{1}{2}\beta+\alpha-Q)-1} \cdot \frac{2}{\gamma} \cdot \overline{G}(\alpha,\beta) d\ell.\end{aligned}$$

When $f(\ell) = e^{-\mu\ell}$, for $\beta < Q$ and $Q - \alpha < \frac{\beta}{2} < \alpha$,

$$\int_0^\infty e^{-\mu\ell} \ell^{\frac{2}{\gamma}(\frac{1}{2}\beta+\alpha-Q)-1} d\ell = \mu^{\frac{2}{\gamma}(Q-\alpha-\frac{1}{2}\beta)} \Gamma\left(\frac{2}{\gamma}\left(\frac{1}{2}\beta+\alpha-Q\right)\right).$$

This completes the proof. □

Special Case: $W = 2$

When $W = 2$, $\Delta_{\beta_6} = \Delta_{\gamma - \frac{4}{\gamma}} = 2 - \frac{8}{\gamma^2}$. By (5.17), we have

$$\left| \mathrm{LF}_{\mathbb{H}}^{(\gamma-\frac{4}{\gamma},0),(\alpha,i)}(1) \right| \cdot \mathbb{E}\left[|\psi_\eta'(i)|^{2\Delta_\alpha - 2} \right] = C_2 \cdot \int_0^\infty |\mathrm{QD}_{1,1}(\alpha,\gamma;\ell)||\mathcal{M}_{0,2}^{\mathrm{disk}}(2;1,\ell)|d\ell. \quad (5.20)$$

Furthermore, we renormalize the moments of the conformal radius of SLE_κ bubbles so that there is no additional multiplicative constant on the right hand side. More specifically, we define the *renormalized moments of the conformal radius* to be

$$\mathrm{CR}_2(\alpha) := \frac{\mathbb{E}\left[|\psi_\eta'(i)|^{2\Delta_\alpha - 2} \right]}{C_2}$$

and therefore have

$$\left| \mathrm{LF}_{\mathbb{H}}^{(\gamma-\frac{4}{\gamma},0),(\alpha,i)}(1) \right| \cdot \mathrm{CR}_2(\alpha) = \int_0^\infty |\mathrm{QD}_{1,1}(\alpha,\gamma;\ell)||\mathcal{M}_{0,2}^{\mathrm{disk}}(2;1,\ell)|d\ell.$$

Proposition 5.2.9 (Moments of the conformal radius of SLE_κ bubbles, same as Proposition 1.1.6)**.** *Fix $W = 2, \rho = 0$ and $\frac{\gamma}{2} < \alpha < Q + \frac{2}{\gamma}$. Suppose η is sampled from $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}[d\eta | i \in D_\eta(0)]$, then we have*

$$\mathbb{E}\left[|\psi_\eta'(i)|^{2\Delta_\alpha - 2} \right] = \frac{\Gamma(\frac{2\alpha}{\gamma})\Gamma(\frac{8}{\kappa} - \frac{2\alpha}{\gamma} + 1)}{\Gamma(\frac{8}{\kappa} - 1)}. \quad (5.21)$$

Consequently,

$$\mathbb{E}\left[\mathrm{Rad}(D_\eta(0), i)^{2\Delta_\alpha - 2} \right] = 2^{2\Delta_\alpha - 2} \cdot \frac{\Gamma(\frac{2\alpha}{\gamma})\Gamma(\frac{8}{\kappa} - \frac{2\alpha}{\gamma} + 1)}{\Gamma(\frac{8}{\kappa} - 1)}. \quad (5.22)$$

Proof. By Lemma 5.2.8, when $\alpha > \frac{2}{\gamma}$,

$$\left| \mathrm{LF}_{\mathbb{H}}^{(\gamma-\frac{4}{\gamma},0),(\alpha,i)}(1) \right| = 2^{-\frac{\alpha^2}{2}} \cdot \frac{2}{\gamma} \cdot \overline{G}(\alpha, \gamma - \frac{4}{\gamma}) \quad (5.23)$$

and when $\alpha > \frac{\gamma}{2}$,

$$|\mathrm{QD}_{1,1}(\alpha,\gamma;r)| = 2^{-\frac{\alpha^2}{2}} r^{\frac{2}{\gamma}(\alpha-Q)} \frac{2}{\gamma}\overline{G}(\alpha,\gamma). \tag{5.24}$$

By [AHS21, Proposition 5.1],

$$|\mathcal{M}_{0,2}^{\mathrm{disk}}(2;1,r)| = \frac{(2\pi)^{\frac{4}{\gamma^2}-1}}{(1-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma^2}{4})^{\frac{4}{\gamma^2}}}(1+r)^{-\frac{4}{\gamma^2}-1}.$$

Notice that when $\frac{2}{\gamma} < \alpha < Q + \frac{2}{\gamma}$,

$$\begin{aligned}\int_0^\infty \frac{r^{\frac{2}{\gamma}\alpha}}{[(1+r)r]^{\frac{4}{\gamma^2}+1}}dr = B\left(\frac{2}{\gamma}(\alpha-Q)+1, \frac{8}{\gamma^2}-\frac{2\alpha}{\gamma}+1\right) &= \frac{\Gamma(\frac{2}{\gamma}(\alpha-Q)+1)\Gamma(\frac{8}{\gamma^2}-\frac{2\alpha}{\gamma}+1)}{\Gamma(\frac{4}{\gamma^2}+1)} \\ &= \frac{\Gamma(\frac{2\alpha}{\gamma}-\frac{4}{\kappa})\Gamma(\frac{8}{\kappa}-\frac{2\alpha}{\gamma}+1)}{\Gamma(\frac{4}{\kappa}+1)},\end{aligned}$$

where $B(x,y)$ is the Beta function with parameter x, y. Therefore, when $\max\{\frac{\gamma}{2},\frac{2}{\gamma}\} < \alpha < Q + \frac{2}{\gamma}$, we have

$$\mathrm{CR}_2(\alpha) = \frac{\overline{G}(\alpha,\gamma)}{\overline{G}(\alpha,\gamma-\frac{4}{\gamma})} \cdot \frac{(2\pi)^{\frac{4}{\kappa}-1}}{(1-\frac{\kappa}{4})\Gamma(1-\frac{\kappa}{4})^{\frac{4}{\kappa}}} \frac{\Gamma(\frac{2\alpha}{\gamma}-\frac{4}{\kappa})\Gamma(\frac{8}{\kappa}-\frac{2\alpha}{\gamma}+1)}{\Gamma(\frac{4}{\kappa}+1)}. \tag{5.25}$$

By shifting relation (2.30) in [RZ22],

$$\frac{\overline{G}(\alpha,\gamma)}{\overline{G}(\alpha,\gamma-\frac{4}{\gamma})} = \frac{\frac{\kappa}{4}\Gamma(1-\frac{\kappa}{4})^{\frac{4}{\kappa}}}{(2\pi)^{\frac{4}{\kappa}}2^{1-\frac{8}{\kappa}}} \cdot \frac{\Gamma(\frac{2\alpha}{\gamma}-1)\Gamma(\frac{4}{\kappa})^2}{\Gamma(\frac{8}{\kappa}-1)\Gamma(\frac{4}{\kappa}-1)\Gamma(\frac{2\alpha}{\gamma}-\frac{4}{\kappa})}.$$

Therefore, when $\frac{2}{\gamma} < \alpha < Q + \frac{2}{\gamma}$, the renormalized moments of the conformal radius is equal to

$$
\begin{aligned}
\mathrm{CR}_2(\alpha) &= \frac{1}{\pi}\frac{\kappa}{4-\kappa}2^{\frac{8}{\kappa}-2}\frac{\Gamma(\frac{2\alpha}{\gamma}-1)\Gamma(\frac{4}{\kappa})^2}{\Gamma(\frac{8}{\kappa}-1)\Gamma(\frac{4}{\kappa}-1)\Gamma(\frac{2\alpha}{\gamma}-\frac{4}{\kappa})}\frac{\Gamma(\frac{2\alpha}{\gamma}-\frac{4}{\kappa})\Gamma(\frac{8}{\kappa}-\frac{2\alpha}{\gamma}+1)}{\Gamma(\frac{4}{\kappa}+1)} \\
&= \frac{2^{\frac{8}{\kappa}-2}}{\pi}\frac{\Gamma(\frac{2\alpha}{\gamma}-1)\Gamma(\frac{4}{\kappa})}{\Gamma(\frac{8}{\kappa}-1)\Gamma(\frac{2\alpha}{\gamma}-\frac{4}{\kappa})}\frac{\Gamma(\frac{2\alpha}{\gamma}-\frac{4}{\kappa})\Gamma(\frac{8}{\kappa}-\frac{2\alpha}{\gamma}+1)}{\Gamma(\frac{4}{\kappa}+1)} \\
&= \frac{2^{\frac{8}{\kappa}-2}}{\pi}\cdot\frac{\kappa}{4}\cdot\frac{\Gamma(\frac{2\alpha}{\gamma}-1)\Gamma(\frac{8}{\kappa}-\frac{2\alpha}{\gamma}+1)}{\Gamma(\frac{8}{\kappa}-1)} \\
&= \frac{\kappa}{4\sqrt{\pi}}\cdot\frac{\Gamma(\frac{2\alpha}{\gamma}-1)\Gamma(\frac{8}{\kappa}-\frac{2\alpha}{\gamma}+1)}{\Gamma(\frac{4}{\kappa})\Gamma(\frac{4}{\kappa}-\frac{1}{2})}.
\end{aligned}
\tag{5.26}
$$

Notice that the lower bound $\alpha > \frac{2}{\gamma}$ comes from $\Gamma(\frac{2\alpha}{\gamma}-\frac{4}{\kappa})$. However, this term is transitory and will be canceled with a term in $\frac{\overline{G}(\alpha,\gamma)}{\overline{G}(\alpha,\gamma-\frac{4}{\gamma})}$. Therefore, by analytic continuation of Gamma function, (5.26) holds when $\frac{\gamma}{2} < \alpha < Q + \frac{2}{\gamma}$. Therefore, when $\alpha = \gamma$,

$$
\mathrm{CR}_2(\gamma) = \frac{1}{C_2} = \frac{\kappa}{4\sqrt{\pi}}\cdot\frac{\Gamma(\frac{8}{\kappa}-1)}{\Gamma(\frac{4}{\kappa})\Gamma(\frac{4}{\kappa}-\frac{1}{2})}
$$

Hence, when $\frac{\gamma}{2} < \alpha < Q + \frac{2}{\gamma}$,

$$
\mathbb{E}\left[|\psi'_\eta(i)|^{2\Delta_\alpha-2}\right] = \frac{\mathrm{CR}_2(\alpha)}{\mathrm{CR}_2(\gamma)} = \frac{\Gamma(\frac{2\alpha}{\gamma})\Gamma(\frac{8}{\kappa}-\frac{2\alpha}{\gamma}+1)}{\Gamma(\frac{8}{\kappa}-1)}. \tag{5.27}
$$

□

Next, we verify the Proposition 5.2.9 by using the Laplace transform of total boundary length $\nu_\phi(\mathbb{R})$. As we will see, it will produce the exact same formula. We mention this computation to motivate our calculation of general weight-W case. From now on, let L_W and R_W denote the left and right quantum boundary length of $\mathcal{M}_{0,2}^{\mathrm{disk}}(W)$ respectively.

Lemma 5.2.10. *Let $\mu > 0$ and we have*

$$
\mathrm{LF}_{\mathbb{H}}^{(\gamma-\frac{4}{\gamma},0),(\alpha,i)}[e^{-\mu\nu_\phi(\mathbb{R})}]\cdot\mathrm{CR}_2(\alpha) = \mathcal{M}_{0,2}^{\mathrm{disk}}(2)[e^{-\mu R_2}|\mathrm{QD}_{1,1}(\alpha,\gamma;L_2)|]
$$

Proof. By definition of welding operation, the L_2 is also equal to outer boundary of $\mathrm{QD}_{1,1}(\alpha,\gamma)$. Therefore,

$$\begin{aligned}
\mathrm{LF}_{\mathbb{H}}^{(\gamma-\frac{4}{\gamma},0),(\alpha,i)}\left[e^{-\mu\nu_\phi(\mathbb{R})}\right]\cdot\mathbb{E}\left[|\psi_\eta'(i)|^{2\Delta_\alpha-2}\right] &= C_2\cdot\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(2;\ell)[e^{-\mu R_2}]\mathrm{QD}_{1,1}(\alpha,\gamma;\ell)d\ell \\
&= C_2\cdot\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(2;\ell)\left[e^{-\mu R_2}|\mathrm{QD}_{1,1}(\alpha,\gamma;\ell)|\right]d\ell \\
&= C_2\cdot\mathcal{M}_{0,2}^{\mathrm{disk}}(2)\left[e^{-\mu R_2}\cdot|\mathrm{QD}_{1,1}(\alpha,\gamma;L_2)|\right].
\end{aligned} \tag{5.28}$$

□

Proof of Proposition 5.2.9 using Laplace transform. We first simplify last line of (5.28). By (5.19), when $Q-\alpha<\frac{\gamma}{2}<\alpha$ and $\gamma<Q$, i.e., $\alpha>\frac{2}{\gamma}$,

$$\begin{aligned}
&\mathcal{M}_{0,2}^{\mathrm{disk}}(2)\left[e^{-\mu R_2}|\mathrm{QD}_{1,1}(\alpha,\gamma;L_2)|\right] \\
&= 2^{-\frac{\alpha^2}{2}}\frac{2}{\gamma}\overline{G}(\alpha,\gamma)\mathcal{M}_{0,2}^{\mathrm{disk}}[e^{-\mu R_2}L_2^{\frac{2}{\gamma}(\alpha-Q)}] \\
&= 2^{-\frac{\alpha^2}{2}}\frac{2}{\gamma}\overline{G}(\alpha,\gamma)\frac{(2\pi)^{\frac{4}{\gamma^2}-1}}{(1-\frac{\gamma^2}{4})\Gamma(1-\frac{\gamma^2}{4})^{\frac{4}{\gamma^2}}}\iint_0^\infty e^{-\mu\ell}r^{\frac{2}{\gamma}(\alpha-Q)}(\ell+r)^{-\frac{4}{\gamma^2}-1}d\ell dr.
\end{aligned}$$

Let $r=\ell\cdot t$ and $dr=dt\cdot\ell$. We have

$$\begin{aligned}
\iint_0^\infty e^{-\mu\ell}r^{\frac{2}{\gamma}(\alpha-Q)}(\ell+r)^{-\frac{4}{\gamma^2}-1}d\ell dr &= \iint_0^\infty e^{-\mu\ell}(\ell t)^{\frac{2}{\gamma}(\alpha-Q)}(\ell+\ell\cdot t)^{-\frac{4}{\gamma^2}-1}\ell dt d\ell \\
&= \left(\int_0^\infty \frac{\ell^{\frac{2}{\gamma}(\alpha-Q)}e^{-\mu\ell}}{\ell^{\frac{4}{\gamma^2}}}d\ell\right)\left(\int_0^\infty \frac{t^{\frac{2}{\gamma}(\alpha-Q)}}{(1+t)^{\frac{4}{\gamma^2}+1}}dt\right).
\end{aligned}$$

When $\frac{2}{\gamma}(\alpha-Q)-\frac{4}{\gamma^2}>-1$, i.e., $\alpha>\frac{4}{\gamma}$,

$$\int_0^\infty \frac{\ell^{\frac{2}{\gamma}(\alpha-Q)}e^{-\mu\ell}}{\ell^{\frac{4}{\gamma^2}}}d\ell = \mu^{\frac{8}{\gamma^2}-\frac{2\alpha}{\gamma}}\Gamma\left(\frac{2\alpha}{\gamma}-\frac{8}{\gamma^2}\right).$$

Furthermore, when $\frac{2}{\gamma} < \alpha < Q + \frac{2}{\gamma}$,

$$\int_0^\infty \frac{t^{\frac{2}{\gamma}(\alpha-Q)}}{(1+t)^{\frac{4}{\gamma^2}+1}} dr = B\left(\frac{2}{\gamma}(\alpha-Q)+1, \frac{8}{\gamma^2} - \frac{2\alpha}{\gamma}+1\right) = \frac{\Gamma(\frac{2}{\gamma}(\alpha-Q)+1)\Gamma(\frac{8}{\gamma^2}-\frac{2\alpha}{\gamma}+1)}{\Gamma(\frac{4}{\gamma^2}+1)}$$
$$= \frac{\Gamma(\frac{2\alpha}{\gamma}-\frac{4}{\kappa})\Gamma(\frac{8}{\kappa}-\frac{2\alpha}{\gamma}+1)}{\Gamma(\frac{4}{\kappa}+1)},$$

where $B(x,y)$ is the Beta function with parameter x, y. To conclude, when $\frac{4}{\gamma} < \alpha < Q + \frac{2}{\gamma}$,

$$\mathcal{M}_{0,2}^{\text{disk}}(2)[e^{-\mu R_2}|\text{QD}_{1,1}(\alpha,\gamma;L_2)|]$$
$$= 2^{-\frac{\alpha^2}{2}} \frac{2}{\gamma} \overline{G}(\alpha,\gamma) \frac{(2\pi)^{\frac{4}{\kappa}-1}}{(1-\frac{\kappa}{4})\Gamma(1-\frac{\kappa}{4})^{\frac{4}{\kappa}}} \mu^{\frac{8}{\kappa}-\frac{2\alpha}{\gamma}} \Gamma\left(\frac{2\alpha}{\gamma}-\frac{8}{\kappa}\right) \frac{\Gamma(\frac{2\alpha}{\gamma}-\frac{4}{\kappa})\Gamma(\frac{8}{\kappa}-\frac{2\alpha}{\gamma}+1)}{\Gamma(\frac{4}{\kappa}+1)}.$$

On the other hand, when $\gamma - \frac{4}{\gamma} < Q$ and $Q - \alpha < \frac{\gamma}{2} - \frac{2}{\gamma} < \alpha$, i.e., $\alpha > \frac{4}{\gamma}$,

$$\text{LF}_{\mathbb{H}}^{(\gamma-\frac{4}{\gamma},0),(\alpha,i)}[e^{-\mu\nu_\phi(\mathbb{R})}] = 2^{-\frac{\alpha^2}{2}} \frac{2}{\gamma} \overline{G}(\alpha,\gamma-\frac{4}{\gamma}) \mu^{\frac{8}{\kappa}-\frac{2\alpha}{\gamma}} \Gamma\left(\frac{2\alpha}{\gamma}-\frac{8}{\kappa}\right).$$

Therefore, when $\frac{4}{\gamma} < \alpha < Q + \frac{2}{\gamma}$, we have

$$\text{CR}(\alpha) = \frac{\overline{G}(\alpha,\gamma)}{\overline{G}(\alpha,\gamma-\frac{4}{\gamma})} \frac{(2\pi)^{\frac{4}{\kappa}-1}}{(1-\frac{\kappa}{4})\Gamma(1-\frac{\kappa}{4})^{\frac{4}{\kappa}}} \frac{\Gamma(\frac{2\alpha}{\gamma}-\frac{4}{\kappa})\Gamma(\frac{8}{\kappa}-\frac{2\alpha}{\gamma}+1)}{\Gamma(\frac{4}{\kappa}+1)},$$

which is identical to our previous calculation (5.25). Notice that by analytic continuation, we can again extend the range of α to $(\frac{\gamma}{2}, Q+\frac{2}{\gamma})$ in the end. □

General weight-W case

In this section, we compute the moments of the conformal radius of $\text{SLE}_{\kappa,0}^{\text{bubble}}(W-2)[d\eta | i \in D_\eta(0)]$ for general $W > 0$.

Lemma 5.2.11. *Let $\mu > 0$ and we have*

$$\text{LF}_{\mathbb{H}}^{(\beta_{2W+2},0),(\alpha,i)}\left[e^{-\mu\nu_\phi(\mathbb{R})}\right] \mathbb{E}\left[|\psi_\eta'(i)|^{2\Delta_\alpha-2}\right] = C_W \cdot \mathcal{M}_{0,2}^{\text{disk}}(W)\left[e^{-\mu R_W} \cdot |\text{QD}_{1,1}(\alpha,\gamma;L_W)|\right]. \tag{5.29}$$

Proof. The proof is identical to that of Lemma 5.2.10. □

Similarly as before, define the *generalized renormalized moments of the conformal radius* $\mathrm{CR}(\alpha, W)$ to be the following:

$$\mathrm{CR}(\alpha, W) := \frac{\mathbb{E}\left[|\psi_\eta'(i)|^{2\Delta_\alpha - 2}\right]}{C_W}. \tag{5.30}$$

Therefore, we have

$$\mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2},0),(\alpha,i)}\left[e^{-\mu\nu_\phi(\mathbb{R})}\right]\cdot \mathrm{CR}(\alpha, W) = \mathcal{M}_{0,2}^{\mathrm{disk}}(W)\left[e^{-\mu R_W}\cdot |\mathrm{QD}_{1,1}(\alpha,\gamma;L_W)|\right]. \tag{5.31}$$

Proposition 5.2.12. *Fix* $\gamma \in (0,2)$. *When* β_{2W+2} *and* α *satisfy* $0 < \beta_{2W+2} < \gamma$ *and* $Q - \frac{\beta_{2W+2}}{2} < \alpha < Q + \frac{\gamma}{2}$, *we have*

$$\begin{aligned}
&\mathbb{E}\left[|\psi_\eta'(i)|^{2\Delta_\alpha - 2}\right] \\
&= \frac{\overline{G}(\alpha,\gamma)}{\overline{G}(\alpha,\gamma-\frac{2W}{\gamma})}\frac{\overline{G}(\gamma,\gamma-\frac{2W}{\gamma})}{\overline{G}(\gamma,\gamma)}\frac{\int_0^\infty \mu_1^{\frac{2}{\gamma}(Q-\alpha)}\left(\frac{\partial}{\partial\mu_1}R(\beta_W;\mu_1,1)\right)d\mu_1}{\Gamma(\frac{2}{\gamma}(Q-\alpha)+1)\Gamma(\frac{2}{\gamma}(\alpha-\frac{W+2}{\gamma}))} \\
&\cdot \frac{\Gamma(\frac{2}{\gamma}(Q-\gamma)+1)\Gamma(\frac{2}{\gamma}(\gamma-\frac{W+2}{\gamma}))}{\int_0^\infty \mu_1^{\frac{2}{\gamma}(Q-\gamma)}\left(\frac{\partial}{\partial\mu_1}R(\beta_W;\mu_1,1)\right)d\mu_1}.
\end{aligned} \tag{5.32}$$

Corollary 5.2.13. *Let* γ, α *be such that* $\alpha > \frac{\gamma}{2}$, *and we have*

$$\mathrm{CR}(\alpha, W)\cdot \mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2},0),(\alpha,i)}\left[e^{-\nu_\phi(\mathbb{R})}\right] = 2^{-\frac{\alpha^2}{2}}\frac{2}{\gamma}\overline{G}(\alpha,\gamma)\mathcal{M}_{0,2}^{\mathrm{disk}}(W)\left[e^{-R_W}\cdot L_W^{\frac{2}{\gamma}(\alpha-Q)}\right].$$

Proof. By (5.18) and definition of $\mathrm{QD}_{1,1}(\alpha,\gamma;\ell)$, when $\frac{\gamma}{2}-\alpha < \frac{\gamma}{2} < \alpha$ and $\gamma < Q$, i.e., $\alpha > \frac{\gamma}{2}$,

$$|\mathrm{QD}_{1,1}(\alpha,\gamma;L_W)| = 2^{-\frac{\alpha^2}{2}}L_W^{\frac{2}{\gamma}(\frac{1}{2}\gamma+\alpha-Q)-1}\frac{2}{\gamma}\overline{G}(\alpha,\gamma).$$

The statement then follows directly from (5.29). □

Lemma 5.2.14. *When $W \in (0, \frac{\gamma^2}{2})$ and $\alpha < Q + \frac{\gamma}{2}$,*

$$\begin{aligned}\mathcal{M}_{0,2}^{\text{disk}}(W)\left[L_W^{\frac{2}{\gamma}(\alpha-Q)}e^{-R_W}\right] \\ = \frac{\gamma}{2(\beta_W - Q)\Gamma\left(\frac{2}{\gamma}(Q-\alpha)+1\right)}\int_0^\infty \mu_1^{\frac{2}{\gamma}(Q-\alpha)}\left(\frac{\partial}{\partial\mu_1}R(\beta_W;\mu_1,1)\right)d\mu_1.\end{aligned}$$

Proof. By [AHS21, Proposition 3.6], when $W \in (0, \frac{\gamma^2}{2})$ and $\beta_W = Q + \frac{\gamma}{2} - \frac{W}{\gamma} \in (Q, Q + \frac{\gamma}{2})$,

$$\mathcal{M}_{0,2}^{\text{disk}}(W)\left[e^{-\mu_1 L_W - R_W}\right] = -\frac{\gamma}{2(\beta_W - Q)}R(\beta_W;\mu_1,1).$$

Taking partial derivatives on both sides with respect to μ_1 and we get that

$$\mathcal{M}_{0,2}^{\text{disk}}(W)\left[Le^{-\mu_1 L_W - R_W}\right] = \frac{\gamma}{2(\beta_W - Q)}\left(\frac{\partial}{\partial\mu_1}R(\beta_W;\mu_1,1)\right).$$

Next, for fixed real number $a > -1$, we integrate the above equation against μ_1^a on both sides. By Fubini's theorem,

$$\begin{aligned}\int_0^\infty \mu_1^a\mathcal{M}_{0,2}^{\text{disk}}(W)\left[L_W e^{-\mu_1 L_W - R_W}\right]d\mu_1 &= \mathcal{M}_{0,2}^{\text{disk}}(W)\left[L_W e^{-R_W}\int_0^\infty \mu_1^a e^{-\mu_1 L_W}d\mu_1\right] \\ &= \Gamma(a+1)\mathcal{M}_{0,2}^{\text{disk}}(W)\left[L_W^{-a}e^{-R_W}\right].\end{aligned}$$

Let $a = \frac{2}{\gamma}(\alpha - Q)$. When $\alpha < Q + \frac{\gamma}{2}$, i.e., $\frac{2}{\gamma}(Q-\alpha) > -1$, we have

$$\begin{aligned}\int_0^\infty \mu_1^{\frac{2}{\gamma}(Q-\alpha)}\mathcal{M}_{0,2}^{\text{disk}}(W)\left[Le^{-\mu_1 L_W - R_W}\right]d\mu_1 \\ = \Gamma\left(\frac{2}{\gamma}(Q-\alpha)+1\right)\mathcal{M}_{0,2}^{\text{disk}}(W)\left[L_W^{\frac{2}{\gamma}(\alpha-Q)}e^{-R_W}\right].\end{aligned}$$

Therefore, when $\alpha < Q + \frac{\gamma}{2}$ and $0 < W < \frac{\gamma^2}{2}$, we have

$$\begin{aligned}\mathcal{M}_{0,2}^{\text{disk}}(W)\left[L_W^{\frac{2}{\gamma}(\alpha-Q)}e^{-R_W}\right] \\ = \frac{\gamma}{2(\beta_W - Q)\Gamma\left(\frac{2}{\gamma}(Q-\alpha)+1\right)}\int_0^\infty \mu_1^{\frac{2}{\gamma}(Q-\alpha)}\left(\frac{\partial}{\partial\mu_1}R(\beta_W;\mu_1,1)\right)d\mu_1.\end{aligned}$$

□

Lemma 5.2.15. *Fix $\gamma \in (0,2)$. When β_{2W+2} and α satisfy $0 < \beta_{2W+2} < \gamma$ and $Q - \frac{\beta_{2W+2}}{2} < \alpha < Q + \frac{\gamma}{2}$, we have*

$$\mathrm{CR}(\alpha, W) = \frac{\overline{G}(\alpha,\gamma)}{\overline{G}(\alpha,\gamma-\frac{2W}{\gamma})} \frac{\gamma \int_0^\infty \mu_1^{\frac{2}{\gamma}(Q-\alpha)} \left(\frac{\partial}{\partial \mu_1} R(\beta_W;\mu_1,1)\right) d\mu_1}{(\gamma - \frac{2W}{\gamma})\Gamma(\frac{2}{\gamma}(Q-\alpha)+1)\Gamma(\frac{2}{\gamma}(\alpha - \frac{W+2}{\gamma}))}. \tag{5.33}$$

Proof. By Lemma 5.2.8, Corollary 5.2.13 and Lemma 5.2.14, when

$$\begin{cases} \frac{\gamma}{2} < \alpha < Q + \frac{\gamma}{2}, \\ 0 < W < \frac{\gamma^2}{2}, i.e., 0 < \beta_{2W+2} < \gamma, \\ \beta_{2W+2} < Q, \\ Q - \alpha < \frac{\beta_{2W+2}}{2} < \alpha, \end{cases} \tag{5.34}$$

we have

$$\begin{aligned} \mathrm{CR}(\alpha, W) &= \frac{\overline{G}(\alpha,\gamma)}{\overline{G}(\alpha,\beta_{2W+2})} \frac{\gamma}{2(\beta_W - Q)\Gamma(\frac{2}{\gamma}(Q-\alpha)+1)\Gamma(\frac{2}{\gamma}(\frac{1}{2}\beta_{2W+2}+\alpha-Q))} \\ &\quad \cdot \int_0^\infty \mu_1^{\frac{2}{\gamma}(Q-\alpha)} \left(\frac{\partial}{\partial \mu_1} R(\beta_W;\mu_1,1)\right) d\mu_1 \\ &= \frac{\overline{G}(\alpha,\gamma)}{\overline{G}(\alpha,\gamma-\frac{2W}{\gamma})} \frac{\gamma \int_0^\infty \mu_1^{\frac{2}{\gamma}(Q-\alpha)} \left(\frac{\partial}{\partial \mu_1} R(\beta_W;\mu_1,1)\right) d\mu_1}{(\gamma - \frac{2W}{\gamma})\Gamma(\frac{2}{\gamma}(Q-\alpha)+1)\Gamma(\frac{2}{\gamma}(\alpha - \frac{W+2}{\gamma}))}. \end{aligned} \tag{5.35}$$

Notice that (5.34) implies $0 < \beta_{2W+2} < \gamma$ and $\frac{Q}{2} < \alpha < Q + \frac{\gamma}{2}$. Since $\frac{W+2}{\gamma} = Q - \frac{\beta_{2W+2}}{2}$, by analytic continuation of $\Gamma(\frac{2}{\gamma}(\alpha - \frac{W+2}{\gamma}))$, the lower bound of α can be extended to $\alpha > Q - \frac{\beta_{2W+2}}{2}$. Therefore, the statement is proved. □

Proof of Proposition 5.2.12. By analytic continuation of $\Gamma(\frac{2}{\gamma}(\alpha - \frac{W+2}{\gamma}))$, we can further relax the range of α and β_{2W+2} to $\alpha \in (\frac{\gamma}{2}, Q+\frac{\gamma}{2})$ and $\beta_{2W+2} \in (0,\gamma)$ as long as $\frac{2}{\gamma}(\alpha - \frac{W+2}{\gamma}) \in \bigcup_{n\geq 0, n\in\mathbb{Z}}(-2n-2,-2n-1)$. Here, we extend to the range of α so that it contains the point

γ. Therefore, by simple computation,

$$
\begin{aligned}
&\mathbb{E}\left[|\psi_\eta'(i)|^{2\Delta_\alpha - 2}\right] \\
&= \frac{\mathrm{CR}(\alpha, W)}{\mathrm{CR}(\gamma, W)} \\
&= \frac{\overline{G}(\alpha,\gamma)}{\overline{G}(\alpha,\gamma - \frac{2W}{\gamma})} \cdot \frac{\overline{G}(\gamma,\gamma - \frac{2W}{\gamma})}{\overline{G}(\gamma,\gamma)}. \\
&\frac{\int_0^\infty \mu_1^{\frac{2}{\gamma}(Q-\alpha)} \left(\frac{\partial}{\partial \mu_1} R(\beta_W;\mu_1,1)\right) d\mu_1}{\Gamma(\frac{2}{\gamma}(Q-\alpha)+1)\Gamma(\frac{2}{\gamma}(\alpha - \frac{W+2}{\gamma}))} \cdot \frac{\Gamma(\frac{2}{\gamma}(Q-\gamma)+1)\Gamma(\frac{2}{\gamma}(\gamma - \frac{W+2}{\gamma}))}{\int_0^\infty \mu_1^{\frac{2}{\gamma}(Q-\gamma)} \left(\frac{\partial}{\partial \mu_1} R(\beta_W;\mu_1,1)\right) d\mu_1}.
\end{aligned}
\tag{5.36}
$$

By analytic continuation of Gamma function, we see that the above equation holds as long as $0 < \beta_{2W+2} < \gamma$ and $Q - \frac{\beta_{2W+2}}{2} < \alpha < Q + \frac{\gamma}{2}$. □

5.2.3. The bulk-boundary correlation function in the LCFT

In this section, we derive an analytic formula linking the bulk-boundary correlation function in the LCFT to the joint law of left, right quantum boundary length and total quantum area of $\mathcal{M}_{0,2}^{\mathrm{disk}}(W)$. First, we recall the definition of the quantum disk with only one bulk insertion point.

Definition 5.2.16 ([ARS22, Definition 4.2])**.** For $\alpha \in \mathbb{R}$, let ϕ be sampled from $\mathrm{LF}_{\mathbb{H}}^{(\alpha,i)}$. We denote $\mathcal{M}_{1,0}^{\mathrm{disk}}(\alpha)$ as the infinite measure described the law of quantum surface $(\mathbb{H},\phi,i)$.

Theorem 5.2.17 ([ARS22, Proposition 2.8],[Rem20])**.** *For $\alpha > \frac{\gamma}{2}$, let h be sampled from $P_{\mathbb{H}}$ and let $\widetilde{\phi}(z) = h(z) - 2Q\log|z|_+ + \alpha G_{\mathbb{H}}(z,i)$. Let $\overline{U}_0(\alpha) := \mathbb{E}\left[\nu_{\widetilde{\phi}}(\mathbb{R})^{\frac{2}{\gamma}(Q-\alpha)}\right]$ where the expectation is taken over $P_{\mathbb{H}}$. Then we have*

$$\overline{U}_0(\alpha) = \left(\frac{2^{-\frac{\gamma\alpha}{2}}2\pi}{\Gamma(1-\frac{\gamma^2}{4})}\right)^{\frac{2}{\gamma}(Q-\alpha)} \Gamma\left(\frac{\gamma\alpha}{2} - \frac{\gamma^2}{4}\right) \qquad \textit{for all } \alpha > \frac{\gamma}{2}. \tag{5.37}$$

Proposition 5.2.18 (Same as Proposition 1.1.8)**.** *Fix $\gamma \in (0,2)$ and $\mu, \mu_\partial > 0$. When*

β_{2W+2} and α satisfy $0 < \beta_{2W+2} < \gamma$ and $Q - \frac{\beta_{2W+2}}{2} < \alpha < Q$, we have

$$\begin{aligned} G_{\mu,\mu_\partial}(\alpha, \beta_{2W+2}) &= \mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2},0),(\alpha,i)}\left[e^{-\mu_\partial \nu_\phi(\mathbb{R}) - \mu\mu_\phi(\mathbb{H})}\right] \\ &= \mathrm{CR}(\alpha, W)^{-1}\frac{2}{\gamma}2^{-\frac{\alpha^2}{2}}\overline{U}_0(\alpha)\frac{2}{\Gamma(\frac{2}{\gamma}(Q-\alpha))}\left(\frac{1}{2}\sqrt{\frac{\mu}{\sin(\pi\gamma^2/4)}}\right)^{\frac{2}{\gamma}(Q-\alpha)} \times \\ &\mathcal{M}_{0,2}^{\mathrm{disk}}(W)\left[e^{-\mu_\partial R_W - \mu A_W} \cdot K_{\frac{2}{\gamma}(Q-\alpha)}\left(L_W\sqrt{\frac{\mu}{\sin(\pi\gamma^2/4)}}\right)\right], \end{aligned} \tag{5.38}$$

where L_W, R_W and A_W denote the left, right quantum boundary length and total quantum area of $\mathcal{M}_{0,2}^{\mathrm{disk}}(W)$ respectively, and $\mathrm{CR}(\alpha, W)$ is the renormalized moments of the conformal radius taking formula (5.33).

Proof. For $\mu_\partial, \mu > 0$, we have that

$$\begin{aligned} &\mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2},0),(\alpha,i)}[e^{-\mu_\partial \nu_\phi(\mathbb{R}) - \mu\mu_\phi(\mathbb{H})}] \cdot \mathrm{CR}(\alpha, W) \\ &= \int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W;\ell)\left[e^{-\mu_\partial R_W - \mu A_W}\right] \cdot \mathrm{QD}_{1,1}(\alpha,\gamma;\ell)\left[e^{-\mu A_{1,1}}\right] d\ell \\ &= \int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W;\ell)\left[e^{-\mu_\partial R_W - \mu A_W} \cdot \mathrm{QD}_{1,1}(\alpha,\gamma;\ell)\left[e^{-\mu A_{1,1}}\right]\right] d\ell \\ &= \mathcal{M}_{0,2}^{\mathrm{disk}}(W)\left[e^{-\mu_\partial R_W - \mu A_W} \cdot \mathrm{QD}_{1,1}(\alpha,\gamma;L_W)\left[e^{-\mu A_{1,1}}\right]\right], \end{aligned} \tag{5.39}$$

where $A_{1,1}$ is the total quantum area of $\mathrm{QD}_{1,1}(\alpha,\gamma,\ell)$. Next, notice that

$$\begin{aligned} \mathrm{QD}_{1,1}(\alpha,\gamma;\ell)\left[e^{-\mu A_{1,1}}\right] &= \left|\mathrm{QD}_{1,1}(\alpha,\gamma;\ell)\right| \cdot \mathrm{QD}_{1,1}(\alpha,\gamma;\ell)^{\#}\left[e^{-\mu A_{1,1}}\right] \\ &= \ell \cdot \left|\mathcal{M}_{1,0}^{\mathrm{disk}}(\alpha;\ell)\right| \cdot \mathcal{M}_{1,0}^{\mathrm{disk}}(\alpha;\ell)^{\#}\left[e^{-\mu A_{1,0}}\right] \\ &= \ell \cdot \mathcal{M}_{1,0}^{\mathrm{disk}}(\alpha;\ell)\left[e^{-\mu A_{1,0}}\right], \end{aligned} \tag{5.40}$$

where $A_{1,0}$ is the total quantum area of $\mathcal{M}_{1,0}^{\mathrm{disk}}(\alpha;\ell)$. The (5.40) follows from the fact that $\mathrm{QD}_{1,1}(\alpha,\gamma;\ell)^{\#}$ and $\mathcal{M}_{1,0}^{\mathrm{disk}}(\alpha;\ell)^{\#}$ are the same probability measure if we ignore the boundary marked point. By [ARS22, Proposition 4.20], when $\alpha \in \left(\frac{\gamma}{2}, Q\right)$,

$$\mathcal{M}_{1,0}^{\text{disk}}(\gamma, \alpha; \ell)\left[e^{-\mu A_{1,0}}\right] = \frac{2}{\gamma} 2^{-\frac{\alpha^2}{2}} \overline{U}_0(\alpha) \ell^{-1} \frac{2}{\Gamma(\frac{2}{\gamma}(Q-\alpha))} \left(\frac{1}{2}\sqrt{\frac{\mu}{\sin(\pi\gamma^2/4)}}\right)^{\frac{2}{\gamma}(Q-\alpha)} K_{\frac{2}{\gamma}(Q-\alpha)}\left(\ell\sqrt{\frac{\mu}{\sin(\pi\gamma^2/4)}}\right), \tag{5.41}$$

where $K_\nu(x)$ is the modified Bessel function of second kind. Precisely,

$$K_\nu(x) := \int_0^\infty e^{-x\cosh t} \cosh(\nu t) dt \qquad \text{for } x > 0 \text{ and } \nu \in \mathbb{R}. \tag{5.42}$$

Therefore, when $\alpha \in \left(\frac{\gamma}{2}, Q\right)$ and $\mu > 0$,

$$\mathrm{QD}_{1,1}(\alpha, \gamma; \ell)\left[e^{-\mu A_{1,1}}\right] \tag{5.43}$$

$$= \frac{2}{\gamma} 2^{-\frac{\alpha^2}{2}} \overline{U}_0(\alpha) \frac{2}{\Gamma(\frac{2}{\gamma}(Q-\alpha))} \left(\frac{1}{2}\sqrt{\frac{\mu}{\sin(\pi\gamma^2/4)}}\right)^{\frac{2}{\gamma}(Q-\alpha)} K_{\frac{2}{\gamma}(Q-\alpha)}\left(\ell\sqrt{\frac{\mu}{\sin(\pi\gamma^2/4)}}\right). \tag{5.44}$$

Finally, together with Corollary 5.2.12, we see that when β_{2W+2} and α satisfy $0 < \beta_{2W+2} < \gamma$ and $Q - \frac{\beta_{2W+2}}{2} < \alpha < Q$,

$$\begin{aligned} &\mathrm{LF}_{\mathbb{H}}^{(\beta_{2W+2},0),(\alpha,i)}\left[e^{-\mu_\partial \nu_\phi(\mathbb{R}) - \mu\mu_\phi(\mathbb{H})}\right] \\ &= \mathrm{CR}(\alpha, W)^{-1} \frac{2}{\gamma} 2^{-\frac{\alpha^2}{2}} \overline{U}_0(\alpha) \frac{2}{\Gamma(\frac{2}{\gamma}(Q-\alpha))} \left(\frac{1}{2}\sqrt{\frac{\mu}{\sin(\pi\gamma^2/4)}}\right)^{\frac{2}{\gamma}(Q-\alpha)} \times \\ &\mathcal{M}_{0,2}^{\text{disk}}(W)\left[e^{-\mu_\partial R_W - \mu A_W} K_{\frac{2}{\gamma}(Q-\alpha)}\left(L\sqrt{\frac{\mu}{\sin(\pi\gamma^2/4)}}\right)\right]. \end{aligned} \tag{5.45}$$

This finishes the proof. □

Remark 5.2.19. For $\beta_W \in (\frac{2}{\gamma}, Q)$ and $W = \gamma(Q + \frac{\gamma}{2} - \beta_W)$, with A_W, L_W and R_W being the total area, left boundary and right boundary of the corresponding weight-W, two-pointed quantum disk $\mathcal{M}_{0,2}^{\text{disk}}(W)$ respectively, define

$$R_{\text{bulk}}(\beta_W;\mu_1,\mu_2) := \frac{2(Q-\beta_W)}{\gamma}\mathcal{M}_{0,2}^{\text{disk}}(W)\left[e^{-A_W-\mu_1 L_W-\mu_2 R_W}-1\right], \tag{5.46}$$

which is the same as [ARSZ23, (1.14)]. Using the exact same argument as in [AHS21, Proposition 3.6], when $W \in (0,\frac{\gamma^2}{2})$ and $\beta_W = Q+\frac{\gamma}{2}-\frac{W}{\gamma} \in (Q, Q+\frac{\gamma}{2})$, we have

$$\begin{aligned}\mathcal{M}_{0,2}^{\text{disk}}(\gamma^2-W)\left[e^{-A_{\gamma^2-W}-\mu_1 L_{\gamma^2-W}-\mu_2 R_{\gamma^2-W}}-1\right]\mathcal{M}_{0,2}^{\text{disk}}(W)\left[e^{-A_W-\mu_1 L_W-\mu_2 R_W}\right]\\ = \frac{-\gamma^2}{4(\beta_W-Q)^2}.\end{aligned} \tag{5.47}$$

Therefore, when $W \in (0,\frac{\gamma^2}{2})$ and $\beta_W \in (Q, Q+\frac{\gamma}{2})$, we have

$$\mathcal{M}_{0,2}^{\text{disk}}(W)\left[e^{-A_W-\mu_1 L_W-\mu_2 R_W}\right] = \frac{\gamma}{2(Q-\beta_W)}R_{\text{bulk}}(2Q-\beta_W;\mu_1,\mu_2)^{-1}. \tag{5.48}$$

Notice that $2Q-\beta_W = \beta_{\gamma^2-W}$. The exact formula of R_{bulk} is obtained in [ARSZ23, Theorem 1.3], which in turn yields the exact formula for $G_{\mu,\mu_\partial}(\alpha,\beta_{2W+2})$ in [ARSZ23, Section 4.3].

CHAPTER 6

OUTLOOK AND FUTURE RESEARCH

In the last chapter, we discuss several conjectures that arise naturally from the contexts of this thesis.

6.1. Generalized SLE bubbles on $\mathbb{H}$: single case

As natural generalizations of Theorem 1.1.1 and Theorem 1.1.3, we can consider the case when $\mathrm{QD}_{0,1}$ has one general boundary insertion, i.e., $\mathrm{QD}_{0,1}(\gamma, \alpha)$ in Definition 4.2.7. For the sake of completeness, we provide two conjectures: one with the bulk insertion and one without. Although our discussion will be centered around the Conjecture 6.1.2.

Conjecture 6.1.1. Fix $W_1 \geq \frac{\gamma^2}{2}$ and $W > 2$. There exist a σ-finite infinite measure $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(W, W_1)$ on $\mathrm{Bubble}_{\mathbb{H}}(0)$ and some constant $C \in (0, \infty)$ such that suppose $\phi \times \eta_{W,W_1}$ is sampled from

$$C \cdot \mathrm{LF}_{\mathbb{H}}^{(\beta_{2W_1+W},0)}(d\phi) \times \mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(W, W_1)[d\eta_{W,W_1} | i \in D_{\eta_{W,W_1}(0)}], \tag{6.1}$$

then the law of $(D_{\eta_{W,W_1}}(0), \phi, 0)$ and $(D_{\eta_{W,W_1}}(\infty), \phi, 0^-, 0^+)$ viewed as a pair of marked quantum surface is equal to

$$\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W_1; \cdot, \ell) \times \mathrm{QD}_{1,1}(\gamma, \beta_W; \ell) d\ell. \tag{6.2}$$

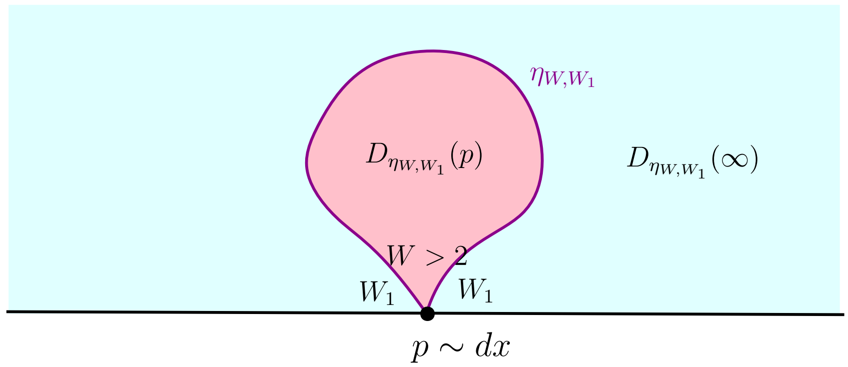


Figure 6.1: Illustration of welding equation (6.3) in Conjecture 6.1.2: first sample a root point p according to Lebesgue measure dx on $\mathbb{R}$, then sample (ϕ, η) according to the product measure $\mathrm{LF}_{\mathbb{H}}^{(\beta_{2W_1+W},p)}(d\phi) \times \mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}(W, W_1)(d\eta)$. The resulting quantum surface $(\mathbb{H}, \phi, \eta, p)/\sim_\gamma$ has the law of $C\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W_1; \cdot, \ell) \times \mathrm{QD}_{0,1}(\gamma, \beta_W; \ell) d\ell$ after uniform embedding.

Conjecture 6.1.2. Fix $W_1 \geq \frac{\gamma^2}{2}$ and $W > 2$. There exist a σ-finite infinite measure $\mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}(W, W_1)$ on $\mathrm{Bubble}_{\mathbb{H}}(p)$ and some constant $C \in (0, \infty)$ such that

$$\mathbf{m}_{\mathbb{H}} \ltimes \left(\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W_1; \cdot, \ell) \times \mathrm{QD}_{0,1}(\gamma, \beta_W; \ell) d\ell\right) = C \cdot \mathrm{LF}_{\mathbb{H}}^{(\beta_{2W_1+W},p)} \times \mathrm{SLE}_{\kappa,p}^{\mathrm{bubble}}(W, W_1) dp. \tag{6.3}$$

Furthermore, there exists some constant $C \in (0, \infty)$ such that

$$\mathbf{m}_{\mathbb{H},0} \ltimes \left(\int_0^\infty \mathcal{M}_{0,2}^{\mathrm{disk}}(W_1; \cdot, \ell) \times \mathrm{QD}_{0,1}(\gamma, \beta_W; \ell) d\ell\right) = C \cdot \mathrm{LF}_{\mathbb{H}}^{(\beta_{2W_1+W},0)}(d\phi) \times \mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(W, W_1), \tag{6.4}$$

where $\mathbf{m}_{\mathbb{H},0}$ is a Haar measure on $\mathrm{conf}(\mathbb{H}, 0)$, i.e., the group of conformal automorphisms of $\mathbb{H}$ fixing 0.

In Conjecture 6.1.1 and 6.1.2, by the quantum triangle welding and the induction techniques developed in Section 4.2, we can show that (1) ϕ has the law of $C \cdot \mathrm{LF}_{\mathbb{H}}^{(\beta_{2W_1+W},0)}$, and (2) the welding interface η_{W,W_1} is independent of ϕ.

However, we have almost zero understanding on the law of η_{W,W_1}, i.e., $\mathrm{SLE}_{\kappa,0}^{\mathrm{bubble}}(W, W_1)$. Recall that in Zhan's limiting constructions of $\mathrm{SLE}_\kappa(\rho)$ bubbles, one takes the weak limit of chordal $\mathrm{SLE}_\kappa(\rho)$ under suitable rescaling. Therefore, in LQG frameworks, we take "quantum version" of the limit by 1) conditioning on the (one-side) quantum boundary length of $\mathcal{M}^{\mathrm{disk}}_{1,2}(2)$ goes to zero 2) constructing a coupling with the limiting picture so that, with high probability, the random domains match.

Nonetheless, this technique will not work in the case of Conjecture 6.1.2, or in a more straightforward way, η_{W,W_1} is not the weak limit of chordal $\mathrm{SLE}_\kappa(W-2, W_1-2)$ under suitable rescaling. Suppose one takes $\mathcal{M}^{\mathrm{disk}}_{1,2}(W)$ and then conditioning on the (one-side) quantum boundary length goes to zero, the limiting quantum surface will always be the same; the boundary marked point is always quantum typical (cf. [MSW21, Appendix A]). In other words, we will always get $\mathrm{SLE}^{\mathrm{bubble}}_{\kappa,0}(W_1-2)$. Therefore, shrinking (one-side) quantum boundary length and coupling will only work for $\mathcal{M}^{\mathrm{disk}}_{1,2}(2)$.

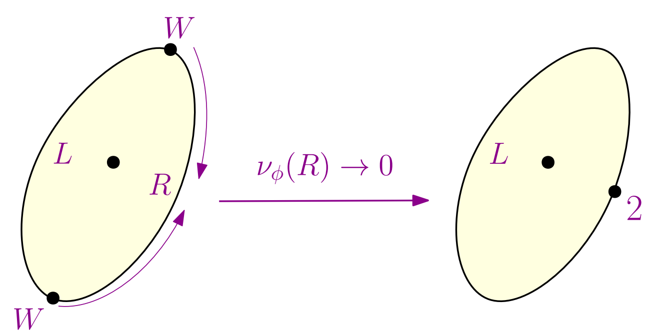


Figure 6.2: On the LHS, we have $\mathcal{M}^{\mathrm{disk}}_{1,2}(W)$. Conditioning on the quantum boundary length of the right arc R shrinks to zero, we will get a $\mathcal{M}^{\mathrm{disk}}_{1,1}(2)$, i.e., $\mathrm{QD}_{1,1}$. Notice that in [MSW21, Appendix A], the weight W is in the restricted range. However, we believe that this is only a technical barrier and will not affect the overall outcome.

Hence, one interesting question is that *how to describe the law of* η_{W,W_1} *in Conjecture 6.1.2?* If better, *what is its corresponding Lowener evolution (driving function)?*

Also, going back to the Euclidean settings, in Zhan's constructions of $\mathrm{SLE}_\kappa(\rho)$ bubbles, one takes the weak limit of $\mathrm{SLE}^{\mathbb{H}}_{\kappa,(\varepsilon;\varepsilon^+)\to 0}(\rho)$ or $\mathrm{SLE}^{\mathbb{H}}_{\kappa,(0;0^-)\to\varepsilon}(\rho)$ under suitable rescaling. Either way, that single force point of SLE_κ is on the outside (see Figure 3.3).

Hence, *what if you have two force points?* In other words, what if we take the weak limit of $\mathrm{SLE}^{\mathbb{H}}_{\kappa,(0;0^-,0^+)\to\varepsilon}(\rho_-,\rho_+)$? I conjecture that it is the $\mathrm{SLE}^{\mathrm{bubble}}_{\kappa,0}(\rho_-)$. Similarly, if we take the weak limit of $\mathrm{SLE}^{\mathbb{H}}_{\kappa,(\varepsilon;\varepsilon^-,\varepsilon^+)\to 0}(\rho_-,\rho_+)$, then it is $\mathrm{SLE}^{\mathrm{bubble}}_{\kappa,0}(\rho_+)$.

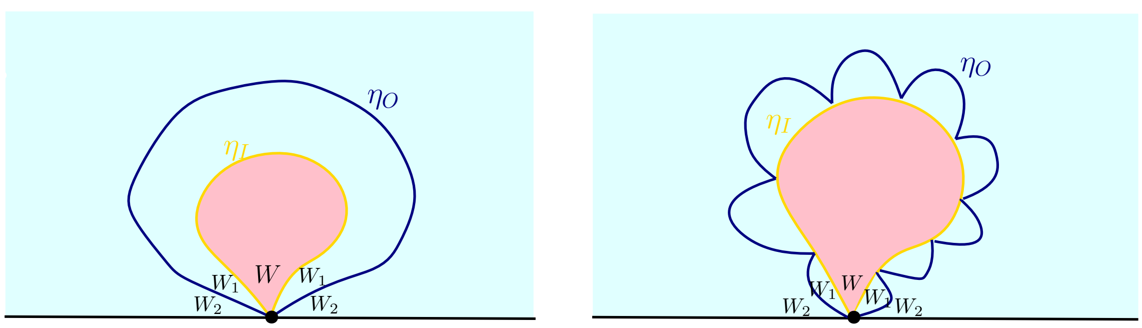
A somewhat similar question as above is *what happens to the inner force point after collapsing the* ε *with* 0*. Do they vanish?* I conjecture that yes, the inner force point vanishes once collapsed.

6.2. Generalized SLE bubbles on $\mathbb{H}$: multiple case

Going one step further, motivated by the induction procedure described in Figure 4.6, we are also interested in understanding the multiple SLE bubbles on $\mathbb{H}$. Specifically, consider welding of three quantum disks

$$\int_0^\infty \int_0^\infty \mathrm{QD}_{0,1}(\gamma,\beta_W;\ell) \times \mathcal{M}^{\mathrm{disk}}_{0,2}(W_1,\ell,r) \times \mathcal{M}^{\mathrm{disk}}_{0,2}(W_2;r,\cdot)dr d\ell \tag{6.5}$$

for $W \geq 2, W_1 > 0$ and $W_2 > 0$.



LHS: $W \geq 2, W_1 \geq \frac{\gamma^2}{2}, W_2 \geq \frac{\gamma^2}{2}$ RHS: $W \geq 2, 0 < W_1 < \frac{\gamma^2}{2}, W_2 \geq \frac{\gamma^2}{2}$

Figure 6.3: Illustration of quantum surface (6.5) when embedded in $(\mathbb{H},\phi,\eta_I,\eta_O)$.

Let $(\mathbb{H},\phi,0,\eta_I,\eta_O)$ be an particular embedding of (6.5) (see Figure 6.3), then it is not hard to show that the joint law of (η_I,η_O) is independent of ϕ. Moreover, the condition law $(\eta_O|\eta_I)$ should equal to $\mathrm{SLE}^{\mathrm{bubble}}_{\kappa,0}(W_1,W_2)$ and the law of $(\eta_I|\eta_O)$ should equal to $\mathrm{SLE}^{\mathrm{bubble}}_{\kappa,0}(W,W_1)$. Recall that $\mathrm{SLE}^{\mathrm{bubble}}_{\kappa,0}(\cdot,\cdot)$ is the welding interface in Conjecture 6.1.2.

The interesting questions to the SLE research communities are *what is the marginal law of*

$\eta_\bullet, \bullet \in \{I, O\}$. *Moreover, what is the Loewner evolution (driving function) of* $\eta_\bullet, \bullet \in \{I, O\}$*?*

6.3. Scaling limits of bubble-decorated quadrangulated disks

Recall that in the SLE loop case [AHS22], $\mathrm{MS}^n \otimes \mathrm{SAW}^n$ is the measure on pairs (M, η), where M is a quadrangulation, η is a self-avoiding loop on M, and each (M, η) has weight $n^{5/2} 12^{-\#\mathcal{F}(M)} 54^{-\#\eta}$, where $\#\mathcal{F}(M)$ denotes the number of faces of M and $\#\eta$ is the number of edges of η. It is proved that the following convergence result holds.

Theorem 6.3.1 ([AHS22, Theorem 1.2])**.** *There exists constant* $c_0 > 0$ *and for all* $c \in (0,1)$*,*

$$\mathrm{MS}^n \otimes \mathrm{SAW}^n|_{A(c)} \xrightarrow{w} c_0 \cdot \mathrm{QS} \otimes \mathrm{SLE}_{8/3}^{\mathrm{loop}}|_{A(c)}, \tag{6.6}$$

where $A(c)$ *is the event that the length of the loop is in* $[c, c^{-1}]$.

In the disk case, we say a planar map D is a *quadrangulated disk* if it is a planar map where all faces have four edges except for the exterior face, which has arbitrary degree and simple boundary. Let ∂D denote the edges on the boundary of the exterior face, and we denote $\#\partial D$ the boundary length of D. Let MD^n be the measure on the quadrangulated disks such that each disk D has weight $n^{5/2} 12^{-\#\mathcal{F}(D)} 54^{-\#\partial D}$, which has the same scaling as MS^n above. Note that here if D is sampled from MD^n, then D is viewed as a metric measure space by considering the graph metric rescaled by $2^{-1/2} n^{-1/4}$ and giving each vertex mass $2(9n)^{-1}$.

If D is a quadrangulated disk, then we say η is a *self-avoiding bubble* on D rooted at $e_r \in \partial D$ if η is an orderer set of edges $e_1, \dots, e_{2k} \in \mathcal{E}(D)$ with $r \in \{1, \dots, 2k\}$ and e_j and e_i share an end-point if and only if $|i - j| \le 1$ or $(i, j) \in \{(1, 2k), (2k, 1)\}$.

Let $\mathrm{MD}^n \otimes \partial \mathrm{MD}^n \otimes \mathrm{SAB}^n$ denote the measure on pairs (D, e, η) where η is a self-avoiding bubble on D rooted at edge $e \in \partial D$ and the pair (D, η) has weight $\#\partial D^{-1} \cdot n^{5/2} 12^{-\#\mathcal{F}(D)} 54^{-\#\eta}$. For (D, e, η) sampled from $\mathrm{MD}^n \otimes \partial \mathrm{MD}^n \otimes \mathrm{SAB}^n$, we view D as a metric measure space and view η as a bubble on this metric measure space rooted at edge e so that the time it takes to traverse each edge on the loop is $2^{-1}n^{-1/2}$.

Conjecture 6.3.2. There exists some $c_0 > 0$ such that for all $c \in (0,1)$,

$$\mathrm{MD}^n \otimes \partial\mathrm{MD}^n \otimes \mathrm{SAB}^n|_{A(c)} \xrightarrow{w} c_0 \cdot \mathrm{LF}_{\mathbb{H}}^{(\beta_6,p)} \times \mathrm{SLE}_{8/3,p}^{\mathrm{bubble}} \times dp|_{A(c)}, \tag{6.7}$$

in Gromov-Hausdorff-Prokhorov-uniform topology, where $A(c)$ is event that the length of the bubble is in $[c, c^{-1}]$.

We can also understand the measure $\mathrm{MD}^n \otimes \partial\mathrm{MD}^n \otimes \mathrm{SAB}^n$ from the welding perspective. Suppose $\overline{\mathrm{MD}^n}$ is a measure on qudrangulated disks such that each disk $\overline{D}$ has weight $n^{5/2}12^{-\#\mathcal{F}(\overline{D})}54^{-2\#\partial\overline{D}}$ and $\underline{\mathrm{MD}^n}$ is a measure on qudrangulated disks with each disk $\underline{D}$ has weight $n^{5/2}12^{-\#\mathcal{F}(\underline{D})}54^{-\#\partial\underline{D}}$. Let $\overline{\mathrm{MD}^n_{0,2}}$ be the measure on $(\overline{D}, e_1, e_2)$ such that we first sample $\overline{D}$ from reweighted measure $(\#\partial\overline{\mathrm{MD}^n})^2\overline{\mathrm{MD}^n}$ and then sample two edges e_1, e_2 uniformly on $\partial\overline{D}$. Similarly, let $\underline{\mathrm{MD}^n_{0,1}}$ be the measure on $(\underline{D}, e)$ such that we first sample $\underline{D}$ from reweighted measure $(\#\partial\underline{\mathrm{MD}^n}) \cdot \underline{\mathrm{MD}^n}$ and then sample an edge e from $\partial\underline{D}$ uniformly.

For $k \in \mathbb{N}$, let $\overline{\mathrm{MD}^n_{0,2}}(\cdot, k)$ denote the restriction of $\overline{\mathrm{MD}^n_{0,2}}$ to the event that right boundary has length $2k$ and let $\underline{\mathrm{MD}^n_{0,1}}(k)$ denote the restriction of $\underline{\mathrm{MD}^n_{0,1}}$ to the event that the total boundary has length $2k$. Let $\overline{\mathrm{MD}^n_{0,2}}(\cdot, k)^\#$ and $\underline{\mathrm{MD}^n_{0,1}}(k)^\#$ denote the corresponding probability measure respectively.

Suppose $(\overline{D}, e_1, e_2)$ is sampled from $\overline{\mathrm{MD}^n_{0,2}}(\cdot, k)^\#$ and $(\underline{D}, e)$ is sampled from $\underline{\mathrm{MD}^n_{0,1}}(k)^\#$, then we can do the "discrete conformal welding" by identifying the right boundary of $\overline{D}$ to the total boundary of $\underline{D}$ such that e_1, e_2 and e are identified. The self-avoiding bubble on the discrete disk represents the welding interface of $\overline{D}$ and $\underline{D}$. We parametrize the bubble so that each edge on the bubble has length $2^{-1}n^{-1/2}$ just like the sphere case. Suppose $(\overline{D}, e_1, e_2)$ is sampled from $\overline{\mathrm{MD}^n_{0,2}}(\cdot, k)^\#$ and $(\underline{D}, e)$ is sampled from $\underline{\mathrm{MD}^n_{0,1}}(k)^\#$, then we denote the measure on the disks decorated with a self-avoiding bubble sampled in this way by $\mathrm{Weld}_d^{\mathrm{bubble}}(\overline{\mathrm{MD}^n_{0,2}}(\cdot, k)^\#, \underline{\mathrm{MD}^n_{0,1}}(k)^\#)$. Similarly, let $\mathrm{Weld}_c^{\mathrm{bubble}}(\mathrm{QD}_{0,2}(\cdot, \ell)^\#, \mathrm{QD}_{0,1}(\ell)^\#)$ denote the measure on bubble-decorated quantum disk obtained by identifying the right boundary of the disk sampled from $\mathrm{QD}_{0,2}(\cdot,\ell)^{\#}$ and the total boundary of the disk sampled from $\mathrm{QD}_{0,1}(\ell)^{\#}$.

Conjecture 6.3.3. For any $\ell > 0$, we have

$$\mathrm{Weld}_d^{\mathrm{bubble}}(\overline{\mathrm{MD}_{0,2}^n}(\cdot,[\ell n^{1/2}])^{\#}, \underline{\mathrm{MD}_{0,1}^n}([\ell n^{1/2}])^{\#}) \xrightarrow{w} \mathrm{Weld}_c^{\mathrm{bubble}}(\mathrm{QD}_{0,2}(\cdot,\ell)^{\#}, \mathrm{QD}_{0,1}(\ell)^{\#}) \tag{6.8}$$

in Gromov-Hausdorff-Prokhorov-uniform topology.

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