1. Let $\{a_i\}$, $\{b_i\}$ be Cauchy sequences of real numbers. Show that the following conditions are equivalent:

   i) The sequence $\{a_i - b_i\}$ approaches 0.

   ii) The sequence $a_1, b_1, a_2, b_2, \ldots$ is Cauchy.

   Solution.

   (i) $\Rightarrow$ (ii): Let $\varepsilon > 0$. Since $\{a_i\}$ and $\{b_i\}$ are each Cauchy, there exists N_1 such that $|a_i - a_j| < \varepsilon/2$ and $|b_i - b_j| < \varepsilon/2$ for $i, j > N_1$. Since $\{a_i - b_i\} \to 0$, there exists N_2 such that $|a_j - b_j| < \varepsilon/2$ for $i, j > N_2$. So if $i, j > N := \max(N_1, N_2)$ then $|a_i - b_j| \leq |a_i - a_j| + |a_j - b_j| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Since also $|a_i - a_j|, |b_i - b_j| < \varepsilon$, (ii) follows.

   (ii) $\Rightarrow$ (i): Let $\varepsilon > 0$. Since $a_1, b_1, a_2, b_2, \ldots$ is Cauchy, there exists N such that $|a_i - b_j| < \varepsilon$ for $i, j > N$. In particular, $|a_i - b_i| < \varepsilon$ for $i > N$. So (i) follows.

2. Let A be the 3×3 real matrix $\begin{pmatrix} 2 & -3 & -1 \\ 0 & 3 & 2 \\ 2 & 3 & 3 \end{pmatrix}$ and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(v) = Av$ (viewing elements of $\mathbb{R}^3$ as column vectors). Find a basis for the kernel of T, and find a basis for the image of T.

Solution.

Applying row reduction, we obtain the sequence of matrices

$$\begin{pmatrix} 1 & -3/2 & -1/2 \\ 0 & 3 & 2 \\ 0 & 6 & 4 \end{pmatrix}, \begin{pmatrix} 1 & -3/2 & -1/2 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{pmatrix}.$$

So the kernel is the one dimensional subspace spanned by $(3, 4, -6)$, and the image is the two dimensional space spanned by the columns of the given matrix. Since neither of the first two columnns is a multiple of the other, a basis for the image is $\{(2, 0, 2), (-3, 3, 3)\}$.

3. Just from the definition, derive the formula for the derivative of the function $f(x) = 1/x$.

Solution.

$$f'(x) = \lim_{h\to 0} \frac{1/(x+h) - 1/x}{h} = \lim_{h\to 0} \frac{x-(x+h)}{hx(x+h)} = \lim_{h\to 0} -1/x(x+h) = -1/x^2.$$

4. (a) Which of the following ideals in $\mathbb{R}[x]$ are prime? maximal? the unit ideal? $(x^2-1), (x^2+1), (5), (3, x-1)$

   (b) Do the same with $\mathbb{R}[x]$ replaced by $\mathbb{Z}[x]$.

   Justify your assertions.

Solution.

(a) Since $x^2 - 1 = (x+1)(x-1)$, the ideal (x^2-1) is not prime in $\mathbb{R}[x]$ and so it is not maximal. Since non-zero multiples of x^2-1 have degree at least 2, the ideal (x^2-1) does not contain 1 and so is not the unit ideal.

The polynomial x^2+1 is irreducible in the PID $\mathbb{R}[x]$, and so the ideal (x^2+1) is prime and maximal. It is not the unit ideal for the same reason as (x^2-1).

The ideals (5) and $(3, x-1)$ are each the unit ideal in $\mathbb{R}[x]$, since they each contain a non-zero constant, which is a unit.

(b) In $\mathbb{Z}[x]$, the ideal (x^2-1) is again not prime, not maximal, and not the unit ideal, by the same reasoning as in $\mathbb{R}[x]$.

The ideal (x^2+1) is prime because it is irreducible in the UFD $\mathbb{Z}[x]$ (or because $\mathbb{Z}[x]/(x^2+1)$ is isomorphic to the integral domain $\mathbb{Z}[i]$). It is not maximal because $\mathbb{Z}[i]$ is not a field. It is not the unit ideal for the same reason as in $\mathbb{R}[x]$.

The ideal (5) is prime but not maximal, because $\mathbb{Z}[x]/(5)$ is isomorphic to $\mathbb{Z}/5\mathbb{Z}[x]$, which is an integral domain but not a field. It is not the unit ideal because 5 is not a unit in $\mathbb{Z}[x]$.

The ideal $(3, x-1)$ is a maximal ideal in $\mathbb{Z}[x]$ because $\mathbb{Z}[x]/(3, x-1)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$, which is a field. It is therefore not the unit ideal.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Suppose that $f''(x) > 0$ for all $x \in \mathbb{R}$. Suppose also that $f(0) = 0$ and that $f'(0) = 1$.

a) Prove that $f(1) > 0$.

b) Find an explicit value of $a > 0$ such that $f(a) > 10$.

Justify your assertions carefully.

Solution.

(a) Since $f'' > 0$, the function f' is increasing (this follows from the Mean Value Theorem applied to f'). So $f'(x) > f'(0) = 1 > 0$ for $x > 0$. Therefore f is increasing on $x \geq 0$, and so $f(1) > f(0) = 0$.

(b) For every $a > 0$ we have $f(a) = f(a) - f(0) = \int_0^a f'(x)\,dx \geq \int_0^a 1\,dx = a$, since $f(0) = 0$ and $f'(x) > 1$ for $x > 0$. So we may take any $a > 10$; e.g., $a = 11$. (One could also take $a = 10$, by using that f' is continuous and that $f'(x) > 1$ for $x > 0$ to get a strict inequality between the integrals.)

6. Let Ω be a non-empty connected open subset of $\mathbb{R}^2$. Suppose that $\partial f/\partial x = \partial f/\partial y = 0$ at all points $(x, y) \in \Omega$. Prove that f is a constant function on Ω. [Hint: What if Ω is an open disc?]

   Solution.

   Since Ω is connected and is a union of open discs, it suffices to show that f is constant on every open disc in Ω. In a disc U of center (a, b), the function f is constant on each horizontal line segment and on every vertical line segment, because $f_x = f_y = 0$ on U (using that a one-variable function on an open interval is constant if its derivative is identically 0, by the Mean Value Theorem). Given any point $(x_0, y_0) \in U$, the horizontal line segment connecting (a, b) to (x_0, b) and the vertical line segment connecting (x_0, b) to (x_0, y_0) are contained in U. Hence $f(a, b) = f(x_0, b) = f(x_0, y_0)$. Thus f is constant on U.

7. (a) Give an example of an open subset $R \subseteq \mathbb{R}^2$; two C^∞ functions $f(x, y), g(x, y)$ on R; and a loop (simple closed curve) C in R such that $\partial f/\partial y = \partial g/\partial x$ on R but $\oint_C f\,dx + g\,dy \neq 0$.

   (b) Explain why there cannot be such an example if $R = \mathbb{R}^2$.

Solution.

(a) Let R be the complement of the origin in $\mathbb{R}^2$, and let C be the unit circle $x^2+y^2 = 1$, oriented counterclockwise. Take $f(x, y) = -y/(x^2+y^2)$ and $g(x, y) = x/(x^2+y^2)$. Then $\partial f/\partial y = (y^2-x^2)/(x^2+y^2)^2 = \partial g/\partial x$. We can parametrize C by $x = \cos\theta$, $y = \sin\theta$, for $0 \leq \theta \leq 2\pi$. Thus $dx = -\sin\theta\,d\theta$ and $dy = \cos\theta\,d\theta$, and so the given integral I is equal to $\int_{\theta=0}^{2\pi} \left(\sin^2\theta + \cos^2\theta\right) d\theta = 2\pi \neq 0$. (Here the differential form $f\,dx + g\,dy$ is equal to $d\theta$, which is not defined at $(0, 0)$; and $\frac{1}{2\pi} I$ computes the winding number of the unit circle around $(0, 0)$.)

(b) There cannot be such an example if $R = \mathbb{R}^2$, because $\mathbb{R}^2$ is simply connected and the integral would equal 0 by Green's Theorem.

8. Let V be a vector space such that $\dim(V) = 3$. Let $T : V \to V$ be a linear transformation.

   (a) Show that if the dimension of the image of $T \circ T$ is equal to 2, then the dimension of the kernel of T is equal to 1.

   (b) Show by example that the converse to (a) is false.

Solution.

(a) If $\dim(\ker T) = 0$, then T is an isomorphism, hence so is $T \circ T$, which is then surjective, contradicting the assumption that $\dim(\text{im } T \circ T) = 2$. If $\dim(\ker T) \geq 2$, then $\dim(\text{im } T) \leq 3 - 2 = 1$. But $\text{im } T \circ T \subseteq \text{im } T$, again contradicting $\dim(\text{im } T) = 2$. So $\dim(\ker T) = 1$.

(b) Define T by $T(x, y, z) = (0, x, y)$. Then $\ker T$ is the span of $(0, 0, 1)$, of dimension 1. But $T \circ T$ takes (x, y, z) to $(0, 0, x)$, and so its image has dimension 1, not 2.

9. Let f be a continuously differentiable increasing function on $\mathbb{R}$, with $f(0) = 1$, $f(1) = 2$, and $f(2) = 6$. For each $x \in \mathbb{R}$ let $g(x)$ be the non-negative square root of $f'(x)$. Let R be the solid region swept out by rotating the graph of $y = g(x)$, from $x = 0$ to $x = 2$, about the x-axis. Compute the volume of R. Explain your assertions.

Solution.

Since f is increasing, $f'(x) \geq 0$, and so $g(x)$ is defined (and continuous). The volume of R is $\int_0^2 \pi g(x)^2 \, dx = \int_0^2 \pi f'(x) \, dx = \pi f(x)|_0^2 = \pi(6-1) = 5\pi$, by the Fundamental Theorem of Calculus.

10. Let G be a group, and let $S \subseteq G$ be the set of elements $g \in G$ such that $g = g^{-1}$.

   (a) Give an example to show that S is not necessarily a subgroup of G.

   (b) Let $H \subseteq G$ be the smallest subgroup of G that contains S. Show that H is a normal subgroup of G.

Solution.

(a) Take $G = S_3$. Then S consists of the identity and the three transpositions. This set has four elements, and so is not a subgroup of G (which has order six).

(b) The group H consists of all finite products of elements of S. (Inverses of elements in S are already in S.) If $s \in S$ and $g \in G$, then $gsg^{-1} \in S$ because $(gsg^{-1})^{-1} = gs^{-1}g^{-1} = gsg^{-1}$. So given an element $h = s_1 \cdots s_n \in H$ with $s_i \in S$, and given an element $g \in G$, the element $ghg^{-1} = (gs_1g^{-1}) \cdots (gs_ng^{-1})$ is also in H. Thus $gHg^{-1} = H$ for all $g \in G$. That is, H is normal.

11. Consider the series $\sum_{n=0}^{\infty}(1-x)x^n = (1-x)+(1-x)x+(1-x)x^2+\cdots$.

   (a) Prove that the series converges pointwise on $[0,1]$ and find its limit.

   (b) Does the series converge uniformly on $[0,1]$? Justify your answer.

Solution.

(a) For every $x \in [0,1]$, if $0 \le x < 1$, then as $n \to \infty$, the partial sum

$$S_n(x) = \sum_{k=0}^{n-1}(1-x)x^k = (1-x)\frac{1-x^n}{1-x} = 1-x^n \to 1.$$

If $x=1$, then it is direct to see that $S_n = 0$. Hence the series converges pointwise on $[0,1]$ to the function

$$S(x) = \begin{cases} 1, & 0 \le x < 1; \\ 0, & x = 1. \end{cases}$$

(b) For every n the function $S_n(x)$ is continuous on $[0,1]$, but the limiting function $S(x)$ is not continuous on $[0,1]$. Therefore, the series doesn't converge uniformly to S on $[0,1]$.

12. For $n \geq 1$, let $P_n[x]$ be the real vector space of polynomials $f(x) \in \mathbb{R}[x]$ having degree at most n, and let $\mathcal{D}$ be the differential operator $\mathcal{D}(f) = f'$ on $P_n[x]$.

   (a) Explain why $\mathcal{D}$ is a linear transformation, and find its characteristic polynomial.

   (b) Prove that $\mathcal{D}$ is not given by a diagonal matrix with respect to any basis of $P_n[x]$.

Solution.

(a) Since $(f+g)' = f' + g'$ for $f, g \in P_n[x]$, and since $(cf)' = cf'$ for $c \in \mathbb{R}$ and $f \in P_n[x]$, the operator $\mathcal{D}$ is a linear transformation. To find its characteristic polynomial, consider the basis $\{1, x, \cdots, x^n\}$ of $P_n[x]$, a vector space of dimension $n+1$. The matrix of $\mathcal{D}$ with respect to this basis is

$$D = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

It has the characteristic polynomial

$$\det(\lambda \mathbf{I} - D) = \lambda^{n+1}.$$

(b) Note that the only eigenvalue of D is 0, and $0 \cdot \mathbf{I} - D = -D$ has rank n. Hence, the eigenspace associated to 0 is one dimensional. Therefore, $\mathcal{D}$ cannot have a diagonal matrix with respect to any basis, as otherwise it should have $n+1$ linearly independent eigenvectors. (Alternatively, if it were diagonalizable, then the associated diagonal matrix would have all zeroes along the diagonal, since 0 is the only eigenvalue. But any matrix that is similar to the zero matrix is itself the zero matrix, whereas D is not the zero matrix. So D is not diagonalizable.)