Signature

Printed Name

Preliminary Examination, Part II

Friday, April 28, 2023 2:00pm-4:30pm

This examination is based on Penn's code of academic integrity

Instructions:

Sign and print your name above.

This part of the examination consists of six problems, each worth ten points. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete — and justify your assertions. Each problem should be given its own page (or more than one page, if necessary).

If a problem has multiple parts, earlier parts may be useful for later parts. Moreover, if you skip some part, you may still use the result in a later part.

Be sure to write your name both on the exam and on any extra sheets you may submit.

Score (for faculty use only)	
7	
8	
9	
10	
11	
12	
Grader	

7. Let $f(z)$ be an analytic function on a connected open set D. Suppose c_1, $c_2 \in \mathbb{C}$, not both zero, such that

$$c_1 f(z) + c_2 \overline{f(z)} = 0, \quad \text{for all} \quad z \in D.$$

Show that $f(z)$ is a constant function on D.

8. Find the contour integral

$$\int_\gamma \overline{z}\, dz$$

for γ the circle $|z - i| = 2$ oriented counter clockwise.

9. (a) Let A be an $m \times n$ matrix and B be an $l \times n$ matrix. Define an $(m+l) \times n$ matrix

$$C = \begin{bmatrix} A \\ B \end{bmatrix}.$$

How is $\operatorname{Ker} C$ related to $\operatorname{Ker} A$ and $\operatorname{Ker} B$?

(b) Consider the vector space V of polynomials of degree up to two with basis $\{1, x, x^2\}$, and the vector space W with basis $\{1, x, x^2, x^3\}$. Consider two operators $\varphi, \psi \colon V \to W$, defined as follows:

$$\varphi(f(x)) = f''(x) - f'(x),$$
$$\psi(f(x)) = \int_0^x f(t)dt.$$

Write the matrices A of φ and B of ψ in the bases above.

(c) Find the nullspace of $C = \begin{bmatrix} A \\ B \end{bmatrix}$ with A and B from the previous part.

10. Let M be a compact, simply connected smooth manifold of dimension n. Prove there is no smooth immersion $f : M \to T^n$, where T^n is an n-torus.

11. Problem: Let (X, d) be a metric space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. If $x \in X$, show that $\lim_{n\to\infty} x_n$ equals x if and only if each subsequence of $(x_n)_{n\in\mathbb{N}}$ has a further subsequence that converges to x.

12. Let F be a field, and let A, B be $n \times n$ matrices with entries in F, where n is a positive integer. Prove that AB and BA have the same characteristic polynomial. (Hint: Consider first the case when A is invertible.)