Hitchin's system and the Geometric Langlands conjecture

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Penn

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Joint work with Tony Pantev and partly with Carlos Simpson.

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The conjecture:

A G-local system V on a Riemann surface C ,

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A fancier version of the Geometric Langlands conjecture predicts the existence of a canonical equivalence of categories

$$\boxed{c : D_{coh}(\mathrm{Loc}; O) \; !^{=} \; D_{coh}({}^L\mathrm{Bun}; D);} \qquad \text{(GLC)}$$

which is uniquely characterized by the property that c sends the structure sheaves of points V in Loc to Hecke eigen D-modules $c(O_V)$ (corresponding to the above c_V) on ${}^L\mathrm{Bun}$:

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Geometric Langlands

Motivation:

The Langlands program is the non-abelian extension of class field theory (CFT). The abelian case is well understood. Its geometric version, or geometric CFT, is essentially the theory of a curve C and its Jacobian $J = J(C)$. This abelian case of the Geometric Langlands Conjecture (GLC) amounts to the well known result that any rank 1 local system (or: line bundle with flat connection) on the curve C extends uniquely to J, and this extension is natural with respect to the Abel-Jacobi map. The structure group of a rank 1 local system is of course just the abelian group $C^* = GL(1; C)$. The GLC is the attempt to extend this classical result from C^* to all complex reductive groups G. This goes as follows.

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Correspondences

A correspondence between varieties $A; B$ is a subscheme $H \quad A \quad B$. It induces a transform from objects on A to objects on B, by pull-push. (May have to specify a kernel.) Each Hecke correspondence $H \quad \mathrm{Bun} \quad \mathrm{Bun} \quad C$ induces a transform from D-modules on Bun to D-modules on $\mathrm{Bun} \quad C$.

Geometric Langlands

The Geometric Langlands Conjecture says that an irreducible G-local system on C determines a perverse sheaf on Bun which is a simultaneous eigensheaf for the action of the Hecke operators - this turns out to be the right generalization of naturality with respect to the Abel-Jacobi map. (A perverse sheaf is, roughly, a local system on a Zariski open subset of Bun, extended in a natural way across the complement.) Fancier versions of the conjecture recast this as an equivalence of derived categories: of D-modules on Bun vs. coherent sheaves on the moduli space Loc of local systems. There are many related conjectures and extensions, notably to punctured curves via parabolic bundles and local systems. Some of these make an appearance below.

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Some known results

Significant progress has been made towards understanding these conjectures, through works of Drinfeld, Laumon, Beilinson, Lafforgue, Frenkel, Gaitsgory, Vilonen, Heinloth, ... Some vesions are known for GL(n). The conjecture is unknown for other groups. There are more recent results of Heinloth in the parabolic case. Even for GL(n), the proof is indirect: no construction of non-abelian Hecke eigensheaves is known.

HMS and Geometric Langlands

Idea: Hitchin's integrable system $\mathsf{Higgs}_{C;G}$ associated to $(C;G)$ is a twistor rotation of $\mathsf{Loc}_{C;G}$. It is a hyper Kahler SYZ fibration. It also is (or: contains) T $\mathsf{Bun}_{C;G}$.

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HMS applied to Hitchin's system $\Rightarrow$)
D_L should be the relative Floer homology between two Lagrangians in $\mathrm{Higgs}_{C;G_-}$, one fixed, the other moving:

- View L as a point of $\mathrm{Higgs}_{C;G}$. Its Fourier-Mukai dual is a Lagrangian (with line bundle) in $\mathrm{Higgs}_{C;G_-}$.
- A general cotangent fiber of $\mathrm{Higgs}_{C;G_-}$.

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As is appropriate for this workshop, the work described here is based on an abelianization of the GLC in terms of Higgs bundles. Higgs bundles have been discussed in many of the talks in this workshop. Let me review very briefly, with a view towards explaining the results of a recent joint work with Pantev and Simpson on operations on Higgs bundles.

Abelianization via Hitchin's system

A Higgs bundle is a pair $(V;\ )$ consisting of a vector bundle V with a K_C-valued endomorphism $\ : V \ ! \ V \ \ K_C$, where $K_C = \Omega^1_C$ is the canonical bundle of C.

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The main result of [DP1] is formulated as a duality of the Hitchin system: There is a canonical isomorphism between the bases $B; {}^{L}B$ of the Hitchin system for the group G and its Langlands dual ${}^{L}G$, taking the discriminant in one to the discriminant in the other. Away from the discriminants, the corresponding fibers are abelian varieties, and we exhibit a canonical duality between them.

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Abelianized Hecke eigensheaves: of degree-0 line bundles on Hitchin fibers h ^{1}(b). We are thinking of Bun as the base space and of h ^{1}(b) as its spectral cover.

Abelianization via Hitchin's system

It is very tempting to try to understand the relationship of this abelianized result to the full GLC. The view of the GLC pursued in [BeDr] is that it is a quantum theory. The emphasis in [BeDr] is therefore on quantizing Hitchin's system, which leads to the investigation of opers One possibility, discussed in [DP1] and [Ar], is to view the full GLC as a quantum statement whose classical limit is the result in [DP1]. The idea then would be to try to prove GLC by deforming both sides of the result of [DP1] to higher and higher orders. Arinkin has carried out some deep work in this direction. But there is another path.

NAHT and GLC

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This leads us to non-abelian Hodge theory.

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NAHT

The projective (or compact Kahler) case and the one dimensional open case were settled by Simpson twenty years ago - but the open case in higher dimension had to await the recent breakthroughs by Saito, Sabbah, Mochizuki, Jost-Yang-Zuo, Biquard, etc. This higher dimensional theory produces an equivalence of parabolic local systems and parabolic Higgs bundles. This is quite analogous to what is obtained in the compact case, except that the objects involved are required to satisfy three key conditions discovered by Mochizuki. Below we review these exciting developments, and outline our proposal for using NAHT to construct the automorphic sheaves required by the GLC.

NAHT, GLC, and QFT

This approach is purely mathematical of course, but it is parallel to physical ideas that have emerged from the collaborations of Witten with Kapustin, Gukov and Frenkel [KW, GW, W3, FW], where the GLC was placed firmly in the context of quantum field theory.

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A bundle V is **very stable** if the only nilpotent Higgs field on V is $= 0$. (In other words, the cotangent fiber $T_{\{V\}}\mathbf{Bun}$ meets the Hitchin fiber over 0 only at the point $= 0$.) Laumon: very stable implies stable.

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A bundle V is **shaky** if it is stable but there is a Higgs bundle $(V^0, \ )$ mapping to it with V^0 unstable. (More details below.)

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We will describe some known results about these loci for rank 2 bundles. These lead in some cases to an explicit construction (modulo solving the differential equations inherent in the non-abelian Hodge theory) of the Hecke eigensheaf demanded by the GLC.

NAHT: the theorems

Originally Corlette and Simpson proved the non-abelian Hodge theorem for projective manifolds:

Theorem [Corlette, Simpson] Let $(X; O_X(1))$ be a smooth complex projective variety. Then there is a natural equivalence of dg -categories:

$$\mathrm{nah}_X : \left\{\begin{array}{l}\text{finite rank } \mathbb{C}\text{-local systems} \\ \text{on } X\end{array}\right\} \to \left\{\begin{array}{l}\text{finite rank } O_X(1)\text{-semistable Higgs bundles on } X \text{ with} \\ \mathrm{ch}_1 = 0 \text{ and } \mathrm{ch}_2 = 0\end{array}\right\}$$

NAHT: the theorems

Mochizuki proved a version of the non-abelian Hodge correspondence which allows for singularities of the objects involved:

Theorem [Mochizuki] Let $(X; O_X(1))$ be a smooth complex projective variety and let $D \subset X$ be an effective divisor. Suppose that we have a closed subvariety $Z \subset X$ of codimension ≥ 3, such that $X \setminus Z$ is smooth and $D \setminus Z$ is a normal crossing divisor.
Then there is a canonical equivalence of dg-categories:

$$\mathrm{nah}_{X;D} : \left(\begin{array}{c} \text{finite rank tame} \\ \text{parabolic } \mathbb{C}\text{-local} \\ \text{systems on } (X; D) \end{array} \right) \to \left(\begin{array}{c} \text{finite rank locally abelian} \\ \text{tame parabolic Higgs} \\ \text{bundles on } (X; D) \\ \text{which are } O_X(1)\text{-} \\ \text{semistable and satisfy} \\ \mathrm{parch}_1 = 0 \text{ and } \mathrm{parch}_2 = 0 \end{array} \right)$$

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(3) a global condition: vanishing of parabolic Chern classes.

NAHT: the theorems

A feature of the non-abelian Hodge correspondence that is specific to the open case is captured in another result of Mochizuki:

Theorem [Mochizuki's Extension Theorem] Let U be a quasi-projective variety and suppose U has two compacti cations

$$: U \; ! \; X; \quad : U \; ! \; Y$$

where:

- X, Y are projective and irreducible;
- X is smooth and $X \; U$ is a normal crossing divisor away from codimension 3;

Then the restriction from X to U followed by the middle perversity extension from U to Y gives an equivalence of abelian categories:

$$! \quad : @ \begin{matrix} 0 \\ \text{irreducible} \\ \text{parabolic} \\ \text{systems on} \end{matrix} \begin{matrix} \\ \text{tame} \\ C\text{-local} \\ (X; D) \end{matrix} \begin{matrix} 1 \\ \\ A \\ \\ \end{matrix} ! \quad \begin{matrix} \text{simple} D\text{-modules on } Y \\ \text{which are smooth on } U \end{matrix}$$

The plan

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As we saw, non-abelian Hodge theory provides a natural approach to constructing the geometric Langlands correspondence c. The big hope is that the known eigensheaf of the abelianized Heckes, which is a Higgs-type object (E; '), extends by non abelian Hodge theory to a twistor eigensheaf on ^{L}Bun. The original Higgs sheaf appears at $z = 0$, while at the opposite end $z = 1$ we can expect to find precisely the Hecke eigensheaf postulated by the GLC.

The plan

The situation is essentially non-compact: In the moduli space ${}^{L}\mathrm{Bun}^{s}$ of stable bundles there is a locus S of shaky bundles along which our Higgs field $'$ blows up. This can be traced back, essentially, to the difference between the notions of stability for bundles and Higgs bundles.

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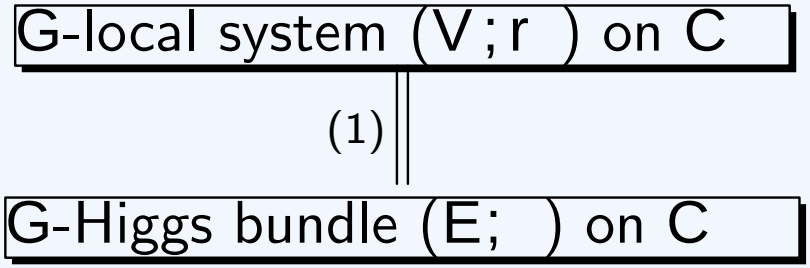
The plan

So the heart of the matter amounts to verification of the Mochizuki conditions: we need to find where the Higgs field blows up, resolve this locus to obtain a normal crossing divisor, lift the objects to this resolution, and verify that the parabolic chern classes of these lifts vanish upstairs. This would provide the crucial third step in the following six step recipe for producing the candidate automorphic sheaf:

G-local system $(V; r)$ on C


G-local system (V ; r) on C
(1)
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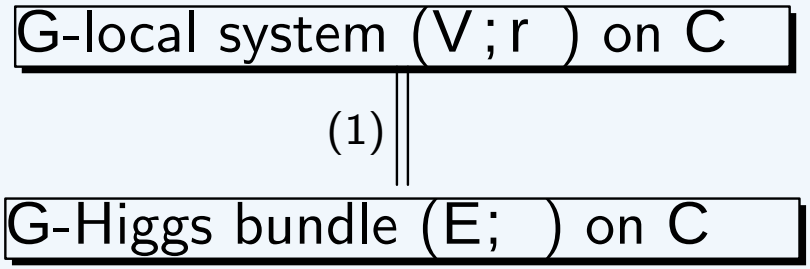
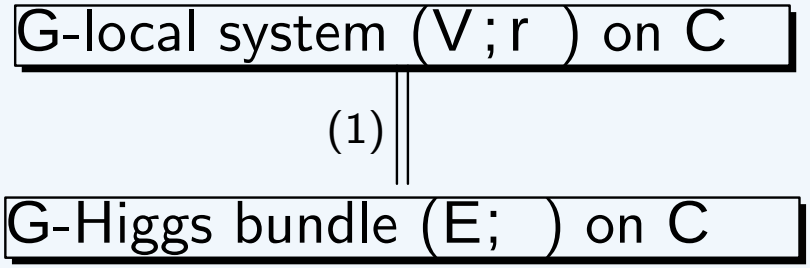


This is the Corlette-Simpson non-abelian Hodge correspondence $(E; \ ) = \mathsf{nah}_C(V; r \ )$ on the smooth compact curve C.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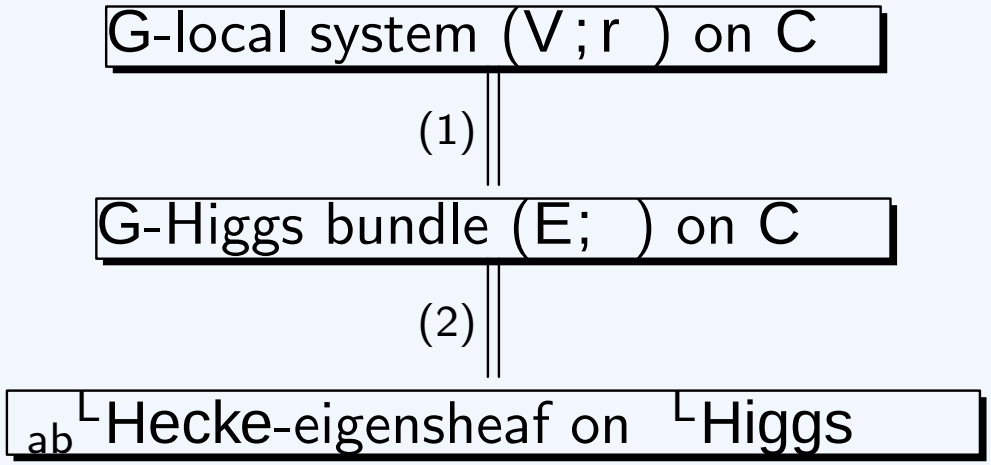
G-local system (V ; r) on C
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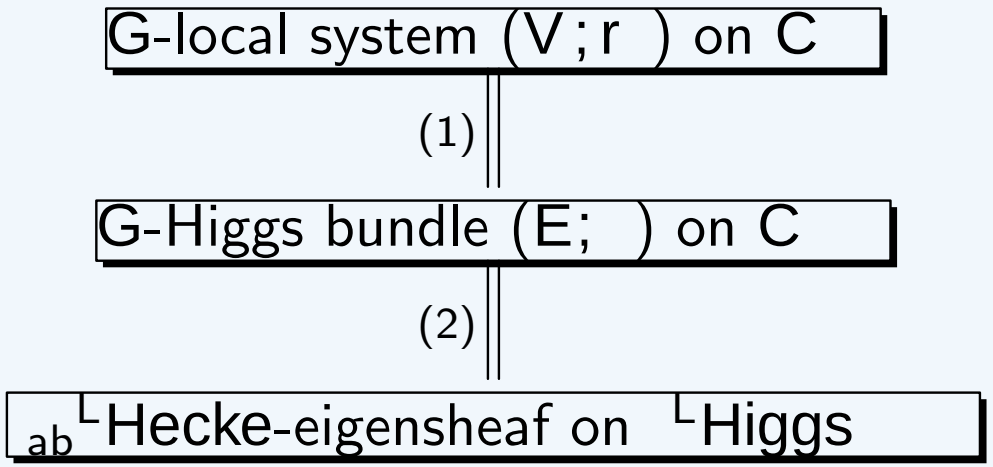
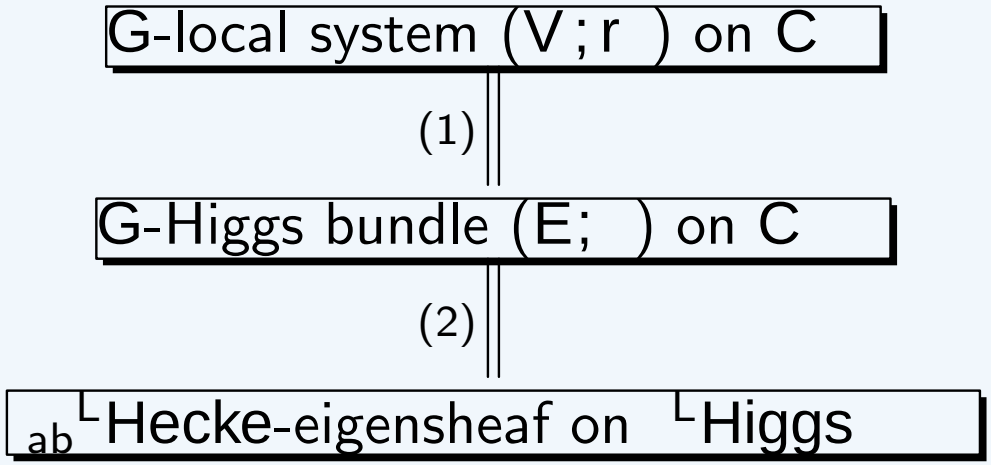


The functor (2) sends $(E; \ ) \ 2$ Higgs to $FM(O_{(E; \ )})$ where FM is a Fourier-Mukai transform for coherent sheaves on $T - Bun = Higgs$. In fact FM is the integral transform with kernel the Poincare sheaf constructed (away from the discriminant) in [DP1]. This sheaf is supported on the fiber product of the two Hitchin fibrations $h : Higgs_0 \ ! \ B$ and $h : Higgs \ ! \ B$.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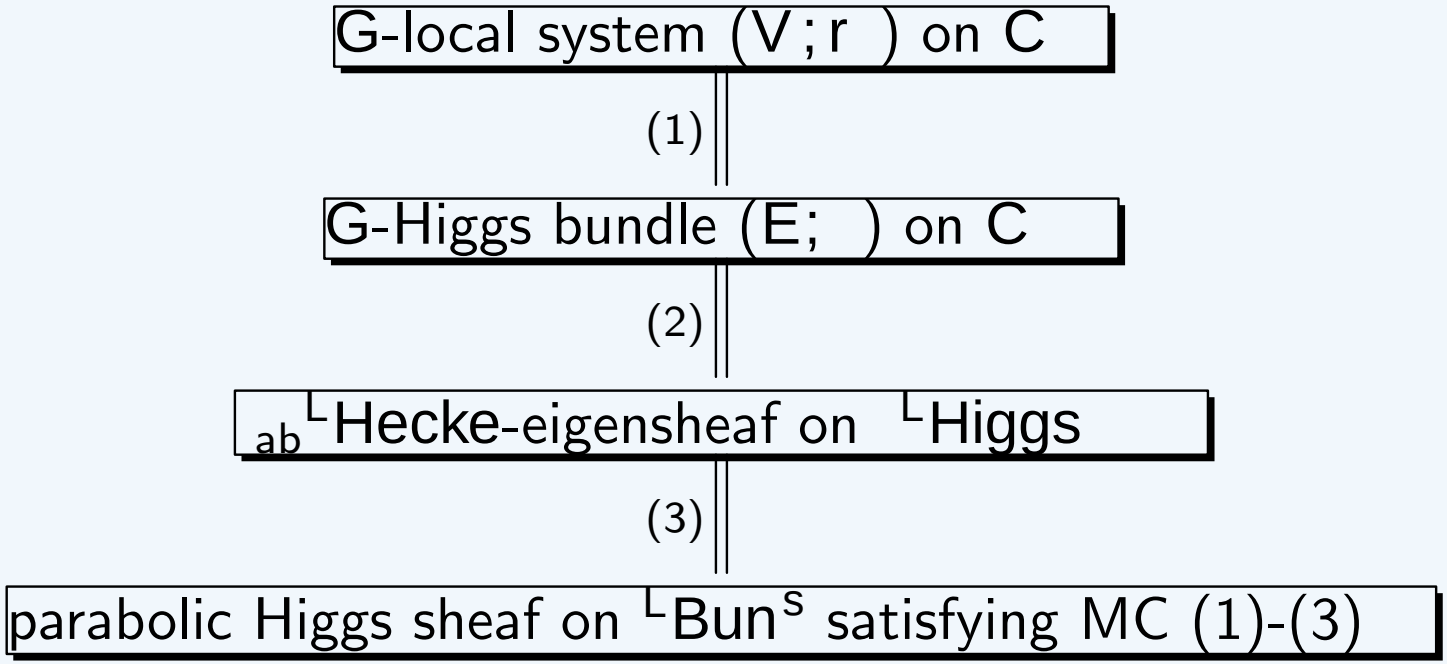
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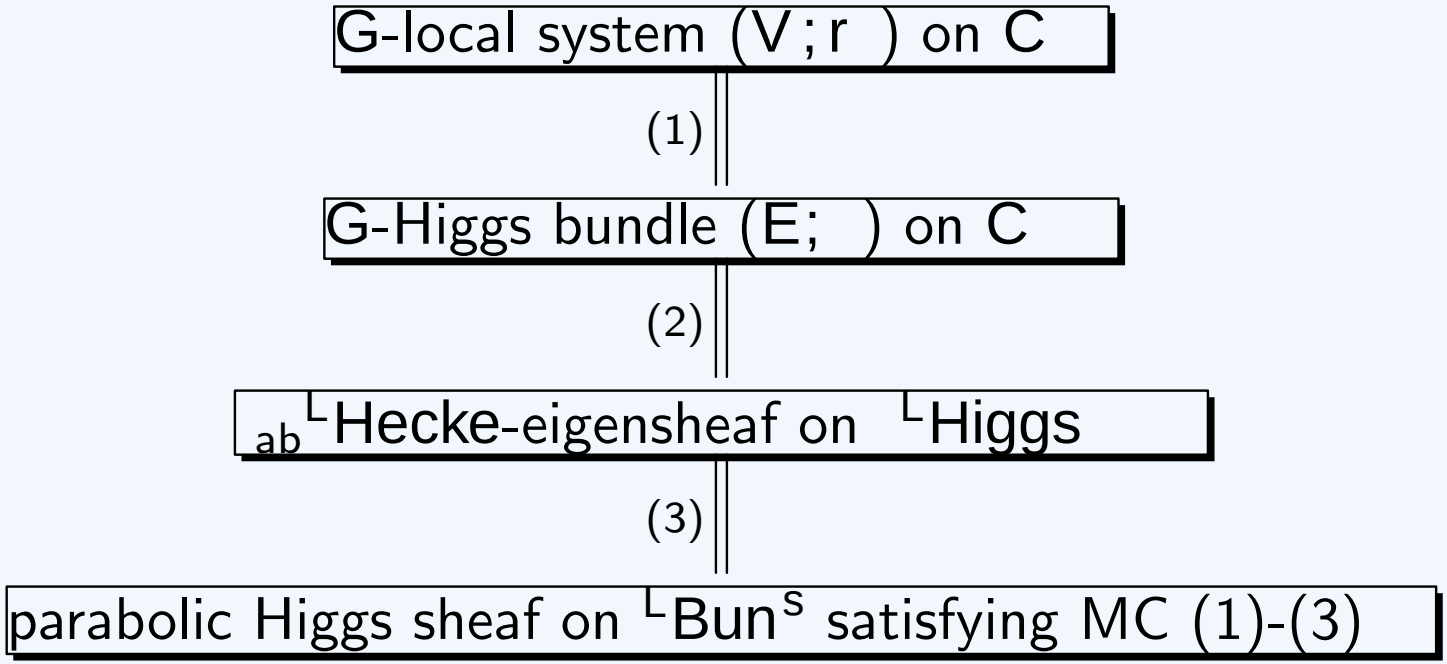
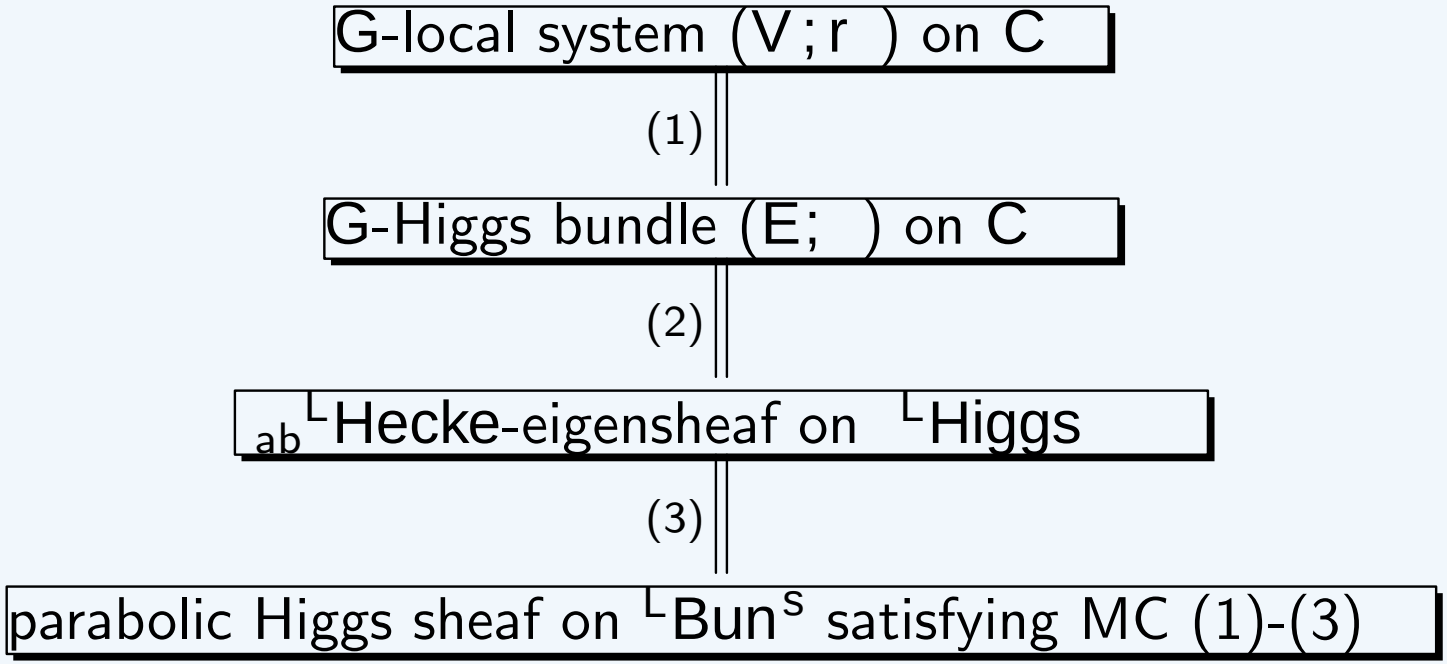
parabolic Higgs sheaf on LBuns satisfying MC (1)-(3)



This part is known only in special cases, see below.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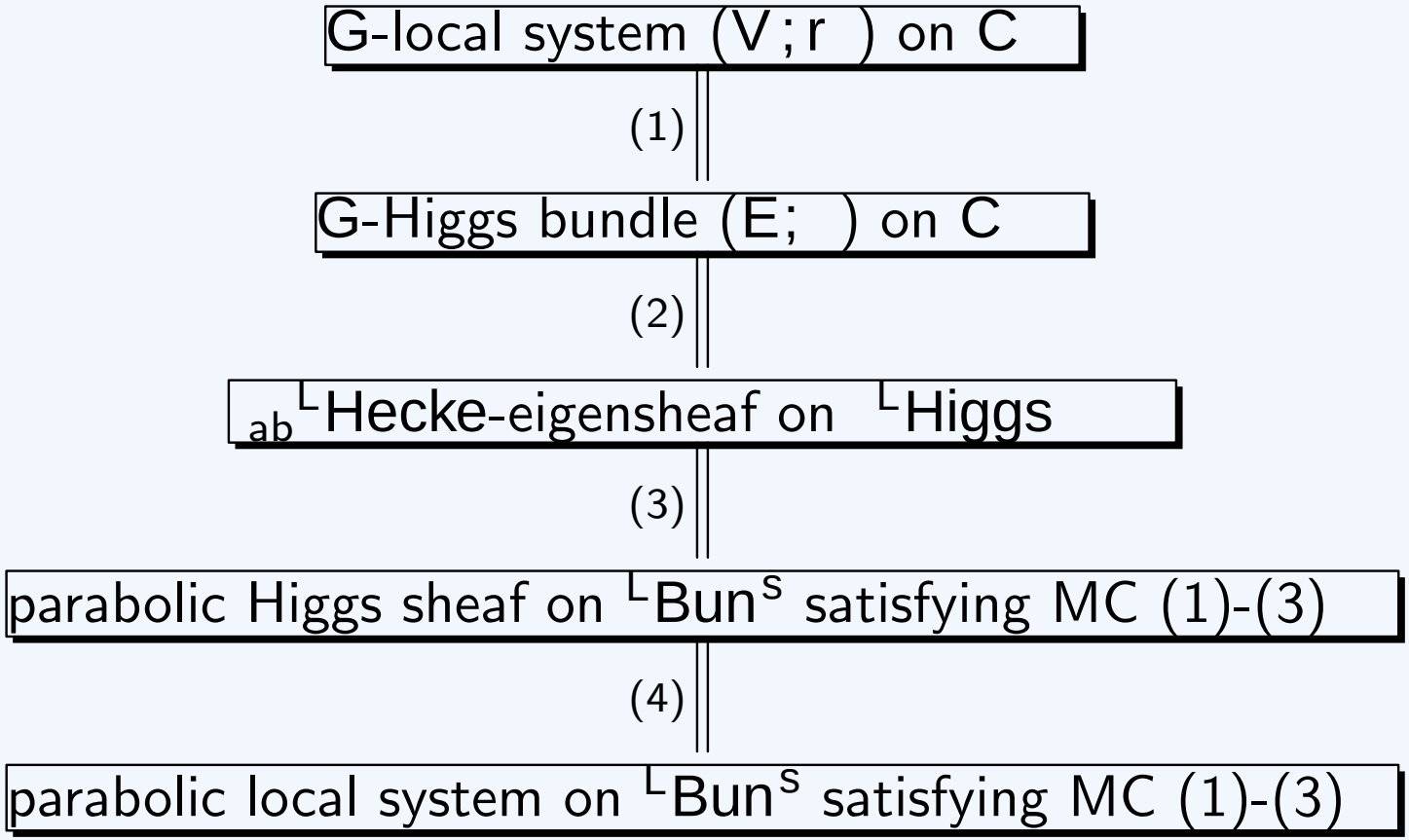
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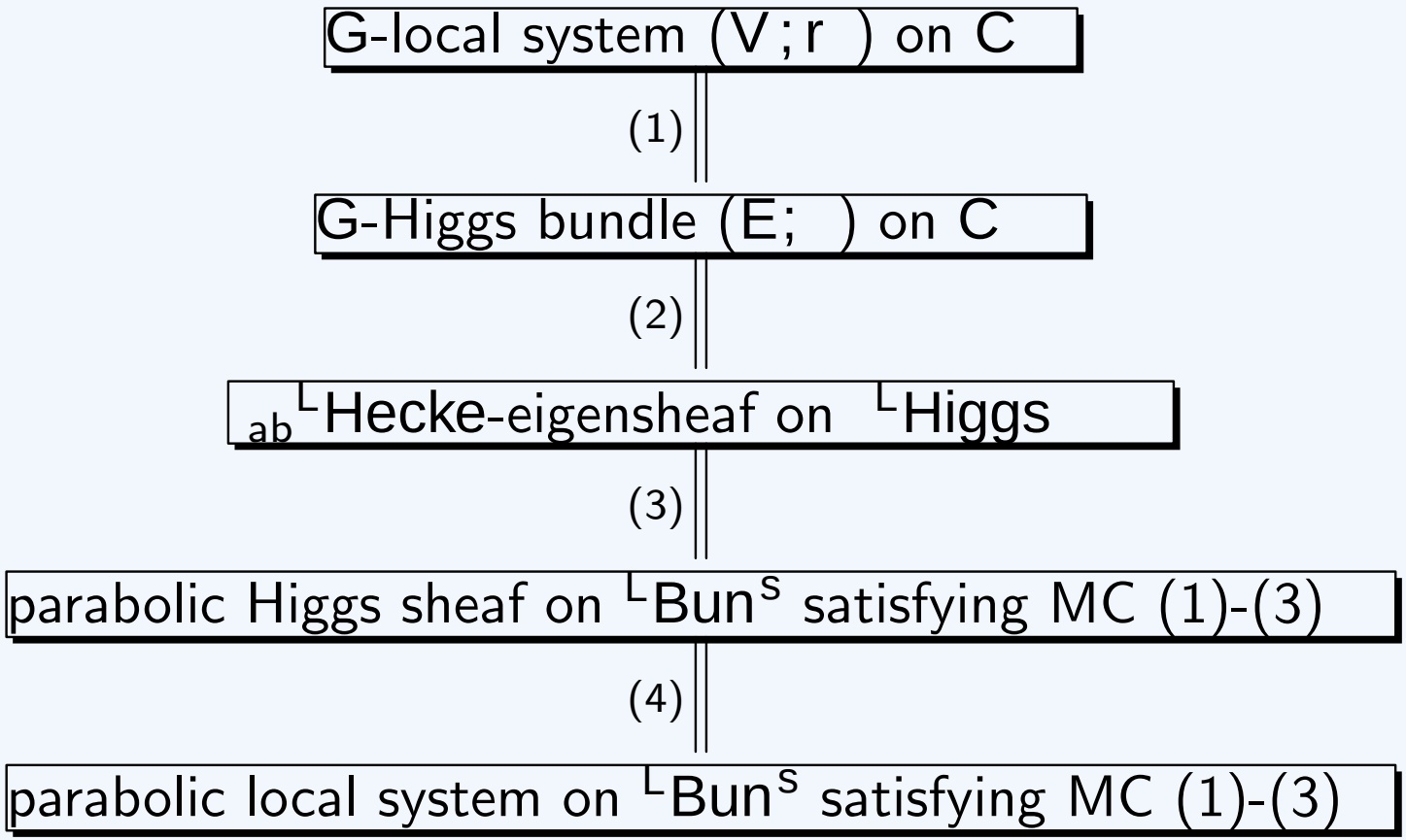
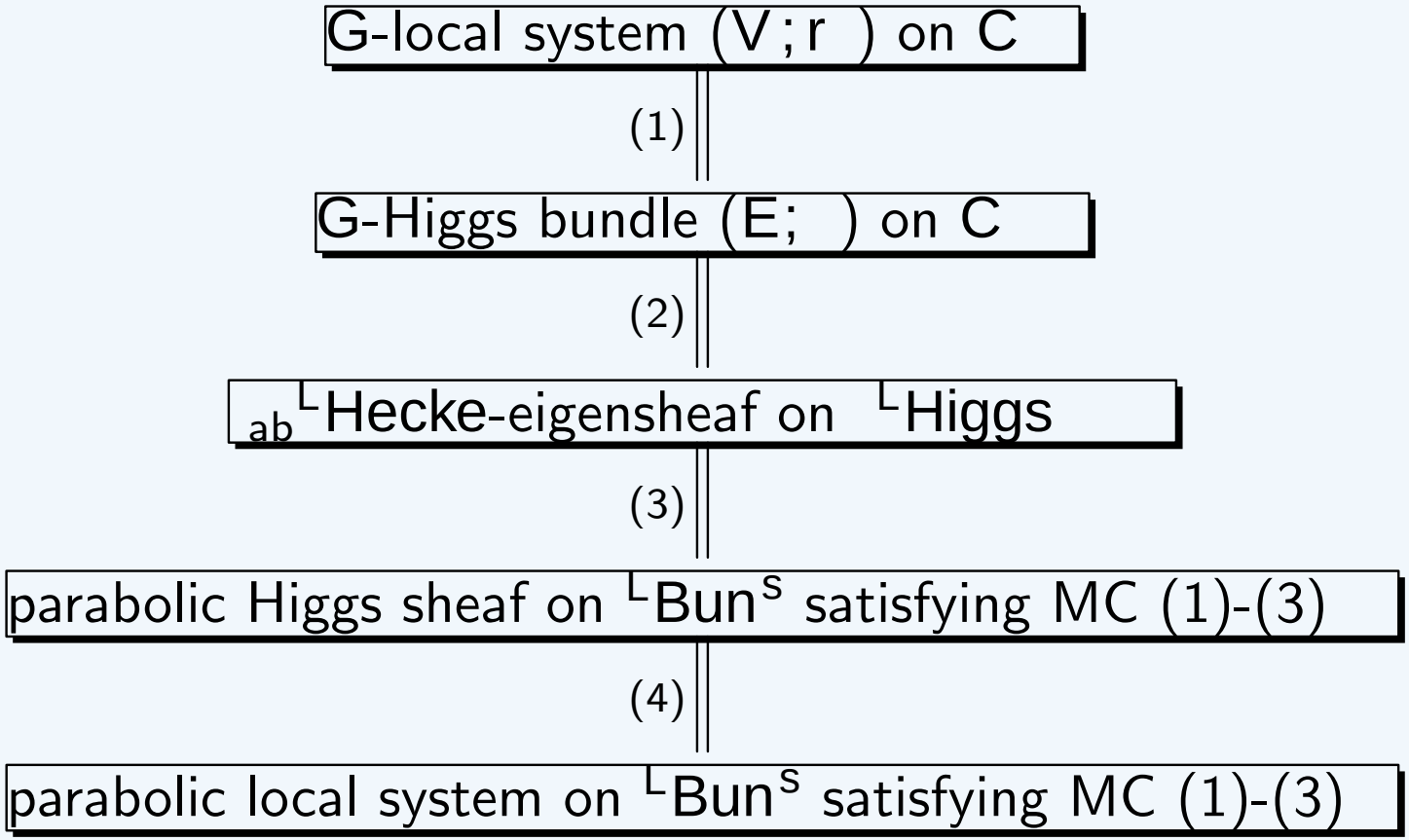


The functor (4) is the parabolic non-abelian Hodge correspondence $\mathsf{nah}_{^{\mathsf{L}}\mathsf{Bun}^{\mathsf{ss}},\mathsf{S}}$ of Mochizuki. Here ${}^{\mathsf{L}}\mathsf{Bun}^{\mathsf{ss}}$ denotes the (rigidified) stack of semistable bundles. Note that here we are applying the first Mochizuki theorem not to a projective variety but to a smooth proper Deligne-Mumford stack with a projective moduli space. In fact Mochizuki's proof works in this generality with no modifications.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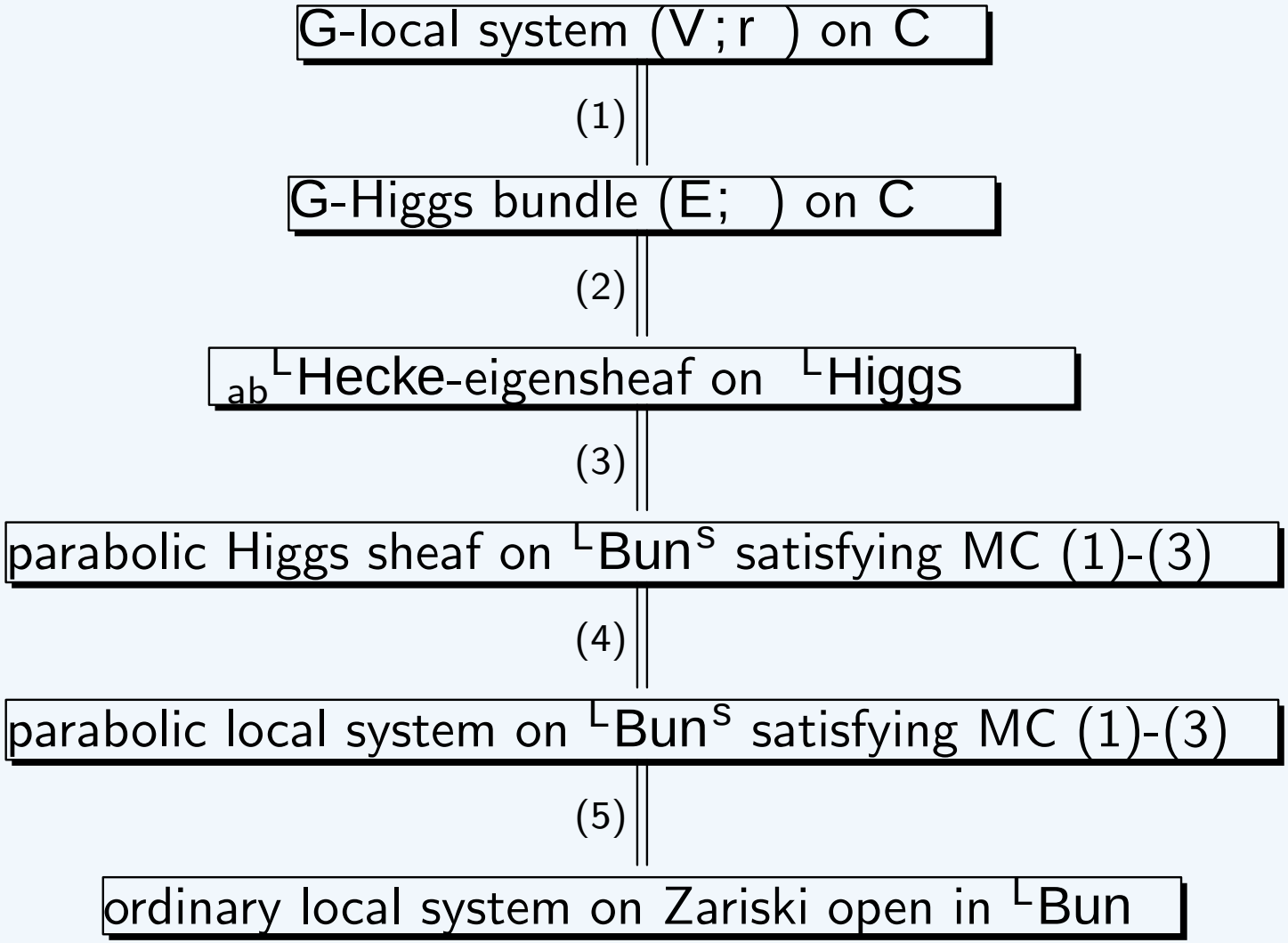
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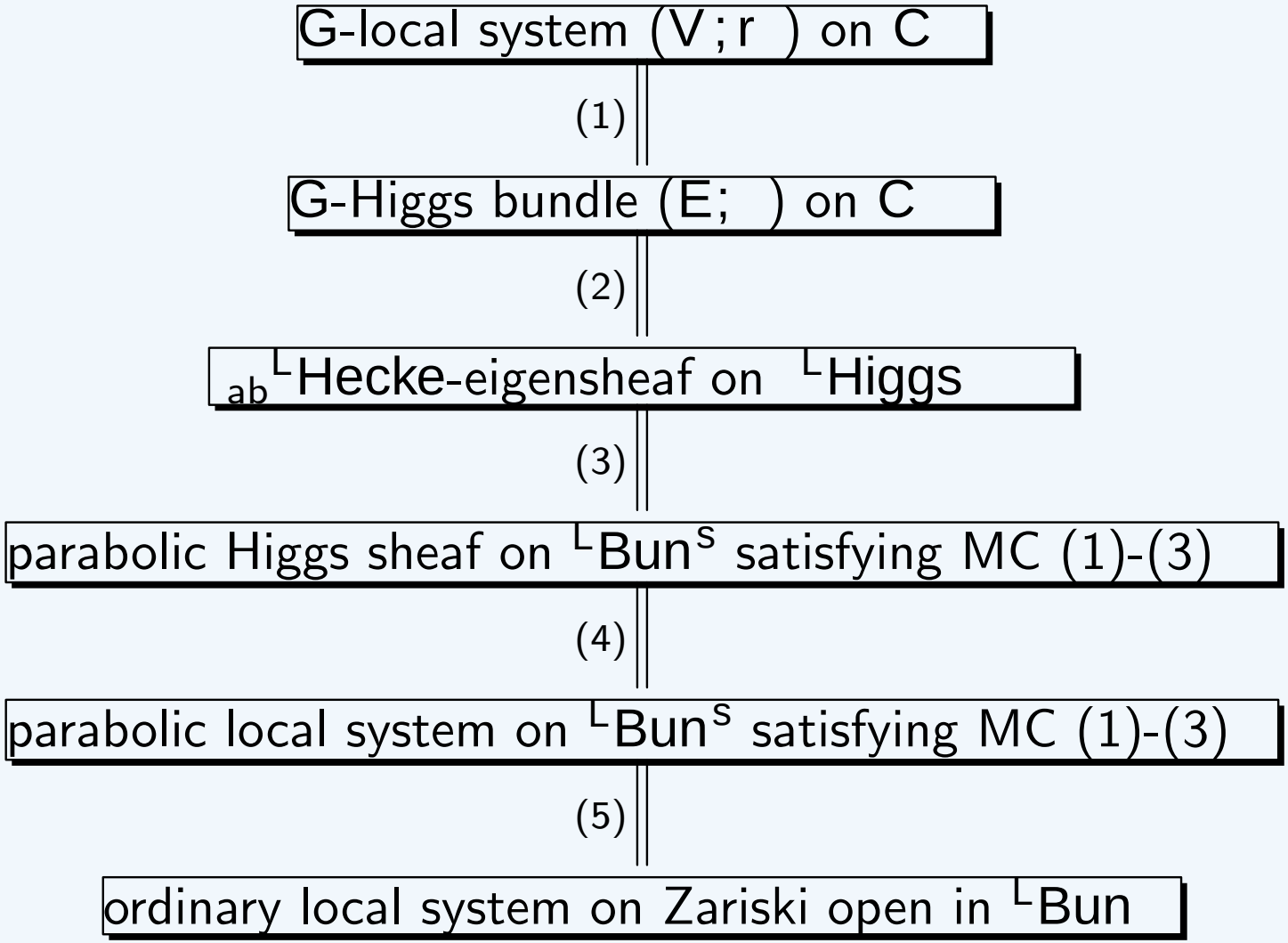
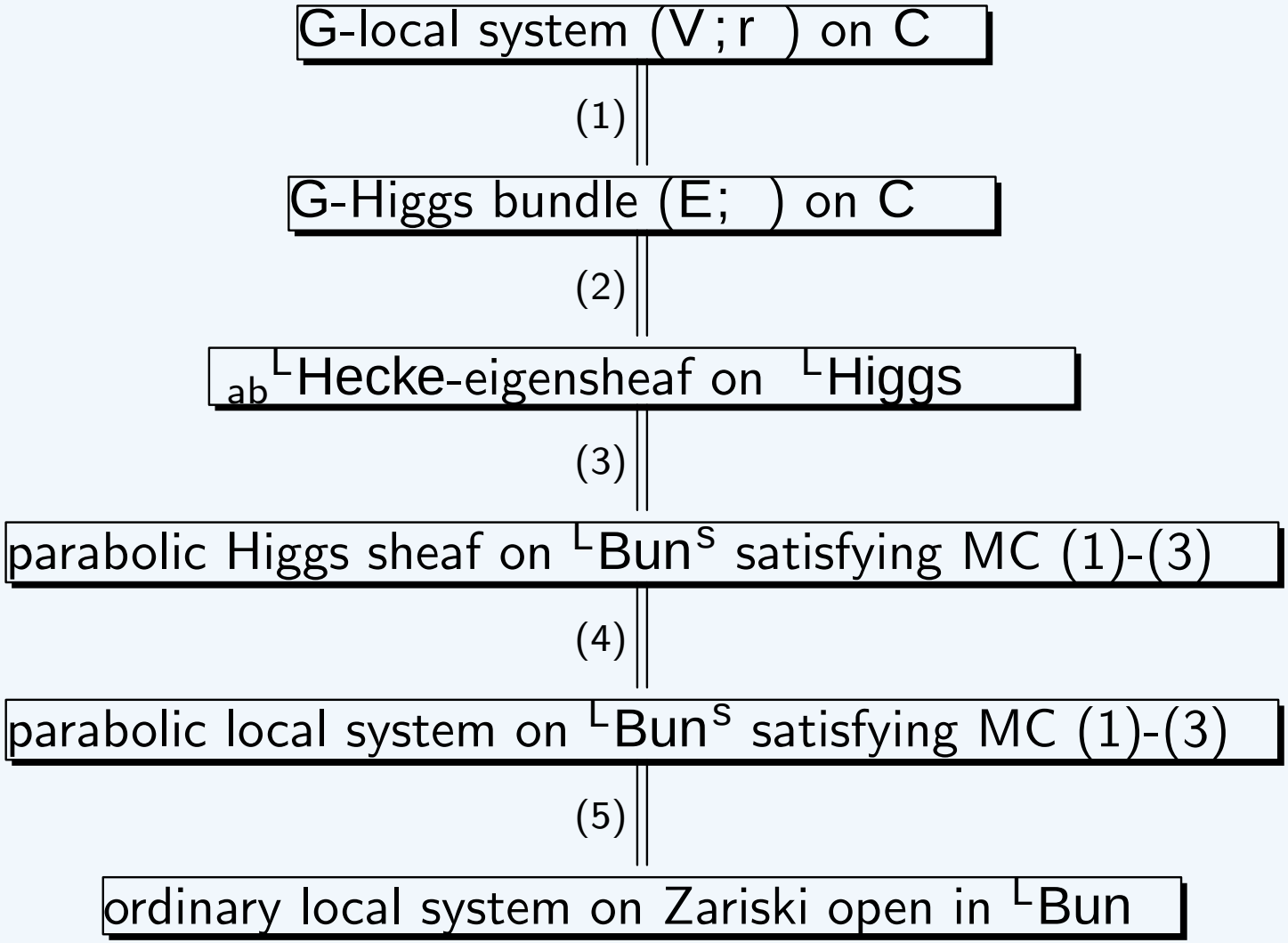
(5)
ordinary local system on Zariski open in LBun



Step 5 is just restriction.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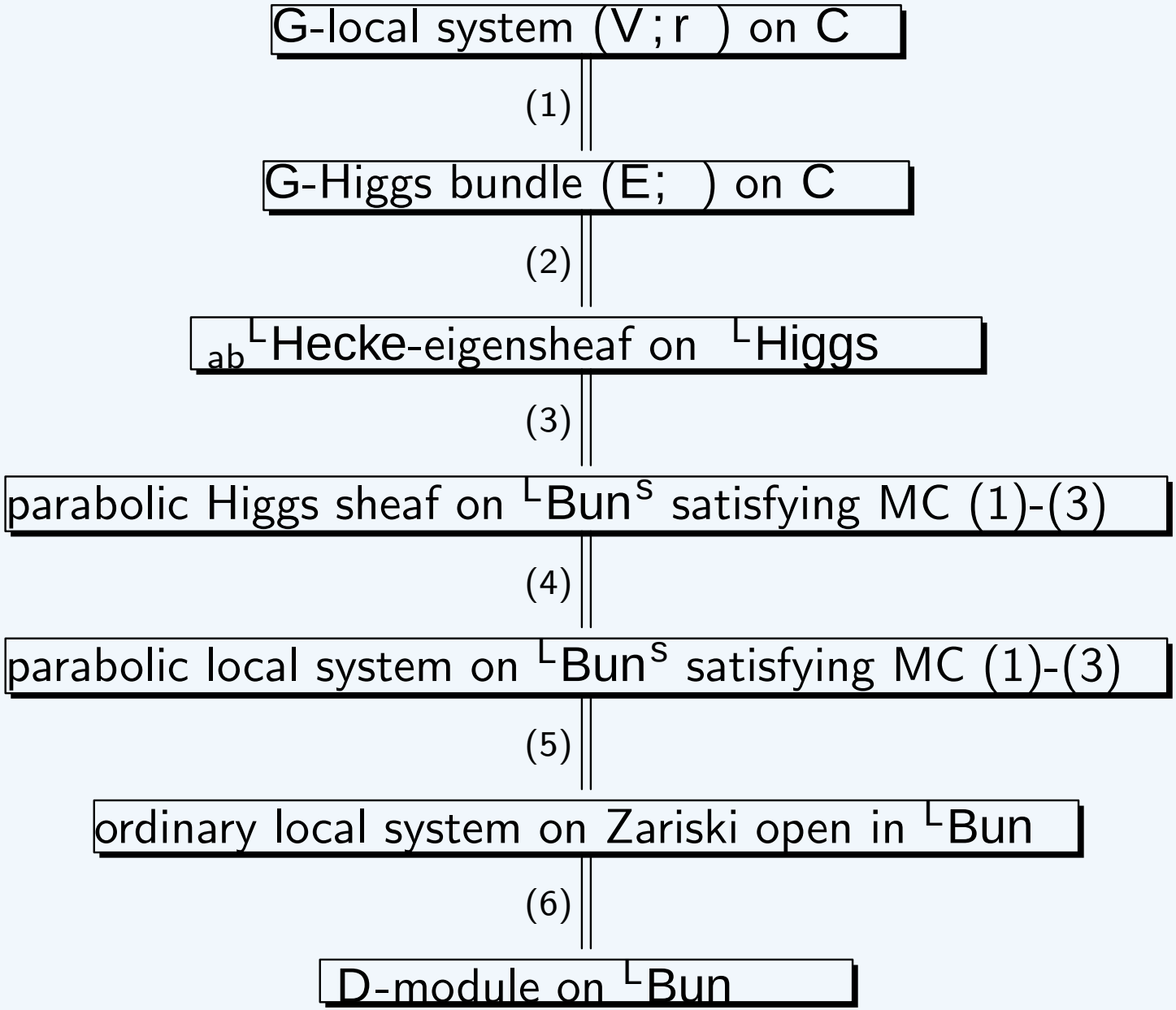
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D-module on LBun

The plan

Note that all of the other steps in this process are essentially already in place. The functor (1) is given by the Corlette-Simpson non-abelian Hodge correspondence $(E;\ ) = \mathsf{nah}_C(V;r\ )$ on the smooth compact curve C.

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The functors (5) and (6) are the pullback and middle extension functors applied to the two compactifications ${}^{L}\mathrm{Bun}^{ss}$ ${}^{L}\mathrm{Bun}^{s}$ ${}^{L}\mathrm{Bun}$. In order to conclude that the composition (6) (5) is an equivalence we need a strengthening of Mochizuki's extension theorem which would allow for Y to be an Artin stack which is only locally of finite type.

Two papers

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The formula generalizes one of Zucker (for maps to a point). The proof uses Mochizuki's machinery, and a careful comparison of R-modules vs. prabolic modules.
This formula allows us, in good situations, to compute the chern classes of parabolic Higgs sheaves on Bun. This is the missing step (3) in the flowchart. The results are then applied in [DP3] (in progress) in a concrete example, namely $\mathbb{P}^1$ with 5 parabolic points, to exhibit explicit solutions of GLC.

THANK YOU!

And see you all next year...

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