ON THE ORBITAL RIGIDITY CONJECTURE AND SUSTAINED P-DIVISIBLE GROUPS

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ABSTRACT

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The orbital rigidity phenomenon for p-divisible groups was first discovered by Ching-Li Chai, motivated by the Hecke orbit conjecture. Later, the general orbital rigidity conjecture was formulated and the second case of this conjecture was proved by Ching-Li Chai and Frans Oort. In this thesis we prove a third case of this conjecture.

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CHAPTER 1

Introduction

1.1. The Orbital Rigidity Conjecture: First Example

The first case of the orbital rigidity conjecture is the following theorem proved in [Cha08].

Theorem 1.1.1. *Let E be a p-divisible formal group over an algebraically closed field k of characteristic p. If W is a reduced irreducible closed formal subscheme of X which is stable under a strongly non-trivial action of a subgroup G of $Aut(E)$, where $Aut(E)$ consists of all group automorphisms of X. Then W is a p-divisible subgroup of E.*

Here the assumption of G acting strongly non-trivially on E means that for every open subgroup $U \subset G$ and every pair $Y_1 \subsetneq Y_2$ of U-invariant p-divisible subgroups of E, the action of U on Y_2/Y_1 is non-trivial.

To better understand how this relates to moduli spaces of Abelian varieties and in which way we can generalize theorem 1.1.1, we need to recall the concept of sustained p-divisible groups as introduced in [CO22].

1.2. What is a Sustained p-Divisible Group

In a nutshell, a p-divisible group $X \to S$ over a base scheme S of characteristic p is ***sustained*** if its p^n-torsion subgroup schemes $X[p^n] \to S$ are constant locally in the flat topology of S, for every natural number n. For a precise definition, see 2.2.1.

Let X be a p-divisible group over the base field κ, and we define the sustained deformation space of X, denoted by $Def_{sus}(X)$, to be the subfunctor of $Def(X)$ that consists of only sustained p-divisible groups. As it turns out:

- $Def_{sus}(X)$ has a natural structure as a smooth formal variety for any p-divisible groups X_0/κ.

- $Def_{sus}(X)$ can be 'built-up' from some p-divisible groups together with some bilinear pairings. Informally speaking, $Def_{sus}(X)$ possesses some 'linear structure'.

To get a better sense of the geometry of $Def_{sus}(X)$, let $K \in \mathbb{N}$, and let $X = \prod_{i=1}^{K} X_i$ where X_i are isoclinic p-divisible groups with slope s_i, and assume that $s_1 > s_2 > ... > s_K$.

Case 1. If $K = 2$, then $Def_{sus}(X)$ is an isoclinic p-divisible of slope $s_1 - s_2$.

Case 2. If $K = 3$, then $Def_{sus}(X)$ can be built up from three p-divisible groups $Def_{sus}(X_i \times X_j), \forall 1 \leq i < j \leq 3$, and these three p-divisible groups are glued together by a family of bilinear pairings one for each $n \in \mathbb{N}$

$$\langle , \rangle_n : Def_{sus}(X_1 \times X_2)[p^n] \times Def_{sus}(X_2 \times X_3)[p^n] \to Def_{sus}(X_1 \times X_3)[p^n]$$

See 3.4.3 for a precise description. In fact, $Def_{sus}(X)$ has a biextension structure in the sense of 3.1.1.

Remark 1.2.1. *In fact, these 'linear structures' on $Def_{sus}(X)$ generalize the Serre-Tate coordinates: if A is an ordinary abelian variety over $k = \bar{k}$ an algebraically closed field of characteristic p, and $X = A[p^\infty]$ the p-divisible group of A, then*

$$Def_{sus}(X) = Def(X)$$

where $Def(X)$ is the deformation space of X. As X has two slopes $\{0, 1\}$, in this case $Def_{sus}(X)$ is a formal torus, and this formal torus structure is precisely the Serre-Tate coordinates on $Def(X)$.

Remark 1.2.2. *The definition of sustained p-divisible groups generalizes the concept of geometrically fiberwise constant p-divisible groups, and helps to illuminate the structural properties of central leaves, for precise definitions of geometrically fiberwise constant p-divisible groups and central leaves, see [Oor04].*

Remark 1.2.3. *The definition of central leaves was motivated by the Hecke orbit conjecture. A special case of the Hecke orbit conjecture says the following: let $\mathcal{M}$ be a PEL type Shimura variety over $\overline{\mathbb{F}_p}$. Let $x_0 \in \mathcal{M}(\overline{\mathbb{F}_p})$. Let $\mathcal{C}(x_0)$ be the central leaf of x_0, that is locus of all points of $\mathcal{M}$ having 'the same p-adic invariants as x_0'. Then the prime-to-p Hecke orbit $\mathcal{H}^{(p)} \cdot x_0$ of x_0 is dense in the central leaf $\mathcal{C}(x_0)$ containing x_0. See [Cha05] for more details.*

The notions of sustained p-divisible groups and sustained deformation spaces provide a connection between 1.1.1 and deformation spaces of p-divisible groups when we substitute the p-divisible group E as in 1.1.1 by $Def_{sus}(X)$ where $X = X_1 \times X_2$ with X_i isoclinic of different slopes.

Somewhat surprising, this 'orbital rigidity' phenomenon as described in 1.1.1 seems to hold in a much broader context. To formulate the general form of 1.1.1, we need to define a family of special subvarieties of $Def_{sus}(X)$. This is the notion of Tate-linear formal subvarieties.

1.3. Tate-linear Formal Subvarieties

Let $K \in \mathbb{N}$, $X = \prod_{i=1}^{K} X_i$ where X_i are isoclinic p-divisible groups with slope s_i over a field κ of characteristic p , and assume that $s_1 > s_2 > ... > s_K$.

- As it turns out, we can associate to X a projective system of finite group schemes

$$Aut^{st}(X) = \varprojlim_{n} Aut^{st}(X)_n$$

where $Aut^{st}(X)_n$ are finite group schemes over the base field κ. Moreover, let

$$Def_{Aut^{st}(X)\text{-torsor}}$$

be the deformation functor of left $Aut^{st}(X)$-torsors, then

$$Def_{Aut^{st}(X)\text{-torsor}} \simeq Def_{sus}(X)$$

- Let $H' \subset Aut^{st}(X)$ be an admissible subgroup. For the precise definition of admissible subgroups see 4.7.3. The contraction product that sends each H' torsor F to the $Aut^{st}(X)$ torsor $Aut^{st}(X) \wedge^{H'} F$ induces a morphism

$$\Phi_{H' \hookrightarrow Def_{Aut^{st}(X)\text{-torsor}}} : Def_{H'\text{-torsors}} \to Def_{Aut^{st}(X)\text{-torsor}}$$

Definition 1.3.1. *A formal subvariety E' of $Def_{sus}(X)$ is called a Tate-linear formal subvariety if there exists an admissible subgroup H' such that the schematic image of*

$$\Phi_{H' \hookrightarrow Def_{Aut^{st}(X)\text{-}torsor}}$$

is E'.

Remark 1.3.2. *We give two examples: let $X = \prod_{i=1}^{K} X_i$ where X_i isoclinic p-divisible groups with slopes s_i such that $s_1 > s_2 ... > s_K$.*

Case 1. *If $K = 2$, then $Def_{sus}(X)$ is a p-divisible group. In this case, the set of Tate-linear formal subvarieties coincides with the set of p-divisible subgroups of $Def_{sus}(X)$.*

Case 2. *If $K = 3$, then $Def_{sus}(X)$ is 'built up' by three p-divisible groups $Def_{sus}(X_i \times X_j), \forall 1 \leq i < j \leq 3$ and a family of bilinear pairings $\langle , \rangle_n$. In this case each Tate-linear formal subvariety is 'built up' by certain p-divisible subgroups H'_{ij} of $Def_{sus}(X_i \times X_i), \forall 1 \leq i < j \leq 3$ that satisfy certain constrains given by $\langle , \rangle_n$.*

Remark 1.3.3. *Readers familiar with the notion of Shimura varieties might find the notion of Tate-linear formal subvarieties similar to the notion of Shimura subvarieties: both Tate-linear subvarieties and Shimura subvarieties come from subgroups (in this case H') of the bigger groups (in this case $Aut^{st}(X)$) that define the ambient spaces.*

Remark 1.3.4. *One way to obtain Tate-linear formal subvarieties of $Def_{sus}(X)$ is to deform not only the p-divisible group X but also some extra structures on X (e.g. a polarization* *of X) in a 'sustained manner' See [CO22] especially Chapter 6 for more information. This provides an extra layer of similarity between Tate-linear formal subvarieties and Shimura subvarieties: Let $\mathbb{A}_g$ be the Shimura variety corresponding to the symplectic group Sp_{2g}, then roughly speaking, each Shimura subvarieties of $\mathbb{A}_g$ is the sublocus on which the restriction of the universal Abelian scheme carries some extra Hodge cycles of given shape, see [Mum69] for the precise statement.*

1.4. The Orbital Rigidity Conjecture: General Form

Now we can state the orbital rigidity conjecture in its general form:

Let $K \in \mathbb{N}$, $X = \prod_{i=1}^{K} X_i$ with X_i isoclinic with slopes s_i over an algebraically closed field κ of charcateristic p, and assume that $s_1 > s_2 > ... > s_K$. Let $E = Def_{sus}(X)$, which is a smooth formal scheme over κ. Let $G \subset \widetilde{Aut}(E)$ be a closed subgroup, acting strongly non-trivially on E. Suppose that W is a reduced irreducible closed formal subscheme of E stable under the action of G. Then W is a Tate-linear formal subvariety of E. Here:

- $\widetilde{Aut}(E)$ is a subgroup of $Aut_{\text{scheme}}(E)$ that consists of automorphisms of E that preserves certain 'linear structure' of E in some sense. For precise definition see 4.4.8.

- The definition of a strongly non-trivial action is given in 3.3.1. Roughly speaking, a strongly-nontrivial action means the following: the action of $\widetilde{Aut}(E)$ acts on all the Jordan-Holder components of $Def_{sus}(X)$, with each component a p-divisible group. The action is strongly non-trivial if the action on each component is strongly non-trivial in the sense of 1.1.1.

When X a p-divisible group with two slopes, the conjecture 1.4 was proved in [Cha08]. When X is a p-divisible group with three slopes, the conjecture 1.4 was proved in [CO22]. The main result of this thesis is to prove the conjecture 1.4 when X has four slopes, that is the following:

Theorem 1.4.1. *Let $X = \prod_{i=1}^{4} X_i$ with X_i isoclinic with slopes s_i and assume that $s_1 >$*

$s_2 > s_3 > s_4$ *over an algebraically closed field* κ *of characteristic* $p \geq 5$. *Let* $E = Def_{sus}(X)$, *which is a smooth formal subvariety over* κ. *Let* G *be a closed subgroup of* $\widetilde{Aut}(E)$, *acting strongly non-trivially on* E. *Suppose that* W *is a reduced irreducible closed formal subscheme of* E *stable under the action of* G. *Then* W *is a Tate-linear formal subvariety of* E.

Remark 1.4.2. *The actual statement of the main result 4.8.2 is slightly more general than 1.4.1.*

1.5. Structure of the Thesis

Some key components of this thesis are:

- In chapter 2, we collect some basic definitions and properties of sustained p-divisible groups, following [CO22].

- In chapter 3 and chapter 5, we discuss the structure of $Def_{sus}(X)$ when $X = X_1 \times X_2 \times X_3$ and the orbital rigidity conjecture in this case. This serves as the 'induction hypothesis' for the case when X has four slopes.

- In chapter 4, we prove the main structural theorem of $Def_{sus}(X)$ when $X = \prod_{i=1}^{4} X_i$, which roughly says that a suitable closed subscheme E_n of $Def_{sus}(X)$ can be trivialized using some p-divisible groups and several families of bilinear pairings. See 4.2.1 for the precise statement. This result serves as the main entry point of analyzing the action for $\widetilde{Aut}(E)$ on E.

- Also in chapter 4, we define the notion of Tate-linear nilpotent groups of type A. Here the name 'type A' is inspired by the notion of simple Lie algebra of type A. The category of Tate-linear nilpotent groups of type A slightly generalized the category of projective system of group schemes of the form $Aut^{st}(X)$ where $X = \prod_{i=1}^{K} X_i$ with X_i isoclinic. Let H be a Tate-linear nilpotent group of type A, we will show that $Def_{H\text{-torsor}}$ possesses geometric structure that is similar to $Def_{sus}(\prod_{i=1}^{K} X_i)$. Hence we may substitute $Def_{sus}(X)$ by $Def_{H\text{-torsor}}$ in the conjecture 1.4. The upshot is that this bigger category (i.e. consists of all the $Def_{H\text{-torsor}}$) is closed under certain operations, thus allowing us to perform some reductions.

- In chapter 6, we recall the definition of tempered perfection as defined in [CO22]. This is a technique that Ching-Li Chai and Frans Oort used in their proof of the orbital rigidity conjecture for the three slopes case. The idea is that for each $n \in \mathbb{N}$ and certain susbcheme $E_n \subset Def_{sus}(X)$, the action of $g_n \in \widetilde{Aut}(E)$ can be written down explicitly for g_n sufficiently closed to the identity. Tempered perfection allows us to 'glue' this family of information together when we vary n. We show that this tempered perfection technique can also be used in our case to prove similar results, in particular theorem 6.3.2 and theorem 6.4.6.

- In chapter 7, we prove that the existence of a formal subvariety W invariant under $G \subset \widetilde{Aut}(X)$ imposes certain Lie bracket conditions, see 7.3.4. Finally, we prove the main result in 7.4.1.

CHAPTER 2

Sustained p-Divisible Groups

We recall the definition and some useful facts of p-divisible groups and collect some definitions and facts about sustained p-divisible groups as given in [CO22].

2.1. p-Divisible Groups

Definition 2.1.1. *Fix a prime number p, a positive integer h, and a commutative ring R. A p-divisible group of height h over R is a codirected diagram* $(G_v, i_v)_{v\in\mathbb{N}}$ *where each* G_v *is a finite commutative group scheme over* S *of order* p^{vh} *that also satisfies the property that*

$$0 \to G_v \xrightarrow{i_v} G_{v+1} \xrightarrow{p^v} G_{v+1}$$

is exact. In other words, the maps of the system identify G_v *with the kernel of multiplication by* p^v *in* G_{v+1}*. Note that these conditions imply that*

$$Im(p^v : G_{v+1} \to G_{v+1}) = ker(p)$$

as subschemes of G_{v+1}*.*

Remark 2.1.2. *We can also define the notion of a p-divisible group over an arbitrary scheme* S*. See for example [Mes72].*

Example 2.1.3. Let R be a commutative ring, and let X be an abelian scheme over R of dimension g, then for each $n \in \mathbb{N}$ the miltiplication map by p^n has kernel $X[p^n]$ which is a finite group scheme pver R of order p^{2gn}. The natural inclusion satisfiy the conditions for the limit $\varinjlim_n X[p^n]$ to be a p-divisible group of height $2g$.

Theorem 2.1.4. ***(Serre-Tate Theorem)*** *Let* κ *be a field of characteristic* p*. Let* A *be an abelian variety over* κ*. Let* Def_A *be the deformation functor of* A*, that is the functor that* *sends every artinian local ring* $(R, m)/\kappa$ *to the set*

$$\left\{(\tilde{A}, \varphi) : \tilde{A} \text{ an abelian scheme over } R, \varphi : A \times_\kappa R/m \xrightarrow{\cong} \tilde{A} \times_R R/m\right\} / \sim$$

Let $A[p^\infty]$ *be the p-divisible group of A, and let* $Def_{A[p^\infty]}$ *be the deformation functor of* $A[p^\infty]$*. Then there is a natural isomorphism of functors between* Def_A *and* $Def_{A[p^\infty]}$*.*

Definition 2.1.5. ***(Isogeny of p-divisible groups)*** *Let* P_1, P *be p-divisible groups over a base scheme S. A homomorphism* $f : P_1 \to P_2$ *is called an isogeny if f is surjective and that* $ker(f)$ *is a finite scheme over S. We say two p-divisible* P_1, P_2 *are isogeneous if there exists an isogeny* $f : P_1 \to P_2$*. Note that if such f exists, then there exists a isogeny* $g : P_2 \to P_1$*.*

Definition 2.1.6. ***(Isoclinic p-divisible groups)*** *A p-divisible group P over a field* κ *of characteristic p is called isoclinic with slope* $\lambda \in [0, 1] \cap \mathbb{Q}$ *if P is isogeneous to another p-divisible* P_1 *such that there exists* $s, t \in \mathbb{N}$ *with*

$$\lambda = \frac{s}{t},$$
$$ker(Frob^t_{P_1}) = ker([p]^s_{P_1})$$

here $Frob_{P_1}$ *is the relative Frobenius of* P_1*.*

Theorem 2.1.7. ***(T. Zink)*** *A p-divisible group P over a field* κ*. Then there exists natural number m and a unique filtration* $0 = P_0 \subset P_1.. \subset P_m = P$ *such that*

- *Each* P_i *is a p-divisible subgroup of P.*
- P_{i+1}/P_i *is an isoclinic p-divisible group over* κ*.*
- *Let* s_i *be the slope of* P_i/P_{i-1}*, then*

$$1 \geq s_1 > ... > s_m \geq 0$$

such a filtration is called the slope filtration of P.

Proof. See [Zin01]. □

Definition 2.1.8. ***(Slopes of a p-divisible group)*** *Let P be a p-divisible group over a field k. Let $0 = P_0 \subset P_1 .. \subset P_m = P$ be the slope filtration of P and s_i be the slope of P_i/P_{i-1}. The slopes of P is the set $\{s_i : 1 \leq i \leq m\}$.*

2.2. Sustained p-Divisible Groups

Definition 2.2.1. *Let $\kappa \supset \mathbb{F}_p$ be a field, and let S be a κ scheme.*

(i) ***(Strongly sustained p-divisible groups)*** *A p-divisible group X/S is κ-strongly sustained if there exists a p-divisible group X_0/κ such that for every $n \in \mathbb{N}$ there exists a faithfully flat morphism $S_{1,n} \to S$ and an $S_{1,n}$-isomorphism*

$$X_0[p^n] \times_{Spec(\kappa)} S_{1,n} \xrightarrow{\sim} X[p^n] \times_S S_{1,n}$$

A p-divisible group $X \to S$ with the above property is said to be strongly κ-sustained over S model on X_0, and X_0 is said to be a κ-model of $X \to S$.

(ii) ***(Sustained p-divisible groups)*** *A p-divisible group X/S is κ-sustained if $\forall n \in \mathbb{N}$ there exists a faithfully flat morphism $S_{2,n} \to S_{\times\kappa}S$ and an $S_{2,n}$ isomorphism*

$$(X[p^n] \times_{S_0} S) \times_{S \times_\kappa S} S_{2,n} \xrightarrow{\sim} (S \times_{S_0} X[p^n]) \times_{S \times_\kappa S} S_{2,n}$$

Lemma 2.2.2. ***(Slope Filtration of Sustained p-divisible group)*** *Let κ be a field of characteristic p. Let X a p-divisible group over κ. Let S an κ scheme and $\mathcal{X}$ a κ-strongly sustained p-divisible group over S modeled on X. Let $0 = X_0 \subsetneq X_1 .. \subsetneq X_m = X$ be the slope filtration of X in the sense of 2.1.7. Then there exists a canonical slope filtration $0 = \mathcal{X}_0 \subsetneq \mathcal{X}_1 .. \subsetneq \mathcal{X}_m = \mathcal{X}$ in the sense that*

- *Each $\mathcal{X}_i$ is a κ-strongly sustained p-divisible subgroup of $\mathcal{X}$ modeled on X_i.*
- *The quotient $\mathcal{X}_{i+1}/\mathcal{X}_i$ is κ-strongly sustained modeled on X_{i+1}/X_i. In fact*

$$\mathcal{X}_{i+1}/\mathcal{X}_i \simeq X_{i+1}/X_i \times_\kappa S$$

Remark 2.2.3. *In fact, slope filtration exists when $\mathcal{X}$ is κ-sustained (instead of κ-strongly sustained). See [CO22] especially Chapter 6 for more details.*

2.3. Stable Homomorphism Schemes

Definition 2.3.1. ***(Stable Hom scheme of p-divisible groups)*** *Let $\kappa \supset \mathbb{F}_p$ be a field and let Y, Z be p-divisible groups over κ. We summarize the definition of $Hom^{st}(Y, Z)$, the stable hom scheme between Y, Z.*

(i) For every n we have a commutative affine group scheme

$$\mathcal{H}om(Y[p^n], Z[p^n])$$

of finite type over κ, which represents the functor

$$S \to Hom_S(Y[p^n]_S, Z[p^n]_S)$$

on the category of all κ-schemes S. In the rest of 2.3.1 we will shorten the notation $\mathcal{H}om(Y[p^n], Z[p^n])$ to $H_n(Y, Z)$.

(ii) There exist natural restriction map

$$r_{n,n+i} : H_{n+i} \to H_n$$

and corestriction map

$$\iota_{n+i,n} : H_n \to H_{n+i}$$

and these maps satisfy

(a) $\iota_{n+i+j,n+i} \circ \iota_{n+i,n} = \iota_{n+i+j,n}$ *and* $r_{n,n+i} \circ r_{n+i,n+i+j} = r_{n,n+i+j}$ *for all* $n, i, j \in \mathbb{N}$.

(b) $r_{n,n+i} \circ \iota_{n+i,n} = [p^i]_{H_n}$, $\iota_{n+i,n} \circ r_{n,n+i} = [p^i]_{H_{n+i}}$ *for all* $n, i \in \mathbb{N}$, *where* $[p^i]_{H_m}$ *denote the endomrophism "multiplication by* p^i*" on* H_m.

(c) $\iota_{n+j,n} \circ r_{n,n+j} = r_{n+j,n+i+j} \circ \iota_{n+i+j,n+i}$ *for all* $n, i, j \in \mathbb{N}$.

(iii) *For any* $m, n \in \mathbb{N}$, *denote by*

$$Im(r_{n,n+m} : H_{n+m}(Y,Z) \to H_n(Y,Z)$$

the image in $H_n(Y,Z)$ *of the homomorphism* $r_{n,n+m}$ *in the sense of fppf sheaves of abelian groups.*

(a) *There exists a natural number* n_0 *such that the image*

$$Im(r_{n,n+m} : H_{n+m}(Y,Z) \to H_n(Y,Z)$$

is a finite subgroup scheme of $H_n(Y,Z)$ *and*

$$Im(r_{n,n+m} : H_{n+m}(Y,Z) \to H_n(Y,Z) = Im(r_{n,n+n_0} : H_{n+n_0}(Y,Z) \to H_n(Y,Z)$$

for all $m \geq n_0$.

(b) *Let* $G_n(Y,Z) := Im(r_{n,n+m} : H_{n+m}(Y,Z) \to H_n(Y,Z)$ *for every* $n \in \mathbb{N}, m \geq n_0$ *where* n_0 *is defined in part (a). For all* $m \geq n$, *the co-restriction homomorphism* $\iota_{n,m} : H_m(Y,Z) \to H_n(Y,Z)$ *induces a monomorphism*

$$j_{n,m} : G_m(Y,Z) \hookrightarrow G_n(Y,Z)$$

Similarly the restriction homomorphism $r_{m,n} : H_n(Y,Z) \to H_m(Y,Z)$ *induces a* *epimorphism*

$$\pi_{m,n} : G_n(Y,Z) \twoheadrightarrow G_m(Y,Z)$$

for all $n \geq m$.

(c) *For all* $n, i \in \mathbb{N}$, *the sequence*

$$0 \to G_i(Y,Z) \stackrel{j_{n+i,i}}{\longrightarrow} G_{n+i}(Y,Z) \stackrel{\pi_{n,n+i}}{\longrightarrow} G_n(Y,Z) \to 0$$

is short exact, and the composition $\text{æ}_{n+i,n} \circ \pi_{n,n+i}$ *is equal to* $[p^i]_{G_n(Y,Z)}$. *In other words the triple*

$$(G_n(Y,Z), j_{n+i,n}, \pi_{n+i,n})_{n,i\in\mathbb{N}} =: \mathcal{H}om'_{div}(Y,Z)$$

is a p-divisible group over κ, *and* $G_n(Y,Z)$ *is the kernel of the endomorphism* $[p^n]$ *of* $\mathcal{H}om'_{div}(Y,Z)$.

Notations 2.3.2. *We will use* $Hom^{st}(Y,Z)$ *to denote the p-divisible group* $\mathcal{H}om_{div}(Y,Z)$.

We collect some properties of $Hom^{st}(Y,Z)$.

Proposition 2.3.3. *Let* $\kappa \supset \mathbb{F}_p$ *be the base field,* Y, Z *be p-divisible groups over* κ. *We further assume that both* Y, Z *are isoclinic with slope* s_Y, s_Z *and of dimension* d_Y, d_Z. *Then*

1. *If* $s_Y > s_Z$, *then* $Hom^{st}(Y,Z) = 0$.

2. *if* $s_Y \leq s_Z$, *then* $Hom^{st}(Y,Z)$ *is isoclinic of slope* $s_Z - s_Y$.

3. *If* $s_Y = s_Z$, *then* $Hom^{st}(Y,Z)$ *is an etale p-divisible group.*

Definition 2.3.4. ***(Stable isomorphism schemes of p-divisible groups)*** *Let* S *be a scheme over* $\kappa \supset \mathbb{F}_p$. *Let* Y, Z *be* κ*-sustained p-divisible groups over* S. *We summarize the definition of* $Isom^{st}(Y,Z)$, *the stable isomorphism scheme between* Y, Z. *This definition is parallel to 2.3.1.*

(i) For every n we have a commutative affine group scheme

$$\mathcal{I}som(Y[p^n], Z[p^n])$$

of finite type over κ, which represents the functor

$$S \to Isom_S(Y[p^n]_S, Z[p^n]_S)$$

on the category of all κ-schemes S. In the rest of 2.3.1 we will shorten the notation $\mathcal{I}som(Y[p^n], Z[p^n])$ to $I_n(Y, Z)$.

(ii) There exist natural restriction map

$$r_{n,n+i} : I_{n+i} \to I_n$$

(iii) For any $m, n \in \mathbb{N}$, denote by

$$Im(r_{n,n+m} : I_{n+m}(Y, Z) \to I_n(Y, Z)$$

the image in $H_n(Y, Z)$ of the homomorphism $r_{n,n+m}$ in the sense of fppf sheaves of abelian groups.

(a) There exists a natural number n_0 such that the image

$$Im(r_{n,n+m} : I_{n+m}(Y, Z) \to I_n(Y, Z)$$

is a finite subgroup scheme of $I_n(Y, Z)$ and

$$Im(r_{n,n+m} : I_{n+m}(Y, Z) \to I_n(Y, Z) = Im(r_{n,n+n_0} : I_{n+n_0}(Y, Z) \to I_n(Y, Z)$$

for all $m \geq n_0$.

(b) Let $K_n(Y,Z) := Im(r_{n,n+m} : I_{n+m}(Y,Z) \to I_n(Y,Z)$ for every $n \in \mathbb{N}$ and $m \geq n_0$. The restriction homomorphism $r_{m,n} : I_n(Y,Z) \to I_m(Y,Z)$ induces a epimorphism

$$\pi_{m,n} : K_n(Y,Z) \twoheadrightarrow K_m(Y,Z)$$

for all $n \geq m$.

(iv) The stable isomorphism scheme of Y, Z, denoted by $Isom^{st}(Y,Z)$ is the projective system

$$Isom^{st}(Y,Z) := \varprojlim_{n} K_n(Y,Z)$$

where the connecting morphisms are $r_{m,n}$. We will also use the notation $Isom^{st}(Y,Z)_n$ to denote $K_n(Y,Z)$.

Notations 2.3.5. *Let X be a p-divisible group over $\kappa \supset \mathbb{F}_p$. Then the stable automorphism scheme of X, that is $Isom^{st}(X,X)$, will be denoted by $Aut^{st}(X)$.*

2.4. Sustained Deformation Spaces

We have the following:

Lemma 2.4.1. ***(Definition and Smoothness of sustained deformation space)*** *Let X be a p-divisible group over $\kappa \supset \mathbb{F}_p$. The function $Def_{sus}(X) : Art_k \to Sets$, sending each Artinian local augmented κ algebra (S,e) to the set*

$$\{(\mathcal{X}_S, \varphi) : \mathcal{X}_S \text{ strongly } \kappa\text{-sustained }, \mathcal{X}_S \times_e Spec(\kappa) \xrightarrow{\varphi} X \text{ an isomorphism}\}/\sim$$

is representable by a smooth formal scheme. We will denote this smooth formal scheme again by $Def_{sus}(X)$.

Proof. For proof see [CO22] Chapter 6. □

Lemma 2.4.2. *(Relation between Def_{sus} and Hom^{st})*

1. *When $X = Y \times Z$ with Y, Z isoclinic, then there is a natural isomorphism*

$$\iota : Hom^{st}(Y, Z) \xrightarrow{\sim} Def_{sus}(X)$$

2. *When there is a exact sequence*

$$0 \to Y \to X \to Z \to 0$$

with Y, Z isoclinic, then $Def_{sus}(X)$ has a natural $Hom^{st}(Y, Z)$ torsor structure.

Proposition 2.4.3. *(**'Kummer theory' construction of stable Hom to sustained deformation**) Let X, Y be isoclinic p-divisible groups over a field κ of characteristic p with slopes s_X, s_Y respectively and that $s_X < s_Y$. Let f be a functorial point of $Hom^{st}(X, Y)$. Let $X \times^{(1,f)} Y$ be the sustained deformation of $X \times Y$ corresponding to $\iota(f) \in Def_{sus}(X \times Y)$. Then:*

(a) *Let $f \in Hom^{st}(X[p^n], Y[p^n])$. Consider the Kummer sequence*

$$0 \to X[p^n] \to X \overset{[p^n]_X}{\to} X \to 0$$

and consider the pushout diagram with respect to the homomorphism

$$f \in Hom^{st}(X[p^n], Y[p^n])$$

Let $X \times^{(1,f)} Y$ be the p-divisible group that fits into this push out diagram, that is

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X[p^n] & \longrightarrow & X & \xrightarrow{[p^n]_X} & X & \longrightarrow & 0 \\ & & \downarrow f & & & & \downarrow = & & \\ 0 & \longrightarrow & Y[p^n] & \longrightarrow & X \times^{(1,f)} Y & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

Then

$$X \times^{(1,f)} Y \simeq X \times Y/\Gamma_{-f}$$

where Γ_{-f} is the graph of $-f$. This is the coproduct of X, Y with respect to $(1, f) : X[p^n] \to X \times Y$ in the category of group schemes, hence the notation. Note that this is well defined for $f \in Hom^{st}(X, Y) = \varinjlim Hom^{st}(X_n, Y_n)$, where $X_n = X[p^n], Y_n = Y[p^n]$. Moreover, if $f \in Hom^{st}(X_n, Y_n)$ for a given n. Then

$$(X \times^{(1,f)} Y)[p^m] = \frac{ker(\phi_{m+n} : X_{n+m} \oplus Y_{n+m} \to Y_n)}{(x, -f(x) : x \in X_n)} \tag{2.1}$$

where $\phi_{m+n}(x, y) = [p^m] \cdot f(x) + [p^m] \cdot y$.

(b) Given $\tilde{f}_{n+m} \in Hom^{st}(X_{m+n}, Y_{m+n})$ a lifting f, that is

$$[p^m]\tilde{f}_{n+m} = f_n$$

we can define a morphism $\Psi^m_{\tilde{f}}$ by the following diagram:

$$\begin{array}{ccc} X[p^{m+n}] \times Y[p^m] & \xrightarrow{(x_{m+n}, y_m) \to (x_{m+n}, -f_{m+n}(x_{m+n}) + y_m)} & ker(\phi_{m+n} : X_{n+m} \oplus Y_{n+m} \to Y_n) \\ \downarrow {\scriptstyle [p^n]_X \times id_Y} & & \downarrow {\scriptstyle /(x, -f(x)) : x \in X_n} \\ X[p^m] \times Y[p^m] & \xrightarrow{\Psi^m_{\tilde{f}}} & (X \times^{(1,f)} Y)[p^m] \end{array}$$

In fact, this morphism $\Psi^m_{\tilde{f}}$ is an isomorphism of truncated p-divisible groups.

Proof. Part (a) follows from the definition of $X \times^{(1,f)} Y$. Part (b) is an easy exercise. □

Definition 2.4.4. *Let G be a group value functor on the big fpqc site over a $Spec(\kappa)$ where $\kappa \supset \mathbb{F}_p$ a field. We define the deformation functor of G-torsors, denoted by $Def_{G\text{-}torsor}$, to be the functor that sends every Artinian local algebra (R, m) over κ to the set*

$$\Big\{(\mathcal{G}, \varphi) : \mathcal{G} \textit{ is a } G\textit{-torsor over } R \textit{ and } \varphi : \mathcal{G} \times_R R/m \overset{\simeq}{\mapsto} G \times_\kappa R/m\Big\} / \sim$$

Theorem 2.4.5. ***(Sustained deformation space and deformation space of Aut^{st}-torsors are isomorphic)*** *Let X_0 be a p-divisible group over a field $\kappa \supset \mathbb{F}_p$. Let*

$$\Phi : Def_{sus}(X_0) \mapsto Def_{Aut^{st}(X)\text{-}torsor}$$

be the morphism that sends every functorial point $\tilde{X}$ over an artinian local algebra R to $Isom^{st}(X_0 \times_\kappa R, \tilde{X})$. Note that there is a natural left $Aut^{st}(X_0)$ torsor structure on $Isom^{st}(X_0 \times_\kappa R, \tilde{X})$ given by precomposing with an element in $Aut^{st}(X_0)$. Then

(a). Φ is an isomorphism of functors.

(b). The inverse of Φ can be described explicitly as: for every $Aut^{st}(X_0)$-torsor $\mathcal{T}$, $\Phi^{-1}(\mathcal{T})$ is given by the contracted product with X_0, that is

$$\Phi^{-1}(\mathcal{T}) = X_0 \times^{Aut^{st}(X_0)} \mathcal{T}$$

Proof. See [CO22]. □

CHAPTER 3

Biextension and 3-Slopes Case

In this chapter, we recall the definition of a biextension, then we show that $Def_{sus}(X)$ is a biextension when $X = \prod_{i=1}^{3} X_i$ with X_i isoclinic with mutually different slopes, see 3.2.2. Finally, we construct a 'trivialization' of such $Def_{sus}(X)$ in 3.4.3. Note that Mumford constructed similar 'trivialization' for general biextensions of p-divisible groups in [Mum68], but our method utilizes the moduli interpretation and allows us to generalize to other cases.

3.1. Biextension Basic

We use the following definition of bi-extensions of abelian groups as given in [Mum68].

Definition 3.1.1. *(bi-extensions of abelian groups) Let A, B, C be 3 abelian groups. A bi-extension of $B \times C$ by A will denote a set F on which A acts freely, together with a map*

$$F \xrightarrow{\pi} B \times C$$

making $B \times C$ into the quotient F/A, together with 2 laws of composition:

$$+_1 : F \times_B F \to F$$

$$+_2 : F \times_C F \to F$$

There are subject to the requirements:

(a) for all $b \in B$, $F'_b := \pi^{-1}(b \times C)$ is an abelian group under $+_1$, π is a surjective homomorphism of F'_b onto C, and via the action of A on F'_b, A is isomorphic to the kernel of π;

(b) for all $c \in C, F^2_c := \pi^{-1}(B \times c)$ is an abelian group under $+_2$, π is a surjective homomorphism of F^2_c onto B, and via the action of A on F^2_c, A is isomorphic to the kernel of π.

(c) given $x, y, u, v \in F$ such that

$$\pi(x) = (b_1, c_1)$$
$$\pi(y) = (b_1, c_2)$$
$$\pi(u) = (b_2, c_1)$$
$$\pi(v) = (b_2, c_2)$$

then

$$(x +_1 y) +_2 (u +_1 v) = (x +_2 u) +_1 (y +_2 v)$$

Definition 3.1.2. ***(bi-extensions of group functors)*** *If F, G, H are three group functors from the category of R-algebras to the category of abelian groups, a biextension of $G \times H$ by F is a fourth functor K such that for every K-algebra S, $K(S)$ is a biextension of $G(S) \times H(S)$ by $F(S)$ and for every R homomorphism $S_1 \to S_2$, the map $K(S_1) \to K(S_2)$ is a homomorphism of bi-extensions (in the obvious sense). In particular, if F, G, H are formal groups, this gives us a biextension of formal groups.*

Example 3.1.3. Let A be an abelian variety over a field k. Let $\hat{A}$ be the dual of A. Let $\mathcal{P}$ be the Poincare line bundle over $A \times \hat{A}$. Let $\mathcal{T}$ be the total space of $\mathcal{P}$ and let $\mathcal{Z}$ be the zero section. Then there is a biextension structure on $\mathcal{T} - \mathcal{Z}$. This is a biextension of $A \times \hat{A}$ by G_m. See [MRM74] for more details.

3.2. Sustained Deformation Spaces as Biextensions

Definition 3.2.1. *Let $X = \prod_{i=1}^{3} X_i$ with X_i isoclinic of slopes s_i and assume $s_1 > s_2 > s_3$. Let $E = Def_{sus}(X)$. We will define a free H_{13} action on E, that is a morphism*

$$*_E : H_{13} \times E \to E$$

which satisfies the axioms of being a H_{13} action, as follows: Let $e \in E(R)$ and let $\mathcal{X}$ be the pullback of the universal sustained p-divisible group by $e : Spf(R) \to E$, that is

$\mathcal{X}$ is a p-divisible group over R that is κ-strongly sustained modeled on X. Let $f_{13} \in Hom^{st}(X_1[p^N], X_3[p^N])$ for some $N \in \mathbb{N}(R)$. Let

$$0 \subset \mathcal{X}_3 \subset \mathcal{X}_2 \subset \mathcal{X}_1 = \mathcal{X}$$

be the slope filtration of $\mathcal{X}$ where $\mathcal{X}_i$ are p-divisible groups over R. That is $\mathcal{X}$ fits in an exact sequence

$$0 \to \mathcal{X}_2 \to \mathcal{X} \to \mathcal{X}/\mathcal{X}_2 \to 0$$

Then there exists $M \in \mathbb{N}$ with $M \geq N$, $\phi \in Hom^{st}((\mathcal{X}/\mathcal{X}_2)[p^M], \mathcal{X}_2[p^M])(R)$ such that

$$\mathcal{X} = \mathcal{X}/\mathcal{X}_2 \times^{(1,\phi)} \mathcal{X}_2$$

As $f_{13} \in Hom^{st}(X_1[p^N], X_3[p^N])(R) \subset Hom^{st}(X_1[p^M], X_3[p^M])(R)$, and that

$$\mathcal{X}/\mathcal{X}_2 \simeq X_1 \times_\kappa R \text{ by a natural isomorphism}$$
$$0 \to X_3 \times_\kappa R \xrightarrow{\iota} \mathcal{X}_2 \to \mathcal{X}_2/\mathcal{X}_3 \to 0$$

Let $\iota \circ f_{13}$ be the composition

$$\iota \circ f_{13} : \mathcal{X}/\mathcal{X}_2 \simeq X_1 \times_\kappa R[p^M] \xrightarrow{f_{13}} X_3 \times_\kappa R[p^M] \xrightarrow{\iota} \mathcal{X}_2$$

Finally, we define the action of f_{13} on e by

$$*_E(f_{13}, e) = \mathcal{X}/\mathcal{X}_2 \times^{(1,\phi+\iota\circ f_{13})} \mathcal{X}_2$$

It is easy to verify that this is a group action, and it is clear that

$$*_E(f_{13}, e) = e \iff f_{13} = 0$$

hence the action is free.

Lemma 3.2.2. ***(Biextension Structure on*** $Def_{sus}(X)$***)*** *Let* $X = \prod_{i=1}^{3} X_i$ *with* X_i *isoclinic of slopes* s_i *over a field* κ *of characteristic* p *and assume* $s_1 > s_2 > s_3$. *Let* $E = Def_{sus}(X)$.

(a). We define a projection map $\pi : E \to Def_{sus}(X_1 \times X_2) \times Def_{sus}(X_2 \times X_3)$ *as follows: let* $\mathcal{X} \in E$ *be a functorial point. Let* $0 \subset \mathcal{X}_3 \subset \mathcal{X}_2 \subset \mathcal{X}_1 = \mathcal{X}$ *be the slope filtration of* $\mathcal{X}$. *We define* π *by sending* $\mathcal{X}$ *to*

$$\mathcal{X}/\mathcal{X}_3 \times \mathcal{X}_2 \in B = Def_{sus}(X_1 \times X_2) \times Def_{sus}(X_2 \times X_3)$$

Then π *is a faithful morphism.*

(b). $\pi : E \to B$ *is invariant under the* H_{14} *action, that is for* $h_{13} \in H_{13}(R), e \in E(R)$,

$$\pi(e) = \pi(*_E(h_{13}, e))$$

Moreover, let $\tilde{\pi} : E/H_{13} \to B = H_{12} \times H_{23}$ *be the morphism induced by* π, *then* $\tilde{\pi}$ *is an isomorphism.*

(c). E *is a biextension of* $Def_{sus}(X_1 \times X_2) \times Def_{sus}(X_2 \times X_3)$ *by* $Def_{sus}(X_1 \times X_3)$.

Proof. For (a), it suffices to show that for R/κ an Artinian local ring, $f = (f_{12}^n, f_{23}^n) \in (H_{12}[p^n] \times H_{23}[p^n])(R)$ there exists an faithfully flat cover R' over R, and $e \in E(R)$ such that

$$\pi(e) = f_{R'}$$

We construct e, R' as follows: let $f_{23}^{2n} \in H_{23}[p^{2n}](R')$ for some Artinian local ring R' faithfully flat over R such that

$$[p^n]_{H_{23}}(f_{23}^{2n}) = (f_{23}^n)_{R'}$$

Let

$$\Psi_{f_{23}^{2n}}^n : X_2[p^n] \times X_3[p^n] \to (X_2 \times^{(1, f_{23}^n)} X_3)[p^n]$$

the isomorphism over R' constructed using f_{23}^{2n} by the procedure in 2.4.3. Let F be the composition

$$F_n : X_1[p^n] \stackrel{((f_{12}^n)_{R'},0)}{\longrightarrow} X_2[p^n] \times X_3[p^n] \stackrel{\Psi^n_{f_{23}^{2n}}}{\longrightarrow} (X_2 \times^{(1,f_{23}^n)} X_3)[p^n]$$

Let $e \in E(R')$ be the R' point that correspond to the p-divisible

$$X_1 \times^{(1,F_n)} (X_2 \times^{(1,f_{23}^n)} X_3)$$

then

$$\pi(e) = f_{R'}$$

We have proved (a).

For (b), to show that $E/H_{13} \simeq H_{12} \times H_{23}$, it suffices to show that for $n \in \mathbb{N}$ and $f = (f_{12}, f_{23}) \in (H_{12}[p^n] \times H_{23}[p^n])(R)$, the set teoretic preimage

$$\pi^{-1}(f) \subset E(R)$$

is a $H_{13}(R)$ torsor. Given $e, e' \in \pi^{-1}(f) \subset E(R)$. Let $\mathcal{X}, \mathcal{X}'$ be the sustained p-divisible groups corresponding to e, e' respectively. Let $0 \subset \mathcal{X}_3 \subset \mathcal{X}_2 \subset \mathcal{X}_1 = \mathcal{X}$ and $0 \subset \mathcal{X}'_3 \subset \mathcal{X}'_2 \subset \mathcal{X}'_1 = \mathcal{X}'$ be the slope filtrations of $\mathcal{X}, \mathcal{X}'$ respectively. As $\pi(e) = \pi(e') = f$,

$$\mathcal{X}_2 \simeq \mathcal{X}'_2$$

Let $M \in \mathbb{N}, \phi, \phi' \in Hom^{st}((\mathcal{X}/\mathcal{X}_2)[p^M], \mathcal{X}_2[p^M])(R)$ such that

$$\mathcal{X} = \mathcal{X}/\mathcal{X}_2 \times^{(1,\phi)} \mathcal{X}_2,$$

$$\mathcal{X}' = \mathcal{X}/\mathcal{X}_2 \times^{(1,\phi)} \mathcal{X}_2$$

As $\pi(e) = \pi(e')$, the morphism $\phi - \phi' : \mathcal{X}/\mathcal{X}_2[p^M] \to \mathcal{X}_2[p^M]$ factors through $\mathcal{X}_3 \hookrightarrow \mathcal{X}$, i.e.

$$\phi - \phi' \in Hom^{st}(\mathcal{X}/\mathcal{X}_2[p^M], \mathcal{X}_3[p^M])(R),$$
$$*_E(\phi - \phi', e') = e$$

We have proved that $\pi^{-1}(f)$ is a $H_{13}(R)$ torsor.

For (c), fix R/κ an Artinian local algebra. Let $\mathcal{X}$ be a κ-strongly sustained p-divisible group over R modeled on X. Let $0 = \mathcal{X}_0 \subset \mathcal{X}_1 \subset \mathcal{X}_2 \subset \mathcal{X}_3 = \mathcal{X}$ be the slope filtration of $\mathcal{X}$. The natural projection

$$\pi_{12} : Def_{sus}(X) \to Def_{sus}(X_1 \times X_2)$$

can be described as sending $\mathcal{X} \in Def_{sus}(X)(R)$ to $\mathcal{X}_2 \in Def_{sus}(X_1 \times X_2)(R)$. Then we have a natural extension of p-divisible groups

$$0 \longrightarrow \mathcal{X}_2 \longrightarrow \mathcal{X} \longrightarrow \mathcal{X}/\mathcal{X}_2 \longrightarrow 0$$

that is

$$\mathcal{X} \in \mathrm{Ext}^1(\mathcal{X}/\mathcal{X}_2, \mathcal{X}_2)(R)$$

thus the Baer sum structure on Ext group induces an relative group structure on $Def_{sus}(X)$ with respect to the projection map π_{12}.

Similarly, we have another relative group structure induced by the Baer sum on

$$\mathrm{Ext}^1(\mathcal{X}_1, \mathcal{X}/\mathcal{X}_1)$$

with respect to the projection map

$$\pi_{23} : Def_{sus}(X) \to Def_{sus}(X_2 \times X_3)$$

Now it is an easy exercise to check that these two relative group structures satisfy the axioms as defined in 3.1.1. $\square$

3.3. Strongly Non-trivial Action

We collect the definition and some basic properties of a strongly non-trivial action, see [CO22] Chapter 7 for proofs and more details.

Definition 3.3.1. *Let X be a p-divisible group over a field $\kappa \supset \mathbb{F}_p$. Let k be an algebraic closure of κ and let $X_k = X \times_\kappa k$. Let G be a finite dimensional p-aidc Lie group. Let $W(k)$ be the Witt ring of k and $D_*(X_k)$ the covariant Dieudonne module of X_k. A continuous homomorphism $\rho : G \to Aut(X) = End(X)^\times$ of G on X is said to be strongly non-trivial if the associated $W(k) \otimes \mathbb{Q}$-linear representation*

$$d\rho : Lie(G) \to End_{W(k)\otimes\mathbb{Q}}(D_*(X_k)_\mathbb{Q})$$

of the Lie algebra of G on $D_*(X_k)_\mathbb{Q}$ does not contain the trivial representation of $Lie(G)$ as a subquotient.

Remark 3.3.2. *In the notation of 3.3.1, a continuous homomorphism $\rho : G \to Aut(X)$ is strongly non-trivial if and only if there exists a finite number of finite sequences $(w_{i,1}, ..., w_{i,n_i})$ in $Lie(G)$, for $i = 1, ..., r$ and $n_i \geq 1$ for all i, such that*

$$\sum_{i=1}^{r} d\rho(w_{i,1}) \circ d\rho(w_{i,n_i}) \in End^0(X)^\times$$

Definition 3.3.3. *Let $X = \prod_{i=1}^{3} X_i$ with X_i isoclinic of slope s_i, and assume that $s_1 > s_2 > s_3$. Let $H_{ij} = Hom^{st}(X_i, X_j), \forall 1 \leq i < j \leq 3$. Let $E = Def_{sus}(X)$, which is a biextension of $H_{12} \times H_{23}$ by H_{13}. Let $G \subset Aut_{bi-ext}(E)$ be a closed p-adic subgroup. We the action of G on E is strongly non-trivial if the induced action on each H_{ij} is strongly non-trivial, in the sense of 3.3.1, for all $1 \leq i < j \leq 3$.*

3.4. Mumford's Trivialization

Definition 3.4.1. *Let $X = X_1 \times X_2 \times X_3$ a p-divisible group over a field κ of characteristic p with X_i isoclinic. Let $s_i = Slope(X_i)$ and we assume that $s_1 > s_2 > s_3$. Let $E = Def_{sus}(X)$.*

Then E has a natural structure as a biextension of $Hom^{st}(X_1, X_2) \times Hom^{st}(X_2, X_3)$ by $Hom^{st}(X_1, X_3)$, as described in 3.2.2. Denote by

$$H_{ij} := Hom^{st}(X_i, X_j), \forall 1 \leq i < j \leq 3$$

see 2.3.1 and 2.3.2 for the definition of Hom^{st}. Let $\pi : E \to H_{12} \times H_{23}$ the natural projection. Let $E_n = \pi^{-1}(H_{12}[p^n] \times H_{23}[p^n])$. We will define a morphism

$$\psi_n : H_{12}[p^n] \times H_{23}[p^{2n}] \times H_{13} \to E_n$$

as follows:

Fix R/κ an Artinian local ring. Let $f = (f_{12}^n, f_{23}^{2n}, f_{13}^n) \in (H_{12}[p^n] \times H_{23}[p^{2n}] \times H_{13})(R)$. We will write down an element of $E(R)$ using f in the following steps:

(a) given $f_{23}^{2n} \in H_{23}[p^{2n}](R)$, denote $f_{23}^n := [p^n] f_{23}^{2n}$

(b) By 2.4.3.(b), we can construct from f_{23}^{2n} an isomorphism of truncated p-divisible groups

$$\Psi^n_{f_{23}^{2n}} : X_2[p^n] \times X_3[p^n] \to X_2 \times^{f_{23}^n} X_3$$

(c) Let $F = (\Psi^n_{f_{23}^{2n}}) \circ (f_{12}^n, f_{13}^n)$ be the morphism from $X_1[p^n]$ to $(X_2 \times^f X_3)[p^n]$ given by the composition

$$F : X_1 \overset{(f_{12}^n, f_{13}^n)}{\longrightarrow} X_2[p^n] \times X_3[p^n] \overset{\Psi^n_{f_{23}^{2n}}}{\longrightarrow} (X_2 \times^f X_3)[p^n] \tag{3.1}$$

(d) Given F, we can define a point in $E(R)$, denote it by X_f, by

$$X_f := X_1 \times^{(1,F)} (X_2 \times^{(1,f_{23}^n)} X_3)$$

(e) We can now define a morphism

$$\psi_n : H_{12}[p^n] \times H_{23}[p^{2n}] \times H_{13}[p^n] \to E_n$$

by sending f to X_f.

(f) It is easy to check that ψ_n is $H_{13}[p^n]$ equivarient, in the sense that if ${f_{13}^n}' \in H_{13}[p^n]$ another functorial point, then

$$\psi((f_{12}^n, f_{23}^{2n}, f_{13}^n + {f_{13}^n}')) = *_E({f_{13}^n}', \psi_n(f))$$

where $*_E$ corresponds to the H_{13} torsor structure on E, see 3.2.1.

(g) Now we extend the source of ψ_n from $H_{12}[p^n] \times H_{23}[p^{2n}] \times H_{13}[p^n]$ to $H_{12}[p^n] \times H_{23}[p^{2n}] \times H_{13}$ by

$$\psi_n((f_{12}^n, f_{23}^{2n}, f_{13})) = *_E(f_{13}, \psi_n((f_{12}^n, f_{23}^{2n}, 0))$$

for $f_{13} \in H_{13}$ a functorial point.

Remark 3.4.2. *We will refer to ψ_n as Mumford's trivialization, as Mumford constructed similar morphisms for biextensions of p-divisible groups in [Mum68].*

Theorem 3.4.3. *Notation as in 3.4.1. Let $f = (f_{12}^n, f_{23}^{2n}, f_{13})$ and $f' = ({f_{12}^n}', {f_{23}^{2n}}', f_{13}')$ be two functorial points of $H_{12}[p^n] \times H_{23}[p^{2n}] \times H_{13}$. Let $E_n \subset E$ and $\psi_n : H_{12}[p^n] \times H_{23}[p^{2n}] \times H_{13} \to E_n$ as defined in 3.4.1. For $n \in \mathbb{N}$, Let*

$$\langle , \rangle_n : H_{12}[p^n] \times H_{23}[p^n] \to H_{13}[p^n]$$

the bihomomorphism given by

$$\langle f_{12}^n, f_{23}^n \rangle_n = f_{23}^n \circ f_{12}^n \in H_{13}[p^n] = Hom^{st}(X_1[p^n], X_3[p^n])$$

for all $f_{12}^n \in H_{12}[p^n] = Hom^{st}(X_1, X_2)[p^n], f_{23}^n \in H_{23}[p^n] = Hom^{st}(X_2, X_3)[p^n]$ *both functorial points. Then:*

*(a). **(Gluing Data)*** $\psi_n(f) = \psi_n(f')$ *if and only if*

$$f_{12}^n = f_{12}^{n\,\prime}, [p^n]f_{23}^{2n} = [p^n]f_{23}^{2n\prime}$$

$$f_{13} - f'_{13} = \langle f_{12}^n, f_{23}^{2n} - f_{23}^{2n\prime} \rangle_n$$

(b). The morphism ψ_n *is faithfully flat.*

Proof. For (a), as ψ_n respect the H_{13} torsor structure, see 3.2.1(f)(g), it suffices to prove (a) under the assumption that $f_{13}, f'_{13} \in H_{13}[p^n]$. Let F, F' as in 3.2.1(d) that corresponds to f, f' respectively, that is

$$\psi_n(f) = X_1 \times^{(1,F)} (X_2 \times^{(1,f_{23}^n)} X_3)$$

$$\psi_n(f') = X_1 \times^{(1,F')} (X_2 \times^{(1,f_{23}^n)} X_3)$$

then $\psi_n(f) = \psi_n(f') \iff F = F'$. By 3.2.1(c), we have the following diagram that defines F:

$$\begin{array}{cccc}
 & X_2[p^{2n}] \times X_3[p^n] & \xrightarrow{(x_2^{2n}, x_3^n) \to (x_2^{2n}, f_{23}^{2n}(x_2^{2n}) + x_3^n)} & ker(X_2[p^{2n}] \times X_3[p^{2n}] \xrightarrow{\Phi_n} X_3[p^n]) \\
 & \Big\downarrow {\scriptstyle [p^n]_{H_{12}} \times id_{H_{23}}} & & \Big\downarrow {\scriptstyle \pi_{f_{23}^n}} \\
X_1 \xrightarrow{(f_{12}^n, f_{23}^n)} & X_2[p^n] \times X_3[p^n] & \xrightarrow{\Psi^n_{f_{23}^{2n}}} & (X_2 \times^{(1,f_{23}^n)} X_3)[p^n]
\end{array}$$

$$X_1 \xrightarrow{F} (X_2 \times^{(1,f_{23}^n)} X_3)[p^n]$$

where

- $\Phi_n : X_2[p^{2n}] \times X_3[p^{2n}] \to X_3[p^n]$ is defined as sending (x_2^{2n}, x_3^{2n}) to $f_{23}^n([p^n]x_2^{2n}) - [p^n]x_3^{2n}$, for $x_2^{2n} \in X_2[p^{2n}], x_3^{2n} \in X_3[p^{2n}]$ both functorial points.

- $\pi_{f_{23}^n} : ker(X_2[p^{2n}] \times X_3[p^{2n}] \xrightarrow{\Phi_n} X_3[p^n]) \to (X_2 \times^{(1,f_{23}^n)} X_3)[p^n]$ the natural projection map, see 2.4.3.

- x_2^{2n} is a p^n root of x_2^n.

We can similarly write down a diagram for F'. Now an easy diagram chasing shows that

$$F = F' \iff f_{13}^n - f_{13}^{n\prime} = \langle f_{12}^n, f_{23}^{2n} - f_{23}^{2n\prime} \rangle_n$$

We have proved (a).

For (b), first we note that the morphism ψ_n is H_{13} equivariant, by 3.2.1(f)(g). By ignoring the H_{13} component, ψ_n induces a morphism

$$\overline{\psi_n} : H_{12}[p^n] \times H_{23}[p^{2n}] \to E_n/H_{13} \simeq H_{12}[p^n] \times H_{23}[p^n]$$

and it is easy to check that $\overline{\psi_n} = id_{H_{12}} \times [p^n]_{H_{23}}$, so $\overline{\psi_n}$ is faithfully flat. Hence ψ_n is also faithfully flat. □

Corollary 3.4.4. *Notation as in 3.4.3. Then for each $n \in \mathbb{N}$, the morphism*

$$\psi_{n,homo} : H_{12}[p^{2n}] \times H_{23}[p^{2n}] \times H_{13} \xrightarrow{([p^n]_{H_{12}}, id, id)} H_{12}[p^n] \times H_{23}[p^{2n}] \times H_{13} \xrightarrow{\psi_n} F_n$$

is faithfully flat, and for $f = (f_{12}^{2n}, f_{23}^{2n}, f_{13}), f' = (f_{12}^{2n\prime}, f_{23}^{2n\prime}, f_{13}') \in H_{12}[p^{2n}] \times H_{23}[p^{2n}] \times H_{13}$,

$$\psi_{n,homo}(f) = \psi_{n,homo}(f') \iff$$
$$f_{12}^{2n} - f_{12}^{2n\prime} \in H_{12}[p^n],\ \ f_{23}^{2n} - f_{23}^{2n\prime} \in H_{23}[p^n] \text{ and } f_{13} - f_{13}' = \langle [p^n] f_{12}^{2n}, f_{23}^{2n} - f_{23}^{2n\prime} \rangle_n$$

Proof. Obvious. □

CHAPTER 4

Tate-linear Nilpotent Groups of type A and 4-Slopes Case

In this chapter we first prove a similar result to 3.4.3 for the case when $X = \prod_{i=1}^{4} X_i$ with X_i isoclinic with mutually different slopes. Then we define the concept of Tate-linear nilpotent groups of type A.

We first set up some notation used throughout this section.

Notations 4.0.1. ***(Set up of Sustained Deformation Space 4 Slopes Case)***

- *Let $X = \prod_{i=1}^{4} X_i$ be a p-divisible group with 4 slopes over a base field κ of characteristic p, here each X_i is isoclinic with slope s_i and we assume that $s_1 > s_2 > s_3 > s_4$.*
- *Let $E = Def_{sus}(X)$, which is a smooth formal scheme over κ by 4.4.6.*
- *Let $B = Def_{sus}(X_1 \times X_2 \times X_3) \times_{Def_{sus}(X_2 \times X_3)} Def_{sus}(X_2 \times X_3 \times X_4)$. Note that both $Def_{sus}(X_1 \times X_2 \times X_3)$ and $Def_{sus}(X_2 \times X_3 \times X_4)$ are biextensions.*
- *Let $H_{14} = Def_{sus}(X_1 \times X_4)$ a p-divisible group.*
- *We will show that E has a natural H_{14} torsor structure and $E/H_{14} \simeq B$ in 4.1.1. Let $\pi : E \to B$ the projection map as defined in 4.1.3.*

We will define for each $n \in \mathbb{N}$ a subscheme $E_n \subset E$ and $B_n \subset B$ that fit into the following diagram:

$$\begin{array}{ccc} E_n & \xrightarrow{\subset} & E \\ \downarrow \pi & & \downarrow \pi \\ B_n & \xrightarrow{\subset} & B \end{array}$$

To define E_n and B_n, we need the following notations/facts:

(a) Let $H_{ij} := Hom^{st}(X_i, X_j) = \varinjlim_{n} Hom^{st}(X_i[p^n], X_j[p^n]) \simeq Def_{sus}(X_i \times X_j), \forall i < j$.

We denote by $H^n_{ij} := H_{ij}[p^n]$*. For all* $1 \leq i < k < j \leq K$ *and* $n \in \mathbb{N}$*, let*

$$\langle , \rangle_{ikj,n} : H_{ik}[p^n] \times H_{kj}[p^n] \to H_{ij}[p^n]$$

the bilinear pairing given by composition.

(b) Note that $Def_{sus}(X_1 \times X_2 \times X_3)$ *is a biextension, same is* $Def_{sus}(X_2 \times X_3 \times X_4)$*.*

(c) For $n \in \mathbb{N}$*, let* $\psi_{13,n} : H^n_{12} \times H^{2n}_{23} \times H_{13} \to Def_{sus}(X_1 \times X_2 \times X_3)_n$

be the trivializations defined in 3.4.3. Denote by $B_{13} := Def_{sus}(X_1 \times X_2 \times X_3)$*,*

$B_{13,n} := img(\psi^n_{13})$*. For* $n, m \in \mathbb{N}$*, denote*

$$B_{13,n}[p^m] := \psi_{13,n}(H^n_{12} \times H^{2n}_{23} \times H^m_{13})$$

Similarly let $\psi_{24,n} : H^{2n}_{23} \times H^n_{34} \times H_{24} \to Def_{sus}(X_2 \times X_2 \times X_4)$*, and* $B_{24}, B_{24,n}$ *and* $B_{24,n}[p^m]$ *are similarly defined.*

(d) With these notations $B = B_{13} \times_{H_{23}} B_{24}$*. We define*

$$B_n := B_{13,n}[p^n] \times_{H_{23}} B_{24,n}[p^n]$$

which is a finite subscheme of B*. Let*

$$E_n = \pi^{-1}(B_n)$$
$$A_n = H^n_{1,2} \times H^n_{1,3} \times H_{1,4} \times H^{3n}_{2,3} \times H^{2n}_{3,4} \times H^{2n}_{2,4}$$

4.1. 4-slopes Case Basic

Definition 4.1.1. ***(Definition of the*** H_{14} ***torsor structure on*** E***)*** *Notation as in 4.0.1. We define an* H_{14} *action on* $E = Def_{sus}(X)$*, that is a morphism*

$$*_E : H_{14} \times E \to E$$

as follows: let R/κ *an Artinian local ring. Let* $N \in \mathbb{N}$ *and* $h_{14} \in H_{14}[p^N](R)$*. Let* $\mathcal{X}$ *be a* κ*-strongly sustained* p*-divisible group over* R *modeled on* X*, that is* $\mathcal{X} \in E(R)$*. Let*

$$0 \subset \mathcal{X}_4 \subset \mathcal{X}_3 \subset \mathcal{X}_2 \subset \mathcal{X}_1 = \mathcal{X}$$

be the slope filtration of $\mathcal{X}$*. We have a short exact sequence*

$$0 \to \mathcal{X}_2 \to \mathcal{X} \to \mathcal{X}/\mathcal{X}_2 \to 0$$

Then there exists $M \in \mathbb{N}$ *with* $M \geq N$ *and*

$$F \in Hom(\mathcal{X}/\mathcal{X}_2[p^M], \mathcal{X}_2[p^M])(R)$$

s.t.

$$\mathcal{X} = \mathcal{X}/\mathcal{X}_2 \times^{(1,F)} \mathcal{X}_2$$

Let

$$\iota_M : \mathcal{X}_4[p^M] \hookrightarrow \mathcal{X}_2[p^M]$$

the natural embedding. As

$$\mathcal{X}_4 \simeq X_4 \times_\kappa R$$

$$\mathcal{X}/\mathcal{X}_2 \simeq X_1 \times_\kappa R$$

the element

$$\begin{aligned} h_{14} &\in Hom^{st}(X_1[p^N], X_4[p^N])(R) \\ &\subset Hom^{st}(X_1[p^M], X_4[p^M])(R) \\ &\subset Hom(X_1[p^M], X_4[p^M])(R) \end{aligned}$$

gives rise to an element

$$\widetilde{h_{14}} : \mathcal{X}/\mathcal{X}_2[p^M] \to \mathcal{X}_4[p^M]$$

let

$$\iota_M \circ \widetilde{h_{14}} : \mathcal{X}/\mathcal{X}_2[p^M] \to \mathcal{X}_2[p^M]$$

and we define the torsor structure

$$*_E : H_{14} \times E \to E$$

by

$$*_E(h_{14}, \mathcal{X}) := \mathcal{X}/\mathcal{X}_2 \times^{(1, F + \iota_M \circ \widetilde{h_{14}})} \mathcal{X}_2 \in E(R)$$

It is easy to check that this gives rise to an action of H_{14} on E, and as

$$*_E(h_{14}, \mathcal{X}) = \mathcal{X} \iff \iota_M \circ \widetilde{h_{14}} \iff h_{14} = 0$$

this action is free.

Remark 4.1.2. *The definition of the H_{14} action on E is a complete analogy of 3.2.1.*

Lemma 4.1.3. *Notation as in 4.0.1. Then the following statements hold:*

(a). Let $\pi : E \to B$ be the morphism defined as follows: Fix R/k an Artinian local ring. Let $\mathcal{X} \in E(R)$, that is $\mathcal{X}$ is a p-divisible group over R strongly sustained modeled on

X. Let

$$0 \subset \mathcal{X}_4 \subset \mathcal{X}_3 \subset \mathcal{X}_2 \subset \mathcal{X}_1 = \mathcal{X}$$

be the slope filtration of $\mathcal{X}$. Define π as sending $\mathcal{X}$ to

$$\mathcal{X}_1/\mathcal{X}_3 \otimes \mathcal{X}_2 \in B(R) = Def_{sus}(X_1 \times X_2 \times X_3) \times_{Def_{sus}(X_2 \times X_3)} Def_{sus}(X_2 \times X_3 \times X_4)(R)$$

Then π is faithful.

(b). Let $\pi : E \to B$ as in (a). Then π is invariant under the H_{14} action. That is $\pi(*_E(h_{14}, e)) = \pi(e)$ for all $h_{14} \in H_{14}(R), E(R)$. Moreover, let $\bar{\pi} : E/H_{14} \to B$ the morphism induced by π, as π is H_{14} invariant. Then $\bar{\pi}$ is an isomorphism.

Proof. The proof is entirely parallel to 3.2.2(a) and (b). □

4.2. Coordinates in 4-slopes Case

The main goal of this section is to prove 4.2.1, which generalizes Mumford's trivialization of biextensions as described in 3.1.

The main result in this section is 4.2.1. We first give a comparison between the result in 4.2.1 and Mumford's trivialization of biextensions given in 3.4.3:

$\mathrm{F} = \mathrm{Def}_{sus}(\prod_{i=1}^{3} X_i)$ a biextension	$\mathrm{E} = \mathrm{Def}_{sus}(\prod_{i=1}^{4} X_i)$
$\mathrm{H}_{12}, H_{23}, H_{13}$ p-divisible groups	$\mathrm{H}_{ij}, 1 \leq i < j \leq 4,$ p-divisible groups
$\pi : F \to H_{12} \times H_{23}$ projection	$\pi : E \to B$ projection
$\mathrm{F}_n \subset F$	$\mathrm{E}_n \subset E$
$\pi(F_n) = H_{12}[p^n] \times H_{23}[p^n]$	$\pi(E_n) = B_n = B_{13,n}[p^n] \times_{H_{23}} B_{24,n}[p^n]$
$\psi_n : H_{12}[p^n] \times H_{23}[p^n] \times H_{13} \to F_n, \forall n \in \mathbb{N}$	$\psi_n : A_n \to E_n, \forall n \in \mathbb{N}$
$\mathrm{F}_n \subset F_{n+1}, \varinjlim F_n = F$	$\mathrm{E}_n \subset E_{n+1}, \varinjlim E_n = E$
gluing data of ψ_n as in 3.4.3	gluing data of ψ_n as in 4.2.1

Table 4.1: Comparison between two 'trivializations'

Now we state the main result of this section:

Theorem 4.2.1. *Let A_n, E_n as in 4.0.1.(d). Then there exists a morphism $\psi_n : A_n \to E_n$.*

Moreover we can write down the gluing data for ψ_n: let $f = (f_{ij}), f' = (f'_{ij}) \in A_n(R)$ for a fixed Artinian local k algebra R/k. then $\psi_n(f) = \psi_n(f')$ if and only if

$$f_{12}^n = f_{12}^{n'}, f_{23}^n = f_{23}^{n'}, f_{34}^n = f_{34}^{n'}, \tag{4.1}$$

$$f_{13}^n - f_{13}^{n'} - \langle f_{23}^{2n} - f_{23}^{2n'}, f_{12}^n \rangle = 0, \tag{4.2}$$

$$f_{24}^n - f_{24}^{n'} - \langle f_{34}^n, f_{23}^{2n} - f_{23}^{2n'} \rangle_n = 0, \tag{4.3}$$

$$f_{14}^n - f_{14}^{n'} - \langle f_{34}^{2n} - f_{34}^{2n'}, f_{13}^n \rangle_n + \langle -(f_{24}^{2n} - f_{24}^{2n'}) + \langle f_{34}^{2n}, f_{23}^{3n} - f_{23}^{3n'} \rangle_{2n}, f_{12}^n \rangle_n = 0 \tag{4.4}$$

Proof. We use the following notations/facts:

(a) We fix R/k an Artinian local ring.

(b) We use x_i^n to denote an element in $X_i[p^n]$ and f_{ij}^n to denote an element in $H_{ij}[p^n]$.

(c) We have natural bilinear pairings

$$<,>_{ikj,n}\colon H_{ik}[p^n] \times H_{kj}[p^n] \to H_{ij}[p^n]$$

given by compositions. These bilinear pairings will sometimes be denoted simply by $\circ$ when it's clear from the context.

We will define a morphism $\psi_n : A_n \to E_n$. The idea here is pretty simple: we use 2.4.3 to construct a trivialization of $(X_1 \times X_2 \times X_3 \times X_4)[p^n]$ one component at a time.

(a) Let

$$g^{2n} : (X_2 \times^{f_{23}^n} X_3)[p^{2n}] \xrightarrow{(\Psi^{2n}_{f_{23}^{3n}})^{-1}} (X_2 \times X_3)[p^{2n}] \xrightarrow{(f_{24}, f_{34})} X_4^{2n} \tag{4.5}$$

where $(\Psi^{2n}_{f_{23}^{3n}}) : (X_2 \times X_3)[p^{2n}] \mapsto (X_2 \times^{f_{23}^n} X_3)[p^{2n}]$ an isomorphism as defined in 2.4.3(b).

(b) Given g^{2n} we can define

$$\Psi^n_{g^{2n}} : (X_2 \times^{f^n_{23}} X_3)[p^n] \times X_4[p^n] \to (X_2 \times^{f^n_{23}} X_3) \times^{g^n} X_4)[p^n]$$

by 2.4.3, here

$$g^n = [p^n]g^{2n} = g^{2n}|_{\text{n-th level}} \tag{4.6}$$

(c) Given

$$f^{2n}_{23} = [p^n]f^{3n}_{23} = f^{3n}_{23}|_{\text{2n-th level}}$$

once again by 2.4.3 we can define

$$\Psi^n_{f^{2n}_{23}} \times id_{X_4} : (X_2 \times X_3)[p^n] \to (X_2 \times^{f^n_{23}} X_3)[p^n]$$

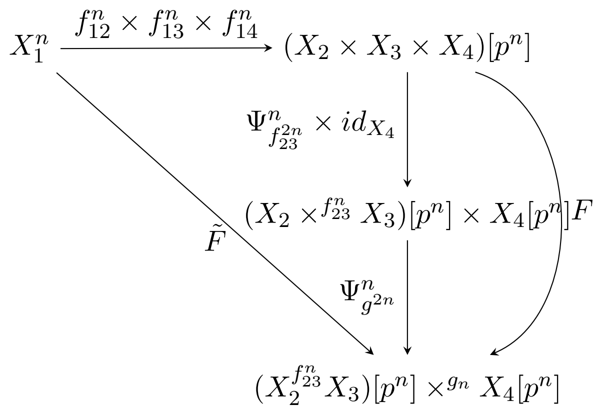
(d) Denote by

$$F = \Psi^n_{g^{2n}} \circ (\Psi^n_{f^{2n}_{23}} \times id_{X_4}) : (X_2 \times X_3 \times X_4)[p^n] \to ((X_2 \times^{f^n_{23}} X_3) \times^{g_n} X_4)[p^n]$$

(e) Let

$$\tilde{F} = (f^n_{12}, f^n_{13}, f^n_{1,4}) \circ F : X_1 \to (X_2 \times^{f^n_{23}} X_3) \times^{g_n} X_4)[p^n] \tag{4.7}$$

(f) To summarize, we have the following diagram.



(g) finally we define ψ_n by sending $f \in A_n(R)$ to

$$X_f := X_1 \times^{\tilde{F}} [(X_2 \times^{f_{23}^n} X_3) \times^{g_n} X_4)] \in E_n(R) \tag{4.8}$$

We then get rid of the restriction $f_{1,4}^n \in H_{1,4}^n$ using the $H_{1,4}$ torsor structure on E.

This finishes the definition of $\psi_n : A_n \to E_n$.

To write down the gluing data: let

$$f, f' \in A_n(R) = (H_{1,2}^n \times H_{1,3}^n \times H_{1,4}^n \times H_{2,3}^{3n} \times H_{3,4}^{2n} \times H_{2,4}^{2n})(R)$$

Let

$$(\tilde{F}, g^{2n}), (\tilde{F}', , g^{2n'})$$

be the data we used to construct $X_f, X_{f'}$, see 4.5, 4.6 and 4.7. Then

$$X_f = X_{f'} \iff \tilde{F} = \tilde{F}', g^n = g^{n'}, f_{23}^n = f_{23}^{n'}$$

Note that the conditions

$$g^n = g^{n'}, f_{23}^n = f_{23}^{n'}$$

are precisely the conditions for$X_f, X_{f'}$ to be isomorphic after modulo the slope filtration corresponding to X_1. In other words, let $\pi_{2,4} : E = Def_{sus}(X) \to Def_{sus}(X_2 \times X_3 \times X_3)$ then

$$g^n = g^{n'}, f_{23}^n = f_{23}^{n'}$$
$$\iff f_{23}^n = f_{23}^{n'}, f_{34}^n = f_{34}^{n'}, f_{24}^n - f_{24}^{n'} = f_{34}^n \circ (f_{23}^{2n} - f_{23}^{2n'})$$
$$\iff \pi_{2,4}(X_f) = \pi_{2,4}(X_{f'})$$

Now we write down the condition for $\tilde{F} = \tilde{F}'$.

We adopt the following notation, if X a p-divisible group and $x^n \in X[p^n]$, then x^m is a lifting of x^n to $X[p^m]$ for $m \geq n$.

By 2.1 we have

$$[(X_2 \times^{f_{23}^n} X_3) \times^{g_n} X_4][p^n] = \frac{ker(\frac{ker(X_2^{3n} \times X_3^{3n}) \to X_3^n}{\Gamma_{-f_{23}^n}} \times X_4^{2n} \to X_4^{2n})}{\Gamma_{-g^n}} \tag{4.9}$$

Let $x_i^m := f_{1i}^m(x_1^m), \forall i \in \{2, 3, 4\}$, then the morphism

$$\tilde{F} : X_1[p^n] \to [(X_2 \times^{f_{23}^n} X_3) \times^{g_n} X_4][p^n]$$

defined in 4.7, can be described as:

$$\tilde{F} : x_1^n \to (x_i^n := f_{1i}^n(x_1))_{i\in\{2,3,4\}} \to (x_2^{3n}, x_3^{2n} - f_{23}^{3n}(x_2^{3n}), x_4^n - f_{34}^{2n}(x_3^{2n}) - f_{24}^{2n}(x_2^{2n}))$$

where $(x_2^{3n}, x_3^{2n} - f_{23}^{3n}(x_2^{3n}), x_4^n - f_{34}^{2n}(x_3^{2n}) - f_{24}^{2n}(x_2^{2n}))$ is understood as an element in the right hand side of 4.9.

Now the it's a matter of elementary algebra to write down the conditions for $\tilde{F} = \tilde{F}'$:

$$\tilde{F} = \tilde{F}' \mod X_3, X_4 \iff f_{12} = f'_{12}$$

which is the first equation of 4.1. We can similarly derive the other two equations of 4.1.

$$\tilde{F} = \tilde{F}' \mod X_4 \iff (x_2^{2n} - x_2^{2n'}, x_3^n - x_3^{n}{}' - f_{23}^{2n}(x_2^{2n}) + f_{23}^{2n}{}'(x_2^{2n}{}')) \in \Gamma_{-f_{23}^n}$$

or equivalently,

$$x_2^{2n} - x_2^{2n}{}' = (f_{12}^{2n} - f_{12}^{2n}{}')(x_1^n) \in [p^n], \tag{4.10}$$

$$-f_{23}^n(x_2^{2n} - x_2^{2n}{}') = x_3^n - x_3^{n}{}' - f_{23}^{2n}(x_2^{2n}) + f_{23}^{2n}{}'(x_2^{2n}{}') \tag{4.11}$$

Rewrite the RHS of 4.11 as

$$x_3^n - x_3^{n}{}' - f_{23}^{2n}(x_2^{2n} - x_2^{2n}{}') - (f_{23}^{2n} - f_{23}^{2n}{}')(x_2^{2n}{}')$$

and notice that

$$f_{23}^{2n}(x_2^{2n} - x_2^{2n}{}') = f_{23}^n(x_2^{2n} - x_2^{2n}{}')$$

as $x_2^{2n} - x_2^{2n}{}' \in [p^n]$ and f_{23}^{2n} is a lifting of f_{23}^n, equation 4.11 becomes

$$x_3^n - x_3^{n}{}' - (f_{23}^{2n} - f_{23}^{2n}{}')(x_2^{n}{}') = 0$$

i.e.

$$f_{13}^n - f_{13}^n{}' - (f_{23}^{2n}{}' - f_{23}^{2n}{}') \circ f_{12}^n = 0$$

which is precisely the second equation of 4.1. Here we use the fact that

$$f_{23}^{2n}(x_2^{2n}{}') - f_{23}^{2n}{}'(x_2^{2n}{}') = (f_{23}^{2n} - f_{23}^{2n}{}')(x_2^{n}{}')$$

where the element $(f_{23}^{2n} - f_{23}^{2n}{}')$ is understood as in $H_{23}[p^n]$. We can similarly derive the third equation of 4.1.

Finally, after unwinding definitions, we have $\tilde{F} = \tilde{F}'$ if and only if

$$
\begin{aligned}
& x_4^n - x_4^{n\prime} - (f_{34}^{2n}(x_3^{2n}) - f_{34}^{2n\prime}(x_3^{2n\prime})) - (f_{24}^{2n}(x_2^{2n}) - f_{24}^{2n\prime}(x_2^{2n\prime})) \\
= & - f_{24}(x_2^{3n} - x_2^{3n\prime}) - f_{23}(x_3^{2n} - x_3^{2n\prime} - f_{23}^{3n}(x_2^{3n}) + f_{23}^{3n\prime}(x_2^{3n\prime}))
\end{aligned}
$$

after some reorganization together with the fact that $x_i^m = f_{1i}^m(x_1^m), \forall i \in \{2, 3, 4\}$ this is precisely the last equation of 4.1. We have proved this lemma. $\square$

Lemma 4.2.2. ***(Basic Properties of ψ_n)*** *Notation as in 4.2.1.*

(a). Let $\tilde{*}$ be the trivial H_{14} torsor structure on $A_n = H_{1,2}^n \times H_{1,3}^n \times H_{1,4} \times H_{2,3}^{3n} \times H_{3,4}^{2n} \times H_{2,4}^{2n}$. Let F_n be the schematic image of ψ_n. Then $\tilde{*}$ descends to a H_{14} torsor structure on F_n, which we denote by $*_{F_n}$, that is

$$*_{F_n} : H_{14} \times F_n \to F_n \text{ a torsor structure}$$

and the diagram

$$
\begin{array}{ccc}
H_{14} \times A_n & \xrightarrow{\tilde{*}} & A_n \\
\Big\downarrow {\scriptstyle id_{H_{14}} \times \psi_n^o} & & \Big\downarrow {\scriptstyle \psi_n^o} \\
H_{14} \times F_n & \xrightarrow{*_{F_n}} & F_n \\
\Big\downarrow {\scriptstyle id_{H_{14}} \times \rho_n} & & \Big\downarrow {\scriptstyle \rho_n} \\
H_{14} \times E_n & \xrightarrow{*_{E_n}} & E_n
\end{array}
$$

where

- *$\rho_n : F_n \hookrightarrow E_n$ the embedding morphism.*
- *$*_{E_n}$ the morphism corresponding to the H_{14} torsor structure on E_n.*
- *ψ_n^o is the morphism $A_n \to F_n$ corresponding to ψ_n, as F_n is defined as the*

schematic image of ψ_n.

(b). The following diagram commutes:

$$
\begin{array}{ccc}
A_{n+1} = H_{1,2}^{n+1} \times H_{1,3}^{n+1} \times H_{1,4} \times H_{2,3}^{3(n+1)} \times H_{3,4}^{2(n+1)} \times H_{2,4}^{2(n+1)} & \xrightarrow{\psi_{n+1}} & E_{n+1} \\
\hookrightarrow \Big\uparrow & & \Big\uparrow = \\
H_{1,2}^{n} \times H_{1,3}^{n} \times H_{1,4} \times H_{2,3}^{3n+2} \times H_{3,4}^{2n+1} \times H_{2,4}^{2n+1} & \xrightarrow{\psi_{n+1}} & E_{n+1} \\
id_{H_{12} \times H_{13}} \times [p^n]_{H_{24} \times H_{34}} \times [p^{2n}]_{H_{23}} \Big\downarrow & & \Big\uparrow \hookrightarrow \\
A_n = H_{1,2}^{n} \times H_{1,3}^{n} \times H_{1,4} \times H_{2,3}^{3n} \times H_{3,4}^{2n} \times H_{2,4}^{2n} & \xrightarrow{\psi_{n}} & E_n
\end{array}
$$

(c). Let $B_{13} = Def_{sus}(X_1 \times X_2 \times X_3)$ *and* $B_{24} = Def_{sus}(X_2 \times X_3 \times X_4)$ *both biextensions. Let* $\psi_{13,n}, \psi_{24,n}, B_{13,n}[p^n], B_{24,n}[p^n]$ *and* B_n *as defined in 4.0.1(c),(d). Then the following diagram commutes:*

$$
\begin{array}{ccc}
A_n = H_{1,2}^{n} \times H_{1,3}^{n} \times H_{1,4} \times H_{2,3}^{3n} \times H_{3,4}^{2n} \times H_{2,4}^{2n} & \xrightarrow{\psi_{n}} & E_n \\
\widehat{\pi_{14}} \Big\downarrow & & \Big\downarrow \pi \\
H_{1,2}^{n} \times H_{1,3}^{n} \times H_{2,3}^{3n} \times H_{3,4}^{2n} \times H_{2,4}^{2n} & \xrightarrow{\overline{\psi_{n}}} & B_n \\
id_{H_{12} \times H_{13}} \times [p^n]_{H_{23} \times H_{24} \times H_{34}} \Big\downarrow & & \Big\downarrow = \\
H_{12}[p^n] \times H_{23}[p^{2n}] \times H_{13}[p^n] \times H_{34}[p^n] \times H_{24}[p^n] & \xrightarrow{\psi_{13,n} \otimes_{H_{23}} \psi_{24,n}} & B_n
\end{array}
$$

where

- $\widehat{\pi_{14}}$ *is the natural projection.*
- $\overline{\psi_n}$ *is the natural morphism induced by* ψ_n.
- $\psi_{13,n} \otimes_{H_{23}} \psi_{24,n}$ *is the tensor product of* $\psi_{13,n}$ *and* $\psi_{24,n}$ *over* H_{24}.

Proof. Proof of (b) and (c) is left as an exercise. We now prove (a).

It suffices to show that

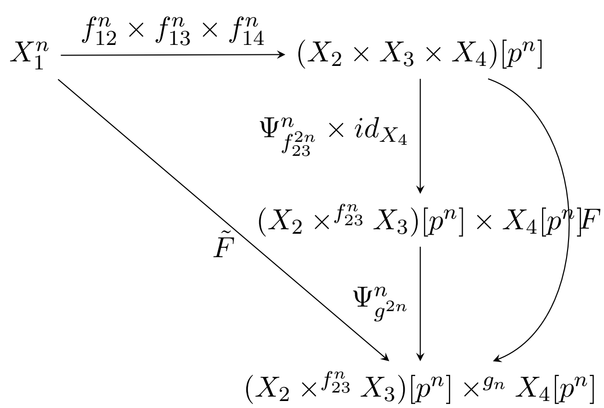
$$\rho_n \circ \psi_n \circ \tilde{*} = *_{E_n} \circ (id_{H_{14}} \times \rho_n) \circ (id_{H_{14}} \times \psi_n)$$

□

Let $h_{14} \in H_{14}[p^n], f = (f_{12}^n, f_{23}^{3n}, f_{34}^{2n}, f_{13}^n, f_{24}^{2n}, f_{14}^n) \in A_n$, both functorial points over the same Artinian local algebra R/κ. Recall that in (f) starting from f we constructed

$$F, \tilde{F}, \Psi_{f_{23}^{2n}}^n, g^n, g^{2n}, \Psi_{g^{2n}}^n$$

that fit into the following diagram:



Note that the vertical sequence of the diagram does not depend on the f_{14}^n component. Now by definition

$$\tilde{*}(h_{14}, f) = (f_{12}^n, f_{23}^{3n}, f_{34}^{2n}, f_{13}^n, f_{24}^{2n}, f_{14}^n + h_{14}^n)$$

Let $F', \tilde{F}', \Psi_{f_{23}^{2n}}^n{}', \Psi_{g^{2n}}^n{}'$ be the morphisms correspond to $\tilde{*}(h_{14}, f) = (f_{12}^n, f_{23}^{3n}, f_{34}^{2n}, f_{13}^n, f_{24}^{2n}, f_{14}^n + h_{14}^n)$. Then we have

$$F = F'$$

hence

$$\tilde{F} - \tilde{F}' = (0, 0, f_{14}^n - (f_{14} + h_{14}^n))_{X_1^n \to (X_2 \times X_3 \times X_4)[p^n]} \circ F$$

Let

$$\Pi_{23} : (X_2 \times^{f_{23}^n} X_3)[p^n] \times^{g_n} X_4[p^n] \to X_2 \times^{f_{23}^n} X_3)[p^n]$$

the natural projection, then it is easy to see that the composition

$$X_4[p^n] \hookrightarrow (X_2 \times X_3 \times X_4)[p^n] \xrightarrow{F} (X_2 \times^{f_{23}^n} X_3)[p^n] \times^{g_n} X_4[p^n] \xrightarrow{\Pi_{23}} (X_2 \times^{f_{23}^n} X_3)[p^n]$$

is the trivial morphism, and that the following diagram commutes

$$\begin{array}{ccccc}
 & & & & X_4[p^n] \\
 & \overset{=}{\nearrow} & & & \downarrow \hookrightarrow \\
X_4[p^n] & \underset{\hookrightarrow}{\longrightarrow} & (X_2 \times X_3 \times X_4)[p^n] & \xrightarrow{F} & (X_2 \times^{f_{23}^n} X_3)[p^n] \times^{g_n} X_4[p^n] \\
 & & & & \downarrow \Pi_{23} \\
 & & & & (X_2 \times^{f_{23}^n} X_3)[p^n]
\end{array}$$

then the morphism $\tilde{F} - \tilde{F}'$, as a morphism from $X_1[p^n]$ to $X_2 \times^{f_{23}^n} X_3)[p^n] \times^{g_n} X_4[p^n]$, factors through $X_4[p^n] \hookrightarrow (X_2 \times^{f_{23}^n} X_3)[p^n] \times^{g_n} X_4[p^n]$; As a morphism from $X_1[p^n]$ to $X_4[p^n]$,

$$\tilde{F} - \tilde{F}' = f_{14}^n - (f_{14} + h_{14})^n = -h_{14}^n$$

This means precisely that

$$\rho_n \circ \psi_n \circ \tilde{*}(h_{14}^n, f) = *_{E_n} \circ (id_{H_{14}} \times \rho_n) \circ (id_{H_{14}} \times \psi_n)(h_{14}^n, f)$$

by the definition of H_{14} torsor structure on E_n, see 4.1.1. We have proved (a).

Theorem 4.2.3. *Notation as in 4.2.1. The morphism $\psi_n : A_n \to E_n$ is faithfully flat.*

Proof. By 4.2.2.(c), we have a commutative diagram

$$\begin{array}{ccc} A_n & \xrightarrow{\quad\psi_n\quad} & E_n \\ {\scriptstyle \widehat{\pi_{14}}}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ H^n_{1,2} \times H^n_{1,3} \times H^{3n}_{2,3} \times H^{2n}_{3,4} \times H^{2n}_{2,4} & \xrightarrow{\overline{\psi_n}} & B_n \end{array}$$

where

- $\widehat{\pi_{14}} : A_n \to H^n_{1,2} \times H^n_{1,3} \times H^{3n}_{2,3} \times H^{2n}_{3,4} \times H^{2n}_{2,4}$ is the natural projection.
- $\overline{\psi_n} : H^n_{1,2} \times H^n_{1,3} \times H^{3n}_{2,3} \times H^{2n}_{3,4} \times H^{2n}_{2,4} \to B_n$ the morphism induced by ψ_n. $\overline{\psi_n}$ is faithfully flat by 4.2.2.(c).
- ψ_n is H_{14} equivariant by 4.2.2.(a).

Hence ψ_n is faithfully flat. □

Corollary 4.2.4. *Recall E_n is a H_{14} torsor over B_n. Denote by $[p^n_{H_{14}}]_* E_n$ the contraction product induced by $[p^n_{H_{14}}]$, that is*

$$[p^n_{H_{14}}]_* E_n = H_{14} \wedge^{H_{14} \overset{[p^n]}{\to} H_{14}} E_n$$

By definition $[p^n_{H_{14}}]_* E_n$ is also a H_{14} torsor over B_n. Then $[p^n_{H_{14}}]_* E_n$ is a trivial H_{14} torsor, that is $[p^n_{H_{14}}]_* E_n = B_n \times H_{14}$.

Proof. By 4.2.1 E_n can be trivialized by

$$A_n = H^n_{1,2} \times H^n_{1,3} \times H_{1,4} \times H^{3n}_{2,3} \times H^{2n}_{3,4} \times H^{2n}_{2,4}$$

with gluing data lies in $H^n_{1,4}$, therefore $[p^n]_* E_n$ can also be trivialized by

$$H^n_{1,2} \times H^n_{1,3} \times H_{1,4} \times H^{3n}_{2,3} \times H^{2n}_{3,4} \times H^{2n}_{2,4}$$

with gluing data in $[p^n]H^n_{1,4} = 0$, i.e. $[p^n]_* E_n$ is trivial, i.e. there exists an morphism

$$T_{can} : B_n \times H_{14} \xrightarrow{\simeq} [p^n]_* E_n$$

□

Corollary 4.2.5. *Let* $\eta_n : E_n \to [p^n]_* E_n \xrightarrow{T_{can}^{-1}} B_n \times H_{14} \xrightarrow{pr_{H_{14}}} H_{1,4}$ *where* $E_n \to [p^n]_* E$ *is the natural map induced by* $[p^n]_{H_{1,4}}$*. Then*

$$\eta_{n+1}|_{E_n} = [p]_{H_{1,4}} \circ \eta_n|_{E_n} \tag{4.12}$$

Proof. An easy corollary of 4.2.2(b). □

We rewrite the trivialization as in 4.2.1 in a more homogeneous way.

Corollary 4.2.6. *Let*

$$\mathcal{A}_n = (H_{12} \times H_{23} \times H_{34})[p^{3n}] \times (H_{13} \times H_{23})[p^{2n}] \times H_{14},$$

let

$$\Pi^n = ([p^{2n}]_{H_{12}}, id_{H_{23}}, [p^n]_{H_{34}}, [p^n]_{H_{13}}, id_{H_{24}}, id_{H_{14}}) : \mathcal{A}_n \longrightarrow A_n$$

the natural morphism. Then the morphism

$$\psi_{n,homo} := \Pi_n \circ \psi_n : \mathcal{A}_n \to E_n$$

is faithfully flat and finite, as both Π_n *and* ψ_n *are. Moreover, for*

$$f = (f^{3n}_{12}, f^{3n}_{23}, f^{3n}_{34}, f^{2n}_{13}, f^{2n}_{24}, f_{14}), f' = (f^{3n\prime}_{12}, f^{3n\prime}_{23}, f^{3n\prime}_{34}, f^{2n\prime}_{13}, f^{2n\prime}_{24}, f'_{14}) \in \mathcal{A}_n$$

$$\psi_{n,homo}(f) = \psi_{n,homo}(f')$$

if and only if

$$f_{12}^n = f_{12}^{n'}, f_{23}^n = f_{23}^{n'}, f_{34}^n = f_{34}^{n'},$$

$$f_{13}^{2n} - f_{13}^{2n'} - \langle f_{23}^{3n} - f_{23}^{3n'}, f_{12}^{3n}\rangle_{3n} = 0,$$

$$f_{24}^{2n} - f_{24}^{2n'} - \langle f_{34}^{3n}, f_{23}^{3n} - f_{23}^{3n'}\rangle_{3n} = 0,$$

$$f_{14} - f'_{14} - \langle f_{34}^{3n} - f_{34}^{3n'}, f_{13}^{2n}\rangle_{3n} + \langle -(f_{24}^{2n} - f_{24}^{2n'}) + \langle f_{34}^{3n}, f_{23}^{3n} - f_{23}^{3n'}\rangle_{3n}, f_{12}^{3n}\rangle_{3n} = 0$$

here we adopt the following notation: supscript means level in the corresponding p-divisible group, i.e. f_{ij}^k *is an element in* $H_{ij}[p^k]$*; If* $m \leq n$ *and* $f_{ij}^n \in H_{ij}[p^n]$*, then* $f_{ij}^m := [p^{n-m}]f_{ij}^n$ *which is an element in* $H_{ij}[p^m]$*.*

Proof. An obvious corollary of 4.1. □

Remark 4.2.7. *The coordinate system in 4.2.6 has the following advantage against 4.1: all the bilinear pairings involve are at level* $3n$*, and the level of* f_{ij} *only depends on* $j - i$*.*

4.3. Trivialization of Universal Torsors

Notations 4.3.1. *We use the following notations in this section:*

(a) $X = \prod_{i=1}^K X_i$ *be a p-divisible group with* X_i *isoclinic of slope* s_i*, and that* $s_1 < s_2.. < s_K$*. Here* $K \in \{3, 4\}$*.*

(b) $E = Def_{sus}(X) = Def_{Aut^{st}(X)\text{-}torsor}$

(c) $Aut^{st}(X)_n := Aut^{st}(X[p^n])$*.*

(d) $H_{ij} := Hom^{st}(X_i, X_j)$*,* $H_{ij}^n := Hom^{st}(X_i, X_j)[p^n]$*.*

(e) Let $\mathcal{X}$ *be the universal sustained p-divisible group over* E *and let* $\mathcal{X}_n := \mathcal{X}[p^n]$*.*

Lemma 4.3.2. *Following the notations as in 4.3.1 and let* $K = 3$*. Let* $\psi_n : H_{12}^n \times H_{23}^{2n} \times$

$H_{13} \to E_n$ be Mumford's trivialization. Denote by

$$E_n[p^n] := \psi(H_{12}^n \times H_{23}^{2n} \times H_{13}^n)$$

Let ϕ_n be the following morphism:

$$\phi_n : H_{12}^{2n} \times H_{23}^{3n} \times H_{13}^{2n} \xrightarrow{([p]^n_{H_{12}}, [p]^n_{H_{23}}, [p]^n_{H_{13}})} H_{12}^n \times H_{23}^{2n} \times H_{13}^n \xrightarrow{\psi_n} E_n[p^n]$$

Then $\mathcal{X}|_{E_n[p^n]} \times_{E_n[p^n], \phi_n} (H_{12}^{2n} \times H_{23}^{3n} \times H_{13}^{2n})$ is isomorphic to $X[p^n] \times (H_{12}^{2n} \times H_{23}^{3n} \times H_{13}^{2n})$. That is the $\mathcal{X}_n|_{E_n[p^n]}$ can be trivialized when pullbacked to $H_{12}^{2n} \times H_{23}^{3n} \times H_{13}^{2n}$ by ϕ_n. Moreover we can compute the gluing data of this trivialization.

Proof. Fix a Artinian local k algebra R. Let

$$f = (f_{12}^{2n}, f_{23}^{3n}, f_{13}^{2n}) \in (H_{12}^{2n} \times H_{23}^{3n} \times H_{13}^{2n})(R)$$

As $\phi_n(f) \in E_n[p^n](R)$, let

$$\mathcal{X}_f := \mathcal{X}_n|_{R, \phi_n(f)}$$

We now trivialize $\mathcal{X}_f$ by the following steps:

1. We first define a morphism F_{2n} as in the following commutative diagram:

$$\begin{array}{ccc} X_1[p^{2n}] & \xrightarrow{f_{12}^{2n} \times f_{13}^{2n}} & X_2[p^{2n}] \times X_3[p^{2n}] \\ & \searrow^{F_{2n}} & \downarrow \Psi_{f_{23}^{3n}} \\ & & (X_2 \times^{f_{23}^{2n}} X_3)[p^{2n}] \end{array}$$

Define F_n as the restriction of F_n to $X_1[p^n]$, that is

$$F_n := [p^n]F_{2n} : X_1[p^n] \to (X_2 \times^{f_{23}^n} X_3)[p^n]$$

2. We can show that

$$\mathcal{X}_f = X_1[p^n] \times^{F_n} (X_2 \times^{f_{23}^n} X_3)[p^n]$$

This part is left as an exercise.

3. Recall the construction Ψ as in 2.4.3. then

$$T_f = (X_1 \times X_2 \times X_3)[p^n] \overset{\Psi^n_{f_{23}^{2n}}}{\to} X_1[p^n] \times (X_2 \times^{f_{23}^n} X_3)[p^n] \overset{(id_{X_1}, \Psi^n_{F_{2n}})}{\to} X_1 \times^{F_n} (X_2 \times^{f_{23}^n} X_3)[p^n] \tag{4.13}$$

is an isomorphism between $(X_1 \times X_2 \times X_3)[p^n] \times R$ and $\mathcal{X}_f$. As these constructions are functorial, we obtain a morphism

$$T : (X_1 \times X_2 \times X_3)[p^n] \times_{E_n[p^n],\phi_n} (H_{12}^{2n} \times H_{23}^{3n} \times H_{13}^{2n}) \to \mathcal{X}_n \times_{E_n[p^n],\phi_n} (H_{12}^{2n} \times H_{23}^{3n} \times H_{13}^{2n})$$

To write down the gluing data, consider another element $f' = (f_{12}^{2n'}, f_{23}^{3n'}, f_{13}^{2n'})$ such that

$$\phi_n(f) = \phi_n(f')$$

we can similarly define $T_{f'}$, and the gluing data between f and f' is

$$T_{f'}^{-1} \circ T_f \in Aut^{st}(X[p^n])$$

some tedious computation similar to 4.2.1 shows that

$$T_{f'}^{-1} \circ T_f = \begin{pmatrix} 1 & f_{12}^{2n} - f_{12}^{2n'} & f_{13}^{2n} - f_{13}^{2n'} - < f_{23}^{3n} - f_{23}^{3n'}, f_{12}^{2n} >_{2n} \\ 0 & 1 & f_{23}^{2n} - f_{23}^{2n'} \\ 0 & 0 & 1 \end{pmatrix}$$

note that as $\phi_n(f) = \phi_n(f')$ we have

$$f_{13}^{n} - f_{13}^{n'} + < f_{23}^{2n} - f_{23}^{2n'}, f_{12}^{n} >_n = 0$$

hence $T_{f'}^{-1} \circ T_f$ is an element in $Aut^{st}(X[p^n])$. □

Lemma 4.3.3. *Notations as in 4.3.1 and let $K = 4$. Let $\psi_n : A_n \to E_n$ be as in 4.2.1, and*

$$\phi_n : H_{12}^{2n} \times H_{13}^{2n} \times H_{14} \times H_{23}^{4n} \times H_{34}^{3n} \times H_{24}^{3n} \overset{([p_{H_{ij}}^n])_{1 \leq i < j \leq 4}}{\longrightarrow} A_n \overset{\psi_n}{\longrightarrow} E_n$$

and

$$E_n[p^n] := \phi_n(H_{12}^{2n} \times H_{13}^{2n} \times H_{14}^{2n} \times H_{23}^{4n} \times H_{34}^{3n} \times H_{24}^{3n})$$

Then

$$\mathcal{X}_n|_{E_n[p^n]} \times_{E_n[p^n], \phi_n} (H_{12}^{2n} \times H_{13}^{2n} \times H_{14}^{2n} \times H_{23}^{4n} \times H_{34}^{3n} \times H_{24}^{3n})$$

is isomorphic to

$$X[p^n] \times_{E_n[p^n], \phi_n} (H_{12}^{2n} \times H_{13}^{2n} \times H_{14}^{2n} \times H_{23}^{4n} \times H_{34}^{3n} \times H_{24}^{3n})$$

Moreover, we can compute the gluing data of this trivialization. This result is an analogy of 4.3.2.

Proof. We sketch the proof, as the proof is pretty similar to 4.3.2.

Given

$$f = (f_{12}^{2n}, f_{13}^{2n}, f_{14}^{2n}, f_{23}^{4n}, f_{24}^{3n}, f_{34}^{3n})$$
$$f' = (f_{12}^{2n'}, f_{13}^{2n'}, f_{14}^{2n'}, f_{23}^{4n'}, f_{24}^{3n'}, f_{34}^{3n'})$$

both elements in $(H_{12}^{2n} \times H_{13}^{2n} \times H_{14} \times H_{23}^{4n} \times H_{34}^{3n} \times H_{24}^{3n})(R)$ for some fixed Artinian local ring R such that $\phi_n(f) = \phi_n(f')$. Let

$$\mathcal{X}_f = \mathcal{X}_{f'} = \mathcal{X}_n|_{R,\phi(f)}$$

Using f, f' we can write down $T_f, T_{f'}$ both isomorphisms

$$X[p^n] \to \mathcal{X}_f$$

in a similar way as in 4.13, and we define

$$h = h(f, f') = (h_{ij})_{4\times 4} = T_{f'}^{-1} \circ T_f$$

h is an element in

$$Aut^{st}(X)_n = \{(h_{ij})_{i,j}, h_{ij} \in H_{ij}[p^n] \, \forall 1 \geq i < j \leq 4, h_{ii} = 1, h_{ij} = 0 \, \forall i > j\}$$

Now similar computation shows:

$$h_{12} = f_{12}^{2n} - f_{12}^{2n}{}', h_{23} = f_{23}^{2n} - f_{23}^{2n'}, h_{34} = f_{34}^{2n} - f_{34}^{2n'} \tag{4.14}$$

$$h_{13} = f_{13}^{2n} - f_{13}^{2n'} - < f_{23}^{3n} - f_{23}^{3n]}, f_{12}^{2n} >_{2n}, \tag{4.15}$$

$$h_{24} = f_{24}^{2n} - f_{24}^{2n'} - < f_{24}^{3n} - f_{24}^{3n]}, f_{34}^{2n} >_{2n}, \tag{4.16}$$

$$h_{14} = f_{14}^{2n} - f_{14}^{2n'} - (f_{34}^{3n} - f_{34}^{3n'}) \circ f_{13}^{2n} + [-(f_{24}^{3n} - f_{24}^{3n'}) + f_{34}^{3n} \circ (f_{23}^{4n} - f_{23}^{4n'})] \circ f_{12}^{2n} \tag{4.17}$$

$\square$

Remark 4.3.4. *As a byproduct,* $X = \prod_{i=1}^{4} X_i$ *a p-divisible group over a field* $k/\mathbb{F}_p$ *with* X_i *isoclinic, we can use the above gluing data to write down the universal sustained deformation of* X *over* $E = Def_{sus}(X)$*. That is, at nth level, we start with the trivial*

$$A_n \times X[p^n]$$

and use ψ_n *as define in 4.2.1 and the gluing data as in 4.3.3 to obtain a 'truncated sustained p-divisible group' over* $E_n = \psi_n(A_n)$*. Let* $n \to \infty$ *we obtain a sustained p-divisible group over* κ *modeled on* X *over the base* E*.*

4.4. Tate-linear Nilpotent Groups of type A

In this section we extend the category of projective systems $Aut^{st}(X) = \varprojlim Aut^{st}(X)_n$ where $X = \prod_{i=1}^{4} X$ p-divisible group with X_i isoclinic of slopes s_i and $s_1 < s_2 < s_3 < s_4$ to a slightly bigger category.

In the following discussion, we use H_{ij} to denote a p-divisible group. In particular, we are **not** assuming that there exists X_i, X_j s.t.

$$H_{ij} = Def_{sus}(X_i \times X_j)$$

Fix $K \in \mathbb{N}$. Let H_{ij} be p-divisible groups over the base field κ of characteristic p, $\forall 1 \leq i < j \leq K$ and let

$$\langle , \rangle_{ikj,n} H_{ik}[p^n] \times H_{kj}[p^n] \to H_{ij}[p^n]$$

bilinear pairings such that

- We have

$$\langle \langle x_{ij,n}, x_{jk,n} \rangle_{ijk,n}, x_{kl,n} \rangle_{ikl,n} = \langle x_{ij,n}, \langle x_{jk,n}, x_{kl,n} \rangle_{jkl,n} \rangle_{ijl,n} \tag{4.18}$$

  for all $1 \leq i < j < k < l \leq K$ and $x_{ij,n}, x_{jk,n}, x_{kl,n}$ functorial points of $H_{ij}[p^n], H_{jk}[p^n]$ and $H_{kl}[p^n]$ respectively.

- the following diagram commutes

$$\begin{array}{ccc} H_{ik}[p^n] \times H_{kj}[p^n] & \xrightarrow{<,>_{ikj,n}} & H_{ij}[p^n] \\ \Big\downarrow [p]^n_{H_{ik}} \times [p]^n_{H_{kj}} & & \Big\downarrow [p]_{H_{ij}} \\ H_{ik}[p^{n+1}] \times H_{kj}[p^{n+1}] & \xrightarrow{<,>_{ikj,n+1}} & H_{ij}[p^{n+1}] \end{array}$$

Consider

$$L_n := \bigoplus_{1 \leq i < j \leq K} H_{ij}[p^n]$$

Then:

- $\langle , \rangle_{ikj,n}$ naturally gives rise to an multiplication on L_n, which will be denoted by $*_n$, as follows: for $h = (h_{ij})_{1 \leq i<j \leq K}, h' = (h'_{ij})_{1 \leq i<j \leq K}$ both functorial points of L_n, we define

  $$h *_n h' = (\widetilde{h_{ij}})_{1 \leq i<j \leq K}$$

  where

  $$\widetilde{h_{ij}} = \sum_{k \text{ s.t. } i<k<j} \langle h_{ik}, h_{kj} \rangle_{ikj,n}$$

  This multiplication structure is associative by 4.18. It is also nilpotent in the sense that for every $x \in L_n$,

  $$x^K = \underbrace{x *_n x ... *_n x}_{\text{K times}} = 0$$

- This ring structure $*_n$ on L_n also induces a Lie algebra structure $[,]_n$ on L_n, by

  $$[h, h']_n = h *_n h' - h' *_n h$$

- Let

  $$L := \varprojlim_n L_n$$

where the transition map $L_{n+1} \to L_n$ is simply $[p]$ and the projective limit takes place in the big fpqc site over $Spec(\kappa)$. Then $*_n$'s induce an associative algebra structure $*$ on L and all the $[,]_n$'s induce a Lie bracket $[,]$ on L.

- The algebra structure $*_n$ on L_n also induces an group structure on L_n, denoted by $\cdot$, by the formula

$$h_1 \cdot h_2 = h_1 + h_2 + h_1 * h_2$$

  for all functorial points $h_1, h_2 \in L_n$. We will denote this group by H_n. The group structure on L induced by $*$ is defined similarly and we denote this group by H.

- Let

$$\pi_{n+1,n} : H_{n+1} \to H_n$$

  given by $[p]_{Lie(H_n)}$. Then $\pi_{n+1,n}$ is a group homomorphism and

$$H = \varprojlim_n H_n$$

  where the transition maps are those induced by $\pi_{n+1,n}$.

- We will use the notation

$$Lie(H_n) := L_n,$$

$$Lie(H) := L$$

Definition 4.4.1. *Let $\mathcal{T}$ be the system that consists of*

- *A family of p-divisible groups $(H_{ij})_{1 \leq i < j \leq K}$*

- *bilinear pairings $\langle , \rangle_{ijk,n}, \forall 1 \leq i < j < k \leq K, n \in \mathbb{N}$*

and assume the conditions as in 4.18 are satisfied; The group H is called the Tate-linear nilpotent group of type A associated to $\mathcal{T}$ and $(Lie(H), [,])$ is called the Lie algebra of H. We will use the notation $H = \varprojlim H_n$ or $H = (H_{ij})_{1\leq i<j\leq K}$ to denote a Tate-linear nilpotent group of type A.

Definition 4.4.2. *A Tate-linear nilpotent group of type A of rank K is called pure if for each (i, j), the p-divisible group H_{ij} is isoclinic.*

Definition 4.4.3. *A pure Tate-linear nilpotent group of type A of rank K is called perfect if $s_{ij} + s_{jk} = s_{ik} \forall 1 \leq i < j < k \leq K$, where s_{ij} is the slope of H_{ij}.*

Example 4.4.4. Let $X = \prod_{i=1}^{4} X_i$ with X_i isoclinic of slope s_i and assume $s_1 < s_2 < s_3 < s_4$. Let

$$H_{ij} = Hom^{st}(X_i, X_j)$$

and

$$\langle,\rangle_{ijk,n} : Hom^{st}(X_i, X_j)[p^n] \times Hom^{st}(X_j, X_k)[p^n] \to Hom^{st}(X_i, X_k)[p^n]$$

the natural bilinear pairing. Then the system $(H_{ij})_{1\leq i<j\leq 4}$ together with $\langle,\rangle_{ijk,n}$ forms a perfect and pure Tate-linear nilpotent group of type A of rank 4.

Remark 4.4.5. *Tate-linear nilpotent groups of type A of rank 3 or 4 that are perfect and pure are the main object of interests in this thesis.*

Given a Tate-linear nilpotent group of type A $H = \varprojlim H_n$, we can consider the universal deformation space of H torsors, and we have the following

Lemma 4.4.6. *The universal deformation space of $\varprojlim H_n$ torsors is smooth.*

Proof. See [CO22], especially Chapter 6. □

Definition 4.4.7. *Let $\varprojlim H_n = ((H_{ij})_{1\leq i<j\leq K}, <,>_{ijk,n})$, $\varprojlim H'_n = ((H'_{ij})_{1\leq i<j\leq K}, <,>'_{ijk,n}$) be two Tate-linear nilpotent groups of type A of rank K. A homomorphism of general sus-*

tained liear groups

$$f : \varprojlim H_n \to \varprojlim H'_n$$

is a family of homomorphisms $(f_{ij})_{1 \leq i < j \leq K}$*:*

$$f_{ij} : H_{ij} \to H'_{ij}$$

that respect all the Weil pairings, that is for all $1 \leq i < j < k \leq K$ *and* $n \in \mathbb{N}$*, we have commutative diagrams*

$$\begin{array}{ccc} H_{ij}[p^n] \times H_{jk}[p^n] & \xrightarrow{<,>_{ijk,n}} & H_{ik}[p^n] \\ \downarrow{\scriptstyle f_{ij} \times f_{jk}} & & \downarrow{\scriptstyle f_{ik}} \\ H'_{ij}[p^n] \times H'_{jk}[p^n] & \xrightarrow{<,>'_{ijk,n}} & H'_{ik}[p^n] \end{array}$$

Note that such a family (f_{ij}) *naturally induces a projective system of group homomorphisms*

$$f_n : H_n \to H'_n$$

Since the construction $H \to Def_{H-tor}$ *is functorial, such a homomorphism also induces*

$$f^* : Def_{H-tor} \to Def_{H'-tor}$$

Definition 4.4.8. *Let* H *be a Tate-linear nilpotent group of type* A*. The automorphism group of* H*, denoted by* $Aut_{sus}(H)$ *or simply* $Aut(H)$ *is the group of automorphisms over* κ*, in the sense of 4.4.7, from* H *to itself.*

To see the geometric meaning of this definition, we have the following:

Theorem 4.4.9. *Let* $X = X_1 \times X_2 \times X_3$ *with* X_i *isoclinic of slope* s_i *and* $s_1 > s_2 > s_3$*. Let*

$$H_{ij} = Hom^{st}(X_i \times X_j), \ \forall 1 \leq i < j \leq 3$$

For all $n \in \mathbb{N}$, let

$$\langle , \rangle_n : H_{12}[p^n] \times H_{23}[p^n] \to H_{13}[p^n]$$

be the natural pairing. Let $H = (H_{ij})_{1 \leq i < j \leq 3}$ be the Tate-linear nilpotent group of type A corresponding to these data.Note that $Def_{sus}(X) = Def_{Aut^{st}(X) torsor} = Def_{H torsor}$. Then

$$Aut_{biext}(E) = Aut_{sus}(H)$$

Proof. See [CO22] Chapter 10. □

4.5. Tate-linear nilpotent groups of type A: Rank = 3 case

Let $H = \varprojlim H_n$ with components $(H_{ij})_{1 \leq i < j \leq 3}$ be a Tate-linear nilpotent group of type A of rank 3.

We will construct a trivialization of $Def_{H\text{-torsor}}$ that is similar to 3.4.3. To do that:

- Let $A_n = H_{12}[p^n] \times H_{23}[p^{2n}] \times H_{13}$, the relations in 3.4.3 gives us a descent data, that is there exists a scheme $E_{H,n}$ and a faithfully flat morphism

  $$\psi_n : A_n \to E_{H,n}$$

  let $E_H := \varinjlim E_{H,n}$.

- Consider $H_{12}[p^{2n}] \times H_{23}[p^{3n}] \times H_{14} \times H_n$. The equation in 4.3 gives us a descent data:

  $$H_{12}[p^{2n}] \times H_{23}[p^{3n}] \times H_{14} \times H_n \to \mathcal{H}_{H,n}$$

  where $\mathcal{T}_{H_n}$ is a H_n torsor over $E_{H,n}$.

- For any fixed $n_0 \in \mathbb{N}$, and all $n \geq n_0$ integers, consider the H_n bundle $\mathcal{T}_{n,n_0} := \mathcal{T}_{H_n}|_{E_{H,n_0}}$ over E_{H,n_0}, where the restriction is via the natural embedding $E_{H,n_0} \hookrightarrow$

$E_{H,n}$. The projective limit

$$\mathcal{T}_{n_0} := \varprojlim_{n} \mathcal{T}_{n,n_0}$$

is then a H torsor over E_{H,n_0}. Finally, let

$$\mathcal{T}_H := \varinjlim_{n_0} \mathcal{T}_{n_0}$$

then $\mathcal{T}_H$ is a H torsor over E_H. Hence we have a natural morphism

$$f : E_H \to Def_{H\text{-torsor}}$$

induced by the H torsor over E_H.

Theorem 4.5.1. *Notation as above. The morphism $f : E_H \to Def_{H\text{-}torsor}$ is an isomorphism of formal schemes. In particular, theorem 3.4.3 is valid when we substitute $Def_{sus}(X)$ with $Def_{H\text{-}torsor}$.*

Proof. We will prove this result in several steps.

Step 1. We first show that E_H is a smooth formal variety. It is easy to see that the trivial H_{13} torsor structure descents to a H_{13} torsor structure to $E_{H,n}$ with $E_{H,n}/H_{13} \simeq H_{12}[p^n] \times H_{23}[p^n]$, hence by taking inductive limit we obtain a H_{13} torsor structure over E_H with $E_H/H_{12} \times H_{23}$. Hence E_H is smooth.

Step 2. As $Def_{H\text{-torsor}}$ is also smooth by 4.4.6, it suffices to show that the the morphism $f : E_H \to Def_{H\text{-torsor}}$ induces an isomorphism between tangent spaces.

Step 3. Consider the following commutative diagram

$$\begin{array}{ccccc}
H_{13} & \xrightarrow{\text{central fiber}} & E_H & \longrightarrow & H_{12} \times H_{23} \\
\Big\downarrow{\scriptstyle f|_{H_{13}} = id_{H_{13}}} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f_\pi} \\
H_{13} & \xrightarrow{\text{central fiber}} & Def_{\varinjlim H_n} & \longrightarrow & H_{12} \times H_{23}
\end{array}$$

To show that f induces an isomorphism between tangent spaces, it suffices to show that f_π is an isomorphism. Note that we are not assuming f is H_{13} equivariant.

Step 4. The morphism f_π is induced by the following $H_{12} \times H_{23}$ bundle over f_π: for each $n \in \mathbb{N}$, let $B_n = H_{12}[p^{2n}] \times H_{23}[p^{3n}]$, consider the trivial $H_{12}[p^n] \times H_{23}[p^n]$ torsor over B_n, together with the gluing data

$$(h_{12}^{2n}, h_{12}^{3n}, \tilde{h}_{12}^n, \tilde{h}_{23}^n) \sim (h_{12}^{2n\prime}, h_{12}^{3n\prime}, \tilde{h}_{12}^{n}{}', \tilde{h}_{23}^n)'$$
$$\Longleftrightarrow h_{12}^n - h_{12}^n{}' = h_{12}^{2n} - h_{12}^{2n\prime} \text{ and } h_{23}^n - h_{23}^n{}' = h_{23}^{2n} - h_{23}^{2n\prime}$$

for all $(h_{12}^{2n}, h_{12}^{3n}, \tilde{h}_{12}^n, \tilde{h}_{23}^n)$ and $(h_{12}^{2n\prime}, h_{12}^{3n\prime}, \tilde{h}_{12}^n{}', \tilde{h}_{23}^n{}')$ functorial points of $B_n \times (H_{12}[p^n] \times H_{23}[p^n])$, thus $f_\pi|_{H_{12}}$ is the natural isomorphism

$$H_{12} \simeq Def_{\varprojlim_n H_{12}[p^n]\text{-torsor}}$$

same with $f_\pi|_{H_{23}}$. Hence f_π is an isomorphism. We have finished the proof.

□

4.6. Tate-linear nilpotent groups of type A: Rank = 4 Case

In this part, we prove an analogy of 4.5.1 for Tate-linear nilpotent groups of type A of rank 4.

Let $H = \varprojlim H_n$ with components $(H_{ij})_{1 \leq i < j \leq K}$ be a Tate-linear nilpotent groups of type A of rank $K = 4$. We will construct a trivialization of $Def_{H\text{-torsor}}$ similar to 4.2.1. To do that:

- Let $A_n = H_{1,2}^n \times H_{1,3}^n \times H_{1,4} \times H_{2,3}^{3n} \times H_{3,4}^{2n} \times H_{2,4}^{2n}$, the relations in 4.2.1 actually gives us a descent data, that is there exists a scheme $E_{H,n}$ and a faithfully flat morphism

$$\psi_n : A_n \to E_{H,n}$$

  We can therefore define

$$E_H := \varinjlim_n E_{H,n}$$

- Similarly, the result in 4.3.3 gives us another descent data: let

$$(H_{12}^{2n} \times H_{13}^{2n} \times H_{14} \times H_{23}^{4n} \times H_{34}^{3n} \times H_{24}^{3n}) \times H_n$$

  the trivial $\mathcal{H}_n$ torsor over $H_{12}^{2n} \times H_{13}^{2n} \times H_{14} \times H_{23}^{4n} \times H_{34}^{3n} \times H_{24}^{3n}$, by 4.3.3, there exists a H_n torsor over $E_{H,n}$, which we denote by $\mathcal{T}_{H_n}$, and a faithfully flat morphism

$$\varphi_n : (H_{12}^{2n} \times H_{13}^{2n} \times H_{14} \times H_{23}^{4n} \times H_{34}^{3n} \times H_{24}^{3n}) \times H_n \to \mathcal{T}_{H_n}$$

- For any fixed $n_0 \in \mathbb{N}$, and all $n \geq n_0$ integers, consider let $\mathcal{T}_{n,n_0} := \mathcal{T}_{H_n}|_{E_{H,n_0}}$, where the restriction is via the natural embedding $E_{H,n_0} \hookrightarrow E_{H,n}$. The projective limit

$$\mathcal{T}_{n_0} := \varprojlim_n \mathcal{T}_{n,n_0}$$

  is then a H torsor over E_{H,n_0}. Finally, let

$$\mathcal{T}_H := \varinjlim_{n_0} \mathcal{T}_{n_0}$$

  then $\mathcal{T}_H$ is a H bundle over E_H.

Theorem 4.6.1. *Notations as above. Then E_H is the universal deformation space of $\varprojlim H_n$ torsors and $\mathcal{T}_H$ is the universal H torsor over E_H. In particular, the theorem 4.2.1 is valid when we substitute $Def_{sus}(X)$ with $Def_{H\text{-}torsor}$.*

Proof. Let E_d be the universal deformation space of $\varprojlim H_n$, which is smooth by the 4.4.6. Notice that $\bar{H} := H/H_{14}$ is also a Tate-linear nilpotent group of type A, and we can similarly define $E_{\bar{H}}, \mathcal{T}_{\bar{H}_n}$. We will denote $B := E_{H/H_{14}} = E_{\bar{H}}$. Let $\pi : E \to B$ the natural morphism induced by $H \to H/H_{14}$.

By construction, E_H has a H_{14} torsor structure and B has a natural $H_{13} \times H_{24}$ torsor structure over $H_{12} \times H_{23} \times H_{34}$, hence B is smooth and therefore E_H is smooth.

Since E_d is the universal deformation space of $\varprojlim H_n$ torsor, and T_H is a $\varprojlim H_n$ bundle $\mathcal{H}$ over E, we have a map

$$f : E \to E_d$$

Similarly we have

$$f_\pi : B \to Def_{H/H_{14}}$$

To prove that f is an isomorphism it suffices to prove that f induces an isomorphism between the tangent spaces.

Consider the following commutative diagram, where both horizontal arrows are given by the natural H_{14} torsor structure on E and E_d respectively. Note that we do not assume the map f preserves the H_{14} torsor structure.

$$\begin{array}{ccccc}
H_{14} & \xrightarrow{\text{central fiber}} & E_H & \longrightarrow & B \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_\pi} \\
H_{14} & \xrightarrow{\text{central fiber}} & E_d = Def_{\varprojlim H_n} & \longrightarrow & Def_{\varprojlim H_n/H_{14}}
\end{array}$$

From this diagram, to prove that f induces an isomorphism between tangent spaces it suffices to prove that f_π induces isomorphism between tangent spaces. But f_π fits into a similar diagram:

$$\begin{array}{ccccc}
H_{13} \times H_{24} & \xrightarrow{\text{central fiber}} & B & \longrightarrow & H_{12} \times H_{23} \times H_{34} \\
\Big\downarrow id & & \Big\downarrow f_{\pi} & & \Big\downarrow g \\
H_{13} \times H_{24} & \xrightarrow{\text{central fiber}} & Def_{(\varprojlim H_n)/H_{14}} & \longrightarrow & H_{12} \times H_{23} \times H_{34}
\end{array}$$

Let g be the right most vertical morphism in the above diagram. In light of the gluing data as in 4.2.1 we use to construct E, this morphism g is obtained as follows: for each $H_{ij} \in \{H_{12}, H_{23}, H_{34}\}$, each $n \in \mathbb{N}$, we consider the $H_{ij}[p^n]$ bundle over $H_{ij}[p^n]$, denote it by $\mathcal{H}_{ij,n}$:

$$\mathcal{H}_{ij,n} = H_n \times H_{2n}/((h_n, h_{2n}) \sim (h'_n, h'_{2n}) \text{ if } h_{2n} - h'_{2n} \in H_{ij}[p^n] \text{ and } h_n - h'_n = h_{2n} - h'_{2n})$$

then it is easy to see that as we let $n \to \infty$ we obtain a universal H_{ij} bundle over H_{ij}, which induces a map

$$g_{ij} : H_{ij} \to H_{ij}$$

as $H_{ij} = Def_{H_{ij}-tor}$ by Kummer theory, and

$$g = \prod g_{ij}$$

By Kummer theory, this map g is an isomorphism. Hence f_{π} in diagram 4.7 induces an isomorphism on tangent space and we have proved the theorem. □

Definition 4.6.2. *Given $H = (H_{ij})_{1 \leq i < j \leq 4}$ a Tate-linear nilpotent group of type A of rank 4. We define:*

- *$E = Def_{H\text{-torsor}}$. For each $n \in \mathbb{N}$ a subscheme $E_n \subset E$, and ψ_n, A_n as in 4.6.*
- *There is naturally a H_{14} action on E, and let $B = E/H_{14}$.*

- *The system* $H^{1,3} := (H_{ij})_{1\leq i<j\leq 3}$ *together with the bilinear pairings* $\langle , \rangle_{123,n}$ *is naturally a Tate-linear nilpotent group of type* A *of rank* 3. *Similarly we define* $H^{2,4}$.

- $B_{13} := Def_{H^{1,3}\text{-}torsor}$ *which is a biextension. Similarly we define* B_{24}. *Note that* $B = B_{13} \times_{H_{23}} B_{24}$

- *Let* $\pi_{123} : E \to B_{13}$ *and* $\pi_{234} : E \to B_{24}$ *the natural projection.*

- *Let* $\pi_{12} : E \to H_{12}$, $\pi_{23} : E \to H_{23}$ *and* $\pi_{34} : E \to H_{34}$ *the natural projections.*

- *As* B_{13} *is a biextension of* $H_{12} \times H_{23}$ *by* H_{13}, *for each* $n \in \mathbb{N}$, *we have a subscheme* $B_{13,n} \subset B$ *and a faithfully flat morphism*

$$\psi_{13,n} : H_{12}[p^n] \times H_{23}[p^{2n}] \times H_{14} \to B_{13,n}$$

  as given in 3.4.3. Similarly we can define

$$\psi_{13,n,homo} : (H_{12} \times H_{23})[p^{2n}] \times H_{13} \to B_{13,n}$$

  as defined in 3.4.4.

4.7. Admissible Subgroups and Tate-Linear Subvarieties

Definition 4.7.1. ***(Nilpotent Filtration)*** *Let* $H = (H_{ij})_{1\leq i<j\leq K}$ *be a Tate-linear nilpotent group of type* A *of rank* K. *For all* $n \in \mathbb{Z}$, *there is a filtration*

$$0 = \mathcal{F}_{K-1,n} \subset \mathcal{F}_{K-2,n}.. \subset \mathcal{F}_{0,n} = Lie(H_n)$$

where

$$\mathcal{F}_{l,n} = \{(h_{ij})_{1\leq i<j\leq K}, with\ h_{ij} \in H_{ij}[p^n]\ s.t.\ h_{ij} = 0, \forall j - i < l\}$$

Lemma 4.7.2. *Notation as in 4.7.1. Then*

(a) Each $\mathcal{F}_{K-1,n}$ *is an ideal of* $(Lie(H_n), [,]_n)$, *as well as a normal subgroup of* H_n.

(b) *For a fixed* $l \in \mathbb{Z}_{\geq 0}$, $\mathcal{F}_{l-1,n}/\mathcal{F}_{l,n} \simeq \bigoplus_{j-i=l-1} H_{ij}[p^n]$. *Let*

$$Gr^k_{nil}(H) := \varprojlim_n (\mathcal{F}_{k,n}/\mathcal{F}_{k+1,n}) = \varprojlim_n (\bigoplus_{j-i=k} H_{ij}[p^n])$$

(c) *By taking projective limit we naturally obtain a filtration*

$$0 = \mathcal{F}_{K-1} \subset \mathcal{F}_{K-2}.. \subset \mathcal{F}_0 = Lie(H)$$

of $Lie(H)$.

(d) $\bigoplus_k Gr^k_{nil}(H) = Lie(H)$ *as sheaves of* $\mathbb{Z}_p$ *modules.*

Proof. Once formulated, the proof of (a)-(d) are easy to check. □

Definition 4.7.3. ***(Definition of Admissible Subgroups).*** *Let* H *be a Tate-linear nilpotent group of type* A *of rank* K *with Lie ring* $Lie(H)$ *associated to the system* $H_{ij}, \forall 1 \leq i < j \leq K$ *and bilinear pairings* $\langle , \rangle_{ijk,n}$. *An admissible subgroup of* H *is a cotorsion free subgroup* G *of* H. *Equivalently, an admissible subgroup of* H *is a family of subgroups* G_n *of* H_n, *for all* $n \in \mathbb{N}$, *such that*

- *The natural homomorphism* $G_{n+1} \hookrightarrow H_{n+1} \stackrel{\pi_{n+1,n}}{\longrightarrow} H_n$ *factors through* G_n *and this morphism* $G_{n+1} \to G_n$ *is surjective.*
- *The projective system* $\varprojlim G_n$ *is cotorsion free as a subgroup of* $\varprojlim H_n$.

Definition 4.7.4. *Let* H *be a Tate-linear nilpotent group of type* A *and let* $G \subset H$ *an admissible subgroup. Then there is a natural morphism* $\Phi_{G \hookrightarrow H} : Def_{G\text{-torsor}} \to Def_{H\text{-torsor}}$ *defined as follows: let* $\mathcal{G}$ *be the universal* G*-torsor over* $Def_{G\text{-torsor}}$ *and let* $\mathcal{H}$ *be the universal* H*-torsor over* $Def_{H\text{-torsor}}$. *Let* $\mathcal{G} \wedge^G H$ *be the contraction product of* $\mathcal{G}$ *with respect to* $G \hookrightarrow H$, *in particular* $\mathcal{G} \wedge^G H$ *is a* H *torsor over* $Def_{G\text{-torsor}}$, *therefore induces a morphism* $\Phi_{G \hookrightarrow H} : Def_{G\text{-torsor}} \to Def_{H\text{-torsor}}$.

Definition 4.7.5. *Notation as in 4.7.3. Let H be a Tate-linear nilpotent group of type A and $Lie(H)$ be it's Lie algebra. Let $G \subset H$ be an admissible subgroup. Let*

$$0 = \mathcal{F}_{K-1} \subset \mathcal{F}_{K-2} .. \subset \mathcal{F}_0 = Lie(H)$$

be the filtration of $Lie(H)$ as defined in 4.7.1. Let

$$0 = \mathcal{G}_{K-1} \subset \mathcal{G}_{K-2} .. \subset \mathcal{G}_0$$

be the induced filtration on G, that is

$$\mathcal{G}_l = G \cap \mathcal{F}_l, \quad \forall l \in \{0, 1, ..., K-1\}$$

Define $Lie(G)$, the Lie ring of G, by

$$Lie(G) := \bigoplus_{l \in \{0,1,...,K-2\}} \mathcal{G}_l / \mathcal{G}_{l+1}$$

Clearly

$$\bigoplus_{l \in \{0,1,...,K-2\}} \mathcal{G}_l / \mathcal{G}_{l+1} \subset \bigoplus_{l \in \{0,1,...,K-2\}} \mathcal{F}_l / \mathcal{F}_{l+1} = Lie(H)$$

It is an easy exercise to check that $Lie(G)$ is indeed a Lie subring of $Lie(H)$.

Definition 4.7.6. *Let H be a Tate-linear nilpotent group of type A and $G \subset H$ an admissible subgroup. Let $Lie(G)$ be the Lie ring of G, which is a sheaf of $\mathbb{Z}_p$ modules over the big fpqc site over $Spec(\kappa)$. The dimension of G, denoted $dim(G)$, is the dimension of the p-divisible group*

$$Lie(G) \otimes \mathbb{Q} / Lie(G)$$

as a smooth formal group.

Lemma 4.7.7. *Notation as in 4.7.4, then* *(a). The schematic image of* $\Phi_{G\hookrightarrow H}$ *is a smooth connected formal subvariety of* $Def_{H\text{-torsor}}$.

(b). $\Phi_{G\hookrightarrow H}$ *is a finite morphism of smooth formal schemes.*

(c). If moreover G *is cotorsion free, then* $\Phi_{G\hookrightarrow H}$ *is a smooth embedding.*

(d). Let $E_G = Im(\Phi_{G\hookrightarrow H})$. *Then*

$$dimE_G = dim(G)$$

where $dim(G)$ *is as defined in 4.7.6.*

Proof. Given in [Cha22]. □

Definition 4.7.8. ***(Definition of Tate-linear formal subvarieties).*** *Let* H *be a Tate-linear nilpotent group of type* A *and* E *the universal deformation space of* H. *A formal subvariety* $W \subset E$ *is called a Tate-linear formal subvariety if there exists an admissible subgroup* $H' \subset H$ *such that the schematic image of* $\Phi_{G\hookrightarrow H}$ *is* W, *see 4.7.4 for the definition of* $\Phi_{G\hookrightarrow H}$.

Lemma 4.7.9. *Let* H *be a Tate-linear nilpotent group of type* A *of rank* K *and let* $G \subset H$ *be an admissible subgroup. Let* E_G *be the Tate-linear formal subvariety corresponding to* G. *Let* $E = Def_{H\text{-torsors}}$ *and* $E' \subset E$ *a formal subvariety. Let* $\mathcal{T}$ *be the universal* H*-torsor over* E. *If the structure group of* $\mathcal{T}|_{E'}$ *can be reduced to* G, *that is, if there is a* G*-torsor* $\mathcal{G}$ *over* E' *such that*

$$H \wedge^G \mathcal{G} \simeq \mathcal{T}|_{E'}$$

where $\wedge$ *denotes the contraction product. Then*

$$E' \subset E_G$$

Proof. The G-torsor $\mathcal{G}$ induces a map $f_{\mathcal{G}} : E' \to Def_{G\text{-torsor}}$ such that $\Phi_{G\hookrightarrow H} \circ f_{\mathcal{G}} = id_{E'}$, thus

$$E' \subset Im(\Phi_{G \hookrightarrow H}) = E_G$$

and we have proved the lemma.

$\square$

Definition 4.7.10. *Let H, G be Tate-linear nilpotent groups of type A of rank K. Let $f : H \to G$ be a homomorphism, in the sense of 4.4.7, and let $f_{ij} : H_{ij} \to G_{ij}$ be the ij component of f, for all $1 \leq i < j \leq K$. We say that f is an isogeny if all f_{ij}, as morphisms between p-divisible groups, are isogenies.*

Lemma 4.7.11. ***(Properties of isogeny)*** *Let H, G be Tate-linear nilpotent groups of type A of rank K. Let $f : H \to G$ be an isogeny. Let $\Phi_f : Def_{H\text{-torsor}} \to Def_{G\text{-torsor}}$ be the morphism induced by f. Then*

(a) f is a finite faithfully flat morphism.

Lemma 4.7.12. ***(Quotient)*** *Let $H = (H_{ij}, \langle , \rangle_{ikj,n})$ be a Tate-linear nilpotent group of type A of rank K as above. Let be i_0, j_0 integers such that $1 \leq i_0 < j_0 \leq K$. Let $H'_{i_0,j_0} \subset H_{i_0,j_0}$ be a p-divisible subgroup. Assume that $\varprojlim H'_{i_0j_0}[p^n]$, as a subgroup of $Lie(H)$, lies in the kernel of $*$; In other words, for all $h'_{i_0j_0} \in \varprojlim H'_{i_0j_0}[p^n], h \in Lie(H)$ functorial points,*

$$h * h'_{i_0j_0} = h'_{i_0j_0} * h = 0, \;\; \forall h'_{i_0j_0} \in H'_{i_0j_0}, h \in Lie(H) \tag{4.19}$$

Condition 4.19 is equivalent to: for all $k, l \in \mathbb{N}$ such that $j_0 < k$ and $1 \leq l < i_0$,

$$\langle h'_{i_0,j_0}, h_{j_0,k} \rangle_{i_0j_0k,n} = 0, \;\; \forall h'_{i_0,j_0} \in H'_{i_0,j_0}[p^n], h_{j_0,k} \in H_{j_0,k}[p^n] \tag{4.20}$$

$$\langle h'_{l,i_0}, h_{i_0,j_0} \rangle_{li_0j_0} = 0, \quad \forall h'_{l,i_0} \in H'_{l,i_0}[p^n], h_{i_0,j_0} \in H_{i_0,j_0}[p^n] \tag{4.21}$$

By abuse of notation, we use $H'_{i_0j_0}$ to denote both $H'_{i_0j_0}$ as a p-divisible group, or $\varprojlim H'_{i_0j_0}[p^n]$, *as a subspace of* $Lie(H)$*, then:*

(a). $H'_{i_0j_0}$ *is an ideal of* $(Lie(H), *)$*, and an ideal of* $(Lie(H), [,])$*, as well as a normal subgroup of* H*.*

(b). The exact sequence

$$1 \to H'_{i_0j_0} \to H \to H/H'_{i_0j_0} \to 1$$

is a central extension of sheaves of groups on the big fpqc site of $Spec(\kappa)$*.*

(c). The quotient group $H/(H'_{i_0j_0})$ *is a Tate-linear nilpotent group of type A with components*

$$H_{ij}, (i,j) \neq (i_0, j_0), \tag{4.22}$$

$$H_{i_0,j_0}/H'_{i_0,j_0} \tag{4.23}$$

and with bilinear pairings descent from that of $<,>_{ikj,n}$*.*

(d). If $K \leq 4$*, then the exact sequence in (b). induces a* $H'_{i_0j_0}$ *action on* $Def_{H\text{-}torsor}$ *and we have an isomorphism of smooth formal schemes*

$$Def_{H\text{-}torsor}/H'_{i_0j_0} \simeq Def_{H/H'_{i_0j_0}\text{-}torsor}$$

(e). If $\tilde{H} \subset H/H_{i_0j_0}$ *an admissible subgroup,* $\pi : H \to H/H_{i_0j_0}$ *the quotient map, then* $\pi^{-1}(\tilde{H})$ *is an admissible subgroup of* H*.*

(f). $dim(\pi^{-1}(\tilde{H})) = dim(\tilde{H}) + dim(H'_{i_0j_0})$

Proof. Part (a)-(c) are trivial.

For (d). Let E be the deformation space of H torsors, and let ψ_n, A_n, E_n as in 4.6. That is

$$A_n = H_{12}[p^n] \times H_{13}[p^n] \times H_{14} \times H_{23}[p^{3n}] \times H_{24}[p^{2n}] \times H_{34}[p^{2n}],$$

$$\psi_n : A_n \to E_n \text{ a faithfully flat morphism}$$

For i, j integers such that $1 \le i < j \le 4, (i,j) \neq (1,4)$, let $e_{ij} \in \{1,2,3\}$ such that we can rewrite

$$A_n = H_{14} \times \prod_{1 \le i < j \le 4, (i,j) \neq (1,4)} H_{ij}[p^{e_{ij} n}]$$

Let $\tilde{E}$ be the deformation space of $H/H'_{i_0 j_0}$ torsors and let $\widetilde{\psi_n}, \widetilde{A_n}, \widetilde{E_n}$ defined similarly but in terms of the group $H/H'_{i_0 j_0}$. Let $\Pi_n : A_n \to \tilde{A}_n$ be the quotient out by the $H'_{i_0 j_0}[p^{e_{i_0 j_0}}]$ component map. As A_n is a product, there is a natural $H'_{i_0 j_0}[p^{e_{i_0 j_0}}]$ torsor structure on A_n that is Π_n invariant. Moreover, given 4.20 and 4.21 and since the gluing data 4.2.1 is in terms of the bilinear pairings $\langle , \rangle_{ikj,n}$, this $H'_{i_0 j_0}[p^{e_{i_0 j_0}}]$ action induces an $H'_{i_0 j_0}[p^{e_{i_0 j_0}}]$ torsor action on E_n. Let $\tilde{\Pi}_n$ be the morphism $E_n \to \widetilde{E_n}$ induced by Π_n, we have a commutative diagram

$$\begin{array}{ccc} A_n & \xrightarrow{\psi_n} & E_n \\ \Pi_n \big\downarrow & & \big\downarrow \widetilde{\Pi_n} \\ \widetilde{A_n} & \xrightarrow[\widetilde{\psi_n}]{} & \widetilde{E_n} \end{array}$$

such that both $\Pi_n, \widetilde{\Pi_n}$ are $H'_{i_0 j_0}[p^{e_{i_0 j_0}}]$ invariant and ψ_n is $H'_{i_0 j_0}[p^{e_{i_0 j_0}}]$ equivariant, and both $\psi_n, \widetilde{\psi_n}$ are faithfully flat, we conclude that

$$E_n / H'_{i_0 j_0}[p^{e_{i_0 j_0}}] \simeq \widetilde{E_n}$$

By taking limit we conclude that:

- There is a $H'_{i_0 j_0}$ torsor structure on E.

- $E/H'_{i_0j_0} \simeq \widetilde{E}$

which is the statement of (d).

For part (e), we first prove that $\pi^{-1}(\tilde{H})$ is torsion free. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H'_{i_0j_0} & \longrightarrow & H & \xrightarrow{\pi} & H/H'_{i_0j_0} & \longrightarrow & 0 \\ & & = \uparrow & & \hookrightarrow \uparrow & & \hookrightarrow \uparrow & & \\ 0 & \longrightarrow & H'_{i_0j_0} & \longrightarrow & \pi^{-1}(\tilde{H}) & \xrightarrow{\pi|_{\pi^{-1}(\tilde{H})}} & \tilde{H} & \longrightarrow & 0 \end{array}$$

then $\pi^{-1}(\tilde{H})$ is cotorsion free follows from an easy diagram chasing: let $h \in H$ an functorial point such that $h^N \in \pi^{-1}(\tilde{H})$ for some N, then $\pi(h)^N \in \tilde{H}$. As $\tilde{H}$ is an admissible subgroup, hence cotorsion free, we conclude that $\pi(h) \in \tilde{H}$, hence $h \in \pi^{-1}(\tilde{H})$.

Part (f) follows directly from the exact sequence

$$0 \longrightarrow H'_{i_0j_0} \longrightarrow \pi^{-1}(\tilde{H}) \xrightarrow{\pi|_{\pi^{-1}(\tilde{H})}} \tilde{H} \longrightarrow 0$$

□

The following two lemmas will be handy when we want to prove some formal subscheme is Tate-linear.

Lemma 4.7.13. ***(Functoriality of being Tate-linear I)*** *Let H be a general sustained linear group with components $(H_{ij})_{1\leq i<j\leq 4}$. Let i_0, j_0 integers such that $1 \leq i_0 < j_0 \leq 4$. Let $H'_{i_0,j_0} \subset H_{i_0,j_0}$ a p-divisible subgroup satisfying the conditions of 4.7.12. Let $G := H/H_{i_0,j_0}$ as given in 4.7.12 and $\pi : H \to G$ the natural map. Let E, F be the deformation space of H and G torsors respectively. Let $\tilde{\pi} : F \hookrightarrow E$ be the morphism induced by π which is a smooth embedding of smooth formal schemes. If a formal subvariety $W \subset F$ is Tate-linear, then $W' := \tilde{\pi}^{-1}(W) \subset E$ is also Tate-linear.*

Proof. Let G' be the admissible subgroup of G corresponding to W and let $H' := \pi^{-1}(G')$.

H' is an admissible subgroup by 4.7.12(e). Let $\tilde{W}'$ be the Tate-linear formal subvariety of F corresponding to H'. As for morphisms between deformation spaces of torsors induced by morphisms between groups are canonical, we have

$$\tilde{\pi}(\tilde{W}') \subset W$$

hence

$$\tilde{W}' \subset W'$$

Moreover, let $Lie(H')$ be the Lie algebra of H', then we have an exact sequence of Lie algebras

$$0 \to H_{i_0 j_0} \to Lie(H') \to Lie(G') \to 0$$

where $H_{i_0 j_0}$ has the trivial Lie algebra structure. Hence

$$\dim(H') = \dim(G') + \dim H_{i_0 j_0}$$

By 4.7.4(d),

$$\dim(H') = \dim(\tilde{W}'),$$

$$\dim(G') = \dim(W)$$

we obtain

$$\dim(\tilde{W}') = \dim(W) + \dim H_{i_0 j_0}$$

By 4.7.12(d)., W' admits a $H_{i_0 j_0}$ torsor structure over W, hence W' is smooth and connected. Moreover,

$$\dim(W') = \dim(W) + \dim(H_{i_0 j_0})$$

hence

$$\dim(W') = \dim(\tilde{W}')$$

As $\tilde{W}' \subset W'$ and both W' and $\tilde{W}'$ are smooth connected and have the same dimension, we conclude that

$$\tilde{W}' = W'$$

as $\tilde{W}'$ is a Tate-linear formal subvariety of F, we have proved the lemma. □

Lemma 4.7.14. ***(Functoriality of being Tate-linear II)*** *Let H, G be Tate-linear nilpotent groups of type A and E, F their universal deformation space respectively. Let $f : G \to H$ an isogeny and $\tilde{f} : F \to E$ the induced morphism between deformation spaces. If $W' \subset F$ a Tate-linear formal subvariety of F and $W := \tilde{f}(W')$, then W is a Tate-linear formal subvariety of E.*

Proof. Let $G' \subset G$ be the admissible subgroup of G corresponding to W' as in 4.7.8. Let $H' = f(G')$ a subgroup of H. Since f is an isogeny, in particular it is surjective, hence H' is also cotorsion free. Therefore H' is an admissible subgroup of H. Let $\tilde{W}$ be the Tate-linear formal subvariety corresponding to H'. Since the morphisms between deformation spaces of torsors induced by morphisms between groups are natural, we have

$$W \subset \tilde{W}$$

By 4.7.11, $\tilde{f}$ is an finite morphism. Hence

$$\dim(W) = \dim(W')$$

and W' is connected, reduced and irreducible.
As f is finite and faithfully flat by 4.7.11,

$$\dim(\tilde{W}) = \dim(H') = \dim(G') = \dim(W')$$

Therefore we conclude that

$$W = \tilde{W}$$

As $\tilde{W}$ is a Tate-linear formal subvariety, so is W. We have proved the lemma. □

4.8. Statement of The Orbital Rigidity Conjecture

Definition 4.8.1. *Let $H = (H_{ij})$ be a Tate-linear nilpotent group of type A and $E = Def_{H-torsor}$, and let $Aut(E) = Aut_{sus}(E)$ as defined in 4.4.8. We say that the action of G on E is strongly non-trivial if the induced action of G on each H_{ij} is strongly non-trivial in the sense of 3.3.1.*

Will all the relevant concepts defined, we state the main result of this thesis.

Theorem 4.8.2. *Let $H = (H_{ij})_{1\leq i<j\leq 4}$ be a Tate-linear nilpotent group of type A of rank 4 over an algebraically closed field κ of characteristic p with $p \geq 5$. Let $G \subset Aut(E)$ be a closed compact p-adic Lie subgroup, acting strongly non-trivially on E in the sense of 3.3.1. Let $W \subset E$ be a closed formal subscheme which is reduced and irreducible. If W is invariant under the action of G, then W is a Tate-linear subvariety.*

Theorem 4.8.2 will be proved in 7.4.1.

CHAPTER 5

The Orbital Rigidity Conjecture: 3-Slopes Case

The main result of this chapter is to state the orbital rigidity conjecture when $X = \prod_{i=1}^{3} X_i$, see 5.2.1 for the precise statement. This result was essentially proved in [CO22] Chapter 10. We rewrite it in a slightly different way and give a short proof based on the results in [CO22] in 5.3.

Notations 5.0.1.

1. *Let $H = (H_{ij})_{1\leq i<j\leq 3}$ be a Tate-linear nilpotent group of type A of rank 3 over an algebraically closed field κ of characteristic $p \geq 3$, we further assume H to be pure and perfect.*

2. *Let*

   - $E = Def_{H\text{-}tor}$ *which is a biextension.*
   - $B = E/H_{13} \simeq H_{12} \times H_{23}$,
   - $\pi : E \to B$, *the natural projection,*
   - $\psi_n : H_{12}[p^n] \times H_{23}[p^{2n}] \times H_{13} \to E_n$, *be Mumford's trivialization. as defined in 3.4.3.*

5.1. Admissible Subgroups and Tate-linear Subvarieties in 3-Slopes Case

Lemma 5.1.1 asserts that, under certain conditions on the bilinear pairing $\langle,\rangle_n$, we can construct a admissible subgroup, and characterize the Tate-linear subvariety associated to it.

Lemma 5.1.1. *Let $H = \varprojlim H_n$ be a Tate-linear nilpotent group of type A with components H_{ij} isoclinic p-divisible groups, $1 \leq i < j \leq 3$. Let $\langle,\rangle_n$ be the Weil pairing(s) $\langle,\rangle_n$:*

$H_{12}^n \times H_{23}^n \to H_{13}^n$. *Let* $P \subset H_{12} \times H_{23}$ *be a p-divisible subgroup satisfying*

$$\langle f_{12}^n, f_{23}^n{}' \rangle = \langle f_{12}^n{}', f_{23}^n \rangle, \ \forall (f_{12}^n, f_{23}^n), (f_{12}^n{}', f_{23}^n{}') \in P[p^n] \tag{5.1}$$

Consider the subscheme $H_{P,n}$ *of* H_n *defined by*

$$H_{P,n} = \left\{ \begin{pmatrix} 1 & f_{12}^n & \frac{1}{2}\langle f_{12}^n, f_{23}^n \rangle_n \\ 0 & 1 & f_{23}^n \\ 0 & 0 & 1 \end{pmatrix} : (f_{12}^n, f_{23}^n) \in P[p^n] \right\}$$

Then

(a). $H_{P,n}$ *is a subgroup scheme.*

(b). *Let* $H_P = \varprojlim H_{P,n} \subset H$, *then* H_P *is an admissible subgroup. Let* E_P *be the schematic image of the following morphism*

$$Def_{H_P\text{-}torsor} \to Def_{H\text{-}torsor}$$

i.e. E_P *is the Tate-linear subvariety corresponding to* H_P *in the sense of 4.7.8.* E^P *can be characterized as follows: let* ϕ_n, E_n *as defined in 4.3.2, then*

$$E_P \cap E_n = \phi_n(\left\{ ([p^n]f_{12}^{3n}, f_{23}^{3n}, \frac{1}{2}\langle f_{12}^{3n}, f_{23}^{3n} \rangle_{3n}) | \ \forall (f_{12}^{3n}, f_{23}^{3n}) \in P[p^{3n}] \right\})$$

(c). *If* $g \in Aut(E)$ *s.t. the restriction of the action of* g *on* $H_{12} \times H_{23}$ *keeps* P *invariant, then* g *acts on* E_P.

Proof. Part (a) is an easy algebra exercise.

Now we prove part (b). From 4.3.2, let $f = (f_{12}^{2n}, f_{23}^{3n}, f_{13}), f' = (f_{12}^{2n'}, f_{23}^{3n'}, f_{13}') \in H_{12}^{2n} \times H_{23}^{2n} \times H_{13}$. Let $\phi_n : H_{12}^{2n} \times H_{23}^{2n} \times H_{13} \to E_n$ as in 4.3.2. Assuming $\phi_n(f) = \phi_n(f')$, by 4.3.2

the gluing data of the universal $Aut^{st}(X)_n$ bundle is given by

$$\begin{pmatrix} 1 & f_{12}^{2n} - f_{12}^{2n'} & f_{13}^{2n} - f_{13}^{2n'} + \langle f_{23}^{3n} - f_{23}^{3n'}, f_{12}^{2n} \rangle_{2n} \\ 0 & 1 & f_{23}^{2n} - f_{23}^{2n'} \\ 0 & 0 & 1 \end{pmatrix} \tag{5.2}$$

note that as

$$f_{13}^{n} - f_{13}^{n'} + \langle f_{23}^{2n} - f_{23}^{2n'}, f_{12}^{n} \rangle_n = 0$$

this is an element in $Aut^{st}(X[p^n])$. When restrict to E_d, we have:

$$f_{13}^{2n} = \frac{1}{2} \langle f_{12}^{3n}, f_{12}^{3n} \rangle_{3n}$$

together the relations between $\langle , \rangle_n$ and $\langle , \rangle_m$, we have

$$f_{13}^{2n} - f_{13}^{2n'} + \langle f_{23}^{3n} - f_{23}^{3n'}, f_{12}^{2n} \rangle_{2n} = \frac{1}{2} (\langle f_{12}^{3n}, f_{12}^{3n} \rangle_{3n} - \langle f_{12}^{3n'}, f_{12}^{3n'} \rangle_{3n}) + \langle f_{23}^{3n} - f_{23}^{3n'}, f_{12}^{2n} \rangle_{2n}$$

$$= \frac{1}{2} \langle f_{12}^{3n} - f_{12}^{3n'}, f_{12}^{3n} - f_{12}^{3n'} \rangle_{3n} - \langle f_{12}^{3n}, f_{12}^{3n} - f_{12}^{3n'} \rangle_{3n} + \langle f_{23}^{3n} - f_{23}^{3n'}, f_{12}^{2n} \rangle_{2n}$$

$$= \frac{1}{2} \langle f_{12}^{3n} - f_{12}^{3n'}, f_{12}^{3n} - f_{12}^{3n'} \rangle_{3n} = \frac{1}{2} \langle f_{12}^{2n} - f_{12}^{2n'}, f_{12}^{2n} - f_{12}^{2n'} \rangle_{n}$$

that is the above matrix 5.2 simplifies to

$$\begin{pmatrix} 1 & f_{12}^{2n} - f_{12}^{2n'} & \frac{1}{2} \langle f_{12}^{2n} - f_{12}^{2n'}, f_{23}^{2n} - f_{23}^{2n'} \rangle_n \\ 0 & 1 & f_{23}^{2n} - f_{23}^{2n'} \\ 0 & 0 & 1 \end{pmatrix}$$

which means the structural group of E_P can be reduced to H_P, by 4.7.9 we have

$$E_P \subset Def_{H_P\text{-torsor}}$$

By dimension consideration we then have

$$dim(Def_{H_P\text{-torsor}}) = dim(P) = dim(E_P)$$

Since both spaces are reduced and irreducible, we conclude that

$$Def_{H_P\text{-torsor}} = E_P.$$

For (c), since E_P is constructed using P and Weil pairings, and every element $g \in Aut(E)$ preserves $\langle , \rangle_n$, hence if moreover g acts on P, g acts on E_P. □

5.2. The Orbital Rigidity Conjecture Three Slopes Case

The following theorem was essentially proved in [CO22] Chapter 10. We rewrite it in this form so that it can be used to prove our main result 7.4.1.

Theorem 5.2.1. *Notation as in 5.0.1. Let $W \subset E$ a closed formal subscheme, reduced and irreducible. Let $G \subset Aut(E)$ a closed p-adic subgroup whose action on E is strongly non-trivial in the sense of 3.3.1. Let $Y = (W \cap H_{13})_{red}$ where $H_{13} = \pi^{-1}(0_B) \subset E$, and let $X = \pi(W) \subset B = H_{12} \times H_{23}$. Both X, Y are p-divisible subgroups by the orbital rigidity conjecture of p-divisible groups. Then*

(a). Let $n \in \mathbb{N}$, let $x = (x_{12}^n, x_{23}^n), x' = ({x_{12}^n}', {x_{23}^n}') \in X[p^n]$, then

$$x_{12}^n {x_{23}^n}' - {x_{12}^n}' x_{23}^n \in Y[p^n]$$

(b). Let $(H_{X,Y})_n$ a subscheme of H defined as follows:

$$(H_{X,Y})_n = \left\{ \begin{pmatrix} 1 & x_{12} & \frac{1}{2}\langle x_{12}, x_{23}\rangle_n + y_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix}, \forall x = (x_{12}, x_{23}) \in X[p^n], y = y_{13} \in Y[p^n] \right\}$$

then $(H_{X,Y})_n$ is a sub group scheme of H_n. Let

$$H_{X,Y} := \varprojlim (H_{X,Y})_n$$

then $W = Image(Def_{H_{X,Y}\, torsor} \hookrightarrow Def_{H\text{-}torsor} = E)$. That is W is the Tate-linear subvariety corresponds to $H_{X,Y}$ in the sense of 4.7.8.

(c). In fact, W can be constructed from X, Y explicitly: let $W \cap E_n$ be the schematic intersection of W and E_n, then

$$W \cap E_n = \psi_{n,homo}(\left\{ (x_{12}^{2n}, x_{23}^{2n}, \frac{1}{2}\langle x_{12}^{2n}, x_{23}^{2n}\rangle_{2n} + y_{13}) | \ \forall (x_{12}^{2n}, x_{23}^{2n}) \in X[p^{2n}], y_{13} \in Y \right\})$$

where $\psi_{n,homo}$ is defined in 3.4.4.

Theorem 5.2.1 will be proved in 5.3.

We collect some results proved in [CO22] that will be used to prove 5.2.1.

Theorem 5.2.2. *Notation as in 5.0.1. Let $\Psi : Y \times E \to E$ be the morphism*

$$\Psi : Y \times E \to E \quad (y, e) \mapsto y * e$$

corresponding to the restriction to Y of the H_{13} action on E. Then

(a). W is invariant under the action of $Y = (W \cap H_{13})_{red}$. That is,

$$\Psi(Y \times W) \subset W$$

(b). Let $\bar{\pi} : E/Y \to B$ the map induced by $\pi : E \to B$. Then

$$\bar{\pi}|_{W/Y} : W/Y \mapsto \bar{\pi}(W/Y)$$

is purely inseparable.

Theorem 5.2.3. *Notation as in 5.0.1. Let $W \subset E$ a reduced irreducible formal subvariety. Let $G \subset Aut_{bi-extension}(E)$ a closed subgroup acting strongly non-trivially on E. If we further assume that*

- *W is invariant under the action of G.*
- *$\pi|_W : W \to \pi(W)$ is an schematic isomorphism.*

Then:

(a). If $\pi(W) \subset H_{12} \times H_{23}$ is a graph that corresponds to a homomorphism $f : H_{12} \to H_{23}$. That is

$$\pi(W) = \{(h_{12}, f(h_{12}))|h_{12} \in H_{12}\}$$

Then the bilinear pairings $\langle -, f(-)\rangle_n : H_{12}[p^n] \times H_{12}[p^n] \to H_{13}[p^n]$ are symmetric for all $n \in \mathbb{N}$. That is, for $h_{12}, h'_{12} \in H_{12}[p^n]$ functorial points,

$$\langle h_{12}, f(h'_{12})\rangle_n = \langle h'_{12}, f(h_{12})\rangle_n$$

(b). If $\pi(W) = H'_{12} \times H'_{23}$ for some $H'_{12} \subset H_{12}, H'_{23} \subset H_{23}$ both p-divisible subgroups, then for all $n \in \mathbb{N}$ and for all $h'_{12} \in H'_{12}[p^n], h'_{23} \in H'_{23}[p^n]$,

$$\langle h'_{12}, h'_{23}\rangle_n = 0$$

Lemma 5.2.4. *Notation as in 5.0.1. Let $P \subset H_{12} \times H_{23}$ a p-divisible subgroup. Let G*

a p-adic Lie group acting strongly non-trivially on E, and $s : P \to E$ a section which is invariant under the action of a G. Let $H'_{12} := (P \cap H_{1,2})_{red}$. Then

$$\langle h_{12}, h_{23} \rangle_n = 0, \forall h_{12} \in H'_{12}, h_{23} \in \pi_{23}(P) \tag{5.3}$$

Moreover, the section s descents to a section $s' : P/H'_{12} \to E/H'_{12}$.

Proof. Recall that E has two relative group law $+_1, +_2$. Let $-_1$ be the inverse group law of $+_1$. Define E' to be the schematic image, as a subscheme of E, of the composition

$$P \times H'_{12} \xrightarrow{(p,h_{12}) \to (s(p), s(p+h_{12}))} E \times E \xrightarrow{\quad -_1 \quad} E$$

Intuitively, given $(x_1, y), (x_2, y) \in P$ where $x_1, x_2 \in H_{12}$ and $y \in H_{23}$, we can consider the 'difference'

$$s(x_1, y) - s(x_2, y)$$

which lies in the fiber $E|_{(x_1 - x_2, y)}$. As we vary x_1, x_2, y we obtain E'.
E' is reduced and irreducible as $P \times H'_{12}$ is. As s is invariant under G, E' is invariant under the action of G. Moreover since

$$s|_{P \cap H_{12}} : H'_{12} \to H_{13}$$

must be trivial by the orbital rigidity theorem of p-divisible group and slope constrains $slope(H'_{12}) \langle slope(H_{13})$,

$$E'|_{(0,0)} = \varphi((H'_{12}, 0) \times 0_{H_{23}})$$

is also trivial. Note that

$$\pi(E') = H'_{12} \times \pi_{23}(P)$$

Therefore by 5.2.3(b).

$$\langle p_1, y\rangle_n = 0, \forall p_1 \in H'_{12}[p^n], y \in \pi_{23}(P)[p^]$$

which is 5.3. E' being trivial also means that s descents to a section

$$s' : P/H'_{12} \to E/H'_{12}$$

. ☐

Corollary 5.2.5. *In the 3-slopes case, if $W \subset E$ a subscheme invariant under the action of G s.t. $\pi : W \to \pi(W)$ is an isomorphism, then for $n \in \mathbb{N}$ and $(x_1, y_1), (x_2, y_2) \in H_{12}[p^n] \times H_{23}[p^n]$, we have*

$$\langle x_1, y_2\rangle_n = \langle x_2, y_1\rangle_n$$

Proof. By applying 5.2.4 we can reduce it to the case when $\pi(W) \subset H_{12} \times H_{23}$ is a graph that corresponds to a homomorphism $f : H_{12} \to H_{23}$. That is

$$\pi(W) = \{(h_{12}, f(h_{12}))|h_{12} \in H_{12}\}$$

then what we need to prove is precisely the statement of 5.2.3(a). ☐

5.3. Proof of 5.2.1

Let $H'_{13} = (W \cap H_{13})_{red}$. By 5.2.2, W is invariant under the action of H'_{13}, and

$$\bar{\pi}|_{W/H'_{13}} : W/Y \mapsto \bar{\pi}(W/H'_{13})$$

is purely inseparable where $\bar{\pi} : E/H_{13} \to B$ is the projection map induced by π.

We can take k_0 big enough such that the morphism

$$\mathcal{L} := [p^{k_0}]_{H_{12}\times H_{23}}$$

dominates $\pi : W/H'_{13} \to \pi(W)$ in the sense that there exists $\xi : \pi(W) \to W/H'_{13}$ such that

$$\pi|_{W/H'_{13}} \circ \xi = \mathcal{L}|_{\pi(W)}$$

Consider

$$E'_{\mathcal{L}} := E/H'_{13} \times_{B,\mathcal{L}} B \tag{5.4}$$

Note that $E'_{\mathcal{L}}$ is also a biextension of H_{13}/H'_{13} by $H_{12} \times H_{23}$, with bilinear pairings $\overline{\langle,\rangle}_n$: $H_{12}[p^n] \times H_{23}[p^n] \to H_{13}/H'_{13}$ induced by $\mathcal{L}$, that is

$$\overline{\langle h_{12}, h_{23}\rangle}_n = \langle [p^{k_0}]h_{12}, [p^{k_0}]h_{23}\rangle_n$$

and the natural morphism $h : E'_{\mathcal{L}} \to E/H'_{13}$ induced by the fiber product structure is a homomorphism in the sense of 4.4.7.

We know that the compact p-adic Lie group G operates on E/H'_{13} and W/H'_{13} is stable under the action of G. There exists a compact open subgroup $G'_{\mathcal{L}} \subset G$ which operates on $E_{\mathcal{L}}$, and the natural map $h : E_{\mathcal{L}} \to E/H'_{13}$ is equivariant with respect the the inclusion $G' \hookrightarrow G$. The morphism $\xi : \pi(W) \to W/H'_{13}$ defines a morphism $\xi_2 : \pi(W) \to E_{\mathcal{L}}$ such that $h \circ \xi_2 = \xi_1$. It follows that

$$\mathcal{L} \circ \pi_{E_{\mathcal{L}}} \circ \xi_2 = \pi_{E/H'_{13}} \circ \xi_1 = \mathcal{L}$$

Therefore

$$\pi_{E_L} \circ \xi_2 = id_{\pi(W)}$$

In other words ξ_2 is a section of the pullback $E_{\mathcal{L}}$ over $\pi(W)$. The following diagram summarizes the relations:

$$
\begin{array}{ccc}
E'_{\mathcal{L}} & \xrightarrow{h} & E/H'_{14} \\
\xi_2 \uparrow \downarrow \pi_{E'_{\mathcal{L}}} & \nearrow \xi_1 & \downarrow \pi' \\
B & \xrightarrow{\mathcal{L}} & B
\end{array}
$$

Moreover ξ_2 is equivariant with respect to the action of G' on E/H'_{13}. Let $W'_{\mathcal{L}}$ denotes the image of this section ξ_2, $\mathcal{G}'_{\mathcal{L}}$ the pullback of $\mathcal{G}$ by $\mathcal{L}$.

To summarize, we have the following diagram

$$
\begin{array}{ccc}
 & & (E, \mathcal{G}, G, W) \\
 & & \downarrow /H'_{13} \\
(E'_{\mathcal{L}}, \mathcal{G}'_{\mathcal{L}}, G'_{\mathcal{L}}, W'_{\mathcal{L}}) & \xrightarrow{\text{pullback by } \mathcal{L}} & (E/H'_{13}, \mathcal{G}/H'_{14}, G, W/H'_{13})
\end{array}
$$

By local rigidity theorem of p-divisible groups, $\pi(W) \subset H_{12} \times H_{23}$ is a p-divisible subgroup. As $\mathcal{L} = [p^{k_0}]$, $\pi(W)$ is preserved by pullback of $\mathcal{L}$, and $\pi(W) = \pi(W'_{\mathcal{L}})$. Recall that $X := \pi(W)$.

As $\xi_2 : X \to E'_{\mathcal{L}}$ a section that is equivariant under the action of G, by 5.2.5, we have

$$\overline{\langle h_{12}, h'_{23} \rangle_n} = \overline{\langle h'_{12}, h'_{23} \rangle_n}$$

for all $n \in \mathbb{N}$ and $(h_{12}, h_{23}), (h'_{12}, h'_{23}) \in X[p^n]$ functorial points. Given that $\overline{\langle h_{12}, h'_{23} \rangle_n} = \langle [p^{k_0}] h_{12}, [p^{k_0}] h'_{23} \rangle_n$ we conclude that

$$\langle h_{12}, h'_{23} \rangle_n = \langle h'_{12}, h'_{23} \rangle_n \tag{5.5}$$

which is precisely 5.2.1(a).

Given 5.5, by 5.1.1 there is an admissible subgroup $H_X \subset H'_{\mathcal{L}}$, where $H'_{\mathcal{L}}$ is the Tate-linear nilpotent group of type A corresponding to the biextension $E'_{\mathcal{L}}$. Let E_X be the Tate-linear formal subvariety corresponding to H_X. By 5.1.1(b),

$$\pi(E_X) = X$$

and by 5.1.1(c), any element $g \in Aut(E)$ that fixes P acts on E_P. In particular, the subgroup $G'_{\mathcal{L}}$ of G acts on E_X.

Let $s_X : X \to E'_{\mathcal{L}}$ be the section corresponding to E_X, as $\pi|_{E_X} : E_X \to X$ is an isomorphism. Then the difference

$$s_X - \xi_2 : X \to H_{13}/H'_{13}$$

is equivariant under the action of $G'_{\mathcal{L}}$. Hence by 5.4.1, it has to be trivial, that is

$$s_X = \xi_2$$

In particular, the schematic image of ξ_2 is a Tate-linear subvariety as E_X is.

As $h(\xi_2) \subset W/H'_{13}$ and both $h(\xi_2)$ and W/H_{13} are reduced, irreducible of dimension $dim(X)$, they must be equal, that is

$$h(\xi_2) = h(E_X) = W/H'_{13} \tag{5.6}$$

Part (c) of 5.2.1 is now an easy consequence of 5.6 and 5.1.1(b).

Given 5.2.1(c), 5.2.1(b) follows from 5.1.1. We have proved 5.2.1.

Remark 5.3.1. *The proof of 7.4.1 follows the same line as the proof of 5.2.1.*

5.4. Equivariant Maps

The following results will be used in the proof of 7.4.1. Roughly speaking, given certain slope constrains, an equivariant homomorphism from a biextension to a p-divisible group has to be trivial.

Theorem 5.4.1. *Let B be a biextension of $X \times Y$ by Z, all isoclinic p-divisible groups. Let P be another isoclinic p-divisible group. Assuming that the slope of P is strictly bigger than the slopes of X, Y, Z. Let G a p-adic Lie group that acts strongly non-trivially on both B and P, $f : B \to P$ an G-equivariant morphism of schemes. Then f is the trivial morphism.*

Proof. Pick $a, r, s \in \mathbb{Z}_{\geq 0}$ such that

$$s_P = \frac{a}{r}, \ \ s > r \text{ and } \frac{a}{s} > max(s_X, s_Y, s_Z)$$

Pick $h_1, ..., h_u$ with $u = dim(P)$ coordinate systems of P. Assuming that $[p^a]_P^*(h_i) = h_i^{p^r}$. Let $(R_B, m_B), (R_P, m_P)$ be the coordinate rings and maximal ideals of B, P respectively. Fix $v \in Lie(G)$, and let $g = exp(p^{na}v)$. Let $\phi_B : G \to Aut_{\text{bi-ext}}(B)$ the natural morphism induced by the action of G on B, and $\phi_P : G \to Aut_{\text{p-div}}(P)$ the natural morphism induced by the action of G on P. Let $\phi_{B,*}, \phi_{P,*}$ be the induced morphisms on Lie algebras. We have

1. $g(z_i) \equiv z_i + \phi_P(v)^*(z_i^{p^{nr}}) + O(z_i^{p^{2nr}})$, by the Taylor expansion of g and the fact that $s_P = \frac{a}{r}$.

2. $g(f^*(z_i))) = f^*(z_i) \mod m_B^{p^{ns}}$ as g acts trivially on $Spf(R_B/m_B^{p^{ns}})$ by **??**.

3. Since f is equivariant under the action of G,

$$g(f^*(z_i)) = f^*(g(z_i)) = f^*(z_i) + \phi_B(v)^*(f(z_i)^{p^{nr}}) \mod m_B^{p^{ns}}$$

   Thus

$$\phi_B(v)^*(f(z_i)^{p^{nr}}) \equiv 0 \mod m_B^{p^{ns}}$$

   as $s > r$, by by taking $n \to \infty$ this implies $\phi)B(v)^* f^*(z_i) = 0$, hence $f^*(z_i) = 0$ as we assume the action of G is strongly non-trivial,

□

5.5. An Auxiliary Result

Lemma 5.5.1. *Let $k \supset \mathbb{F}_p$ the base field, let $H = (H_{ij})_{1 \leq i < j \leq 3}$ be a Tate-linear nilpotent group of type A of rank 3 over k. Let $E = Def_{H\text{-}torsor}$ which is a bi-extension. Let $B = E/H_{13} = H_{12} \times H_{23}$, $B_n := (H_{12} \times H_{23})[p^n]$ and $\pi : E \to B$ the projection map. Let $(R_E.m_E), (R_B, m_B), (R_{H_{13}}, m_{H_{13}})$ be the coordinate ring and maximal ideal of E, B, H_{13} respectively. If N is an integer s.t.*

$$Spf(R_{H_{13}}/m_{H_{13}}^{(p^N)}) \subset H_{13}[p^n]$$
$$Spf(R_B/m_B^{(p^N)}) \subset B_n$$

then

$$Spf(R_E/m_E^{(p^N)}) \subset E_n[p^n]$$

Proof. Given the condition it's obvious that $\pi(R_E/m_E^{p^{(N)}}) \subset B_n$. Let

$$\psi_n : H_{12}^n \times H_{23}^{2n} \times H_{13} \to E_n$$

be Mumford's trivialization. Consider the following diagram:

$$\begin{array}{ccccc}
E_n & \xrightarrow{[Frob_F^N]} & (Frob_F^N)_* E_n & \dashrightarrow & b_0 \in E_n^{(p^N)} \\
\uparrow{\scriptstyle \psi_n} & & \uparrow & & \uparrow{\scriptstyle \psi_n^{(p^N)}} \\
H_{12}^n \times H_{23}^{2n} \times H_{13} & \xrightarrow{Frob_F^N} & H_{12}^n \times H_{23}^{2n} \times H_{13}^{(p^N)} & \longrightarrow & H_{12}^n \times (H_{23}^{2n})^{p^N} \times H_{13}(p^N)
\end{array}$$

where a supscript (p^N) denotes base changed by Frobenius to the Nth power.

Note that we used the following identity:

$$\langle Frob_{H_{12}}^N(-), Frob_{H_{23}}^N(-) \rangle_n = Frob_{H_{13}}(\langle -, - \rangle_n^{(p^N)})$$

The composition of the top arrows is the relative Frobenius of E_n, same with the bottom arrows. Let b_0 be the based point of $E^{(p^N)}$, we want to show that

$$(Frob_{E_n}^N)^{-1}(b_0) \subset E_n[p^n] = \psi_n(H_{12}^n \times H_{23}^{2n} \times H_{13}^n)$$

Let

$$\mathcal{F} = \psi_n^{(p^N)} \circ Frob^N_{H_{12}^n \times H_{23}^{2n} \times H_{13}}$$

Using the commutative diagram, if suffices to show that

$$\mathcal{F}^{-1}(b_0) \subseteq \psi_n^{-1}(E_n[p^n])$$

but this is obvious given that

$$\psi^{-1}(E_n[p^n]) = H_{12}^n \times H_{23}^{2n} \times H_{13}^n$$

and

$$(\psi_n^{(p^N)})^{-1}(b_0) = 0_{H_{12}} \times (H_{23}^n)^{(p^N)} \times 0_{H_{13}}$$

and combining these two we have

$$\begin{aligned} F^{-1}(b_0) =& (Frob^N_{H_{12}^n \times H_{23}^{2n} \times H_{13}})^{-1}(0_{H_{12}} \times (H_{23}^n)^{(p^N)} \times 0_{H_{13}}) \\ \subseteq& H_{12}^n \times H_{23}^{2n} \times Ker(Frob^{p^N}_{H_{13}}) \\ \subset& H_{12}^n \times H_{23}^{2n} \times H_{13} \\ =& \psi^{-1}(E_n[p^n]) \end{aligned}$$

□

We also need an analogy of 5.5.1 in the 4 slopes case.

Lemma 5.5.2. *Let $k \supset \mathbb{F}_p$ the base field. Let $H = (H_{ij})_{1 \le i < j \le 4}$ a Tate-linear nilpotent* *group of type A and* $E = Def_{H\text{-}tor}$. *Let* $\pi : E \to B$ *the natural projection. Let* $F = H_{14}$. *Let* $R_E, R_B, R_F, m_X, m_B, m_F$ *be the formal power series rings and maximal ideals corresponding to* E, B, F *respectively. Fix an integer* n *and let* B_n *as defined in 6.3.1. If* N *is an integer s.t.* $Spf(R_F/m_F^{(p^N)}) \subset F_n$ *and* $Spf(R_B/m_B^{(p^N)}) \subset B_n$, *then* $Spf(R_E/m_E^{(p^N)}) \subset E_n[p^n]$. *Equivalently, let* η_n *as defined in 4.2.5, then* $\eta_n \equiv 0 \mod m_E^{(p^N)}$.

Proof. The proof is an analogy of the proof of 5.5.1 hence omitted. □

5.6. Inseparable Isogenies That Dominante A Purely Inseparable Morphism

We proof the following results for later use. For this section E is a biextension with components H_{12}, H_{23}, H_{13} where H_{13} is the fiber. Recall that we have

$$\psi_n : H_{12}[p^n] \times H_{23}[p^{2n}] \times H_{13} \to E_n$$

We define a subscheme of E_n, for each $m \in \mathbb{N}$

$$E_n[p^m] := \psi(H_{12}[p^n] \times H_{23}[p^{2n}] \times H_{13}[p^m]$$

Theorem 5.6.1. *Let* E *be a biextension of p-divisible groups over a field* k *of characteristic* p, $\pi : F \to E$ *a finite purely inseparable cover with* F *reduced and irreducible. Then we can find an morphism of bi-extension* $f : E \to E$ *s.t.* f *factors through* $\pi : F \to E$.

Proof. let R_F be the ring of regular functions of F. By assumption R_F is a integral domain. Let $R_F = R_E[a_1, .., a_m]$ and $N > 0$ s.t. $a_i^{p^N} \in R_E \; \forall i$. Let n be a a big enough integer and $F_n : E \to E$ be defined as in 5.6.2 s.t.

$$F_n^*(R_E) \subseteq R_E^{(p^N)}$$

then F_n factors through f and we have proven the theorem. □

Lemma 5.6.2. *Let E be a biextension of $X \times Y$ with fiber Z. For any $n \in \mathbb{N}$, $([p_X^n], [p_Y^n], [p_Z^{2n}])$ induce an isogeny $F_n : E \to E$. Moreover, let R_E be the ring of regular functions of E and $F_n^* : R_E \to R_E$ the induced ring homomorphism of F_n, then for a fixed $N \in \mathbb{N}$ we have $F_n^*(R_E) \subset R_E^{(p^N)}, \forall n \gg 0$.*

Proof. The fact that $[p_X^n], [p_Y^n], [p_Z^{2n}]$ induces an isogeny follows easily from the identity

$$\langle [p^n]x_m, [p^n]y_m \rangle_m = [p_Z^{2n}]\langle x_m, y_m \rangle_m, \forall n, m, x_m \in X[p^m], y_m \in Y[p^m]$$

and the characterization of $End_{\text{bi-ext}}(E)$ as a subset of $End(X) \times End(Y) \times End(Z)$.
For the second part, let b_0 be the base point of E that corresponds to the maximal ideal $m_E \subseteq R_E$. By the construction we have

$$F_n^{-1}(b_0) = E_n[p^{2n}]$$

By 5.5.1, when n is big enough, we have

$$Spf(R_E/m_E^{(p^N)}) \subset E_n[p^n] \subset E_n[p^{2n}] = F_n^{-1}(b_0)$$

which implies that for such n

$$F_n^*(m_E) \subset m_E^{(p^N)}$$

□

Corollary 5.6.3. *If $\tilde{E} \subset E$ a Tate linear subvariety, then the above homomorphism $([p_X^n], [p_Y^n], [p_Z^{2n}])$ preserves $\tilde{E}$. Moreover, for each purely inseparable morphism $p : Y \to E'$, we can find a $n_0 \in \mathbb{N}$ s.t. the restriction of $([p_X^n], [p_Y^n], [p_Z^{2n}])$ to E' dominates p.*

Proof. The first part holds given that $([p_X^n], [p_Y^n], [P_Z^n])$ preserves the Weil pairing and $\pi(E') \subset X \times Y$ as $\pi(E')$ is a p-divisible subgroup of $X \times Y$.
The second part follows from the same argument as 5.6.1 and 5.6.2. □

CHAPTER 6

The Orbital Rigidity 4 Slopes Case: First Result

The main result of this chapter is 6.3.2 and 6.4.6. Similar results are proved in [CO22] Chapter 10, and we show that the techniques used in [CO22], especially the tempered perfections as discussed in 6.2, can also be used in our cases.

Notations 6.0.1.

1. *Let* $H = (H_{ij})_{1 \leq i > j \leq 4}$ *be a Tate-linear nilpotent group of type* A *of rank* 4 *that is pure and perfect over an algebraically closed field. For definitions see 4.4.1, 4.4.2 and 4.4.3. In particular we have*

$$s_{ij} + s_{jk} = s_{ik}, \forall 1 \leq i < j < k \leq 4$$

   where s_{ij} = *slope of* H_{ij}*.*

2. *Let* $E = Def_{H\text{-}tor}$*,* $\pi : E \to B$ *the natural projections. We also use the definitions of* B_n, E_n, A_n *as in 4.0.1.*

3. *Let* $s_{ij} = slope(H_{ij})$*. Let* $a_{ij}, r \in \mathbb{N}$ *satisfying*

$$s_{ij} = \frac{a_{ij}}{r}, \forall 1 \leq i < j \leq 4$$

   This implies

$$a_{ij} + a_{jk} = a_{ik}, \forall 1 \leq i < j < k \leq 4$$

4. *Let* $End_{sus}(H)$ *and* $Aut_{sus}(H)$ *be the ring of homomorphisms and group of automorphisms of* H*, respectively. See 4.4.7 and 4.4.8.*

5. *Let* $\psi_n : A_n \to E_n$ *as in 4.2.1.*

6. $\psi_n : A_n \to E_n$ *induces an projection* $\eta_n : [p^n]_* E_n \to H_{14}$*, see 4.2.4.*

7. *Let* $v = (A_{ij}) \in Lie(Aut_{biext}(E)) \subset \prod Lie(Aut(H_{ij})$*. Moreover we assume that* $A_{ij} \in End(H_{ij}) \subset Lie(Aut(H_{ij})$ *for all* $1 \leq i < j \leq 4$*.*

8. $\tilde{A}_{na_{14}} = (H_{12}^{na_{12}} \times H_{23}^{2na_{14}+na_{23}} \times H_{34}^{na_{14}+na_{34}} \times H_{13}^{na_{13}} \times H_{24}^{na_{14}+na24} \times H_{14})$.

9. $\tilde{E}_{na_{14}} = \psi_{na_{14}}(\tilde{A}_{na_{14}})$.

10. $\tilde{E}_{na_{14}}[p^m] = \psi_{na_{14}}((H_{12}^{na_{12}} \times H_{23}^{2na_{14}+na_{23}} \times H_{34}^{na_{14}+na_{34}} \times H_{13}^{na_{13}} \times H_{24}^{na_{14}+na_{24}} \times H_{14}[p^m]))$.

6.1. A Closed Form Formula for the Action on E: 4-Slopes Case

The main result of this section is 6.1.1, which states that when we restrict to a small enough subscheme $\tilde{E}_n \subset E_n \subset E$, then the action of certain $g \in Aut(E)$ is a 'torsor action', and in fact this action can be described explicitly.

Lemma 6.1.1. *Notations as in 6.0.1.*

a). *For every* $n \geq 2$*, the infinite series*

$$\sum_{j\geq 2} \frac{p^{n(j-1)}}{j!} A_{14}^j$$

converges to an element of $End(H_{14})$*.*

b). *For* $x \in \tilde{E}_{na_{14}}$ *a functorial point and* $n \geq 2$*,*

$$exp(p^{na_{14}}v)(x) = ((\sum_{j=1}^{\infty} \frac{p^{(j-1)na_{14}}}{j!} A_{14}^j \eta_{na_{14}}(x)) + e_{na_{14}}^v(x)) * x$$

where $*$ *denotes the torsor structure of* $H_{1,4}$ *on* E*, and* $e_{na_{14}}^v(x)$ *is a point of* $H_{14}[p^{na_{14}}]$ *that depends only on* $\pi(x)$*,* na_{14} *and* $v = (A_{ij})$*.*

c). *For all* $m \leq 2n$ *and for* $x \in \tilde{E}_{na_{14}}[p^{ma_{14}}]$ *a functorial point, we have*

$$exp(p^{na_{14}}v)(x) = (A_{14}\eta_{na_{14}}(x)) + e_{na_{14}}^v(x)) * x \tag{6.1}$$

Proof. Part a). follows from the easy estimate that

$$ord_p(k!) \leq \frac{2k}{p} <= k$$

Now we prove part b). Fix an Artinian local ring R, let

$$x \in \tilde{E}_{na_{14}}(R)$$

a R point and let

$$(x_{ij}) \in \tilde{A}_{na_{14}}(R')$$

be a 'preimage' of x in $\tilde{A}_{na_{14}}$, for some faithfully flat cover R' of R, i.e.

$$\psi_n((x_{ij})) = x_{R'}$$

Since the group $Aut(E)$ also acts on $\tilde{A}_{na_{14}}$ and this action on $\tilde{A}_{na_{14}}$ descents to $\tilde{E}_{na_{14}}$ via the faithfully flat morphism $\psi_{na_{14}}$, therefore if

$$g = exp(p^{na_{14}}v)$$

then

$$g(x)_{R'} = \psi_{na_{14}}(g(x_{ij})_{1\leq i<j\leq 4})$$

where

$$(g(x_{ij}) = (exp(p^{na_{14}} \cdot A_{ij}) \cdot x_{ij}) \equiv (x_{ij} +_{H_{ij}} A_{ij}p^{na_{14}}x_{ij}) \mod A_{na_{14}}, \text{for}(i,j) \neq (2,3),(1,4),$$
$$g(x_{23}) \equiv x_{23} + A_{23}p^{na_{14}}x_{23} + \frac{A_{23}p^{2na_{14}}x_{23}}{2} \mod A_{na_{14}},$$
$$g(x_{14}) = x_{14} + \sum_{j=1}^{\infty}(\frac{p^{na_{14}j}}{j!})A_{14}^{j}x_{14}$$

Using 4.2.1 we can further show that

$$\psi_{na_{14}}(g(x_{ij})_{1\leq i<j\leq 4}) = \psi_{na_{14}}((x_{ij}+f_{ij}(x))_{1\leq i<j\leq 4})$$

where

$$f_{ij}(x) = 0,\ \ \forall 1\leq i<j\leq 4, (i,j)\neq(1,4)$$

$$f_{14}(x) = (\sum_{j=1}^{\infty}\frac{p^{jna_{14}}}{j!}A_{14}^{j})x_{14} + \langle p^{na_{14}}A_{34}x_{34}, x_{13}\rangle_{na_{14}} +$$
$$\langle p^{na_{14}}x_{24} + \langle x_{34}, p^{na_{14}}A_{23}x_{23} + \frac{p^{2na_{14}}A_{23}x_{23}}{2}\rangle_{2na_{14}}, x_{12}\rangle_{na_{14}}$$

Since $(x_{ij})\in\tilde{A}_{na_{14}}$, we have

$$\langle x_{34}, p^{2na_{14}}x_{23}\rangle_{2na_{14}} = 0$$

as $x_{34}\in H_{34}^{na_{14}+na_{34}}$ and $x_{23}\in H_{23}^{2na_{14}+na_{23}}$ and $a_{14}>a_{24}=a_{34}+a_{23}$. Hence f_{14} simplifies to

$$f_{1,4}(x) = (\sum_{j=1}^{\infty}\frac{p^{jna_{14}}}{j!}A_{14}^{j})x_{14} + \langle p^{n}A_{34}x_{34}, x_{13}\rangle_{n} + \langle p^{n}x_{24} + \langle x_{34}, p^{n}A_{23}x_{23}\rangle_{2n}, x_{12}\rangle$$

Therefore

$$\begin{aligned} g(x) =& f_{14}(x) * x \\ =& (\sum_{j=1}^{\infty}(\frac{p^{na_{14}(j-1)}}{j!}A_{14}^{j})\eta_{n}(x) + \langle p^{n}A_{34}x_{34}, x_{13}\rangle_{n} + \langle p^{n}x_{24} + \langle x_{34}, p^{n}A_{23}x_{23}\rangle_{2n}, x_{12}\rangle) * x \end{aligned}$$

we will adopt the notation

$$e_{na_{14}}^{v}(x) := f_{14}(x) - \sum_{j=1}^{\infty}(\frac{p^{j-1na_{14}}}{j!}A_{14})\eta_{na_{14}}(x) \tag{6.2}$$

and rewrite the above equation as

$$g(x) = (\sum_{j=1}^{\infty}(\frac{p^{j-1}na_{14}}{j!}A_{14})\eta_{na_{14}}(x) + e^v_{na_{14}}(x)) * id_{\tilde{E}_{na_{14}}}(x)$$

note that as $e^v_{na_{14}}(x)$ is calculated with $(x_{12}, x_{13}, p^{na_{14}}x_{23}, p^{na_{14}}x_{24}, p^{na_{14}}x_{34})$, it depends only on $\pi(x), v, na_{14}$ where $\pi : E \to B$ the natural projection.

Finally part c). follows from the fact that

$$p^{na_{14}}\eta_{na_{14}}(x) \equiv 0$$

for $x \in \tilde{E}_{na_{14}}[p^{2na_{14}}]$. □

Lemma 6.1.2. *Let e^v_n as defined in 6.1.1, see 6.2. In particular e^v_n is a function $e^v_n : E_n \to H_{14}$ that factors through $\pi : E_n \to B$. Let $x \in E_n$. Then $e^v_{n+1}(x) = [p]_{H_{14}} \cdot e^v_n(x)$ and $e^v(0_{E_n}) = 0_{H_{14}}$.*

Proof. We have the following commutative diagram

$$\begin{array}{ccc}
\tilde{E}_n & \xrightarrow{\hookrightarrow} & \tilde{E}_{n+1} \\
\uparrow \psi_n & & \uparrow \psi_{n+1} \\
\tilde{A}_n & \xleftarrow{\mathcal{P}} & \widetilde{A'_{n+1}} \subset \tilde{A}_{n+1}
\end{array}$$

where

$$\widetilde{A'_{n+1}} = H^n_{12} \times H^n_{13} \times H_{14} \times H^{3n+2}_{23} \times H^{2n+1}_{24} \times H^{2n+1}_{34}, \tag{6.3}$$

$$\mathcal{P} = (id_{12}, id_{13}, id_{14}, [p^2]_{23}, [p]_{24}, [p]_{34}) \tag{6.4}$$

For given a preimage (x_{ij}) of x in $\widetilde{A_n}$, a preimage of x in $\tilde{A}_{n+1}$ can be taken as (x'_{ij}) s.t.

$p(x'_{ij}) = (x_{ij})$. Then we have

$$e_n^v(x) = \langle p^n A_{34} x_{34}, x_{13}\rangle_n + \langle p^n x_{24} + _{34}, p^n A_{23} x_{23}\rangle_{2n}, x_{12}\rangle) * id_{E_n}(x), \tag{6.5}$$

$$e_{n+1}^v(x) = \langle p^{n+1} A_{34} x'_{34}, x'_{13}\rangle_{n+1} + \langle p^{n+1} x'_{24} + \langle x'_{34}, p^{n+1} A_{23} x'_{23}\rangle_{2n+2}, x'_{12}\rangle_{n+1} \tag{6.6}$$

using

$$\langle x, y\rangle_{n+1} = p\langle x, y\rangle_n, \forall x, y \in [p^n], \tag{6.7}$$

$$x'_{12} = x_{12}, x'_{13} = x_{13}, x'_{14} = x_{14}, \tag{6.8}$$

$$[p^2]x'_{23} = x_{23}, [p]x'_{34} = x_{34}, [p]x'_{24} = x_{24} \tag{6.9}$$

it's easy to see that

$$e_n^v(x) = [p]_{14} \cdot e_{n+1}^v(x)$$

□

6.2. Tempered Perfection

We collect some definitions and results as given in [CO22] Chapter 10. These tempered perfection rings are used in the proof of 6.3.2 and 6.4.6.

Definition 6.2.1. *Let κ be a perfect field of characteristic p and let $t_1, ..., t_m$ be m variables, $m \geq 1$. Let $r, s \in \mathbb{Z}_{\geq 0}$ be two positive integers with $r < s$, and let n_0 be a natural numbers. The perfection of the formal power series ring $\kappa[[t_1, .., t_m]]$ is naturally isomorphic to*

$$\bigcup_{n \in \mathbb{N}} \kappa[[t_1^{p^{-n}}, .., t_m^{p^{-n}}]]$$

Denote by ϕ the Frobenius automorphism of this perfect ring.

(a) *Consider the following subring*

$$(\kappa\langle\langle t_1^{p^{-n}},..,t_m^{p^{-n}}\rangle\rangle^{\#}_{s:\phi^r;[i_0]})_{fin} := \sum_{n\in\mathbb{N}} \phi^{-nr}((\underline{t})^{(p^{ns-i_0})})$$

of the perfection of the formal power series ring $\kappa[[t_1^{p^{-n}},..,t_m^{p^{-n}}]]$*, where our convention is that* $(\underline{t})^{(p^{ns-i_0})} = R$ *if* $ns - i_0 \leq 0$*.*

- *Define a decreasing filtration* $Fil^{\#,p^{\bullet}}_{s:\phi^r,[i_0]}$ *on* $(\kappa\langle\langle t_1^{p^{-n}},..,t_m^{p^{-n}}\rangle\rangle^{\#}_{s:\phi^r;[i_0]})_{fin}$ *by ideals*

$$Fil^{\#,p^j}_{s:\phi^r,[i_0]} := \left\{x \in (\kappa\langle\langle t_1^{p^{-n}},..,t_m^{p^{-n}}\rangle\rangle^{\#}_{s:\phi^r;[i_0]})_{fin} \mid \quad \exists n \in \mathbb{N}_{>0} \text{ s.t. } n+j \geq 0 \text{ and } x^{p^n} \in (\underline{t})^{(p^{n+j})}\right\}$$

of $(\kappa\langle\langle t_1^{p^{-n}},..,t_m^{p^{-n}}\rangle\rangle^{\#}_{s:\phi^r;[i_0]})_{fin}$*, where* $(\underline{t})$ *is the maximal ideal of* $\kappa[[t_1,..,t_m]]$*.*

- *Define* $\kappa\langle\langle t_1^{p^{-n}},..,t_m^{p^{-n}}\rangle\rangle^{\#}_{s:\phi^r;[i_0]}$ *to be the completion of the ring*

$$(\kappa\langle\langle t_1^{p^{-n}},..,t_m^{p^{-n}}\rangle\rangle^{\#}_{s:\phi^r;[i_0]})_{fin}$$

with respect to the filtration $Fil^{\#,p^{\bullet}}_{s:\phi^r,[i_0]}$*.*

(b) *Consider the following subring*

$$(\kappa\langle\langle t_1^{p^{-n}},..,t_m^{p^{-n}}\rangle\rangle^{b}_{s:\phi^r;[i_0]})_{fin} := \sum_{n\in\mathbb{N}} \phi^{-nr}((\underline{t})^{p^{ns-i_0}})$$

of the perfection of the formal power series ring $\kappa[[t_1^{p^{-n}},..,t_m^{p^{-n}}]]$*, where our convention is that* $(\underline{t})^{(p^{ns-i_0})} = R$ *if* $ns - i_0 \leq 0$*.*

- *Define a decreasing filtration* $Fil^{b,p^{\bullet}}_{s:\phi^r,[i_0]}$ *on* $(\kappa\langle\langle t_1^{p^{-n}},..,t_m^{p^{-n}}\rangle\rangle^{b}_{s:\phi^r;[i_0]})_{fin}$ *by ideals*

$$Fil^{b,p^j}_{s:\phi^r,[i_0]} := \left\{x \in (\kappa\langle\langle t_1^{p^{-n}},..,t_m^{p^{-n}}\rangle\rangle^{b}_{s:\phi^r;[i_0]})_{fin} \mid \quad \exists n \in \mathbb{N}_{>0} \text{ s.t. } n+j \geq 0 \text{ and } x^{p^n} \in (\underline{t})^{(p^{n+j})}\right\}$$

of $(\kappa\langle\langle t_1^{p^{-n}},..,t_m^{p^{-n}}\rangle\rangle^{b}_{s:\phi^r;[i_0]})_{fin}$*, where* $(\underline{t})$ *is the maximal ideal of* $\kappa[[t_1,..,t_m]]$*.*

- *Define* $\kappa\langle\langle t_1^{p^{-n}},..,t_m^{p^{-n}}\rangle\rangle^b_{s:\phi^r;[i_0]}$ *to be the completion of the ring*

$$(\kappa\langle\langle t_1^{p^{-n}},..,t_m^{p^{-n}}\rangle\rangle^b_{s:\phi^r;[i_0]})_{fin}$$

with respect to the filtration $Fil^{b,p^\bullet}_{s:\phi^r,[i_0]}$.

Definition 6.2.2. *Let* $\kappa \supset \mathbb{F}_b$ *be a perfect field and let* $t_1,...,t_m$ *be variables. Let* $C>0, d\geq 0, E>0$ *be real numbers.*

1. *Define a commutative algebra*

$$\kappa\langle\langle t_1^{p^{-\infty}},...,t_m^{p^{-\infty}}\rangle\rangle^{E,\#}_{C,d}$$

whose underlying abelian group is the set of all formal series $\sum_I b_I \underline{t}^I$ *with* $b_I\in\kappa$ *for all* I*, here* I *runs through all elements in* $\mathbb{N}[\frac{1}{p}]^m$ *such that*

$$|I|_p \leq Max(C\cdot(|I|_\infty+d)^E,1)$$

here for any multi-index $I=(i_1,...,i_m)\in\mathbb{Z}[1/p]^m_{\geq 0}$*,* $|I|_p$ *is the p-adic norm of* I *and* $|I|_{\infty,\max}$ *is the archimedean norm of* I*, defined by*

$$|I|_p := max(p^{-ord_p(i_1)},...,p^{-ord_p(i_m)})$$
$$|I|_{\infty,max} := \max(i_1,i_2,...,i_m)$$

2. *Define a commutative algebra*

$$\kappa\langle\langle t_1^{p^{-\infty}},...,t_m^{p^{-\infty}}\rangle\rangle^{E,\flat}_{C,d}$$

whose underlying abelian group is the set of all formal series $\sum_I b_I \underline{t}^I$ *with* $b_I\in\kappa$ *for* *all I, where I runs through all elements in $\mathbb{N}[\frac{1}{p}]^m$ such that*

$$|I|_p \leq Max(C \cdot (|I|_\sigma + d)^E, 1)$$

where

$$|I|_\sigma := |i_1| + |i_2| ... + |i_m|$$

Definition 6.2.3. *Let (R, m) be an augmented complete Noetherian local domain over a perfect field κ characteristic p. Let R^{perf} be the perfection of R, and let ϕ be the Frobenius automorphism on R. Let A, b, d be real numbers, $A, b > 0$ and $d \geq b$.*

(a) Define a decreasing filtration $(Fil^{\bullet}_{R^{perf},deg})_{\bullet \in \mathbb{R}}$ on R^{perf} indexed by real numbers u by

$$Fil^u_{R^{perf},deg} := \begin{cases} \{x \in R^{perf} | \exists j \in s.t. x^{p^j} \in m^{u \cdot p^j}\}, & \text{if } u \geq 0 \\ R^{perf}, & \text{if } u \leq 0 \end{cases}$$

It is easy to see that $Fil^u_{R^{perf},deg}$ is an ideal of R^{perf} for evrey $u \in \mathbb{R}$.

(b) Define a subring $((R, m)^{perf,b}_{A,b;d})_{fin}$ of R^{perf} by

$$((R, m)^{perf,b}_{A,b;d})_{fin} := \sum (\phi^{-n} R \bigcap Fil^{b \cdot p^{An} - d}_{R^{perf},deg})$$

It is not difficult to see that $((R, m)^{perf,b}_{A,b;d})_{fin}$ is a subring of R^{perf}.

(c) Define

$$(R, m)^{perf,b}_{A,b;d}$$

to be the completion of $((R, m)^{perf,b}_{A,b;d})_{fin}$ with respect to the filtration induced by the filtration $(Fil^{\bullet}_{R^{perf},deg})$ of R^{perf}:

$$(R, m)^{perf,b}_{A,b;d} = \lim_{u \to \infty} ((R, m)^{perf,b}_{A,b;d})_{fin} / (Fil^u_{R^{perf},deg} \bigcap ((R, m)^{perf,b}_{A,b;d})_{fin})$$

(d) *Define a filtration* $(Fil^{\bullet}_{(R,m)^{perf,b}_{A,b;d}})_{\bullet}$ *on* $(R,m)^{perf,b}_{A,b;d}$ *by*

$$Fil^{u}_{(R,m)^{perf,b}_{A,b;d}} := \lim_{v\to\infty}(Fil^{u}_{R^{perf},deg}\cap((R,m)^{perf,b}_{A,b;d})_{fin})/(Fil^{v}_{R^{perf},deg}\cap((R,m)^{perf,b}_{A,b;d})_{fin})$$

To state 6.2.5, we set up some notations.

Notations 6.2.4. ***(The setup for 6.2.5)***

1. *Let* (R,m) *be an augmented complete Noetherian local domain over a perfect field* κ *of characteristic* p*. Let* $(R,m)^{perf,b}_{A,b:d}$ *be a tempered perfection of* R*, where* A,b,d *are real numbers,* $A,b>0, d\geq b$*. See 6.2.3 for the definition of* $(R,m)^{perf,b}_{A,b:d}$*.*

2. *The tempered perfection* $(R,m)^{perf,b}_{A,b:d}$ *carries a filtration*

$$(Fil^{\bullet}_{(R,m)^{perf,b}_{A,b:d},deg})_{\bullet}$$

which is induced by the filtration $Fil^{\bullet}_{R^{perf}_{deg}}$ *on the perfection* R^{perf} *of* R*.*

3. *Let* $m,m'>0$ *be positive integers, and let*

$$\kappa\langle\langle \underline{u}^{p^{-\infty}},\underline{v}^{p^{-\infty}}\rangle\rangle^{E,b}_{C;d}=\kappa\langle\langle u_1^{p^{-\infty}},...,u_m^{p^{-\infty}},v_1^{p^{-\infty}},v_{m'}^{p^{-\infty}}\rangle\rangle^{E,b}_{C;d}$$

be a tempered perfection of $\kappa[[\underline{u},\underline{v}]]=\kappa[[u_1,...,u_m,v_1,..,v_{m'}]]$*, where* E,C,d *are real numbers,* $E,C>0$ *and* $d\geq 0$*.*

4. *Let* $g_1,...,g_m,h_1,...,h_{m'}$ *be elements of the maximal ideal of* $(R,m)^{perf,b}_{A,b:d}$*.*

5. *Let* $A'>0, b'>0, d'\geq b'$ *be real numbers such that the following conditions hold.*

   - *The continuous ring homomorphism*

$$ev_{\underline{g}\otimes 1,1\otimes\underline{h}}:\kappa[[u_1,...,u_m,v_1,..,v_{m'}]]\longrightarrow(R\hat{\otimes}_{\kappa}R,m_{R\hat{\otimes}_{\kappa}R})^{perf,b}_{A,b:d}$$

which sends a typical formal power series

$$f(u_1, ..., u_m, v_1, ..., v_{m'}) \in \kappa[[u_1, ..., u_m, v_1, ..., v_{m'}]]$$

to

$$f(g_1 \otimes 1, ..., g_m \otimes 1, 1 \otimes h_1, ..., 1 \otimes h_{m'}) \in (R\hat{\otimes}_\kappa R, m_{R\hat{\otimes}_\kappa R})^{perf,b}_{A,b:d}$$

extends to a continuous ring homomorphism

$$ev_{\underline{g}\otimes 1, 1\otimes \underline{h}} : \kappa\langle\langle \underline{u}^{p^{-\infty}}, \underline{v}^{p^{-\infty}} \rangle\rangle^{E,b}_{C;d} \longrightarrow (R\hat{\otimes}_\kappa R, m_{R\hat{\otimes}_\kappa R})^{perf,b}_{A,b:d}$$

The existence of such a triple (A', b', d') is straight-forward from the definitions. See [CO22] Chapter 9 for case when (R, m) is a formal power series ring.

- *The continuous ring homomorphism*

$$ev_{\underline{g}\otimes 1, 1\otimes \underline{h}} : \kappa[[u_1, ..., u_m, v_1, .., v_{m'}]] \longrightarrow (R, m)^{perf,b}_{A,b:d}$$

which sends a typical formal power series

$$f(u_1, ..., u_m, v_1, ..., v_{m'}) \in \kappa[[u_1, ..., u_m, v_1, ..., v_{m'}]]$$

to

$$f(g_1, ..., g_m, h_1, ..., h_{m'}) \in (R, m)^{perf,b}_{A,b;d}$$

extends to a continuous ring homomorphism

$$ev_{\underline{g},\underline{h}} : \kappa\langle\langle \underline{u}^{p^{-\infty}}, \underline{v}^{p^{-\infty}} \rangle\rangle^{E,b}_{C;d} \longrightarrow (R, m)^{perf,b}_{A',b':d'}$$

- *The diagram*

$$\begin{array}{ccc} \kappa\langle\langle \underline{u}^{p^{-\infty}}, \underline{v}^{p^{-\infty}}\rangle\rangle^{E,b}_{C;d} & \xrightarrow{ev_{\underline{g}\otimes 1, 1\otimes \underline{h}}} & (R\hat{\otimes}_\kappa R, m_{R\hat{\otimes}_\kappa R})^{perf,b}_{A,b:d} \\ = \Big\downarrow & & \Big\downarrow \Delta^* \\ \kappa\langle\langle \underline{u}^{p^{-\infty}}, \underline{v}^{p^{-\infty}}\rangle\rangle^{E,b}_{C;d} & \xrightarrow{ev_{\underline{g},\underline{h}}} & (R,m)^{perf,b}_{A',b':d'} \end{array}$$

commutes, where the vertical arrow Δ^* is induced by the multiplication map $\Delta : R \otimes R \to R$ for the κ-algebra R.

6. *For every element $f \in \kappa\langle\langle \underline{u}^{p^{-\infty}}, \underline{v}^{p^{-\infty}}\rangle\rangle^{E,b}_{C;d}$, define elements*

$$f(\underline{g}, \underline{f}) \in (R,m)^{perf,b}_{A',b':d'} \text{ and } f(\underline{g}\otimes 1, 1\otimes \underline{h}) \in (R\hat{\otimes}_\kappa R, m_{R\hat{\otimes}_\kappa R})^{perf,b}_{A,b:d}$$

by

$$f(\underline{g}, \underline{h}) f(g_1, ..., g_m, h_1, ..., h_{m'}) := ev_{\underline{g},\underline{h}}(f)$$
$$f(\underline{g}\otimes 1, 1\otimes h) = f(g_1\otimes 1, ..., g_m \otimes 1, 1\otimes h_1, ..., 1\otimes h_{m'}) := ev_{\underline{g}\otimes 1, 1\otimes \underline{h}}(f)$$

Theorem 6.2.5. ***(Hypocotyl elongation for tempered virtual functions).*** *We use the notation in 6.2.4. Let (R, m) be an augmented complete Noetherian local domain over a perfect field κ of characteristic p.*

- *Let $g_1, .., g_m, h_1, ..., h_{m'}$ be elements of the maximal ideal of $(R, m)^{perf,b}_{A,b;d}$.*

- *Let $f(i_1, ..., u_m, v_1, ..., v_{m'})$ be an element of*

$$\kappa\langle\langle u_1^{p^{-\infty}}, ..., u_m^{p^{-\infty}}, v_1^{p^{-\infty}}, v_{m'}^{p^{-\infty}}\rangle\rangle^{E,b}_{C;d}$$

which lies in the closure of the image of

$$\kappa\langle\langle \underline{u}^{p^{-\infty}}\rangle\rangle_{C;d}^{E,b} \otimes \kappa\langle\langle \underline{v}^{p^{-\infty}}\rangle\rangle_{C;d}^{E,b} \longrightarrow \kappa\langle\langle \underline{u}^{p^{-\infty}}, \underline{v}^{p^{-\infty}}\rangle\rangle_{C;d}^{E,b}$$

- *Let $q = p^r$ be a power of p for some positive integer r. Let $(d_n)_{n\in\mathbb{N}, n\geq n_0}$ be a sequence of positive integers such that $\lim_{n\to\infty} \frac{q^n}{d_n} = 0$.*

Suppose that

$$f(g_1, .., g_m, h_1^{q^n}, ..., h_{m'}^{q^n}) \equiv 0, \quad \mathrm{mod}\ Fil^{d_n}_{(R,m)^{perf,b}_{A',b';d'},deg}$$

in $(R,m)^{perf,b}_{A',b';d'}$ for all $n \geq n_0$. Then

$$f(g_1 \otimes 1, ..., g_m \otimes 1, 1 \otimes h_1, .., 1 \otimes h_{m'}) = 0$$

in the completed tempered perfection $(R\hat{\otimes}_\kappa R, m_{R\hat{\otimes}_\kappa R})^{perf,b}_{A',b':d'}$ of $R\hat{\otimes}_\kappa R$.

Proof. See [CO22] Chapter 10. □

6.3. Proof of The First Result

Notations 6.3.1. *We set up some notations for 6.3.2. Note that these notations are compatible with 6.0.1.*

1. *We use all the notations as in 6.0.1.*

2. *Let $H = (H_{ij})_{1\leq i>j\leq 4}$ be a Tate-linear nilpotent group of type A of dimension 4 that is pure and perfect. For definitions see 4.4.1, 4.4.2 and 4.4.3.*

3. *Let $E = Def_{H\text{-}tor}$, $\pi : E \to B$ the natural projections. We also use the definitions of B_n, E_n, A_n as in 4.0.1. Let*

$$R_E = \kappa\langle\langle t_1, ..., t_m\rangle\rangle$$

where R_E is the ring of regular functions of E. Note that E is formally smooth.

4. *Let* $s_{ij} = slope(H_{ij})$. *Let* $a_{ij}, r, s \in \mathbb{N}$ *satisfying*

   (a) $s_{ij} = \frac{a_{ij}}{r}, \forall 1 \leq i < j \leq 4$

   (b) $r < s < 2r$, *hence* $s_{14} = \frac{a_{14}}{r} > \frac{a_{14}}{s} > \frac{a_{14}}{2r}$

   (c) $s_{14} = \frac{a_{14}}{r} > \frac{a_{14}}{s} > s_{kl}, \forall (k,l) \neq (1,4)$.

5. *Fix* $n_2, c_2 \in \mathbb{N}$ *such that* $H_{ij}[p^n] \supset H_{ij}[Frob_p^{\frac{n}{s_{ij}} - c_2}]$ *for all* $1 \leq i < j \leq 4$ *and* $n \geq n_2$.

6. *Let* $n_3 \in \mathbb{N}$ *such that* $H_{ij}[p^{na_{ij}}] \supset H_{ij}[Frob_p^{ns}]$ *for all* $n \geq n_3, (i,j) \neq (1,4)$, *and that* $H_{14}[p^{na_{14}}] \subset H_{14}[Frob_p^{ns}] \subset H_{14}[p^{2na_{14}}]$ *for all* $n \geq n_3$.

Theorem 6.3.2. *Notations as in 6.3.1. Further, Let* G *a p-adic Lie group acting strongly non-trivially on* E *and* $W \subset E$ *a reduced irreducible formal subscheme of* E *that is invariant under the action of* G. *Let* $H'_{14} = (W \cap H_{1,4})_{red}$ *be the intersection of* W *with* $H_{1,4}$ *endowed with reduced structure. By orbital rigidity theorem of p-divisible group 1.1.1 we know* H'_{14} *is a p-divisible subgroup of* $H_{1,4}$. *Let*

$$\mathcal{Y} : H'_{14} \times E \to E$$

$$(h'_{14}, e) \to h'_{14} * e$$

corresponding to the restriction to H'_{14} *of the action of* H_{14} *on* E. *Let* $v = (A_{ij}) \in Lie(G)$ *be an element of the Lie algebra of* G *such that* $A_{ij} \in End(H_{ij})$.

a) *Then*

$$(\mathcal{Y} \circ (A_{14}|_{H'_{14}} \times id_W))(H'_{14} \times W) \subset W$$

b) *Assume in addition that the action of* G *on* H'_{14} *is strongly non-trivial. Then*

$$\mathcal{Y}(H'_{14} \times W) \subset W$$

Proof. We first show that 6.3.2.a) $\implies$ 6.3.2.b).

By 3.3.2, the assumption that the action of G on H'_{14} is strongly non-trivial implied that there exists elements $h^{kl} = (h^{kl}_{ij}) \in Lie(G)$, indexed by a finite subset

$$\{(k,l) \in \mathbb{N}^2 : k \in \{1, ..., m\}, l \in \{1, ..., n_k\}\}$$

where $n_k \in \mathbb{N}_{\geq 1}$ for each $k = 1, .., m$, such that

$$\sum_{1 \leq k \leq m} h^{k1}_{14} \circ h^{k2}_{14} ... \circ h^{kn_k}_{14} \in End(H'_{14})^{\times}_{\mathbb{Q}}$$

Hence the statement 6.3.2.b) follows from statement 6.3.2.a) and the above linear algebra consequence of the assumption that G operates strongly non-trivially on H'_{14}. Now we prove statement 6.3.2.a).

Step 1. Preliminary reduction steps

(a) It suffices to prove the statement after extending the base field to an algebraic closure of k. So we may and do assume that k is algebraically closed.

(b) If $E \to E'$ is an isogeny of triple-extensions, the statement holds for E if and only if it holds for E'. Modify E by suitable isogeny, we may and do assume that H_{ij} are p-divisible groups such that H_{14} with slope$(H_{14}) = \frac{a_{14}}{r}$, we have

$$H_{14}[p^{a_{14}}] = H_{14}[Frob^r_{H_{14}}]$$

(c) Choose a suitable regular system of parameters $(u_1, .., u_b)$ for H_{14} such that $H_{14} = Spf(k[[u_1, .., u_b]]$ and

$$[p^{a_{14}}]^*(u_i) = u_i^{p^r}$$

Step 2. Recall the definition of $\tilde{E}_N$ and $\tilde{E}_N[p^M]$ as in 6.1.1 for $N, M \in \mathbb{N}$.

By 6.1.1, especially 6.1,

$$\psi(exp(p^{na_{14}}v)) \equiv (A_{14} \circ \eta_{na_{14}} + e_v^{na_{14}}) * id_E \mod \tilde{E}_{na}[p^{2na_{14}}]$$

By 5.5.2 and the definition of n_3 in 6.3.1, we have

$$H_{14}[p^{na_{14}}] \subset R_E/(m_E^{p^{(sn)}}) \subset \tilde{E}_{na_{14}}[p^{2na_{14}}], \forall n \geq n_3$$

Hence

$$\psi(exp(p^{na_{14}}v)) \equiv (A_{14} \circ \eta_{na_{14}} + e_v^{na_{14}}) * id_E \mod m_E^{p^{ns}}, \forall n \geq n_3 \tag{6.10}$$

For each $j = 1, ..., b$ define

$$a_{j,n} = (A_{14} \circ \eta_{na} + e_v^{na_{14}})^*(u_j) \in R_E/m_E^{p^{(ns)}}$$

for all $n \geq n_3$. Then by 4.12 and 6.1.2 it is easy to see that $\{a_{j,n}\}_{n \geq n_3}$ are ϕ^r compatible sequences for all $j = 1, 2, ..., m$. Let $i_1 := max(s-r, \frac{n_3}{r})$ Then by [CO22] especially 6.8.3.3 and 6.8.3.4, each $\{a_{j,n}\}_{n \geq n_3}$ gives rise to an element $\tilde{a}_j \in \kappa\langle\langle t_1^{p^{-\infty}}, ..., t_m^{p^{-\infty}}\rangle\rangle^{\flat}_{s:\phi^r;[i_1]}$.

Step 3. Elements $\tilde{a}_1, ..., \tilde{a}_m \in (R_E, m_E)^b_{s:\phi^r;[i_1]}$ defines a ring homomorphism

$$\tilde{\eta}[v] : R_{H_{14}} = k[[u_1, ..., u_m]] \to (R_E, m_E)^{perf,b}_{s:\phi^r;[i_1]}$$

Let

$$\omega_1 : (R_E, m_E)^{perf,b}_{s:\phi^r;[i_1]} \to (R_{H14}, m_{H_{14}})^{perf,b}_{s:\phi^r;[i_1]}$$

be the ring homomorphism induced by the inclusion $H_{14} \hookrightarrow E$. Because the restriction to H_{14} of the morphism $\eta_n|_{H_{14}} \to H_{14}$ equal to $[p^n]_{H_{14}}$ for every $n \in \mathbb{N}$, and that

$e_v^n|_{H_{14} \text{ as a subscheme of } E} = 0$, we see that

$$\omega_1 \circ \tilde{\eta}[v] = A_{14}^* \circ j_{R_{H_{14}}}$$

where $jR_{H_{14}} : R_{Z_1} \hookrightarrow (R_{H_{14}}, m_{H_{14}})^{perf,b}_{s:\phi^r;[i_1]}$ is the natural injection from $R_{H_{14}}$ to its tempered perfection and A_{14}^* is the ring homomorphism induced by A_{14} on H_{14}.

Step 4. We also have the following ring homomorphisms

(a) The canonical homomorphism $R_E \to R_E/I_W$ gives rise to a homomorphism

$$\tau : (R_E, m_E)^{perf,b}_{s:\phi^r;[i_1]} \to (R_{H_{14}}, m_{H_{14}})^{perf,b}_{s:\phi^r;[i_1]}$$

(b) Continuous ring homomorphisms

$$\Delta_1 : R_E \to R_{H'_{14}} \hat{\otimes} R_E,$$
$$\Delta_2 : R_E \to R_{H'_{14}} \hat{\otimes} R_W$$

(c) The ring endomorphism

$$\omega_2 : (R_W, m_W)^{perf,b}_{s:\phi^r;[i_1]} \to (R_{H'_{14}}, m_{H'_{14}})^{perf,b}_{s:\phi^r;[i_1]}$$

induced by

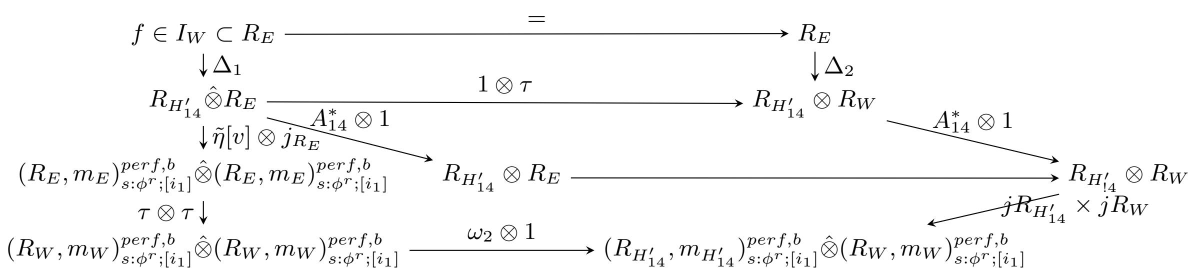
$$H'_{14} \hookrightarrow H_{14} \hookrightarrow W$$

(d) The ring endomorphism

$$A_{14}^* = A_{14}^*|_{H'_{14}} : R_{H'_{14}} \to R_{H'_{14}}$$

corresponding to the endomorphism H_{14} of the p-divisible group H_{14}.

It follows that the following diagram commutes



Step 5. Recall that I_W is the prime ideal of the coordinate ring of E. We want to show that for all $f \in I_W$,

$$(A^*_{14} \times 1_{R_E}) \circ \Delta_2(f) = 0$$

Because $jR_{H'_{14}}$ and jR_W are both injective, it suffices to show that for all $f \in I_W$,

$$(jR_{H'_{14}} \otimes jR_W) \circ (A^*_{14} \times 1_{R_E}) \circ \Delta_2(f) = 0$$

From the commutative diagram we see that it suffices to show (a stronger statement) that

$$(\tau \otimes \tau) \circ (\tilde{\eta}[v] \otimes j_{R_E}) \circ \Delta_1(f) = 0, \forall f \in I_W \tag{6.11}$$

Step 6. Let $f \in I_W$. Define an element

$$\tilde{f} \in (R_E, m_E)^{perf,b}_{s:\phi^r;[i_1]}\hat{\otimes}(R_E, m_E)^{perf,b}_{s:\phi^r;[i_1]} \tag{6.12}$$

by

$$\tilde{f} := ((\tilde{\eta}[v] \otimes j_{R_E}) \circ \Delta_1)(f)$$

where $(\tilde{\eta}[v] \otimes j_{R_E}) \circ \Delta_1$ is the composition

$$R_{H_{14}}\hat{\otimes}R_E \xrightarrow{\Delta_1} R_E\hat{\otimes}R_E \xrightarrow{\tilde{\eta}[v]\otimes j_{R_E}} (R_E, m_E)^{perf,b}_{s:\phi^r;[i_1]}\hat{\otimes}(R_E, m_E)^{perf,b}_{s:\phi^r;[i_1]}$$

We want to show that the image of $\tilde{f}$ under the map

$$(R_E, m_E)^{perf,b}_{s:\phi^r;[i_1]}\hat{\otimes}(R_E, m_E)^{perf,b}_{s:\phi^r;[i_1]} \xrightarrow{\tau^b\otimes\tau^b} (R_W, m_W)^{perf,b}_{s:\phi^r;[i_1]}\hat{\otimes}(R_W, m_W)^{perf,b}_{s:\phi^r;[i_1]}$$

is zero.

Step 7. Letϕ be the Frobenius endomorphism $x \to x^p$ on $(R_W, m_W)^b$. Let

$$\nu_W : (R_W, m_W)^b\hat{\otimes}(R_W, m_W)^b \to (R_W, m_W)^b$$

be map which defines multiplication for the ring $(R_W, m_W)^{perf,b}_{s:\phi^r;[i_1]}$. Geometrically ν_W corresponds to the diagonal morphism from $Spec((R_W, m_W)^{perf,b}_{s:\phi^r;[i_1]})$ to its self-product. Because the formal subvariety $W \subset E$ is assumed to be stable under G, therefore stable under $\psi(exp(p^{na_{14}}v)$. Hence 6.10 implies that

$$\nu_W(\phi^{nr} \otimes 1)((\tau^b \otimes \tau^b)(\tilde{f})) \equiv 0 \mod Fil^{p^{ns}}$$

where $\phi^{nr} \otimes 1$ is the ring homomorphism

$$\phi^{nr} \otimes 1 : (R_W, m_W)^{perf,b}_{s:\phi^r;[i_1]}\hat{\otimes}(R_W, m_W)^{perf,b}_{s:\phi^r;[i_1]} \to (R_W, m_W)^{perf,b}_{s:\phi^r;[i_1]}\hat{\otimes}(R_W, m_W)^{perf,b}_{s:\phi^r;[i_1]}$$

Applying theorem 6.2.5, also note that $r < s$ which implies $\lim \frac{p^{nr}}{p^{ns}} = 0$, we conclude that

$$(\tau^b \otimes \tau^b)(\tilde{f}) = 0$$

in $(R_W, m_W)^{perf,b}_{s:\phi^r;[i_1]}\hat{\otimes}(R_W, m_W)^{perf,b}_{s:\phi^r;[i_1]}$, for every element $f \in I_W$, which is precisely 6.11. As we have seen, this implies that

$$(A_{14}|_{H'_{14}} \otimes 1)(\Delta_2(\tilde{f})) = 0$$

in $R_{H'_{14}} \otimes R_W$ for every element $f \in I_W$. We have proved the result.

□

6.4. Further Consequences

The following result 6.4.1 is proved in [CO22] Chapter 10. The main purpose of this section is to prove 6.4.6, which is an analogy of 6.4.1.

Lemma 6.4.1. *Let $\pi : E \to X \times Y$ be a biextension of $X \times Y$ by Z over k. Assume that X, Y, Z all isoclinic with the slope(Z) strictly bigger than slope(Y), slope(Z). Let G be a closed subgroup of $Aut_{biext}(E)$ such that the action of G on Z is strongly non-trivial. Let W be a reduce irreducible subscheme of E stable under G. The closed formal subscheme $Z' = (W \cap Z)_{red}$ is a p-divisible subgroup of Z, and W is stable under the translation action by Z'. Let $W' = W/Z'$, a reduced irreducible closed formal subscheme of the biextension*

$$E/Z' = (Z \twoheadrightarrow Z/Z')_* E$$

of $X \times Y$ by Z/Z'. Then the natural map

$$q_{W'} : W' \to E/Z$$

is finite purely inseparable formal morphism. In other words the affine coordinate ring $R_{W'}$ of W' is finite over the subring $R_{Im(q_{W'})}$, the affine coordinate ring of the schematic image of $q_{W'}$, and there exists a natural number m such that $x^{p^m} \in R_{Im(q_{W'})}$ for every $x \in R_{W'}$.

Proof. See [CO22], Chapter 10. □

The rest of this section will be devoted into proving an analogy of 6.4.1. We first setup notations.

Notations 6.4.2. ***(Notations and assumptions for the rest of this subsection)***

1. *We continue with the notations as in 6.3.2: let $H = (H_{ij})$ be a general sustained linear*

group, pure and perfect. Let $G \subset Aut(H)$. *Let* $W \subset Def_{H-torsor}$ *reduced irreducible closed formal subscheme of* E *stable under the action of* G. *Let* $v = (A_{ij})$ *be an element of the Lie algebra of* G *with components* $A_{ij} \in End(H_{ij})$.

2. *There exists positive integers* a_{14}, r, s, n_3 *such that*

   (a) $0 < a_{14} \leq r < s$

   (b) *slope*$(H_{14}) = \frac{a_{14}}{r}$, $H_{14}[p^{a_{14}}] = H_{14}[F^r]$.

   (c) *the congruence condition 6.10 holds.*

3. *Recall that in Step 3 of 6.3.2 we pick a regular system of parameters* $u_1, u_2, ..., u_b$ *of the complete local ring* $R_{H_{14}}$ *with* $[p^a]^*_{H_{14}} = u_i^{p^r}$ *for all* $i = 1, ..., b$, *and constructed a continuous ring homomorphism*

$$\tilde{\eta}(v) : R_{H_{14}} \to (R_E, m_E)^{perf,b}_{s:\phi^r;[i_1]}$$

Define the schematic image $Im(\tilde{\eta}[v]|_W)$ *of the restriction to* W *of* $\tilde{\eta}[v]$ *by*

$$Im(\tilde{\eta}[v]|_W) := Spf(R_{H_{14}}/Ker(\tau^b \circ \tilde{\eta}[v]))$$
$$= Spf(R_{H_{14}}/ker(R_{H_{14}} \xrightarrow{\tilde{\eta}[v]^*} (R_E, m_E)^{perf,b}_{s:\phi^r;[i_1]} \xrightarrow{\tau^b} (R_W, m_W)^{perf,b}_{s:\phi^r;[i_1]})$$

Lemma 6.4.3. *We continue with the notations of 6.3.2. For every element* $v = (A_{ij})_{1 \leq i < j \leq 4}$, *the diagram*

$$
\begin{array}{ccccc}
R_{H_{14}} & \xrightarrow{\Delta_Z} & R_{H_{14}}\hat{\otimes}R_{H_{14}} & \xrightarrow{j_{R_{H_{14}}}\circ A_{14}^*\otimes\tilde{\eta}} & (R_{H_{14}})^{perf,b}_{s:\phi^r;[i_1]}\hat{\otimes}(R_E)^{perf,b}_{s:\phi^r;[i_1]} \\
\downarrow{\scriptstyle\tilde{\eta}} & & & & \downarrow{\scriptstyle j} \\
(R_E)^{perf,b}_{s:\phi^r;[i_1]} & & \xrightarrow{\Delta^b} & & (R_{H14}\hat{\otimes}R_E)^{perf,b}_{s:\phi^r;[i_1]} \\
\uparrow{\scriptstyle j_{R_E}} & & & & \uparrow{\scriptstyle j_{H_{14}\hat{\otimes}R_E}} \\
R_E & & \xrightarrow{\Delta} & & R_{H14}\hat{\otimes}R_E
\end{array}
$$

commutes. The arrows $\Delta_{H_{14}}, \Delta^b, j_{R_{H_{14}}}, j_E, j_{R_{H_{14}}\hat{\otimes}R_E}, j$ *are as follows:*

- $\Delta_{H_{14}}$ *corresponds to the group law on* H_{14}*.*

- $\Delta : R_E \to R_{H_{14}} \otimes R_E$ *corresponds to the* H_{14} *torsor structure* $Z \times E \to E$ *on* E*, which induces a ring homomorphism* $\Delta^b : (R_E)^{perf,b}_{s:\phi^r;[i_1]} \to (R_{H14}\hat{\otimes}R_E)^{perf,b}_{s:\phi^r;[i_1]}$

- $j_{R_{H_{14}}}, j_E, j_{R_{H_{14}}\hat{\otimes}R_E}$ *are the inclusions maps from* $R_{H_{14}}, R_E, R_{H_{14}}\hat{\otimes}R_E$ *to their tempered perfections*

- *The downward vertical arrow* j *on the right is the natural ring homomorphism, from the tensor product* $(R_{H_{14}})^{perf,b}_{s:\phi^r;[i_1]}\hat{\otimes}(R_E)^{perf,b}_{s:\phi^r;[i_1]}$ *of tempered perfections to tempered perfection* $(R_{H14}\hat{\otimes}R_E)^{perf,b}_{s:\phi^r;[i_1]}$ *of* $R_{H_{14}}\hat{\otimes}R_E$*.*

Proof. Left as exercise. □

Proposition 6.4.4. *We use the notations and assumptions in 6.4.2. Then*

(a) The formal subvariety W *of* E *is stable under the translation by the smallest p-divisible subgroup of* H_{14} *which contains the schematic image* $Im((\tilde{\eta}[v])|W)$ *of the restriction to* W *of the morphism* $\tilde{\eta}[v] : E \to H_{14}$*, for every element* $v \in Lie(G) \cap (\prod End(H_{ij}))$*.*

(b) Let $H_{14,\tilde{\eta}}$ be the smallest p-divisible subgroup of H_{14} which contains the schematic image $Im((\tilde{\eta}[v])|_W)$ for every $v \in Lie(G) \cap \prod(End(H_{ij}))$. Then W is stable under the translation action by $H_{14,\tilde{\eta}}$.

Proof. We will show that W is stable under the translation action of $Im(\tilde{\eta}[v]|_W)$. The statement (a) follows easily from this apparently weaker statement.
Let $I_W = ker(\tau : R_E \to R_W)$ be the ideal of R_W corresponds to W. Let

$$J[v] := Ker(\tau^b \circ \tilde{\eta}[v] : R_E \circ \tilde{\eta}[v] : R_{H_{14}} \to (R_W, m_W)^{perf,b}_{s:\phi^r;[i_1]})$$

We need to show that the kernel of the composition

$$R_E \xrightarrow{\Delta} R_{H_{14}} \otimes R_E \xrightarrow{q_{[v]} \otimes \tau} (R_{H_{14}}/J[v]) \otimes R_W$$

contains I_W, where $q_{[v]} : R_{H_{14}} \twoheadrightarrow R_{H_{14}}/J[v]$ is the quotient map. Let

$$J_{[v]} : R_{H_{14}}/J[v] \to (R_W, m_W)^{perf,b}_{s:\phi^r;[i_1]}$$

be the injective ring homomorphism such that

$$\tau^b \circ \tilde{\eta}[v]^* = J_{[v]} \circ q_{[v]}$$

We have a commutative diagram

$$\begin{array}{ccccc}
R_E & \xrightarrow{\Delta} & R_{H_{14}} \otimes R_E & \xrightarrow{q_{[v]} \otimes \tau} & (R_{H_{14}}/J[v]) \otimes R_W \\
 & & \downarrow{\scriptstyle \tilde{\eta}[v] \otimes j_{R_E}} & & \downarrow{\scriptstyle J_{[v]} \otimes j_{R_W}} \\
 & & (R_E, m_E)^{perf,b}_{s:\phi^r;[i_1]} \hat{\otimes} (R_E, m_E)^{perf,b}_{s:\phi^r;[i_1]} & \xrightarrow{\tau^b \otimes \tau^b} & (R_W, m_W)^{perf,b}_{s:\phi^r;[i_1]} \hat{\otimes} (R_W, m_W)^{perf,b}_{s:\phi^r;[i_1]}
\end{array}$$

In step 4 of 6.3.2 we showed that $I_W \subset Ker((\tau^b \otimes \tau^b) \circ (\tilde{\eta}[v] \otimes J_{R_E}) \otimes \Delta$. Therefore

$$I_W \subset Ker((q_{[v]} \otimes \tau) \circ \Delta_1)$$

because $J_{[v]} \otimes J_{R_W}$ is an injective ring homomorphism. We have prove the statement (a). The statement (b) follows from (a). □

Corollary 6.4.5. *In 6.4.4, assume in addition that G operates strongly non-trivially on H_{14}. Then the intersection $w \cap H_{14}$ with reduced structure is equal to $H_{14,\tilde{\eta}}$, the smallest p-divisible subgroup which contains all schematic images $Im((\tilde{\eta}[v]|_W))$, where v runs through all elements of $Lie(G) \cap (\prod_{1\le i<j\le 4} End(H_{ij}))$.*

Now we prove the main result of this section.

Theorem 6.4.6. *Let H be a Tate-linear nilpotent group of type A of dimension 4 that is pure and perfect, let $E = Def_{H\text{-}tor}$, $\pi : E \to B$ the natural projection. Recall that E admits a H_{14} torsor structure over B. Assuming that $s_{14} > s_{ij}, \forall (i,j) \neq (1,4)$. Let G be a closed subgroup of $Aut(H) = Aut(E)$, in the sense of 4.4.8. Let W be a reduced irreducible closed formal subscheme of E stable under the action of G. Suppose that the action of G on H_{14} is strongly non-trivial. By 6.3.2 the reduced formal subscheme $H'_{14} = (W \cap H_{14})_{red}$ is a p-divisible subgroup of H_{14}, and W is stable under the translation action by H'_{14}. Let $W' = W/H'_{14}$, a reduced irreducible closed formal subscheme of the biextension $E/H'_{14} = (H_{14} \twoheadrightarrow H_{14}/H'_{14})_* E$. Then the natural map*

$$q_{W'} : W' \to E/H_{14}$$

is a finite purely inseparable formal morphism.

Proof. Extend the perfect base field k if neccessary, we may and do assume that the base field k is algebraically closed. Recall $B = E/H_{14}$. As the closed fiber of the formal morphism $\pi|_{W/H'_{14}} : W/H'_{14} \to B$ is finite over k, therefore $\pi|_{W/H'_{14}}$ is finite. Denote by $\bar{W}$ the schematic image of $\pi|_W$, a reduced irreducible formal subscheme of B stable under the action of G. We need to show that W is purely inseparable over $\bar{W}$.

Now W.L.O.G. assume H'_{14} is trivial, hence $W = W'$. Let $R_W, R_{\bar{W}}$ be the coordinate rings of $W, \bar{W}$ respectively, and let $j : R_{\bar{W}} \to R_W$ be the continuous injective ring homomorphism induced by $\pi|_W$. We know that R_W is finite over $R_{\bar{W}}$, and must show that there exists $N \in \mathbb{N}$ such that $x^{p^N} \in R_W$ for all $x \in R_{\bar{W}}$. Suppose no such natural number N exists. Then there exist continuous ring homomorphisms $h_1, h_2 : R_W \to k[[u]]$ from R_W to the power series ring in one variable u, such that $h_1 \circ j = h_2 \circ j$ but $h_1 \neq h_2$. Since the projection $E \to B = E/H_{14}$ is a H_{14} torsor, there exists a continuous k-linear ring homomorphism $\delta : R_{H_{14}} \to k[[u]]$ such that

$$\mu_{k[[u]]} \circ (\delta \otimes h_1) \circ \Delta = h_2$$

where

- $\Delta : R_E \to R_{H_{14}} \hat{\otimes} R_E$ corresponds to the action of H_{14} on E,
- $\mu_{k[[u]]} : k[[u]] \hat{\otimes} k[[u]] \to k[[u]]$ is the multiplication map on $k[[u]]$,
- $Ker(\delta) \subseteq m_{H_{14}}$, or equivalently $k[[u]]$ is a finite module over the subring $Im(\delta)$, because $h_1 \neq h_2$.

We know from that for every $v = (A_{ij})_{1 \leq i < j \leq 4} \in Lie(G)$ with components $A_{ij} \in End(H_{ij})$, the kernel of the composition $\tau^b \circ \tilde{\eta}[v]$ of the continuous ring homomorphism

$$R_{H_{14}} \xrightarrow{\tilde{\eta}[v]} (R_E, m_E)^{perf,b}_{s:\phi^r;[i_1]} \xrightarrow{\tau^b} (R_W, m_W)^{perf,b}_{s:\phi^r;[i_1]}$$

contains the maximal ideal $m_{H_{14}}$ of $R_{H_{14}}$. In other words $\tau^b \circ \tilde{\eta}[v]$ is equal to the composition $R_{H_{14}} \twoheadrightarrow k \hookrightarrow (R_W, m_W)^{perf,b}_{s:\phi^r;[i_1]}$, the trivial k-linear ring homomorphism.

Consider the following diagram,

$$
\begin{array}{ccccc}
R_{H_{14}} & \xrightarrow{\Delta_{H_{14}}} & R_{H_{14}}\hat{\otimes}R_{H_{14}} & \xrightarrow{(j_{R_{H_14}} \circ A_{14}) \otimes \tilde{\eta}[v]} & (R_{H_{14}})^{perf,b}_{s:\phi^r;[i_1]}\hat{\otimes}(R_E)^{perf,b}_{s:\phi^r;[i_1]} \\
\downarrow{\scriptstyle \tilde{\eta}[v]} & & & & \downarrow{\scriptstyle j} \\
(R_E)^{perf,b}_{s:\phi^r;[i_1]} & & \xrightarrow{\Delta^b} & & (R_{H_{14}}\hat{\otimes}R_E)^{perf,b}_{s:\phi^r;[i_1]} \\
\downarrow{\scriptstyle (h_2 \circ \tau)^b} & & & & \downarrow{\scriptstyle (1 \otimes \tau)^b} \\
(k[[u]])^{perf,b}_{s:\phi^r;[i_1]} & \xleftarrow{\mu^b_{k[[u]]}} & (k[[u]]\hat{\otimes}k[[u]])^{perf,b}_{s:\phi^r;[i_1]} & \xleftarrow{(\delta \otimes h_1)^b} & (R_{H_{14}}\hat{\otimes}R_W)^{perf,b}_{s:\phi^r;[i_1]}
\end{array}
$$

The Commutativity of the top half of the diagram follows from 6.4.3, while the bottom half commutes because $\mu_{k[[u]]} \circ (\delta \otimes h_1) \circ \Delta = h_2$. The homomorphism

$$R_{H_{14}} \overset{(h_2\circ\tau)^b\circ\tilde{\eta}[v]}{\longrightarrow} (k[[u]]\hat{\otimes}k[[u]])^{perf,b}_{s:\phi^r;[i_1]}$$

is the trivial k-linear ring homomorphism because $\tau^b \circ \tilde{\eta}[v]$ is. On the other hand, we have

$$(h_2 \circ \tau)^b \circ \tilde{\eta}[v] = \mu^b_{k[[u]]} \circ (\delta \otimes h_1)^b \circ (1 \otimes \tau)^b \circ j \circ ((j_{R_E} \circ A^*_{14}) \otimes \tilde{\eta}[v]) \circ \Delta_{H_{14}}$$

The right hand side of the above equality is equal to the following composition

$$R_{H_{14}} \xrightarrow{A_{14}} R_{H_{14}} \xrightarrow{\delta} k[[u]] \overset{j_{k[[u]]}}{\longrightarrow} (k[[u]])^{perf,b}_{s:\phi^r;[i_1]}$$

Therefore the non-trivial $k[[u]]$-point δ^* of H_{14} lies in the kernel of the endomorphism A_{14} for every element $v = (A_{ij}))_{1\leq i<j\leq 4} \in Lie(G) \cap (\prod End(H_{ij})$. Since the action of G on H_{14} is strongly non-trivial, the point $\delta^* \in H_{14}(k[[u]])$ is 0. This is a contradiction. We have proved that W is purely inseparable over $\bar{W}$. $\square$

CHAPTER 7

The Main Theorem

Notations 7.0.1. ***Setup of This Section***

i) *Let H be a Tate-linear nilpotent group of type A of rank 4 over an algebraically closed field κ of characteristic p with $p \geq 5$. We further assume H to be perfect and pure.*

ii) $E = Def_{H\text{-}torsor}$

iii) *Let $B = E/H_{14}, \pi : E \to B$ the projection map.*

iv) *Let $G \subset Aut(E)$ a closed p-adic Lie subgroup acting strongly non-trivially on E.*

v) *Let $W \subset E$ a reduced irreducible formal subscheme. Assume that W is invariant under the action of G.*

vi) *Let $Y := \pi(W) \cap (H_{13} \times H_{24})$ where $H_{13} \times H_{24} \subset B$ as a subscheme. $X = (\pi_{12} \times \pi_{23} \times \pi_{34})(W) \subset H_{12} \times H_{23} \times H_{34}$. Since W is invariant under the action of G, hence both X, Y are p-divisible subgroups by 5.2.1.*

vii) *For $n \in \mathbb{N}$, let $B_{13}, B_{24}, B_{13,n}, B_{24,n}, B_n, E_n, \pi_{123}, \pi_{234}$ as defined in 4.6.2. Note that both B_{13}, B_{24} are bi-extensions and that $B = B_{13} \times_{H_{23}} B_{24}$.*

viii) *Let A_n, E_n and $\psi : A_n \to E_n$ as in 4.2.1.*

7.1. Compatibility of Trivialization

The following result will serve as the 'induction hypothesis' in the proof of orbital rigidity of 4 slopes case.

Theorem 7.1.1. *Notation as in 7.0.1, and let $W \subset E$ a reduced irreducible invariant under the action of $G \subset Aut(E)$. The action of G induces action on both B_{13} and B_{24}. Let $\psi^{1,3}_{n.homo}, \psi^{2,4}_{n,homo}$ be (homogeneous) Mumford's trivialization of B_{13} and B_{24} respectively, see*

3.4.4. Then

(a). the following diagram

$$
\begin{array}{ccc}
(H_{12} \times H_{23} \times H_{34})[p^{3n}] \times (H_{13} \times H_{24})[p^{2n}] \times H_{14} & \xrightarrow{\psi_n} & E_n \\
\Big\downarrow \Pi_n & & \Big\downarrow \pi|_{E_n} \\
(H_{12} \times H_{23} \times H_{34})[p^{3n}] \times (H_{13} \times H_{24})[p^{2n}] & \xrightarrow{\overline{\psi_n}} & B_n \\
\Big\downarrow [p^n]_{H_{12}\times H_{23}\times H_{34}\times H_{13}\times H_{24}} & & \Big\downarrow \hookrightarrow \\
(H_{12} \times H_{23} \times H_{34})[p^{2n}] \times (H_{13} \times H_{24}) & \xrightarrow{\psi^{1,3}_{n,homo} \otimes_{H_{23}} \psi^{2,4}_{n,homo}} & B
\end{array}
$$

commutes, where

- $\psi^{1,3}_{n,homo} \otimes_{H_{23}} \psi^{2,4}_{n.homo} : (H_{12}[p^{2n}] \times H_{23}[p^{2n}] \times H_{13} \times_{H_{23}} (H_{23} \times H_{34})[p^{2n}] \times H_{24} \longrightarrow B^{1,3}_n \otimes_{H_{23}} B^{2,4}_n \hookrightarrow B$

- $\overline{\psi_n}$ *is the morphism from* $H_{12} \times H_{23} \times H_{34})[p^{3n}] \times (H_{13} \times H_{24})[p^{2n}]$ *to* B_n *induced by* ψ_n.

- Π_n *is the natural projection from* $(H_{12} \times H_{23} \times H_{34})[^{3n}] \times (H_{13} \times H_{24})[p^{2n}] \times H_{14}$ *to* $(H_{12} \times H_{23} \times H_{34})[p^{3n}] \times (H_{13} \times H_{24})[p^{2n}]$.

(b). Let $\overline{W_n} := \pi(W) \cap B_n$. *Let* X, Y *as defined in 7.0.1(vi). Let* $\mathcal{S}_n$ *be the morphism* $\mathcal{S}_n : X[p^{2n}] \times Y[p^n] \to B$ *that sends* (x, y)

to

$$\psi^{1,3}_n \otimes_{H_{23}} \psi^{2,4}_n(x^{2n}_{12}, x^{2n}_{23}, x^{2n}_{34}, \frac{1}{2}x^{2n}_{12}x^{2n}_{23} + y^n_{13}, \frac{1}{2}x^{2n}_{23}x^{2n}_{34} + y^n_{24})$$

where

$$x = (x^{2n}_{12}, x^{2n}_{23}, x^{2n}_{34}) \in X, y = (y^n_{13}, y^n_{24})$$

Then $\mathcal{S}_n$ *factors through* $\overline{W_n}$.

Proof. Part (a) follows from the construction of ψ_n as in 4.2.1 an easy diagram chasing. Part (b) is a direct consequence of 5.2.1 which says that $\pi(W)$ is a Tate-linear formal subvariety of B, given that $\pi(W)$ is a reduced irreducible subscheme of B invariant under the induced action of G on B.

□

7.2. Existence of Admissible Subgroups

Lemma 7.2.1. *Let $H = (H_{ij})_{1\leq i<j\leq 4}$ be a Tate-linear nilpotent group of type A. Let $X \subset H_{12} \times H_{23} \times H_{34}$, $Y \subset H_{13} \times H_{24}$ p-divisible subgroups. If we further assume that*

$$(x_{12}^n {x_{23}^n}' - {x_{12}^n}' x_{23}^n, x_{23}^n {x_{34}^n}' - {x_{23}^n}' x_{34}^n) \in Y, \forall x = (x_{12}^n, x_{23}^n, x_{34}^n), x' = ({x_{12}^n}', {x_{23}^n}', {x_{34}^n}') \in X[p^n] \tag{7.1}$$

$$x_{12}^n y_{24}^n = y_{13}^n x_{34}^n, \forall x = (x_{12}^n, x_{23}^n, x_{34}^n) \in X[p^n], y = (y_{13}^n, y_{24}^n) \in Y[p^n] \tag{7.2}$$

then there is an admissible subgroup $H = H_{X,Y} \subset H$ such that $Lie(H) = X \oplus Y \oplus e_{H14}$. For the definition of $Lie(H)$ see 4.7.5.

Proof. Consider the subschemes

$$H_n = \left\{ \begin{pmatrix} 1 & x_{12} & \frac{1}{2}x_{12}x_{23} + y_{13} & \frac{1}{6}x_{12}x_{23}x_{34} + x_{12}y_{24} \\ 0 & 1 & x_{23} & \frac{1}{2}x_{23}x_{34} + y_{24} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \forall x = (x_{12}, x_{23}, x_{34}) \in X[p^n], y = (y_{13}, y_{24}) \in Y[p^n] \right\}$$

It is a simple algebra exercise to check that H_n is indeed a group scheme and that the natural morphism $H_{n+1} \to H_n$ is faithfully flat. □

7.3. The Case When $\pi|_W$ is Isomorphic

The main result of this section is 7.3.4, which is a special case of the main result of this thesis 4.8.2.

Lemma 7.3.1. *Let $\mathcal{A}_n, E_n, B_n, \psi_{n,homo} : \mathcal{A}_n \to E_n$ as in 4.2.6. Note that $B_n = \pi(E_n)$.*

Then $\psi_{n,homo}$ induces a faithfully flat morphism

$$\overline{\psi_n} : (H_{12} \times H_{23} \times H_{34})[p^{3n}] \times (H_{13} \times H_{24})[p^{2n}] \to B_n$$

Moreover, let W, X, Y as in 7.0.1. For $n \in \mathbb{N}$, let $\overline{W_n} := \pi(W) \cap B_n$ a finite subscheme of W. Let $\mathcal{J}_n$ be morphism from

$$X[p^{3n}] \times Y[p^{2n}]$$

to

$$(H_{12} \times H_{23} \times H_{34})[p^{3n}] \times (H_{13} \times H_{24})[p^{2n}]$$

that sends

$$(x, y) = (x_{12}^{3n}, x_{23}^{3n}, x_{34}^{3n}), (y_{13}^{2n}, y_{24}^{2n})$$

to

$$(x_{12}^{3n}, x_{23}^{3n}, x_{34}^{3n}, \frac{1}{2}\langle x_{12}^{3n}, x_{23}^{3n}\rangle_{123,3n} + y_{13}^{2n}, \frac{1}{2}\langle x_{23}^{3n}, x_{34}^{3n}\rangle_{234,3n} + y_{24}^{2n})$$

Then

$$\mathcal{J}_n \circ \overline{\psi_n} : X[p^{3n}] \times Y[p^{2n}] \to B_n$$

factors through $\overline{W_n}$ and as a morphism from $X[p^{3n}] \times Y[p^{2n}]$ to $\overline{W_n}$ it is faithfully flat.

Proof. This is a reformulation of the result in 5.2.1 and 7.1.1. □

Remark 7.3.2. *The significance of 7.3.1 is that this coordinate system, that is trivializing $\pi(W)$ using X and Y, is more natural and easier to handle.*

Corollary 7.3.3. *Notation as 7.0.1. Let $\mathcal{J}_n$ be the morphism defined in 7.3.1. We further assume that $\pi|_W : W \to \pi(W)$ is an isomorphism. Then for each $n \in \mathbb{N}$, there exists a morphism*

$$f_n : X[p^{3n}] \times Y[p^{2n}] \to H_{14}$$

s.t. for all $x \in X[p^{3n}], y \in Y[p^{2n}]$,

$$\psi_{n,homo}(\mathcal{J}_n(x,y), f_n(x,y)) \in W_n$$

where $(\mathcal{J}_n(x,y), f_n(x,y))$ *is an element in* $(H_{12} \times H_{23} \times H_{34})[p^{3n}] \times (H_{13} \times H_{24})[p^{2n}] \times H_{14}$. *Moreover, we have the following compatibility between different* n*'s: for* $x' \in X[p^{3n+2}], y' \in Y[p^{2n+1}]$,

$$f_{n+1}(x', y') = f_n([p^2] \cdot x', [p] \cdot y')$$

as elements in H_{14}.

Proof. This is a direct consequence of 7.3.1 and the fact that $\pi|_W : W \to \pi(W)$ is an isomorphism. □

Theorem 7.3.4. *Notation as in 7.0.1. Assume that* $\pi|_W : W \to \pi(W)$ *is an isomorphism. Let* X, Y *and* $f_n : X[p^{3n}] \times Y[p^{2n}] \to H_{14}$ *as in 7.3.3. Let*

$$\tilde{f}_n(x, y, \Delta) = f_n(x, y) - f_n(x, y + \Delta)$$

where

$$x = (x_{12}^{3n}, x_{23}^{3n}, x_{34}^{3n}) \in X[p^{3n}] \subset (H_{12} \times H_{23} \times H_{34})[p^{3n}],$$

$$y = (y_{13}^{2n}, y_{24}^{2n}) \in Y[p^{2n}] \subset (H_{13} \times H_{24})[p^{2n}],$$

$$\Delta = (\Delta_{13}^{2n}, \Delta_{24}^{2n}) \in Y[p^{2n}] \subset (H_{13} \times H_{24})[p^{2n}]$$

Then

(a) $\tilde{f}_n(x, y, \Delta + \Delta') - \tilde{f}_n(x, y, \Delta) - \tilde{f}_n(x, y, \Delta') = 0$.

(b) $\tilde{f}_n(x, y, \Delta)$ *is independent of* y.

(c) $\tilde{f}_n(x_1 + x_2, y, \Delta) - \tilde{f}_n(x_1, y, \Delta) - \tilde{f}_n(x_2, y, \Delta) = 0, \forall x_1, x_2 \in X[p^{3n}], y, \Delta \in Y[p^{2n}]$.

(d) *For* $x = (x^n_{12}, x^n_{23}, x^n_{34}), x' = ({x^n_{12}}', {x^n_{23}}', {x^n_{34}}') \in X[p^n]$, $y = (y^n_{13}, y^n_{24}) \in Y[p^n]$, *we have*

$$(x^n_{12}{x^n_{23}}' - {x^n_{12}}'x^n_{23}, x^n_{23}{x^n_{34}}' - {x^n_{23}}'x^n_{34}) \in Y$$

$$\langle x^n_{12}, y^n_{24}\rangle_n = \langle y^n_{13}, x^n_{34}\rangle_n$$

(e) *There exists an admissible subgroup* $H_{X,Y}$ *of* H *such that*

$$Lie(H_{X,Y}) = X \oplus Y \oplus e_{H_{14}}$$

and that

$$\pi(E_{X,Y}) = \pi(W) \text{ as subschemes of } B$$

where $E_{X,Y}$ *is the Tate-linear subvariety of* E *that corresponds to* $H_{X,Y}$. *For the definition of Lie algebra of an admissible subgroup see 4.7.5. For the definition of Tate-linear subvariety that corresponds to an admissible subgroup, see 4.7.8.*

(f) W *is a Tate-linear subvariety.*

Proof. Let $F(x, y, \Delta, \Delta') = \tilde{f}_n(x, y, \Delta + \Delta') - \tilde{f}_n(x, y, \Delta) - \tilde{f}_n(x, y, \Delta')$. We prove the result in several steps:

Step 1. We show that for all $\forall(x, y), (x', y')$ such that $\overline{\psi_n}(x, y) = \overline{\psi_n}(x', y')$, and all $\Delta, \Delta' \in Y[p^{2n}]$,

$$F(x, y, \Delta, \Delta') = F(x', y', \Delta, \Delta')$$

First notice that $\overline{\psi_n}(x, y) = \overline{\psi_n}(x', y') \iff \overline{\psi_n}(x, y+\Delta) = \overline{\psi_n}(x', y'+\Delta), \forall\Delta \in Y[p^{2n}]$.

By 4.2.6 we have

$$\tilde{f}(x,y,\Delta)-\tilde{f}(x',y',\Delta)=[f_n(x,y)-f_n(x',y')]-[f_n(x,y+\Delta)-f_n(x',y'+\Delta)]$$
$$=[y_{13}^{3n}(x_{34}^{2n}-x_{34}^{2n\prime})+x_{12}^{3n}(y_{24}^{2n}-y_{24}^{2n\prime})]-[(y_{13}^{3n}+\Delta)(x_{34}^{2n}-x_{34}^{2n\prime})+x_{12}^{3n}(y_{24}^{2n}-y_{24}^{2n\prime})]$$
$$=-\Delta_{13}^{2n}(x_{34}^{3n}-x_{34}^{3n\prime})=0$$

here terms involving only x's cancel out in two brackets hence omitted, and all the 'multiplication' refers to bilinear pairings at level $3n$, for example $y_{13}^{3n}(x_{34}^{2n}-x_{34}^{2n\prime})=\langle y_{13}^{3n},x_{34}^{2n}-x_{34}^{2n\prime}\rangle_{134,3n}$. Hence

$$F(x,y,\Delta,\Delta')-F(x',y',\Delta,\Delta')$$
$$=(\tilde{f}(x,y,\Delta+\Delta')-\tilde{f}(x',y',\Delta+\Delta'))-(\tilde{f}(x,y,\Delta)-\tilde{f}(x',y',\Delta))-(\tilde{f}(x,y,\Delta')-\tilde{f}(x',y',\Delta'))$$
$$=(-\Delta_{13}^{2n}-\Delta_{13}^{2n\prime}+\Delta_{13}^{2n}+\Delta_{13}^{2n\prime})(x_{34}^{3n}-x_{34}^{3n\prime})$$
$$=0$$

Step 2. Let $x\in X[p^{3n}]$, $Y,\Delta\in Y[p^{2n}]$, $\delta=(\delta_{13}^n,\delta_{24}^n)\in Y[p^n]$. Again by 4.2.6, we have

$$\overline{\psi_n}(x,y)=\overline{\psi_n}(x,y+\delta),\overline{\psi_n}(x,y+\Delta)=\overline{\psi_n}(x,y++\Delta+\delta)$$

Moreover,

$$\tilde{f}(x,y,\Delta+\delta)-\tilde{f}(x,y,\Delta)$$
$$=[f(x,y)-f(x,y+\delta)]-[f(x,y+\Delta)-f(x,y+\Delta+\delta)]$$
$$=[-x_{12}^{3n}\cdot\delta_{24}^n]-[-x_{12}^{3n}\cdot\delta_{24}^n]=0$$

Hence it's also easy to see that

$$F(x, y, \Delta + \delta, \Delta') = F(x, y, \Delta, \Delta'),$$
$$F(x, y, \Delta, \Delta' + \delta) = F(x, y, \Delta, \Delta')$$

Step 3. Combining results in Step 1 and Step 2, we know F_n descent to a morphism

$$\overline{F}_n : B_n \times Y[p^n] \times Y[p^n] \to H_{14}$$

together with compatibility condition in 7.3.3 we obtain a morphism of schemes

$$F := \varinjlim \overline{F}_n : B \times Y \times Y \to H_{14} \tag{7.3}$$

As W is invariant under the action of G, F is equivariant under G, hence by 5.4.1 we have $F \equiv 0$. This proves (a).

Step 4. By (a) we have

$$\tilde{f}_n(x, y, \Delta + \Delta') = \tilde{f}_n(x, y, \Delta) + \tilde{f}_n(x, y, \Delta') \tag{7.4}$$

On the other hand

$$\begin{aligned}
&\tilde{f}_n(x, y, \Delta + \Delta') \\
&= f(x, y) - f(x, y + \Delta + \Delta') \\
&= f(x, y) - f(x, y + \Delta) + f(x, y + \Delta) - f(x, y, \Delta + \Delta') \\
&= \tilde{f}_n(x, y, \Delta) + \tilde{f}_n(x, y + \Delta, \Delta')
\end{aligned}$$

Hence

$$\tilde{f}_n(x, y, \Delta') = \tilde{f}_n(x, y + \Delta, \Delta') \tag{7.5}$$

that is $\tilde{f}_n(x, y, \Delta)$ is independent of y. This proves (b).

Step 5. The prove of (c) is similar to the prove of (a). Consider the function

$$K_n : X[p^{3n}] \times X[p^{3n}] \times Y[p^{2n}] \times Y[p^{2n}] \to H_{14}$$

defined as

$$K_n(x, x', y, \Delta) = \tilde{f}_n(x+x', y, \Delta) - \tilde{f}_n(x, y, \Delta) - \tilde{f}_n(x', y, \Delta), \forall x, x' \in X[p^{3n}], y, \Delta \in Y[p^{2n}]$$

We first show that for all $\delta_x \in X[p^{2n}]$,

$$K_n(x + \delta_x, x', y, \Delta) - K_n(x, x', y, \Delta) = 0$$

Pick any $y' \in Y[p^{2n}]$ such that $\overline{\psi_n}(x + \delta_x, y) = \overline{\psi_n}(x, y')$,, By 4.2.1 and (b),

$$\begin{aligned}
&\tilde{f}_n(x + \delta_x, y, \Delta) - \tilde{f}_n(x, y, \Delta) \\
&= \tilde{f}_n(x + \delta_x, y, \Delta) - \tilde{f}_n(x, y', \Delta) \\
&= [f_n(x + \delta_x, y) - f_n(x, y')] - [f_n(x + \delta_x, y + \Delta) - f_n(x, y' + \Delta)] \\
&= [y_{13}^{2n}\delta_{x,34}^{2n} + x_{12}^{3n}(y_{24}^{2n} - y_{24}^{2n\prime})] - [(y_{13}^{2n} + \Delta_{13}^{2n})\delta_{x,34}^{2n} + x_{12}^{3n}(y_{24}^{2n} - y_{24}^{2n\prime})] \\
&= -\Delta_{13}^{2n}\delta_{x,34}^{2n}
\end{aligned}$$

Hence

$$\begin{aligned}
&K_n(x + \delta_x, x', y, \Delta) - K_n(x, x', y, \Delta) \\
&= [\tilde{f}_n(x + x' + \delta_x, y, \Delta) - \tilde{f}_n(x + x', y, \Delta)] - [\tilde{f}_n(x + \delta_x, y, \Delta) - \tilde{f}_n(x, y, \Delta)] \\
&= -\Delta_{13}^{2n}\delta_{x,34}^{2n} - (-\Delta_{13}^{2n}\delta_{x,34}^{2n}) \\
&= 0
\end{aligned}$$

Similarly we can show that for $\delta_y \in Y[p^n]$,

$$K_n(x, x', y, \Delta + \delta_y) - K_n(x, x', y, \Delta) = 0$$

hence K_n descents to a morphism

$$\overline{K_n} : X[p^n] \times X[p^n] \times Y[p^n] \times Y[p^n] \to H_{14}$$

and together with the compatibility conditions as in 7.3.3 we obtain a function

$$K := \varinjlim K_n : X \times X \times Y \times Y \to H_{14}$$

which has to be trivial by the orbital rigidity of p-divisible groups 1.1.1. This proves (c).

Step 6. The first equation of (d) follows from 5.2.1 and that $\pi(W)$ is invariant under the induced action of G on B.

Let $x \in X[p^{3n}], \Delta \in Y[p^{2n}]$, on one hand, by (b) we have

$$\begin{aligned} & [p^n]\tilde{f}_n(x, 0, \Delta) \\ = & \tilde{f}_n(x, 0, [p^n]\Delta) \\ \overset{(x,0)\sim(x,[p^n]\Delta)}{=} & -\langle x_{12,3n}, [p^n]\Delta\rangle_{3n} \\ = & -\langle x_{12,n}, [p^n]\Delta\rangle_n \end{aligned}$$

On the other hand, by (c) and the fact that $\tilde{f}_n(0,0,\Delta) = 0, \forall \Delta \in Y[p^{2n}]$ we have

$$\begin{aligned}
&[p^n]\tilde{f}_n(x,0,\Delta)\\
=&\tilde{f}_n([p^n]x,0,\Delta) - \tilde{f}_n(0,0,\Delta)\\
=&(f_n([p^n]x,0) - f_n(0,0)) - (f_n([p^n]x,\Delta) - f_n(0,\Delta))\\
=&-\langle \Delta_{13}, [p^n]x_{34}\rangle_{3n}\\
=&-\langle [p^n]\Delta_{13}, x_{34,n}\rangle_n
\end{aligned}$$

That is for all $\delta \in Y[p^n], x_n = (x_{12,n}, x_{23,n}, x_{34,n}) \in X[p^n]$, we have

$$\langle \delta_{13,n}, x_{34,n}\rangle_n = \langle x_{12,n}, \delta_{24,n}\rangle_n$$

This proves the second equation in (d).

Step 7. Part (e) is a direct consequence of 7.2.1.

Step 8. Let $E_{X,Y}$ as defined in Step 7. Then there exists a morphism $T : \pi(W) \to H_{14}$ s.t.

$$\xi_W(w) = T(w) * \xi_{E_{X,Y}}(w)$$

where

- $\xi_W : \pi(W) \to E$ is the section from $\pi(W)$ to E that corresponds to W.
- $\xi_{E_{X,Y}} : \pi(E_{X,Y}) = \pi(W) \to E$ is the section from $\pi(W)$ to E that corresponds to $W_{\mathcal{H}}$.
- $w \in W(R)$ any R point of W for any Artinian local algebra R over k.

By 5.4.1, T is a trivial morphism. That is $\xi_W = \xi_{E_{X,Y}}$, which is equivalent to $W = E_{X,Y}$. As $E_{X.Y}$ is a Tate-linear subvariety by definition, W is also a Tate-linear subvariety. This proves (f).

□

Lemma 7.3.5. ***(Functoriality of Tate-linear Subvarieties)*** *Let $H = (H_{ij})_{1\leq i<j\leq 4}$ be a sustained nilpotent linear group of rank 4. Let $H^{1,3} = (H_{ij})_{1\leq i<j\leq 3}$ and let $\pi_{123} : H \to H^{1,3}$ the natural group scheme hommorphism. Let $B_{13} = Def_{H^{1,3}\text{-}torsor}$. Let $H' \subset H$ an admissible subgroup, and let $E_{H'}$ be the Tate-linear subvariety of E corresponds to H'. Let $\pi^{1,3}(H') \subset H^{1,3}$ and $E_{\pi^{1,3}(H')} \subset B_{13}$. Then the group homomorphism π_{123} induces a morphism*

$$\Pi_{123} : E \to B_{13}$$

s.t.

$$\Pi_{123}(E_{H'}) = E_{\pi^{1,3}(H')}$$

Proof. Left as exercise. □

7.4. Proof of Main Theorem

Theorem 7.4.1. ***(Orbital Rigidity Conjecture 4 Slopes Case).*** *Notation as in 7.0.1. Let $W \subset E$ a closed formal subvariety, reduced and irreducible, let G be a compact p-adic Lie subgroup of $Aut_{sus}(E)$ that acts strongly non-trivially on E. If W is invariant under G, then W is a Tate-linear formal subvariety of E.*

Proof. Let $H'_{14} = (W \cap H_{14})_{red}$, which is a p-divisible groups by the orbital rigidity theorem of p-divisible groups. Let $\pi' : E/H'_{14} \to B$ induced by the natural projection $\pi : E \to B$. Let $W' = W/H'_{14}$, where the H'_{14} action on W is guaranteed by 6.3.2. By 6.4.6 the map $W' \to \pi(W) \subset B$ is a finite purely inseparable morphism.

Recall $B = B_{13} \times_{H_{12}} B_{24}$. By the orbital rigidity theorem in three slopes case 5.2.1, both $\pi_{13}(W) \subset E_{13}, \pi_{24}(W) \subset E_{24}$ are Tate-linear subvarieties. Then by 5.6.3 we can find an homomorphism

$$\mathcal{L} : B \to B$$

that preserves $\pi(W)$ and $\mathcal{L}|_{\pi(W)}$ dominates $\pi'|_{W'} : W' \to \pi(W)$. That is, there exists $\xi_1 : \pi(W) \to W/H'_{14}$ such that $\pi|_{W'} \circ \xi_1 = \mathcal{L}|_{\pi(W)}$

Consider

$$E'_{\mathcal{L}} := E/H'_{14} \times_{B,\mathcal{L}} B$$
$$H'_{\mathcal{L}} := (H/H'_{14})_{\mathcal{L}}$$

where $H'_{\mathcal{L}} = ((H'_{\mathcal{L}})_{ij})_{1\leq i<j\leq 4}$ is the Tate-linear nilpotent group of type A with the same components as H/H'_{14}, that is

$$(H'_{\mathcal{L}})_{ij} = \begin{cases} H_{ij} & (i,j) \neq (1,4) \\ H_{14}/H'_{14} & (i,j) = (1.4) \end{cases}$$

but with bilinear pairings induced by $\mathcal{L}$. Then $H'_{\mathcal{L}}$ corresponds to $E'_{\mathcal{L}}$, that is

$$Def_{H'_{\mathcal{L}}\text{-torsor}} = E'_{\mathcal{L}}$$

As the compact p-adic Lie group G operates on E/H'_{14} and that W/H'_{14} is stable under the action of G, there exists a compact open subgroup

$$G'_{\mathcal{L}} \subset G$$

which operates on $E'_{\mathcal{L}}$, and the natural map $h : E_{\mathcal{L}} \to E/H'_{14}$ is equivariant with respect the the inclusion $G'_{\mathcal{L}} \hookrightarrow G$. The morphism $\xi_1 : \pi(W) \to W/H'_{14}$ defines a morphism $\xi_2 : \pi(W) \to E_{\mathcal{L}}$ such that $h \circ \xi_2 = \xi_1$. It follows that

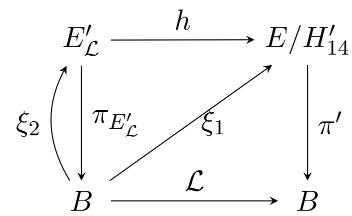
$$\mathcal{L} \circ \pi_{E'_{\mathcal{L}}} \circ \xi_2 = \pi' \circ \xi_1 = \mathcal{L}$$

Therefore

$$\pi_{E'_{\mathcal{L}}} \circ \xi_2 = id_{\pi(W)}$$

In other words ξ_2 is a section of the pullback $E_{\mathcal{L}}$ over $\pi(W)$.

The following graph demonstrates the relations between various maps:

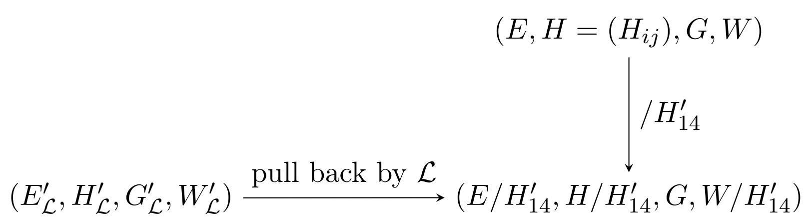


Moreover ξ_2 is equivariant with respect to the action of G' on E/H'_{14}. Let

$$W'_{\mathcal{L}} = W/H_{14} \times_{B,\mathcal{L}} B$$

the pullback of W/H'_{14} by $\mathcal{L}$, and let W'_{ξ_2} be the subscheme of $E'_{\mathcal{L}}$ that corresponds to the section ξ_2. Apparently $W'_{\xi_2} \subset W'_{\mathcal{L}}$. As $\dim(W'_{\xi_2}) = \dim(W'_{\mathcal{L}})$ and both are reduced and irreducible, we know that $W'_{\xi_2} = W'_{\mathcal{L}}$.

The following diagram illustrates above constructions.



Applying 7.3.4.f). to $(E'_{\mathcal{L}}, H'_{\mathcal{L}}, W'_{\xi_2} = W'_{\mathcal{L}}, G'_{\mathcal{L}})$, we conclude that $W'_{\xi_2} \subset E'_{\mathcal{L}}$ is Tate-linear subvariety. Hence by 4.7.14 and 4.7.13 we conclude that $W \subset E$ is also Tate-linear.

□

BIBLIOGRAPHY

[Cha05] C.-L. Chai. "Hecke orbits on Siegel modular varieties." In: *Geometric methods in algebra and number theory*. Springer, 2005, pp. 71–107.

[Cha08] C.-L. Chai. "A rigidity result for p-divisible formal groups." In: *Asian Journal of Mathematics* 12.2 (2008), pp. 193–202.

[Cha22] C.-L. Chai. *Personal letter*. March 2022.

[CO22] C.-L. Chai and F. Oort. *The Hecke orbit conjecture (Draft)*. 2022.

[Mes72] W. Messing. "The crystals associated to Barsotti-Tate groups." In: *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes* (1972), pp. 112–149.

[Mum68] D. Mumford. *Biextension of formal groups, in the proceedings of the Bombay Colloquium on Algebraic Geometry, Tata Institute of Fundamental Research Studies in Mathematics 4*. 1968.

[Mum69] D. Mumford. "A note of Shimura's paper "Discontinuous groups and abelian varieties"." In: *Mathematische Annalen* (1969).

[MRM74] D. Mumford, C. P. Ramanujam, and J. I. Manin. *Abelian varieties*. Vol. 3. Oxford university press Oxford, 1974.

[Oor04] F. Oort. "Foliations in moduli spaces of abelian varieties." In: *Journal of the American Mathematical Society* 17.2 (2004), pp. 267–296.

[Zin01] T. Zink. "On the slope filtration." In: *Duke Mathematical Journal* 109.1 (2001), pp. 79–95.