Signature	PRINTED NAME

PRELIMINARY EXAMINATION, PART I

Monday, May 3, 2021 — 9:15am-12:45pm

This examination is based on Penn's code of academic integrity

Instructions:

Sign and print your name above.

This part of the examination consists of six problems, each worth ten points. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete — and justify your assertions. Each problem should be given its own page (or more than one page, if necessary).

If a problem has multiple parts, earlier parts may be useful for later parts. Moreover, if you skip some part, you may still use the result in a later part.

You can either work on this exam in electronic form (e.g., using a tablet) or else print it out and then save it to a *single* pdf file with your name on the first page; e.g., by using a scanner or by taking photos, and then merging them into one pdf file. (As an alternative to printing it, you can write your answers on paper, making sure to sign and print your name at the top of the first page, after which you would do the scanning and merging.) After you have completed the exam, upload the pdf of your exam to the folder in Penn Box with your name on it. This upload should be completed within 30 minutes of finishing your work on the exam.

During this exam, you should be logged into Zoom for proctoring purposes, using the link you have been provided. Your camera should be on, but you should mute your microphone and use the private chat feature if you need to ask a question.

Score	
1	
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6	

1. Let $\{a_i\}$, $\{b_i\}$ be Cauchy sequences of real numbers. Show that the following conditions are equivalent:

   i) The sequence $\{a_i - b_i\}$ approaches 0.

   ii) The sequence $a_1, b_1, a_2, b_2, \ldots$ is Cauchy.

Extra page for work for problem 1.

2. Let A be the 3×3 real matrix $\begin{pmatrix} 2 & -3 & -1 \\ 0 & 3 & 2 \\ 2 & 3 & 3 \end{pmatrix}$ and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(v) = Av$ (viewing elements of $\mathbb{R}^3$ as column vectors). Find a basis for the kernel of T, and find a basis for the image of T.

Extra page for work for problem 2.

3. Just from the definition, derive the formula for the derivative of the function $f(x) = 1/x$.

Extra page for work for problem 3.

4. (a) Which of the following ideals in $\mathbb{R}[x]$ are prime? maximal? the unit ideal? $(x^2-1), (x^2+1), (5), (3, x-1)$

   (b) Do the same with $\mathbb{R}[x]$ replaced by $\mathbb{Z}[x]$.

   Justify your assertions.

Extra page for work for problem 4.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Suppose that $f''(x) > 0$ for all $x \in \mathbb{R}$. Suppose also that $f(0) = 0$ and that $f'(0) = 1$.

   (a) Prove that $f(1) > 0$.

   (b) Find an explicit value of $a > 0$ such that $f(a) > 10$.

   Justify your assertions.

Extra page for work for problem 5.

6. Let Ω be a non-empty connected open subset of $\mathbb{R}^2$. Suppose that $\partial f/\partial x = \partial f/\partial y = 0$ at all points $(x, y) \in \Omega$. Prove that f is a constant function on Ω. [Hint: What if Ω is an open disc?]

Extra page for work for problem 6.

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Preliminary Examination, Part II

Monday, May 3, 2021 | 1:45-5:15pm

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Score	
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12	

7. (a) Give an example of an open subset $R \subseteq \mathbb{R}^2$; two C^∞ functions $f(x, y), g(x, y)$ on R; and a loop (simple closed curve) C in R such that $\partial f/\partial y = \partial g/\partial x$ on R but $\oint_C f\,dx + g\,dy \neq 0$.

   (b) Explain why there cannot be such an example if $R = \mathbb{R}^2$.

Extra page for work for problem 7.

8. Let V be a vector space such that $\dim(V) = 3$. Let $T : V \to V$ be a linear transformation.

   (a) Show that if the dimension of the image of $T \circ T$ is equal to 2, then the dimension of the kernel of T is equal to 1.

   (b) Show by example that the converse to (a) is false.

Extra page for work for problem 8.

9. Let f be a continuously differentiable increasing function on $\mathbb{R}$, with $f(0) = 1$, $f(1) = 2$, and $f(2) = 6$. For each $x \in \mathbb{R}$ let $g(x)$ be the non-negative square root of $f'(x)$. Let R be the solid region swept out by rotating the graph of $y = g(x)$, from $x = 0$ to $x = 2$, about the x-axis. Compute the volume of R. Explain your assertions.

Extra page for work for problem 9.

10. Let G be a group, and let $S \subseteq G$ be the set of elements $g \in G$ such that $g = g^{-1}$.

   (a) Give an example to show that S is not necessarily a subgroup of G.

   (b) Let $H \subseteq G$ be the smallest subgroup of G that contains S. Show that H is a normal subgroup of G.

Extra page for work for problem 10.

11. Consider the series $\sum_{n=0}^{\infty}(1-x)x^n = (1-x) + (1-x)x + (1-x)x^2 + \cdots$.

(a) Prove that the series converges pointwise on $[0, 1]$ and find its limit.

(b) Does the series converge uniformly on $[0, 1]$? Justify your answer.

Extra page for work for problem 11.

12. For $n \geq 1$, let $P_n[x]$ be the real vector space of polynomials $f(x) \in \mathbb{R}[x]$ having degree at most n, and let $\mathcal{D}$ be the differential operator $\mathcal{D}(f) = f'$ on $P_n[x]$.

    (a) Explain why $\mathcal{D}$ is a linear transformation, and find its characteristic polynomial.

    (b) Prove that $\mathcal{D}$ is not given by a diagonal matrix with respect to any basis of $P_n[x]$.

Extra page for work for problem 12.