L^2 DECAY OF CERTAIN BILINEAR OSCILLATORY INTEGRAL OPERATORS

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ABSTRACT

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In this thesis, we study bilinear oscillatory integral operators of the form

$$I_\lambda(f_1, f_2) = \int_M e^{i\lambda\Phi(x)} f_1(x^1) f_2(x^2) a(x) d\sigma(x)$$

where $x^1 := (x_1, \ldots, x_d)$, $x^2 := (x_{d+1}, \ldots, x_{2d})$, and ρ, Φ, a are smooth functions on an open box B_1 with a compactly supported, $\partial_i \rho$ nonvanishing on B_1 for each i, and $M := \{x \in B_1 \,|\, \rho(x) = 0\}$. Under an additional determinant condition that has similarities to both a mixed Hessian condition on Φ and a Phong-Stein rotational curvature condition on ρ, we prove that this operator has optimal L^2 decay, namely that

$$|I_\lambda(f_1, f_2)| \leq C|\lambda|^{-\frac{d-1}{2}} ||f_1||_{L^2(\mathbb{R}^d)} ||f_2||_{L^2(\mathbb{R}^d)}$$

The proof uses a frequency space decomposition which is a higher-dimensional analogue of one developed in earlier work with Gressman, and applies this to the functions f_1 and f_2 to generate a kernel which captures the oscillatory behavior of the phase and can be analyzed using stationary phase arguments, among others. The constant C in the bound depends continuously on parameters based on a, Φ, ρ, the dimension d, and the size of the support of the integrand, and so the result is stable under small perturbations of these objects.

We then study two specific bilinear operators which have polynomial phase, and show how the results of the main theorem can be leveraged to prove decay even when the determinant condition in the hypothesis does not hold. We also use these examples to show that the decay of the operator is affected by the precise way in which the determinant condition fails.

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Introduction

1.1. Background

1.1.1. Oscillatory Integrals

The study of oscillatory integrals and oscillatory integral operators is a key area of harmonic analysis with many connections to other fields of mathematics. An oscillatory integral can take many forms, but a unifying feature is that the integrand is a product with an oscillating function in the form of a complex exponential, which oscillates at a "speed," in some sense, that is controlled by a parameter. Often when studying oscillatory integrals, we are seeking to quantify "smallness" of the output in terms of this speed parameter.

The prototypical example of an oscillatory integral is the Fourier transform:

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi}dx$$

There is a basic sense in which the Fourier transform exhibits the type of relationship between speed of oscillation and size of output that we are looking for: the Riemann-Lebesgue Lemma states that if $f \in L^1(\mathbb{R}^d)$, then $\lim_{|\xi|\to\infty} |\hat{f}(\xi)| = 0$.

A major area of study in harmonic analysis is the Fourier restriction problem. This is a broad question which asks: when can we take an L^p function f, restrict its Fourier transform to a set S, and have this be an L^q function? (In many cases, the set S is taken to be a hypersurface.) The Fourier restriction problem is related to other problems in harmonic analysis as well as problems in other fields such as PDEs. A given Fourier restriction problem can also often be viewed as an oscillatory integral problem, as the Fourier restriction operator can often be written as the adjoint of a particular oscillatory integral operator.

Stein [32] divides oscillatory integrals into two types: oscillatory integrals of the first kind, and oscillatory integrals of the second kind. We detail each type and list some major results below.

Oscillatory Integrals of the First Kind. Stein defines an oscillatory integral of the first kind to be an integral of the form

$$I(\lambda) = \int e^{i\lambda\phi(x)}\psi(x)dx$$

where $\lambda \in \mathbb{R}$ is a parameter, ϕ is a real-valued smooth function, and ψ is complex-valued and smooth, and usually assumed to have compact support. We seek to prove that $I(\lambda) \to 0$ as $\lambda \to \infty$, and specifically to determine how fast $I(\lambda)$ decays with respect to λ. A key property of oscillatory integrals of the first kind is that the limiting factor determining this decay is the portion of the integral where $\phi(x)$ is stationary, i.e., where $\nabla\phi(x) = 0$. In other words, if $\nabla\phi(x) \neq 0$ on the support of ψ, then $I(\lambda)$ decays rapidly. This is the principle of non-stationary phase, which we state and give a proof for below, as a similar type of argument is used in a proposition later on in the proof of Theorem 1. We will state and prove the result for $d = 1$, but the result holds in any dimension.

Proposition (Principle of Non-Stationary Phase [32, p. 331])**.** *Let ϕ and ψ be smooth functions on $\mathbb{R}$, with ψ compactly supported in $[a, b]$ and $\phi'(x) \neq 0$ on $[a, b]$. Then for any $N \geq 0$,*

$$|I(\lambda)| \leq C_{N,\phi,\psi}|\lambda|^{-N}$$

Proof. Define the following differential operator for functions on $[a, b]$:

$$Df(x) := \frac{1}{i\lambda\phi'(x)} \cdot \frac{df}{dx}$$

and note that its transpose is

$${}^{t}Df(x) = \frac{-d}{dx}\left(\frac{f(x)}{i\lambda\phi'(x)}\right)$$

By construction, for any $x \in [a, b]$ and any $N \geq 0$, we have $D^N(e^{i\lambda\phi}) = e^{i\lambda\phi}$. Thus, by integration by parts, since ψ smooth and supp $(\psi) \subset [a,b]$ implies $\psi(a) = \psi(b) = 0$, we have

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)}\psi(x)dx = \int_a^b D^N(e^{i\lambda\phi(x)})\psi(x)dx = \int_a^b e^{i\lambda\phi(x)}({}^tD)^N(\psi(x))dx \qquad (\star)$$

Then, because $|({}^tD)^N(\psi(x))| = |\lambda|^{-N}|(\tilde{D})^N(\psi(x))|$ where

$$\tilde{D}f(x) := \frac{-d}{dx}\left(\frac{f}{\phi'(x)}\right),$$

we can complete the proof by applying the triangle inequality to $(\star)$ and using the fact that ϕ smooth and $\phi'(x) \neq 0$ on $[a,b]$ implies that $|\phi'(x)| \geq c$ for some $c > 0$ on $[a,b]$. □

If we do not necessarily have that $\nabla\phi \neq 0$ on the support of the integral, we can still obtain estimates for $I(\lambda)$. One such result is the van der Corput lemma, which doesn't necessarily require any knowledge about whether ϕ is stationary or non-stationary. The van der Corput lemma says that if the absolute value of the k^{th} derivative of the phase is uniformly bounded below by a positive constant, then $I(\lambda)$ is $O(|\lambda|^{-1/k})$.

Proposition (van der Corput lemma [32, p. 332])**.** *Suppose ϕ is real-valued and smooth in (a,b), and that $|\phi^{(k)}(x)| \geq 1$ for all $x \in (a,b)$. Then*

$$\left|\int_a^b e^{i\lambda\phi(x)}dx\right| \leq c_k|\lambda|^{-1/k}$$

provided $k \geq 2$ or $k = 1$ and $\phi'(x)$ is monotone. The bound c_k is independent of ϕ and λ.

There is also an analogue of the van der Corput lemma for higher dimensional cases.

Proposition (van der Corput lemma analogue [32, p. 342])**.** *Suppose ψ is smooth and supported in the unit ball, and suppose ϕ is a real-valued function such that for some multi-index α with $|\alpha| = k > 0$, we have $|\partial^\alpha\phi| \geq 1$ throughout the support of ψ. Then*

$$\left|\int_{\mathbb{R}^d} e^{i\lambda\phi(x)}\psi(x)dx\right| \leq c_k(\phi)\cdot|\lambda|^{-1/k}\cdot(||\psi||_{L^\infty} + ||\nabla\psi||_{L^1})$$

Finally, another useful result for oscillatory integrals of the first kind deals with the case when the phase has a single nondegenerate critical point, i.e., $\nabla\phi(x_0) = 0$ but the Hessian of ϕ at x_0 is invertible. This result is due to Hörmander [21].

Theorem (Hörmander [21, p. 220])**.** *Suppose $K \subset \mathbb{R}^d$ is compact, $X \supset K$ is open, and $N > 0$ is an integer. If $\psi \in C_0^{2N}(K)$, $\phi \in C^{3N+1}(X)$, $\operatorname{Im}\phi \geq 0$ in X, $\operatorname{Im}\phi(x_0) = 0$, $\nabla\phi(x_0) = 0$, $\det[\mathcal{H}(\phi)(x_0)] \neq 0$, $\nabla\phi \neq 0$ in $K \setminus \{x_0\}$, then*

$$\int e^{i\lambda\phi(x)}\psi(x)dx = \left(\frac{2\pi i}{\lambda}\right)^{d/2} e^{i\lambda\phi(x_0)}(\det\mathcal{H}(\phi)(x_0))^{-1/2}\sum_{k<N}\lambda^{-k}L_k\psi + O(|\lambda|^{-N})$$

where L_k is a differential operator of order $2k$ acting on ψ at x_0.

The definition of L_k and details on the constants in the term $O(|\lambda|^{-N})$ can be found in [21]. One way of viewing this result is that it gives us an asymptotic expansion for the λ decay of $I(\lambda)$; it essentially tells us that $I(\lambda) \sim \lambda^{-d/2}\sum c_k\lambda^{-k}$.

Oscillatory Integrals of the Second Kind. For Stein, oscillatory integrals of the second kind are oscillatory integral operators, which he separates into three different types. The first type is an operator from functions on $\mathbb{R}^d$ to functions on $\mathbb{R}^d$ of the form

$$(T_\lambda f)(\xi) = \int_{\mathbb{R}^d} e^{i\lambda\Phi(x,\xi)}\psi(x,\xi)f(x)dx, \qquad \xi \in \mathbb{R}^d$$

The second type is an operator from functions on $\mathbb{R}^{d-1}$ to functions on $\mathbb{R}^d$ of the form

$$(T_\lambda f)(\xi) = \int_{\mathbb{R}^{d-1}} e^{i\lambda\Phi(x,\xi)}\psi(x,\xi)f(x)dx, \qquad \xi \in \mathbb{R}^d$$

where here, if we have the special case $\Phi(x,\xi) = x \cdot \overline{\xi} + \phi(x)\xi_d$ where $x \in \mathbb{R}^{d-1}$ and $\overline{\xi} = (\xi_1, \ldots, \xi_{d-1})$, then the adjoint T_λ^* is essentially the operator which restricts the Fourier transform of a function on $\mathbb{R}^d$ to the surface $x_n = \phi(x)$ in $\mathbb{R}^d$.

The third type is a Fourier integral operator, which has connections to the study of Radon transforms, but is less directly relevant to our results in this thesis, so we omit its description here.

For the first type of operator above, we have another result from Hörmander [20] which guarantees decay, provided a "mixed Hessian" condition is satisfied. This condition is related to one of the assumptions in the hypothesis of Theorem 1.

Theorem (Hörmander [20])**.** *Let $a \in C_0^\infty(\mathbb{R}^{2d})$, let $\Phi \in C^\infty(\mathbb{R}^{2d})$ be real-valued, and for $\lambda > 1$ define*

$$T_\lambda f(x) = \int e^{i\lambda\Phi(x,y)} a(x,y) f(y) dy$$

for $f \in C_0^\infty(\mathbb{R}^d)$. If $\det \partial^2\Phi/\partial x \partial y \neq 0$ in $\operatorname{supp} a$ and $1 \leq p \leq 2$, $1/p + 1/p' = 1$, then

$$||T_\lambda f||_{L^{p'}(\mathbb{R}^d)} \leq C\lambda^{-d/p'} ||f||_{L^p(\mathbb{R}^d)}$$

We note that with $p = p' = 2$, this becomes

$$||T_\lambda f||_{L^2(\mathbb{R}^d)} \leq C\lambda^{-d/2} ||f||_{L^2(\mathbb{R}^d)}$$

Outside of oscillatory integrals of the first and second kind, there has been significant work done to understand the decay properties of multilinear oscillatory integrals. Phong, Stein, and Sturm studied multilinear oscillatory integral operators with polynomial phase in [30] and proved that decay was tied to the reduced Newton polyhedron of the phase. Carbery and Wright also studied multilinear oscillatory integral operators on measurable real-valued functions in [3] while proving a higher dimensional analogue of the van der Corput lemma.

Soon after, Christ, Li, Tao, and Thiele [7] introduced a general framework for studying multilinear oscillatory integral operators:

$$\Lambda_\lambda(f_1, \ldots, f_n) = \int_{\mathbb{R}^d} e^{i\lambda\Phi(x)} \prod_{j=1}^{n} f_j(\pi_j(x)) a(x) dx$$

where $\lambda \in \mathbb{R}$ is a parameter, $\Phi : \mathbb{R}^d \to \mathbb{R}$ is a measurable real-valued function, $d \geq 2$, and $a \in C_0^1(\mathbb{R}^d)$ is compactly supported. Each π_j denotes orthogonal projection from $\mathbb{R}^d$ to a linear subspace $V_j \subset \mathbb{R}^d$ of dimension $\kappa < d$, and $f_j : V_j \to \mathbb{C}$ is locally integrable with respect to Lebesgue measure on V_j. The authors were interested in finding conditions on P and V_j that allowed them to prove an inequality of the form

$$|\Lambda_\lambda(f_1, \ldots, f_n)| \leq C(1 + |\lambda|)^{-\epsilon} \prod_{j=1}^{n} ||f_j||_{L^\infty(V_j)}.$$

for $\epsilon > 0$ (bounds with other exponents on the f_j can then be obtained through a simple interpolation argument). Note that for any function P, this inequality is automatic with $\epsilon = 0$. The authors studied polynomial phases P and were able to prove results for $\kappa = d - 1$ and $\kappa = 1$, provided that the polynomial phase P had bounded degree and satisfied a nondegeneracy condition with respect to the subspaces V_j.

More recent progress in this area includes work by Christ and Silva [9], Deng, Shi, and Yan [11], Dong, Maldague, and Villano [12], Gilula, Gressman, and Xiao [13], Greenblatt [14], Gressman and Xiao [19], Niepla, O'Neill, and Zeng [24, 25], Xiao [38], and Zeng [39].

1.1.2. Radon Transforms and Rotational Curvature

We can define the classical Radon transform for measurable functions on $\mathbb{R}^d$ to be the following

$$Rf(\omega, t) = \int_{x \cdot \omega = t} f(x) d\sigma(x), \;\; \omega \in S^{d-1}, t \in \mathbb{R}$$

where S^{d-1} is the unit sphere in $\mathbb{R}^d$ and $d\sigma(x)$ is Lebesgue measure on the hyperplane $\{x \in \mathbb{R}^d \,|\, x \cdot \omega = t\}$. Oberlin and Stein fully determined the boundedness properties of the Radon transform in [26]:

Theorem (Oberlin and Stein, [26]). *For $d \geq 2$,*

$$||Rf||_{L^q_\omega L^r_t} \leq C_{p,q,r} ||f||_{L^p_x}$$

holds if and only if $1 \leq p < \frac{d}{d-1}$*,* $q \leq p'$*, and* $\frac{1}{r} = \frac{d}{p} - d + 1$*, where* p' *is the dual exponent to* p *(*$\frac{1}{p} + \frac{1}{p'} = 1$*).*

(The above uses the definition $||Rf||_{L^q_\omega L^r_t} = (\int_{S^{d-1}} ||Rf(\omega, \cdot)||^q_{L^r(\mathbb{R})} d\sigma(\omega))^{1/q}$.)

The Radon transform can be thought of as a type of geometric averaging operator, as it is averaging the input function f over a family of submanifolds of $\mathbb{R}^d$. Consider a more general geometric averaging operator or Radon-like transform of the form

$$Tf(x) = \int_{M_x} f(y) d\sigma_x(y)$$

where $M_x = \{y \mid \rho(x, y) = 0\}$ for ρ a real-valued and smooth function defined on $U \subset \mathbb{R}^d \times \mathbb{R}^d$, and $d\sigma_x$ is Lebesgue measure on M_x. Note that if $z \in \mathbb{R}^d$ and $\Psi_r(z) := z_1^2 + \cdots + z_d^2 - r$, then setting $\rho(x, y) = \Psi_r(x - y)$ means that

$$Tf(x) = \int_{\partial B_r(x)} f(y) d\sigma_x(y)$$

i.e., T is just a classical spherical averaging operator. In general, T can be shown to be bounded from $L^2(\mathbb{R}^d)$ to $L^2_{\frac{d-1}{2}}(\mathbb{R}^d)$ [33], provided that the function ρ defining the submanifolds M_x satisfies a condition known as rotational curvature.

Rotational curvature was introduced by Phong and Stein in [27–29], in part to study singular Radon transforms, which can be defined as being operators of the form:

$$\widetilde{T}f(x) = \int_{M_x} f(y) K_x(y) d\sigma_x(y) \tag{$\star$}$$

where $x \to M_x$ is any smooth mapping from $\mathbb{R}^d$ to smooth submanifolds, $K_x(y)$ is a Calderón-Zygmund kernel with singularity at $x = y$, and $d\sigma_x(y)$ is any measure on M_x with smooth density.

There are several equivalent ways of defining rotational curvature, but we will use the definition that most closely relates to the setting of the results of this thesis, as there are connections between this definition and the hypothesis of our first main result, Theorem 1. (The definition below is also the one used in the proof of the result for the Radon-like transform T defined above.)

Suppose $\rho(x,y)$ is a real-valued and smooth function defined on $U \subset \mathbb{R}^d \times \mathbb{R}^d$. We define the rotational curvature of ρ to be the determinant of the following block matrix

$$J(\rho) = \det \left[\begin{array}{c|c} \rho(x,y) & \partial\rho(x,y)/\partial y_j \\ \hline \dfrac{\partial\rho(x,y)}{\partial x_k} & \dfrac{\partial^2\rho(x,y)}{\partial x_k \partial y_j} \end{array} \right]$$

We say that ρ has rotational curvature if $J(\rho) \neq 0$ when $\rho = 0$. Note that this is equivalent to defining

$$J(\rho) = \det \left[\begin{array}{c|c} 0 & \partial\rho(x,y)/\partial y_j \\ \hline \dfrac{\partial\rho(x,y)}{\partial x_k} & \dfrac{\partial^2\rho(x,y)}{\partial x_k \partial y_j} \end{array} \right]$$

and requiring $J(\rho) \neq 0$ when $\rho = 0$.

To help understand this definition, we give two examples which come from Stein [32]. For these examples, we consider the setting where we have a mapping from $\mathbb{R}^d$ to hypersurfaces, defined by

$$x \mapsto M_x := \{y \in \mathbb{R}^d \,|\, \rho(x,y) = 0\}$$

where ρ is smooth and real-valued as above. If ρ has rotational curvature, i.e., $J(\rho) \neq 0$ when $\rho = 0$, then this immediately implies that $\nabla_y \rho(x,y) \neq 0$ when $\rho = 0$ (the first row of an invertible matrix cannot vanish), and therefore the surfaces M_x are locally smooth submanifolds and vary smoothly with x.

First, consider the case where M_x is given by $M_x = x + M_0$, i.e., all of the hypersurfaces M_x are given by some translate of a fixed hypersurface. One way that this might happen is if we have $\rho(x,y) = \psi(y-x)$, so that $M_0 = \{y \,|\, \psi(y) = 0\}$. If $y \in M_0$, then $\nabla_y \rho(x,y) = \nabla_y \psi(y)$

is a normal vector to M_0 at y, and the requirement that $J(\rho) \neq 0$ when $\rho = 0$ is equivalent to requiring that the $(n-1) \times (n-1)$ symmetric matrix given by restricting

$$\left(\frac{\partial^2 \psi}{\partial_{y_i} \partial_{y_j}}\right)_{1 \leq i,j \leq n}$$

to the the tangent plane to M_0 at y (the plane perpendicular to $\nabla_y \psi(y)$) is invertible. Thus in this case, ρ having rotational curvature is equivalent to M_0 having nonvanishing Gaussian curvature.

Second, consider the case where $\rho(x, y) = x \cdot y + c$, for some constant $c \neq 0$. If we calculate $J(\rho)$, we see that $J(\rho) = c$ by the first definition and $J(\rho) = -x \cdot y$ by the second definition. In either case, $J(\rho) = c \neq 0$ whenever $\rho = 0$, so ρ has rotational curvature. In this situation, each hypersurface M_x is just an affine hyperplane, so has zero Gaussian curvature, but the idea is that ρ still has rotational curvature because the hypersurfaces M_x rotate as x varies.

After the introduction of rotational curvature by Phong and Stein to study singular Radon transforms, a key development was made by Christ, Nagel, Stein, and Wainger [8], who used vector field techniques to define a curvature condition on the submanifolds M_x which implies L^p boundedness for the singular Radon transform defined as above in $(\star)$. Around the same time, Christ [6] studied an operator that averages a function along translates of the curve $\gamma(t) := (t, t^2, \ldots, t^d)$, and established optimal $L^p - L^q$ mapping behavior (outside of two endpoint cases, in which restricted weak-type estimates are obtained), by iterating the operator and using geometric and combinatorial techniques now known as the method of refinements. Tao and Wright [36] then built on the techniques in these papers to study Radon-like transforms which average over smooth families of curves, and also obtained optimal $L^p - L^q$ bounds up to endpoints.

Other key developments were due to Stovall [35], who generalized the result of Tao and Wright, moving from a bilinear result to a multilinear result, and Gressman [17], who developed an alternate idea of curvature (similar to but distinct from rotational curvature), which he used to study Radon-like transforms of intermediate dimension and obtain sharp L^p-improving estimates up to endpoints. Finally, other notable papers in this area include works by Bak [1], Choi [4, 5], Dendrinos, Laghi, and Wright [10], Erdoğan and Oberlin [2], Gressman [15, 16], Lee [23], Iosevich and Sawyer [22], Seeger [31], and Stovall [34], although this is by no means an exhaustive list.

1.2. Definitions and Assumptions

In this thesis, we will be looking at bilinear operators of the form

$$I_\lambda(f_1, f_2) = \int_M e^{i\lambda\Phi(x)} f_1(x^1) f_2(x^2) a(x) d\sigma(x) \tag{1.1}$$

where

- $x^1 := (x_1, \ldots, x_d)$ and $x^2 := (x_{d+1}, \ldots, x_{2d})$
- f_1, f_2 are measurable functions on $\mathbb{R}^d$
- ρ, Φ are smooth and real-valued functions on $B_1 := (-b_1, b_1)^{2d}$
- a is smooth and compactly supported in $B_0 \subset B_1$ with $B_0 := [-b_0, b_0]^{2d}$
- The gradient of ρ is nonvanishing on B_1 and $M = \{x \in B_1 \,|\, \rho(x) = 0\}$
- $d\sigma$ is Lebesgue measure on M

Furthermore, we assume that $|\lambda|$ is bounded below, specifically that:

$$|\lambda|^{-1/2} \leq \min\{b_1 - b_0, 1\} \tag{1.2}$$

and that $|\partial_i \rho|$ is uniformly bounded below by a positive constant on B_1 for $i = 1, \ldots, 2d$.

Throughout this thesis, we will use the notation $a \lesssim b$ to mean that there exists some constant c such that $a \leq cb$, with c depending only on the admissible positive constants d, b_0, b_1, C_ρ, C'_ρ, C_Φ, C_a, and the constant c appearing in (1.7). Here, d is the dimension,

b_0 and b_1 are the constants from the definition of (1.1), and the remaining constants are defined as follows:

$$\max_{|\alpha|\leq 2d+2} \sup_{x\in B_0} |\partial^\alpha a(x)| \leq C_a \tag{1.3}$$

$$\max_{|\alpha|\leq 2d+2} \sup_{x\in B_1} |\partial^\alpha \Phi(x)| \leq C_\Phi \tag{1.4}$$

$$\max_{|\alpha|\leq 2d+3} \sup_{x\in B_1} |\partial^\alpha \rho(x)| \leq C_\rho \tag{1.5}$$

$$\min_{1\leq i\leq 2d} \inf_{x\in B_1} |\partial_i \rho(x)| \geq (C'_\rho)^{-1} \tag{1.6}$$

1.3. Main Results

The first result we will prove is a bilinear version of the main result in [18]. (The main result in [18] is discussed in Section 1.4.) This bilinear version is analogous to the multilinear version, with the main difference being in the formulation of the determinant condition (1.7) in the hypothesis.

Theorem 1. *Suppose that $I_\lambda(f_1, f_2)$ is the operator defined by* (1.1)*, satisfying all of the conditions* (1.2)*,* (1.3)*,* (1.4)*,* (1.5)*, and* (1.6)*. Suppose also that there exists a constant $c > 0$ such that for every $x \in B_1$ and every $(\omega_1, \omega_2) \in S^1$, we have*

$$\left| \det \begin{bmatrix} \partial_{x_1}\rho(x) & \partial^2_{x_1 x_{d+1}}(\omega_1\Phi(x) + \omega_2\rho(x)) & \cdots & \partial^2_{x_1 x_{2d}}(\omega_1\Phi(x) + \omega_2\rho(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_d}\rho(x) & \partial^2_{x_d x_{d+1}}(\omega_1\Phi(x) + \omega_2\rho(x)) & \cdots & \partial^2_{x_d x_{2d}}(\omega_1\Phi(x) + \omega_2\rho(x)) \\ 0 & \partial_{x_{d+1}}\rho(x) & \cdots & \partial_{x_{2d}}\rho(x) \end{bmatrix} \right| \geq c \tag{1.7}$$

Then for any $f_1, f_2 \in L^2(\mathbb{R}^d)$, we have

$$|I_\lambda(f_1, f_2)| \lesssim |\lambda|^{-\frac{d-1}{2}} ||f_1||_{L^2(\mathbb{R}^d)} ||f_2||_{L^2(\mathbb{R}^d)} \tag{1.8}$$

and this decay is optimal in the sense that for any λ, there exist $f_1, f_2 \in L^2(\mathbb{R}^d)$ such that $|I_\lambda(f_1, f_2)| \geq c'|\lambda|^{-\frac{d-1}{2}} ||f_1||_{L^2(\mathbb{R}^d)} ||f_2||_{L^2(\mathbb{R}^d)}$, with c' independent of $|\lambda|$.

The determinant condition (1.7) in the hypothesis of Theorem 1 is only used in the final stages of the proof, so we can leverage much of the machinery of the proof of Theorem 1 to prove results for two specific operators which satisfy all of the hypotheses of Theorem 1 except for the determinant condition (1.7). The first operator fails the determinant condition only on the plane $x_2 = 0$.

Theorem 2. *Suppose that $\tilde{I}_\lambda(f_1, f_2)$ is the operator defined as follows:*

$$\tilde{I}_\lambda(f_1, f_2) = \int_M e^{i\lambda(\frac{1}{2}x_2^2 x_4 + x_3 x_5)} f_1(x^1) f_2(x^2) a(x) d\sigma(x) \tag{1.9}$$

where f_1, f_2 are measurable functions on $\mathbb{R}^3$, B_0, B_1 are boxes as in (1.1), *$|\lambda|$ satisfies* (1.2), *a is smooth and compactly supported in B_0, and $M = \{x \in B_1 \,|\, \rho(x) = 0\}$ for*

$$\rho(x) = -\tfrac{1}{2}x_2^2 x_5 + x_3 x_4 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

Then for any $f_1, f_2 \in L^2(\mathbb{R}^3)$, we have

$$|\tilde{I}_\lambda(f_1, f_2)| \lesssim |\lambda|^{-1/2} \log|\lambda| \, ||f_1||_{L^2(\mathbb{R}^3)} ||f_2||_{L^2(\mathbb{R}^3)} \tag{1.10}$$

The second operator fails the determinant condition only on the plane $x_2 + x_5 = 0$, and despite the similarities to the first operator, we get a slightly better result for the second operator, with a slightly simpler proof.

Theorem 3. *Suppose that $\tilde{\tilde{I}}_\lambda(f_1, f_2)$ is the operator defined as follows:*

$$\tilde{\tilde{I}}_\lambda(f_1, f_2) = \int_M e^{i\lambda(x_2 x_4 + x_2 x_3 x_5 + \frac{1}{2}x_3 x_5^2)} f_1(x^1) f_2(x^2) a(x) d\sigma(x) \tag{1.11}$$

where f_1, f_2 are measurable functions on $\mathbb{R}^3$, B_0, B_1 are boxes as in (1.1), *$|\lambda|$ satisfies* (1.2), *a is smooth and compactly supported in B_0, and $M = \{x \in B_1 \,|\, \rho(x) = 0\}$ for*

$$\rho(x) = \tfrac{1}{2}x_2^2 x_5 + \tfrac{1}{2}x_2 x_5^2 - x_3 x_4 - \tfrac{1}{2}x_3^2 x_5 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

Then for any $f_1, f_2 \in L^2(\mathbb{R}^3)$*, we have*

$$|\tilde{\tilde{I}}_\lambda(f_1, f_2)| \lesssim |\lambda|^{-1/2} \, ||f_1||_{L^2(\mathbb{R}^3)} ||f_2||_{L^2(\mathbb{R}^3)} \tag{1.12}$$

Note that for both of these specific operators, we are assuming that $B_1 = (-b_1, b_1)^6$ is not too large (for example, $b_1 < 1$), so that we have $|\partial_i \rho|$ uniformly bounded below on B_1, per the general requirement for these operators.

1.4. Motivation

In earlier work with Gressman [18], we proved the following theorem for multilinear oscillatory integral operators.

Theorem 4. *Suppose* $I'_\lambda(f_1, \ldots, f_{2d})$ *is a multilinear operator on measurable functions* f_j *on* $\mathbb{R}$*, given by*

$$I'_\lambda(f_1, \ldots, f_{2d}) = \int_M e^{i\lambda\Phi(x)} \left[\prod_{j=1}^{2d} f_j(x_j)\right] a(x) d\sigma(x) \tag{1.13}$$

where we have the same definitions and assumptions as in the definition of (1.1)*. Suppose that* λ, a, Φ, ρ *satisfy the bounds* (1.2), (1.3), (1.4), (1.5), *and* (1.6)*. Further suppose that there exists some* $c > 0$ *such that for every* $x \in B_1$ *and every* $(\omega_1, \omega_2) \in S^1$*, the indices* $\{1, \ldots, 2d\}$ *can be partitioned into two sets* $\{i_1, \ldots, i_d\}$ *and* $\{j_1, \ldots, j_d\}$ *such that*

$$\left| \det \begin{bmatrix} \partial_{i_1}\rho(x) & \partial^2_{i_1 j_1}(\omega_1\Phi(x) + \omega_2\rho(x)) & \cdots & \partial^2_{i_1 j_d}(\omega_1\Phi(x) + \omega_2\rho(x)) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{i_d}\rho(x) & \partial^2_{i_d j_1}(\omega_1\Phi(x) + \omega_2\rho(x)) & \cdots & \partial^2_{i_d j_d}(\omega_1\Phi(x) + \omega_2\rho(x)) \\ 0 & \partial_{j_1}\rho(x) & \cdots & \partial_{j_d}\rho(x) \end{bmatrix} \right| \geq c$$

Then for all $f_1, \ldots, f_{2d} \in L^2(\mathbb{R})$*,*

$$|I'_\lambda(f_1, \ldots, f_{2d})| \lesssim |\lambda|^{-\frac{d-1}{2}} \prod_{j=1}^{2d} ||f_j||_{L^2(\mathbb{R})}$$

A natural next step is to ask whether a bilinear version of this theorem is possible. The answer is yes: this is Theorem 1. The proof of Theorem 1 is similar in structure and content to the proof of the multilinear result, although small but pervasive changes are required to deal with the fact that for the bilinear operator, there is now a fixed partition of the x variables into $x^1 := (x_1, \ldots, x_d)$ and $x^2 := (x_{d+1}, \ldots, x_{2d})$.

Once we have proved Theorem 1, another natural next step is to ask what happens when the determinant condition (1.7) fails. It turns out that if the determinant condition (1.7) fails, the way in which it fails, i.e., the nature of the zeroes of the determinant, greatly influences the decay of the resulting operator. To see this, consider the following two examples.

Example. An operator in dimension $d = 3$ which fails the determinant condition is

$$\Phi(x) = x_1x_4 + x_2x_5$$

$$\rho(x) = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

as then the determinant would be

$$\det \begin{bmatrix} 1 & \omega_1 & 0 & 0 \\ 1 & 0 & \omega_1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \omega_1^2$$

and this vanishes only at $(\omega_1, \omega_2) = (0, \pm 1) \in S^1$. We now show that this exhibits optimal decay as a type of Fourier transform. To make the following estimates simpler, we assume that the amplitude function a has the form $a(x) = a_1(x_1, x_2, x_3)a_2(x_4, x_5, x_6)$. First, note that

$$\begin{aligned}&\int_M e^{i\lambda(x_1x_4+x_2x_5)} f(x_1,x_2,x_3)g(x_4,x_5,x_6)a(x)d\sigma(x)\\&\quad=\int_{x_3+x_6=0} e^{i\lambda(x_1x_4+x_2x_5)}\tilde{f}(x_1,x_2,x_3)\tilde{g}(x_4,x_5,x_6)\tilde{a}(x)d\sigma(x)\end{aligned}$$

when we make the change of variables $x_1+x_2+x_3 \to x_3$ and $x_4+x_5+x_6 \to x_6$ (keeping all other variables unchanged), as this has Jacobian determinant 1. Here, $\tilde{f}(x_1,x_2,x_3) = f(x_1,x_2,x_3-x_1-x_2)$ and $\tilde{g}(x_4,x_5,x_6) = g(x_4,x_5,x_6-x_5-x_4)$, and by abuse of notation we will drop the tildes going forward, as $||\tilde{f}||_{L^2} = ||f||_{L^2}$ and $||\tilde{g}||_{L^2} = ||g||_{L^2}$. From here, we do another change of variables:

$$\begin{aligned}&\int_M e^{i\lambda(x_1x_4+x_2x_5)} f(x_1,x_2,x_3)g(x_4,x_5,x_6)a(x)d\sigma(x)\\&\quad=\int_{x_3+x_6=0} e^{i\lambda(x_1x_4+x_2x_5)} f(x_1,x_2,x_3)g(x_4,x_5,x_6)\tilde{a}(x)d\sigma(x)\\&\quad=\sqrt{2}\int_{\mathbb{R}^5} e^{i\lambda(x_1x_4+x_2x_5)} f(x_1,x_2,x_3)g(x_4,x_5,-x_3)\tilde{\tilde{a}}(\overline{x})d\overline{x}\end{aligned}$$

where $\overline{x} := (x_1,x_2,x_3,x_4,x_5)$ and we're using the change of variables $x_6 = \Psi(\overline{x}) = -x_3$ which has Jacobian determinant $(1+|\nabla\Psi(\overline{x})|^2)^{1/2} = \sqrt{2}$. Going forward, we treat x_3 as a parameter, and use Fubini's Theorem and the definition of the Fourier transform:

$$\begin{aligned}&\int_M e^{i\lambda(x_1x_4+x_2x_5)} f(x_1,x_2,x_3)g(x_4,x_5,x_6)a(x)d\sigma(x)\\&\quad=\sqrt{2}\int_{\mathbb{R}^5} e^{i\lambda(x_1x_4+x_2x_5)} f(x_1,x_2,x_3)g(x_4,x_5,-x_3)\tilde{\tilde{a}}(\overline{x})d\overline{x}\\&\quad=\sqrt{2}\int_{\mathbb{R}}\left[\int_{\mathbb{R}^4} e^{i\lambda(x_1x_4+x_2x_5)} f(x_1,x_2;x_3)g(x_4,x_5;-x_3)\tilde{\tilde{a}}(\overline{x})dx_1dx_2dx_4dx_5\right]dx_3\\&\quad=\sqrt{2}\int\left[\int\left[\int e^{i\lambda(x_1x_4+x_2x_5)} f(x_1,x_2;x_3)\tilde{a_1}(x_1,x_2;x_3)dx_1dx_2\right]\right.\\&\qquad\qquad\left.\cdot\, g(x_4,x_5;-x_3)\tilde{a_2}(x_4,x_5;-x_3)dx_4dx_5\right]dx_3\\&\quad=\sqrt{2}\int\left[\int\widehat{(f\tilde{a_1})}(\lambda x_4,\lambda x_5;x_3)g(x_4,x_5;-x_3)\tilde{a_2}(x_4,x_5;-x_3)dx_4dx_5\right]dx_3\end{aligned}$$

By $\widehat{(f\tilde{a}_1)}(\cdot,\cdot;x_3)$ we mean the situation in the center brackets on the fourth line above; a Fourier transform of $f\tilde{a}_1$ in only the first two variables. To finish, we apply Cauchy-Schwarz, another change of variables, and Plancherel's Theorem:

$$\begin{aligned}
&\left|\int_M e^{i\lambda(x_1x_4+x_2x_5)} f(x_1,x_2,x_3)g(x_4,x_5,x_6)a(x)d\sigma(x)\right| \\
&\quad\approx \left|\int\left[\int \widehat{(f\tilde{a}_1)}(\lambda x_4,\lambda x_5;x_3)g(x_4,x_5;-x_3)\tilde{a_2}(x_4,x_5;-x_3)dx_4dx_5\right]dx_3\right| \\
&\quad\leq \left(\int\left[\int |\widehat{(f\tilde{a}_1)}(\lambda x_4,\lambda x_5;x_3)|^2 dx_4dx_4\right]dx_3\right)^{1/2} ||g\tilde{a_2}||_{L^2(\mathbb{R}^3)} \\
&\quad\lesssim |\lambda|^{-1}\left(\int\left[\int |\widehat{(f\tilde{a}_1)}(x_4,x_5;x_3)|^2 dx_4dx_4\right]dx_3\right)^{1/2} ||g||_{L^2(\mathbb{R}^3)} \\
&\quad= |\lambda|^{-1}||f\tilde{a_1}||_{L^2(\mathbb{R}^3)}||g||_{L^2(\mathbb{R}^3)} \\
&\quad\lesssim |\lambda|^{-1}||f||_{L^2(\mathbb{R}^3)}||g||_{L^2(\mathbb{R}^3)}
\end{aligned}$$

Note also that this decay is optimal; if, without loss of generality, we assume $a(0)$ is positive and $a(x)$ is real on a neighborhood of $x=0$, and if we let

$$\begin{aligned}
f(x_1,x_2,x_3) &= \chi_{|\cdot|<c_1|\lambda|^{-1}}(x_1)\chi_{|\cdot|<c_2|\lambda|^{-1}}(x_2)\chi_{|\cdot|<c_3}(x_3) \\
g(x_4,x_5,x_6) &= \chi_{|\cdot|<c_4}(x_4)\chi_{|\cdot|<c_5}(x_5)\chi_{|\cdot|<c_6}(x_6)
\end{aligned}$$

then as long as $c_1,\dots,c_6$ are small enough depending only on a (as well as on the specific choice of ρ,Φ for this example), we can guarantee that $\mathrm{Re}[e^{i\lambda(x_1x_4+x_2x_5)}] \geq \frac{1}{2}$ and $a(x) \geq a(0)/2 > 0$ on the support of the integrand. This implies

$$\begin{aligned}
&\left|\int_M e^{i\lambda(x_1x_4+x_2x_5)} f(x_1,x_2,x_3)g(x_4,x_5,x_6)a(x)d\sigma(x)\right| \\
&\quad\geq \left|\mathrm{Re}\left[\int_M e^{i\lambda(x_1x_4+x_2x_5)} f(x_1,x_2,x_3)g(x_4,x_5,x_6)a(x)d\sigma(x)\right]\right| \\
&\quad\geq C\int_M f(x_1,x_2,x_3)g(x_4,x_5,x_6)d\sigma(x) \\
&\quad= C'|\lambda|^{-2} = C''|\lambda|^{-1}||f||_{L^2(\mathbb{R}^3)}||g||_{L^2(\mathbb{R}^3)}
\end{aligned}$$

for some constant C'' that is independent of $|\lambda|$. Thus the decay of $|\lambda|^{-1}$ proved above for this operator is sharp; this operator cannot have decay greater than $|\lambda|^{-1}$. (Very similar arguments to the one above are explained in more detail at the end of Section 4.1 and in Section 4.2.1.)

Example. As another example for $d = 3$, we can also consider the operator

$$\Phi(x) = 0$$

$$\rho(x) = x_1x_4 + x_2x_5 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

as then the determinant would be

$$\det \begin{bmatrix} 1 + x_4 & \omega_2 & 0 & 0 \\ 1 + x_5 & 0 & \omega_2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 + x_1 & 1 + x_2 & 1 \end{bmatrix} = \omega_2^2$$

which vanishes only at $(\omega_1, \omega_2) = (\pm 1, 0) \in S^1$. As $\Phi = 0$, there is no λ dependency in the integral, so the operator cannot have any λ decay.

Later in this thesis, we will deal with two more specific examples, also for $d = 3$, where the determinant in the condition (1.7) has zeroes that only depend on the x variables: these are the results Theorem 2 and Theorem 3. In both of these cases, where the zeroes are relatively simple, we show how the machinery developed in the proof of Theorem 1 can be used to prove decay on regions outside of the determinant's zeroes.

One thing to note is that despite the fact that we are unable to prove optimal decay for the specific operators in this thesis, we will prove that both operators decay at a rate of at least $|\lambda|^{-1/2} \log |\lambda|$ and that both operators cannot decay at a rate greater than $|\lambda|^{-3/4}$, which makes them distinct from the examples above.

CHAPTER 2

Frequency Space Decomposition

In this section we will first develop a decomposition of frequency space $\mathbb{R}^d$, and then define a transformation which decomposes a function in frequency space.

The following results are heavily adapted from the $d = 1$ case outlined in earlier work with Gressman [18]. The structure of the decomposition requires only minor modifications to translate the $d = 1$ case, while the transformation has a slightly different construction that retains all the key properties that the $d = 1$ transformation has, but uses smooth cutoff functions to localize on the frequency side as opposed to characteristic functions. This change gives us the option of more easily relating the x variables in the original integral with $f_1(x^1)$ and $f_2(x^2)$ to the y variables in the integral of the transformed functions $Vf_1(y^1, \xi^1)$ and $Vf_2(y^2, \xi^2)$. This is done via a Schwartz tails argument, which we briefly outline at the end of this section.

2.1. Construction of the Decomposition

For each $n \in \mathbb{Z}_{>0}$, define a cube B_n by the following.

$$B_n := \left[-\frac{n(n+1)}{2}, \frac{n(n+1)}{2}\right]^d$$

Now consider what happens when we subdivide B_n into $(n+1)^d$ smaller cubes with side length n in every direction. If we take the union of the interior smaller cubes (that is, all smaller cubes that do not intersect the boundary of B_n), then this union is exactly the cube B_{n-1}. Let $\mathcal{B}_n$ be the collection of all cubes in the subdivision of B_n which are not contained in B_{n-1}, and define $\mathcal{Q} = \bigcup_n \mathcal{B}_n$.

For our purposes, it is helpful to create a frequency space division where there is a fixed smallest side length of the cubes. To this end, define $\mathcal{Q}_{n_0(n_0+1)/2}$ to be the collection of boxes which either belong to the subdivision of B_{n_0} or are in $\mathcal{B}_n$ for $n > n_0$, and for any $|\lambda| \geq 2$,

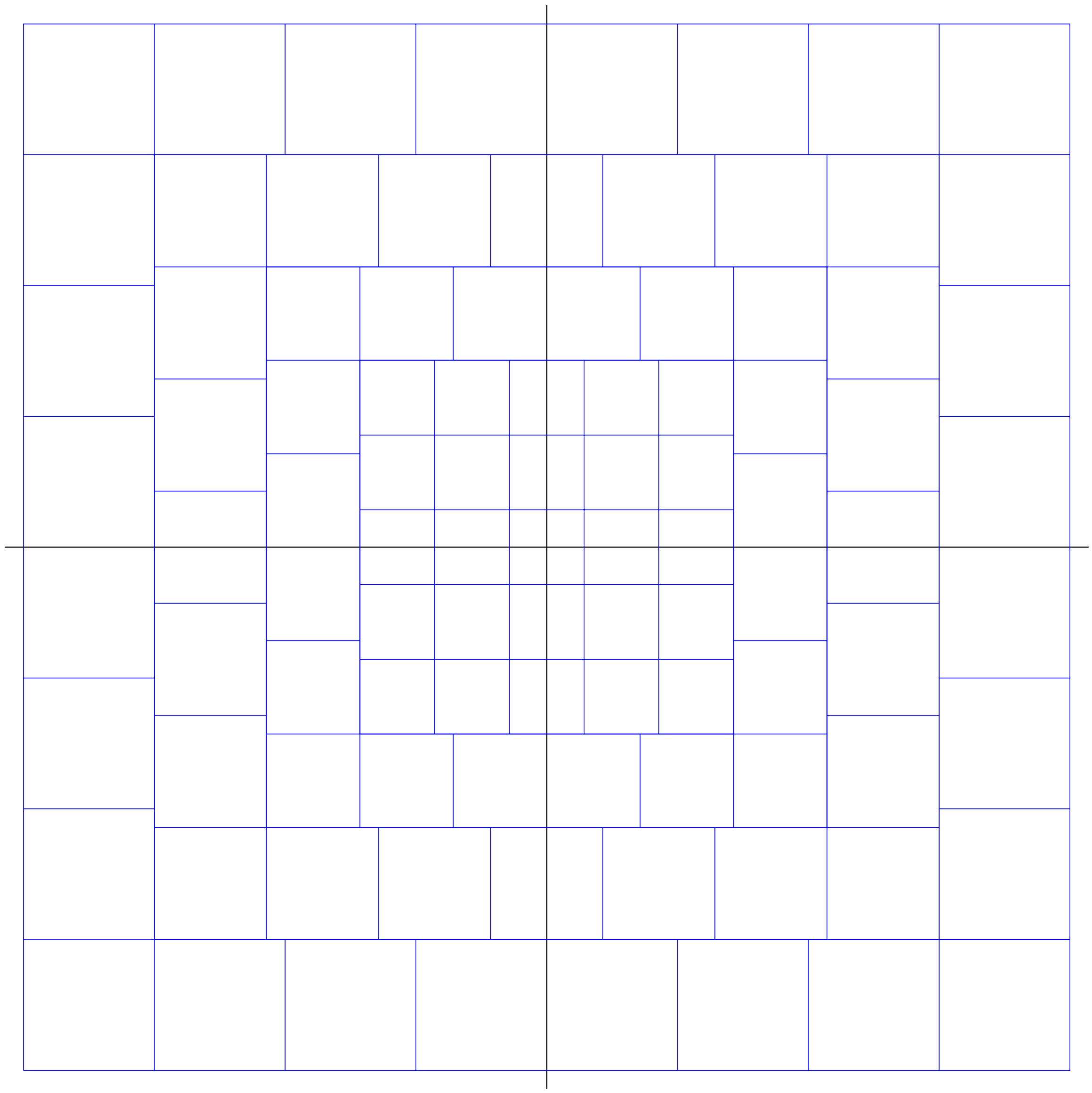

Figure 2.1: An illustration of the frequency space decomposition $\mathcal{Q}_\lambda$ of $\mathbb{R}^2$ for $\lambda = 10$; the smallest cubes have side length $n_0 = 4$.

define $\mathcal{Q}_\lambda$ to be equal to $\mathcal{Q}_{n_0(n_0+1)/1}$ for n_0 the maximal integer satisfying $n_0(n_0+1)/2 \le |\lambda|$.

We now prove a result that gives an estimate on the side length of a box in the decomposition $\mathcal{Q}_\lambda$. In the proposition below, and in the rest of this thesis, we denote by $|Q|$ the measure of the cube Q. Note that this means that $|Q|^{1/d}$ is the side length of the cube Q.

Proposition 1. *Let $\xi \in \mathbb{R}^d$ satisfy $\xi \in Q \in \mathcal{Q}_\lambda$. Then*

$$\tfrac{1}{2}|Q|^{2/d} \le \max\{|\lambda|, 3|\xi|\} \le 3d^{1/2}|Q|^{2/d} \tag{2.1}$$

Proof. As above, let n_0 be the maximal integer satisfying $n_0(n_0+1)/2 \le |\lambda|$.

Case 1. Consider first the case that $|\lambda| > 3|\xi|$. Then

$$|\xi| < \frac{1}{3}|\lambda| < \frac{1}{3} \cdot \frac{(n_0+1)(n_0+2)}{2} = \frac{1}{3}\left[\frac{n_0(n_0+1)}{2} + \frac{2(n_0+1)}{2}\right] \le \frac{n_0(n_0+1)}{2}$$

Thus $\xi \in B_{n_0}$, and therefore by construction of the decomposition we must have $|Q|^{1/d} = n_0$, so:

$$\begin{aligned} |\lambda| &\ge \frac{n(n_0+1)}{2} \ge \frac{n_0^2}{2} = \tfrac{1}{2}|Q|^{2/d} \\ |\lambda| &< \frac{(n_0+1)(n_0+2)}{2} = \tfrac{1}{2}[n_0^2 + 3n_0 + 2] \le 3n_0^2 = 3|Q|^{2/d} \end{aligned}$$

Since $|\lambda| > 3|\xi|$, we have $\max\{|\lambda|, 3|\xi|\} = |\lambda|$, and therefore this case gives us the chain of inequalities:

$$\tfrac{1}{2}|Q|^{2/d} \le \max\{|\lambda|, 3|\xi|\} \le 3|Q|^{2/d} \tag{$\star$}$$

Case 2. Next, suppose $|\lambda| \le 3|\xi|$ and also $\xi \in Q$ with $|Q|^{1/d} = n_0$. By construction of the decomposition, this means $\xi \in B_{n_0}$, and therefore $|\xi_i| \le n_0(n_0+1)/2$ for all i. Using these inequalities, we get:

$$3|\xi| \geq |\lambda| \geq \frac{n_0(n_0+1)}{2} \geq \frac{n_0^2}{2} = \tfrac{1}{2}|Q|^{2/d}$$
$$3|\xi| = 3\left(\sum_{i=1}^{d} |\xi_i|^2\right)^{1/2} \leq 3d^{1/2} \cdot \frac{n_0(n_0+1)}{2} = \tfrac{3}{2}d^{1/2}[n_0^2 + n_0] \leq 3d^{1/2}n_0^2 = 3d^{1/2}|Q|^{2/d}$$

Summarizing, since in this case $3|\xi| = \max\{|\lambda|, 3|\xi|\}$,

$$\tfrac{1}{2}|Q|^{2/d} \leq \max\{|\lambda|, 3|\xi|\} \leq 3d^{1/2}|Q|^{2/d} \qquad (\star)$$

Case 3. Finally, suppose $|\lambda| \leq 3|\xi|$ and $\xi \in Q$ with $|Q|^{1/d} = n > n_0$. By construction of the decomposition, this implies $\xi \in B_n \setminus B_{n-1}$, and therefore by definition of the boxes B_n, all components ξ_i of $\xi = (\xi_1, \ldots, \xi_d)$ satisfy $|\xi_i| \leq \frac{n(n+1)}{2}$, and there is at least one component ξ_j satisfying $|\xi_j| \geq \frac{(n-1)n}{2}$. Using this information, we can deduce the inequalities we need for $3|\xi|$:

$$3|\xi| \geq 3|\xi_j| \geq 3 \cdot \frac{(n-1)n}{2} \geq 3 \cdot \frac{n}{2} \cdot \frac{n}{2} = \tfrac{3}{4}|Q|^{2/d}$$
$$3|\xi| = 3\left(\sum_{i=1}^{d} |\xi_i|^2\right)^{1/2} \leq 3d^{1/2} \cdot \frac{n(n+1)}{2} \leq 3d^{1/2} \cdot n^2 = 3d^{1/2}|Q|^{2/d}$$

where the third inequality in the first line above follows because $n > n_0 \geq 1$ and hence $(n-1) \geq n/2$. Again, summarizing, this gives us

$$\tfrac{3}{4}|Q|^{2/d} \leq \max\{|\lambda|, 3|\xi|\} \leq 3d^{1/2}|Q|^{2/d} \qquad (\star)$$

To combine the three starred inequalities from the three exhaustive and mutually exclusive cases, we take the minimum of the lower bounds, and the maximum of the upper bounds, giving us our desired result. □

This result highlights a key feature of how the decomposition $\mathcal{Q}_\lambda$ scales: at small frequencies, the side lengths of the boxes are all the same and are all comparable to $|\lambda|^{1/2}$, and at large frequencies, the side lengths of the boxes begin to grow and are comparable to the square root of their distance to the origin.

2.2. Construction of the Transformation

Before we define the transformation, we first introduce the frequency cutoff functions and note a few of their key properties.

Lemma 1. *Let ϕ be a positive radial mollifier on $\mathbb{R}^d$ supported in $B_{1/16}(0)$. Let $\psi = \chi_{[-1/2,1/2]^d} * \phi$, and for any cube $Q \in \mathcal{Q}_\lambda$ with center ξ_Q and volume $|Q|$, define $\psi_Q(\xi) := \psi(|Q|^{-1/d}(\xi - \xi_Q))$. Then ψ_Q has the following properties:*

1. $\psi_Q = \chi_Q * \phi_Q$, *where* $\phi_Q(\xi) = |Q|^{-1}\phi(|Q|^{-1/d}\xi)$
2. ψ_Q *is smooth*
3. *supp* $\psi_Q \subset \frac{9}{8}Q$
4. $\psi_Q \equiv 1$ *on* $\frac{7}{8}Q$
5. $||\psi_Q||_{L^\infty} = 1$

where for any cube Q and any positive number r, rQ is the cube with the same center as Q and side length $r|Q|^{1/d}$.

Proof. Property 2 follows from property 1 and the fact that ϕ is a mollifier, and properties 3 and 4 follow from property 1 and the fact that ϕ_Q is a mollifier supported in $B_{|Q|^{1/d}/16}(0)$. Property 5 follows from properties 1 and 4 and Young's convolution inequality, because $||\chi_Q||_{L^\infty} = 1$ and as a mollifier $||\phi_Q||_{L^1} = 1$. It suffices to prove property 1, which is just a statement about the compatibility of convolution with scaling and translation:

$$\begin{aligned}\psi_Q(\xi) &= \psi(|Q|^{-1/d}(\xi - \xi_Q)) \\ &= \int \chi_{[-1/2,1/2]^d}(|Q|^{-1/d}(\xi - \xi_Q) - y)\phi(y)dy \\ &= \int \chi_{[-1/2,1/2]^d}(|Q|^{-1/d}(\xi - |Q|^{1/d}y - \xi_Q))\phi(y)dy \\ &= \int \chi_Q(\xi - |Q|^{1/d}y)\phi(y)dy \\ &= \int \chi_Q(\xi - z) \cdot |Q|^{-1}\phi(|Q|^{-1/d}z)dz \\ &= \chi_Q * \phi_Q(\xi)\end{aligned}$$

□

Now we define the transformation.

Lemma 2. *Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ be such that $\widehat{\varphi}$ is bounded below on $[-\frac{9}{16}, \frac{9}{16}]^d$ for any $Q \in \mathcal{Q}_\lambda$, let ξ_Q be the center of Q, let $|Q|$ be the volume of Q, and define*

$$\varphi_Q(x) := |Q|^{1/2}e^{2\pi i\langle\xi_Q,x\rangle}\varphi(|Q|^{1/d}x)$$

For any $\xi \in \mathbb{R}^d$ in the interior of Q, let $\varphi_\xi := \varphi_Q$, and for any $\xi \in \mathbb{R}^d$ on the boundary of $Q \in \mathcal{Q}_\lambda$, let $\varphi_\xi := 0$. Then there exists a dense subspace of $L^2(\mathbb{R}^d)$ and a bounded map V from this subspace into $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ such that

$$f(x) = \int_{\mathbb{R}^d\times\mathbb{R}^d} Vf(y,\xi)\varphi_\xi(x-y)dyd\xi \tag{2.2}$$

for all f in the dense subspace.

Proof. Let the dense subspace be functions with Fourier transform in $C_c^\infty(\mathbb{R}^d)$. Let ψ_Q be defined as in Lemma 1, and let

$$\omega = \sum_{Q\in\mathcal{Q}_\lambda} \psi_Q$$

Note that the sum is everywhere nonzero and everywhere finite, as the support condition in Lemma 1 guarantees that ψ_Q is only supported in the union of Q and its neighbors (more precisely, Q and any cube Q' which intersects $\frac{9}{8}Q$). Finally, define $\widetilde{\psi_Q} := \psi_Q/\omega$ so that $\sum_{Q \in Q_\lambda} \widetilde{\psi_Q} = 1$, i.e., the functions $\widetilde{\psi_Q}$ form a partition of unity. Then

$$\widehat{f} = \sum_{Q \in \mathcal{Q}_\lambda} \widehat{f}\,\widetilde{\psi_Q}$$

and as the sum is finite at any given point, and all functions in the sum are Schwartz functions, we get the following equality on the physical side.

$$f = \sum_{Q \in \mathcal{Q}_\lambda} \mathcal{F}^{-1}\left[\widehat{f}\,\widetilde{\psi_Q}\right]$$

Since the Fourier transform of φ is bounded below on $[-\frac{9}{16}, \frac{9}{16}]^d$ and each $\widetilde{\psi_Q}$ is supported in $\frac{9}{8}Q$, if we let

$$\widehat{T_Q f}(\xi) := \widehat{f}(\xi)\frac{\widetilde{\psi_Q}(\xi)}{\widehat{\varphi}(|Q|^{-1/d}(\xi - \xi_Q))}$$

then the operators T_Q are uniformly bounded:

$$\begin{aligned}
||T_Q f||_{L^2} &= ||\widehat{T_Q f}||_{L^2} \\
&= \left(\int |\widehat{f}(\xi)|^2 \frac{|\widetilde{\psi_Q}(\xi)|^2}{|\widehat{\varphi}(|Q|^{-1/d}(\xi - \xi_Q))|^2} d\xi\right)^{1/2} \\
&\leq \left(\int |\widehat{f}(\xi)|^2 \frac{1}{C_\varphi^2} d\xi\right)^{1/2} \\
&= C_\varphi^{-1}||f||_{L^2}
\end{aligned}$$

where C_φ is just the lower bound of φ on $[-\frac{9}{16}\frac{9}{16}]^d$ in the hypothesis, and is independent of Q. Note also that the operators $\{T_Q\}_{Q \in \mathcal{Q}_\lambda}$ are almost orthogonal in the sense of Stein [32], as

$$||T_Q^* T_{Q'}||_{L^2 \to L^2} \leq \begin{cases} C_\varphi^{-2} & Q = Q' \text{ or } Q \text{ adjacent to } Q' \\ 0 & \text{else} \end{cases}$$

Also, since

$$\begin{aligned}\mathcal{F}\left[T_Q f * \varphi_Q\right](\xi) &= \widehat{T_Q f}(\xi)\widehat{\varphi_Q}(\xi) \\ &= \widehat{f}(\xi)\frac{\widetilde{\psi_Q}(\xi)}{\widehat{\varphi}(|Q|^{-1/d}(\xi-\xi_Q))} \cdot |Q|^{-1/2}\widehat{\varphi}(|Q|^{-1/d}(\xi-\xi_Q)) \\ &= |Q|^{-1/2}\widehat{f}(\xi)\widetilde{\psi_Q}(\xi)\end{aligned}$$

we have the identity

$$\int (T_Q f)(y)\varphi_Q(x-y)dy = |Q|^{-1/2}\mathcal{F}^{-1}\left[\widehat{f}\,\widetilde{\psi_Q}\right](x)$$

for every $Q \in \mathcal{Q}_\lambda$ and every f in the dense subspace. Now, for every $\xi \in \mathbb{R}^d$, define

$$Vf(y,\xi) := |Q|^{-1/2}(T_Q f)(y) \tag{2.3}$$

Then $Vf(y,\xi)$ is a Schwartz function in y and constant in ξ within the interior of any $Q \in \mathcal{Q}_\lambda$. Also,

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d} Vf(y,\xi)\varphi_\xi(x-y)dyd\xi &= \sum_{Q\in\mathcal{Q}_\lambda}\int_{\mathbb{R}^d\times Q} Vf(y,\xi)\varphi_\xi(x-y)dyd\xi \\ &= \sum_{Q\in\mathcal{Q}_\lambda}\int_{\mathbb{R}^d\times Q} |Q|^{-1/2}(T_Q f)(y)\varphi_Q(x-y)dyd\xi \\ &= \sum_{Q\in\mathcal{Q}_\lambda} |Q| \int_{\mathbb{R}^d} |Q|^{-1/2}(T_Q f)(y)\varphi_Q(x-y)dy \\ &= \sum_{Q\in\mathcal{Q}_\lambda} |Q|^{1/2} \int_{\mathbb{R}^d} T_Q f(y)\varphi_Q(x-y)dy \\ &= \sum_{Q\in\mathcal{Q}_\lambda} \mathcal{F}^{-1}\left[\widehat{f}\widetilde{\psi_Q}\right](x) \\ &= f(x)\end{aligned}$$

by earlier observations. Finally, note that

$$
\begin{aligned}
\int_{\mathbb{R}^d\times\mathbb{R}^d} |Vf(y,\xi)|^2 dy d\xi &= \sum_{Q\in\mathcal{Q}_\lambda} \int_{\mathbb{R}^d\times Q} |Vf(y,\xi)|^2 dy d\xi \\
&= \sum_{Q\in\mathcal{Q}_\lambda} \int_{\mathbb{R}^d\times Q} |Q|^{-1}|T_Q f(y)|^2 dy d\xi \\
&= \sum_{Q\in\mathcal{Q}_\lambda} \int_{\mathbb{R}^d} |T_Q f(y)|^2 dy \\
&= \sum_{Q\in\mathcal{Q}_\lambda} \int_{\mathbb{R}^d} \left| \widehat{f}(\zeta) \frac{\widetilde{\psi_Q}(\zeta)}{\widehat{\varphi}(|Q|^{-1/d}(\zeta-\zeta_Q))} \right|^2 d\zeta \\
&\leq \sum_{Q\in\mathcal{Q}_\lambda} \frac{1}{C_\varphi^2} \int_{\mathbb{R}^d} |\widehat{f}(\zeta)\widetilde{\psi_Q}(\zeta)|^2 d\zeta \\
&\leq \sum_{Q\in\mathcal{Q}_\lambda} \frac{1}{C_\varphi^2} \int_{\mathbb{R}^d} |\widehat{f}(\zeta)|^2 \widetilde{\psi_Q}(\zeta) d\zeta \\
&= C_\varphi^{-2} ||\widehat{f}||_{L^2}^2 \\
&= C_\varphi^{-2} ||f||_{L^2}^2
\end{aligned} \tag{2.4}
$$

The third-to-last line follows from the fact that $0 \leq \widetilde{\psi_Q}(\zeta) \leq 1$ per Lemma 1 and the construction of $\widetilde{\psi_Q}$ from ψ_Q. The second-to-last line follows from the fact that as a member of the dense subspace, $\widehat{f}$ is compactly supported, and therefore the sum is finite and can be exchanged with the integral to apply the fact that $\{\widetilde{\psi_Q}\}_{Q\in\mathcal{Q}_\lambda}$ is a partition of unity. □

One nice feature of the function φ_ξ is that in addition to being supported in a box with side length proportional to the inverse of the side length of Q, it also gains roughly a factor of the side length of Q every time we take a derivative.

Proposition 2. *Suppose that in addition to the hypotheses of Lemma 2, φ is supported in $[-1/(2d^{1/4}\sqrt{3}), 1/(2d^{1/4}\sqrt{3})]^d$. For $\xi \in \mathbb{R}^d$, let $r = (\max\{|\lambda|, 3|\xi|\})^{-1/2}$. Then for any* *multi-index α, there exists a constant C_α depending only on α and φ such that*

$$|\partial^\alpha(e^{-2\pi i\langle\xi,x\rangle}\varphi_\xi(x))| \leq \begin{cases} C_\alpha r^{-d/2-|\alpha|} & x \in [-r/2, r/2]^d \\ 0 & \textit{else} \end{cases} \tag{2.5}$$

Proof. If φ is supported in $[-1/(2d^{1/4}\sqrt{3}), 1/(2d^{1/4}\sqrt{3})]^d$, then φ_ξ is supported in $[-1/(2d^{1/4}|Q|^{1/d}\sqrt{3}), 1/(2d^{1/4}|Q|^{1/d}\sqrt{3})]^d$. Furthermore, the inequality (2.1) implies that $|Q|^{-1/d} \leq rd^{1/4}\sqrt{3}$, and so the support of φ_ξ is contained in $[-r/2, r/2]^d$. By the product rule,

$$\begin{aligned} |\partial^\alpha(e^{-2\pi i\langle\xi,x\rangle}\varphi_\xi(x))| &= ||Q|^{1/2}\partial^\alpha(e^{2\pi i\langle\xi_Q-\xi,x\rangle}\varphi(|Q|^{1/d}x))| \\ &\leq |Q|^{1/2}\sum_{\beta\leq\alpha} c_{\alpha,\beta}|2\pi i(\xi_Q-\xi)|^\beta|Q|^{(|\alpha|-|\beta|)/d}|(\partial^{\alpha-\beta}\varphi)(|Q|^{1/d}x)| \\ &\lesssim |Q|^{1/2+|\alpha|/d}\sum_{\beta\leq\alpha} c_{\alpha,\beta}\frac{|\xi_Q-\xi|^\beta}{|Q|^{|\beta|/d}}|(\partial^{\alpha-\beta}\varphi)(|Q|^{1/d}x)| \\ &\lesssim_{\alpha,\varphi} |Q|^{1/2+|\alpha|/d} \\ &\lesssim_{\alpha,\varphi} r^{-d/2-|\alpha|} \end{aligned}$$

where the second-to-last line follows from the fact that $\xi \in Q$ and therefore $|\xi_{Q,j}-\xi_j| \leq |Q|^{1/d}/2$ for any index $j \in \{1,\dots,d\}$, and the last line follows from the fact that $|Q|^{-1/d} \approx r$ due to the inequality (2.1). $\square$

Before we proceed, we expand slightly on the comment at the beginning of the section: using smooth cutoff functions to localize in frequency space (as opposed to characteristic functions) allows us to more easily translate between localization on the frequency side and localization on the physical side.

As an example, consider a Schwartz function f on $\mathbb{R}^d$. Let η be a smooth cutoff function with $\eta = 1$ on $|x| < \epsilon$ and $\eta = 0$ on $|x| > 2\epsilon$, and let $\tilde{f}(x) = f(x)(1-\eta(x))$, so that $\tilde{f} \equiv 0$ on $|x| < \epsilon$. We will now sketch how one can show that applying the transformation in some sense preserves the fact that $\tilde{f}$ is "small" near $x = 0$, by getting a (small) upper bound on $|V(\tilde{f})(y,\xi)\chi_{|y|<\epsilon/2}|$.

Using the definition of the transformation, we note that

$$\begin{aligned} V(\tilde{f})(y,\xi) &= |Q|^{-1/2}(T_Q\tilde{f})(y) \\ &= |Q|^{-1/2}\mathcal{F}^{-1}\left(\hat{\tilde{f}}(\cdot)\cdot\frac{\widetilde{\psi_Q}(\cdot)}{\widehat{\varphi}(|Q|^{-1/d}(\cdot-\xi_Q))}\right)(y) \\ &=: |Q|^{-1/2}\tilde{f}*\check{g}_\xi(y) \end{aligned}$$

where $\check{g}_\xi(y) = \mathcal{F}^{-1}[g_\xi](y)$ and

$$g_\xi(\zeta) = \frac{\widetilde{\psi_Q}(\zeta)}{\widehat{\varphi}(|Q|^{-1/d}(\zeta-\xi_Q))} = \frac{\psi_Q(\zeta)}{\left(\sum_{\{Q' \text{ adj. } Q\}}\psi_{Q'}(\zeta)\right)\widehat{\varphi}(|Q|^{-1/d}(\zeta-\xi_Q))}$$

with the second equality coming from the definition of $\widetilde{\psi_Q}$ and the support conditions on ψ_Q. Here, the sum in the denominator is over cubes Q' that are adjacent to Q, more precisely cubes Q' that intersect the slightly expanded cube $\frac{9}{8}Q$.

Now, define $g_{\xi_0}(\zeta) = g_\xi(|Q|^{1/d}\zeta + \xi_Q)$, so that $g_{\xi_0}(\zeta)$ is g_ξ translated and scaled to make the numerator become our base function from Lemma 1, $\psi(\zeta) = \xi_{[-1/2,1/2]^d} * \phi$. That is, we have

$$g_{\xi_0}(\zeta) = \frac{\psi(\zeta)}{\left(\sum_{\{Q' \text{ adj. } Q\}}\psi_{Q'}(|Q|^{1/d}\zeta+\xi_Q)\right)\widehat{\varphi}(\zeta)}$$

One then shows by the quotient rule and properties of ψ, φ (with some effort) that for any multi-index α and any ξ, we have $||\partial^\alpha g_{\xi_0}||_{L^1} \le C_{\alpha,\varphi,\psi}$, for some C that depends only on $\partial^\beta\psi, \partial^\gamma(\hat{\varphi})$ for $\beta,\gamma \le \alpha$, along with dimensional constants. Importantly, C_α can be chosen to be independent of ξ and $|Q|$. By properties of the Fourier transform, this implies that

$$|\check{g}_\xi(y)| \lesssim_k \frac{|Q|}{(1+|Q|^{1/d}|y|)^k}$$

for any non-negative integer k, with $k = |\alpha|$ in the above reasoning. Recall also that (2.1) tells us that $|Q|^{1/d} \approx \max\{|\lambda|^{1/2}, |\xi|^{1/2}\}$. Putting everything together, we have

$$\begin{aligned}
|V\tilde{f}(y,\xi)\chi_{|y|<\epsilon/2}| &= \left|\chi_{|y|<\epsilon/2}|Q|^{-1/2} f * \check{g}_\xi(y)\right| \\
&\leq \chi_{|y|<\epsilon/2}|Q|^{-1/2} \int |\tilde{f}(y-z)| \cdot |\check{g}_\xi(z)| dz \\
&\lesssim \chi_{|y|<\epsilon/2}|Q|^{1/2} \int_{|y-z|>\epsilon} \frac{|\tilde{f}(y-z)|}{(1+|Q|^{1/d}|z|)^k} dz \\
&\leq |Q|^{1/2} ||\tilde{f}||_{L^\infty(\mathbb{R}^d)} \int_{|z|>\epsilon/2} \frac{dz}{(1+|Q|^{1/d}|z|)^k}
\end{aligned}$$

where the fourth line follows from the third because the integrand is non-negative and because $|y - z| > \epsilon$ and $|y| < \epsilon/2$ imply $|z| > \epsilon/2$ by the triangle inequality. Note also that in the last line above, we could use Hölder's inequality with different exponents to introduce a different L^p norm on $\tilde{f}$ in the upper bound; here we have chosen the L^∞ norm for simplicity's sake.

All that remains is to bound the integral in the last line above. Simplifying the denominator and changing variables to polar coordinates gives us that

$$\int_{|z|>\epsilon/2} \frac{dz}{(1+|Q|^{1/d}|z|)^k} \leq |Q|^{-k/d} \int_{|z|>\epsilon/2} \frac{dz}{|z|^k} \lesssim |Q|^{-k/d} \epsilon^{d-k}$$

provided $k > d$. Combining this with the above, we see

$$\begin{aligned}
|V\tilde{f}(y,\xi)\chi_{|y|<\epsilon/2}| &\lesssim |Q|^{1/2-k/d} \epsilon^{d-k} ||\tilde{f}||_{L^\infty(\mathbb{R}^d)} \\
&= |Q|^{(d-2k)/2d} \epsilon^{d-k} ||\tilde{f}||_{L^\infty(\mathbb{R}^d)} \\
&\approx (\max\{|\lambda|, |\xi|\})^{(d-2k)/4} \epsilon^{d-k} ||\tilde{f}||_{L^\infty(\mathbb{R}^d)}
\end{aligned}$$

provided $k > d$. As long as ϵ is not too small in terms of λ, we can take k large to get an upper bound that is as "small" as we like, in terms of both ξ decay and λ decay.

CHAPTER 3

Rapid Decay Cases

In the original multilinear version of Theorem 1, the decomposition is applied to each of the $2d$ single-variable L^2 functions and the kernel of the resulting operator is analyzed using, in part, the stationary phase results that follow. There are regions where this kernel can be shown to exhibit rapid decay without using the determinant condition in the hypothesis.

All of these kernel estimates and rapid decay results translate to the case of a bilinear operator on $L^2(\mathbb{R}^d)$ functions with only minor modifications. The following results and proofs are adapted heavily from the results in [18].

3.1. Stationary Phase

In this section, we restate several results from [18], as they are used in the proofs in the following sections, and outline the key points of their proofs.

Lemma 3. *For each positive integer N, there is a constant C_N such that for every smooth manifold M with measure $d\sigma$ of smooth positive density, every pair (φ, ψ) of C^N real-valued functions on M with ψ compactly supported, and every nonzero complex number K,*

$$\left| \int e^{i\varphi} \psi d\sigma \right| \leq \frac{C_N}{|K|^N} \sum_{j=0}^{N} \int |(X^*)^j \psi| \left[|X\varphi - K|^{N-j} + \sum_{\ell=2}^{N-j} |X^\ell \varphi|^{\frac{N-j}{\ell}} \right] d\sigma, \tag{3.1}$$

where X is any C^N vector field on M and X^* is the first-order differential operator dual to X.

The proof begins by applying a type of stationary phase argument (similar in concept to the argument outlined in Section 1.1.1) with integration by parts and a differential operator that when applied to ψ in the integral generates a multiple of the original integral. One can then obtain an expression for N^{th} powers of this differential operator, and apply the inequality for arithmetic and geometric means to get the desired result.

The next result is not a stationary phase result, but a version of the simple size inequality for the integral of a bounded function of compact support, translated so that it can be applied to the integral over a smooth manifold of a bounded function of compact support.

Proposition 3. *Suppose that $M \subset B_1 := (-b_1, b_1)^{2d}$ is the zero set of a function ρ which satisfies the inequalities* (1.5) *and* (1.6) *and that f is a measurable function on B_1 supported in a product of intervals $I := I_1 \times \cdots \times I_{2d}$. Then for any $j_0 \in \{1, \ldots, 2d\}$,*

$$\left| \int_M f d\sigma \right| \mid \lesssim ||f||_{L^\infty(M)} \prod_{\substack{j=1 \\ j \neq j_0}}^{2d} |I_j| \tag{3.2}$$

where $d\sigma$ is Lebesgue measure on M.

The main features in the proof are an application of the Implicit Function Theorem, and a change of variables to transform the integral into an integral over standard Euclidean space $\mathbb{R}^{2d-1}$.

Lastly, the previous two results are combined in a helpful corollary.

Corollary 1. *For M as in Proposition 3 and φ, ψ, and X as in Lemma 3, if ψ is supported on a product of intervals $I := I_1 \times \cdots \times I_{2d} \subset B_1$ and if N is any fixed positive integer, then*

$$\begin{aligned} \left| \int_M e^{i\varphi} \psi d\sigma \right| \lesssim \frac{C_N |I|}{|K|^N |I_{j_0}|} \sum_{k=0}^{N} ||(X^*)^k \psi||_{L^\infty(M\cap I)} \Bigg[&||X\varphi - K||^{N-k}_{L^\infty(M\cap I)} \\ &+ \sum_{\ell=2}^{N-k} ||X^\ell \varphi||^{\frac{N-k}{\ell}}_{L^\infty(M\cap I)} \Bigg] \end{aligned} \tag{3.3}$$

for any complex number K and any $j_0 \in \{1, \ldots, 2d\}$.

3.2. Kernel Estimates

As indicated earlier, we will now apply the frequency space decomposition to the functions in our operator:

$$I_\lambda(f_1, f_2) = \int_{\rho=0} e^{i\lambda\Phi(x)} f_1(x^1) f_2(x^2) a(x) d\sigma(x)$$

We again emphasize that the results in this subsection and the following subsection still hold in general terms and only require the assumptions in the definition of the operator (1.1) along with (1.2), (1.3) (1.4), (1.5), and (1.6).

When we apply the frequency space decomposition, i.e., we use the identity (2.2), we get

$$I_\lambda(f_1, f_2) = \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \mathcal{I}(y,\xi) f_1(y^1,\xi^1) f_2(y^2,\xi^2) dy d\xi$$

where above, we are somewhat abusing notation by renaming $f_j(y^j,\xi^j) := Vf_j(y^j,\xi^j)$, but there is no issue in doing this because per Lemma 2, $||Vf_j||_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} \lesssim ||f_j||_{L^2(\mathbb{R}^d)}$. The kernel here is

$$\mathcal{I}(y,\xi) := \int_M e^{i\lambda\Phi(x)} \varphi_{\xi^1}(x^1 - y^1)\varphi_{\xi^2}(x^2 - y^2) a(x) d\sigma(x) \tag{3.4}$$

Before proceeding, we first reiterate some notation that will be used going forward. For each $\xi \in \mathbb{R}^{2d}$, write $\xi^1 := (\xi_1, \ldots, \xi_d)$ and $\xi^2 := (\xi_{d+1}, \ldots, \xi_{2d})$, let $r_j := (\max\{|\lambda|, 3|\xi^j|\})^{-1/2}$ (for $j = 1, 2$), and let $r = \min\{r_1, r_2\}$. Note that by assumption and by (2.5), each φ_{ξ^j} is supported in $[-r_j/2, r_j/2]^d$, so we must have $x^j - y^j \in [-r_j/2, r_j/2]^d$. Also note that in the integral $\mathcal{I}(y,\xi)$ we are only integrating over points $x \in \operatorname{supp} a \subset B_0$. Combining these two observations with the condition (1.2), we have that for any $k = 1, \ldots, 2d$, $|x_k - y_k| \leq \max_j r_j \leq |\lambda|^{-1/2} \leq b_1 - b_0$, and $|x_k| \leq b_0$, which, along with the triangle inequality, shows that $\mathcal{I}(y,\xi)$ is supported in $B_1 \times \mathbb{R}^{2d}$.

Next, we will apply the results of the previous subsection to get a preliminary estimate on the size of the kernel $\mathcal{I}(y,\xi)$. We will be using the smooth vector fields X_i defined as follows for $i = 1, \ldots, 2d$.

$$X_i = \partial_i - \frac{\partial_i \rho}{|\nabla\rho|^2} \sum_{j=1}^{2d} (\partial_j \rho)\partial_j \tag{3.5}$$

The result that follows is adapted from a multilinear version in [18].

Proposition 4. *Let $N \leq 2d + 2$ be a positive integer. Assuming* (1.2), (1.3), (1.4), (1.5), *and* (1.6), *we have*

$$|\mathcal{I}(y,\xi)| \lesssim \left(1 + \frac{r^2}{\max_j r_j}\left(\sum_{i=1}^{2d} |X_i(\lambda\Phi + 2\pi\xi\cdot x)|_y|^2\right)^{1/2}\right)^{-N} r_{j_0}^{-1} r_1^{d/2} r_2^{d/2} \tag{3.6}$$

for $j_0 = 1, 2$. *Also, if* $\mathcal{I}(y,\xi) \neq 0$ *then* $|\rho(y)| \lesssim \max_j r_j$.

Proof. If we first use the simple size estimate (3.2), we get

$$\begin{aligned} |\mathcal{I}(y,\xi)| &= \left|\int_M e^{i\lambda\Phi(x)} \varphi_{\xi^1}(x^1 - y^1)\varphi_{\xi^2}(x^2-y^2) a(x) d\sigma(x)\right| \\ &\lesssim ||e^{i\lambda\Phi}\varphi_{\xi^1}(\cdot - y^1)\varphi_{\xi^2}(\cdot - y^2)||_{L^\infty(M)} r_1^d r_2^d r_{j_0}^{-1} \\ &\lesssim r_1^{d/2} r_2^{d/2} r_{j_0}^{-1} \end{aligned} \tag{3.7}$$

because each φ_{ξ^j} is supported in a product of d intervals of length r_j and $||\varphi_{\xi^j}||_{L^\infty} \lesssim r_j^{-d/2}$ by (2.5).

Next, we rewrite $\mathcal{I}(y,\xi)$ to pull out expressions that match the left hand side of (2.5):

$$\mathcal{I}(y,\xi) = \int_M e^{i(\lambda\Phi(x) + 2\pi\xi\cdot x)} \underbrace{\left[e^{-2\pi i\xi\cdot y} a(x) \prod_{j=1}^{2} e^{-2\pi i \xi^j\cdot(x^j - y^j)} \varphi_{\xi^j}(x^j - y^j)\right]}_{=: a_{y,\xi}(x)} d\sigma(x)$$

and use $a_{y,\xi}(x)$ to denote the function in brackets above. We will apply the stationary phase and size estimate in Corollary 1 with $\psi = a_{y,\xi}$, $\varphi = \lambda\Phi(x) + 2\pi\xi\cdot x$, and $X = \tilde{r}X_i$, where $\tilde{r} := r^2/\max_j r_j$ and i is any index in $\{1,\ldots,2d\}$. This gives

$$\begin{aligned} |\mathcal{I}(y,\xi)| \lesssim \frac{C_N r_1^d r_2^d}{|K|^N r_{j_0}} \sum_{k=0}^{N} ||(\tilde{r}X_i^*)^k a_{y,\xi}||_{L^\infty} \Bigg[&||\tilde{r}X_i(\lambda\Phi + 2\pi\xi\cdot x) - K||_{L^\infty}^{N-k} + \\ &\sum_{\ell=2}^{N-k} ||(\tilde{r}X_i)^\ell(\lambda\Phi + 2\pi\xi\cdot x)||_{L^\infty}^{(N-k)/\ell}\Bigg] \end{aligned} \tag{3.8}$$

for any $K \neq 0, K \in \mathbb{C}$, and where the L^∞ norm is taken over

$$M \cap \operatorname{supp}\{\varphi_{\xi^1}(\cdot - y^1)\varphi_{\xi^2}(\cdot - y^2)\} \subset M \cap \left[y + [-r_1/2, r_1/2]^d \times [-r_2/2, r_2/2]^d\right] =: M \cap I_{y,\xi}$$

Now, we begin simplifying the right hand side of (3.8). First, we can calculate $X_i^* g$ for an arbitrary function g:

$$\begin{aligned}
\int X_i f g &= \int \left[\partial_i f - \frac{\partial_i \rho}{|\nabla\rho|^2}\sum_{j=1}^{2d}(\partial_j\rho)(\partial_j f)\right] g \\
&= \int \left[(\partial_i f)g - \frac{\partial_i \rho}{|\nabla\rho|^2}\sum_{j=1}^{2d}(\partial_j\rho)(\partial_j f)g\right] \\
&= \int \left[-f\partial_i g + f\sum_{j=1}^{2d}\partial_j\left(\frac{\partial_i\rho}{|\nabla\rho|^2}(\partial_j\rho)g\right)\right] \\
&= \int f\left[-\partial_i g + \sum_{j=1}^{2d}\frac{\partial_i\rho}{|\nabla\rho|^2}\partial_j\rho\partial_j g + g\sum_{j=1}^{2d}\partial_j\left(\frac{\partial_i\rho\partial_j\rho}{|\nabla\rho|^2}\right)\right]
\end{aligned}$$

And thus

$$X_i^* g = -\partial_i g + \sum_{j=1}^{2d}\frac{\partial_i\rho}{|\nabla\rho|^2}\partial_j\rho\partial_j g + g\sum_{j=1}^{2d}\partial_j\left(\frac{\partial_i\rho\partial_j\rho}{|\nabla\rho|^2}\right). \tag{3.9}$$

By the product rule and quotient rule, $X_i^* g$ is a linear combination of g and $\partial_j g$ for $j = 1, \ldots, 2d$ with coefficients that are polynomials in $|\nabla\rho|^{-2}$ and the first and second derivatives of ρ. Thus $(X_i^*)^k g$ will be a linear combination of $\partial^\alpha g$ for multiindices $|\alpha| \leq k$, with coefficients that are polynomials in $|\nabla\rho|^{-2}$ and $\partial^\beta\rho$ for multiindices $|\beta| \leq k+1$, as the coefficients are differentiated each time we apply X_i^*.

With all this in mind, we now find an estimate for $(\tilde{r}X_i^*)^k a_{y,\xi}$, where we recall the vector fields X_i are given by (3.5) and $\tilde{r} = r^2/\max_j r_j$. By the reasoning above, $(X_i^*)^k a_{y,\xi}$ will be a linear combination of $\partial^\alpha a_{y,\xi}$ for $|\alpha| \leq k$ with coefficients that are polynomials in $|\nabla\rho|^{-2}$ and $\partial^\beta\rho$ for $|\beta| \leq k+1$. By the initial assumptions (1.5) and (1.6) on ρ, these coefficients are bounded by uniform constants, so applying the triangle inequality gives

$$|(\tilde{r}X_i^*)^k a_{y,\xi}| \lesssim \sum_{|\alpha|\le k} |\tilde{r}\partial^\alpha a_{y,\xi}| \tag{3.10}$$

By the product rule, triangle inequality, and (2.5), we have for any given multiindex α,

$$\begin{aligned} |\partial^\alpha a_{y,\xi}| &\le \sum_{\gamma\le\alpha} c_{\gamma,\alpha}|\partial^{\alpha-\gamma}a|\cdot|\partial^\gamma(\varphi_{\xi^1}(\cdot - y^1)\varphi_{\xi^2}(\cdot - y^2)| \\ &\lesssim \sum_{\gamma\le\alpha} r_1^{-d/2} r_2^{-d/2} r^{-|\gamma|} \end{aligned} \tag{3.11}$$

because $r = \min\{r_1, r_2\}$ and thus $r^{-|\gamma|}$ is an upper bound for the additional negative powers of r_1, r_2 which come from differentiating $\varphi_{\xi^1}, \varphi_{\xi^2}$ a combined total of $|\gamma|$ times. This then gives us

$$|\tilde{r}^k\partial^\alpha a_{y,\xi}| \lesssim r_1^{-d/2} r_2^{-d/2} \sum_{\gamma\le\alpha} r^{-|\gamma|}\tilde{r}^k \lesssim r_1^{-d/2} r_2^{-d/2} \tag{3.12}$$

if $|\alpha| \le k$, because $\gamma \le \alpha$, $\tilde{r} = r^2/\max_j r_j \le r$, and $r \approx \min\{|\lambda|^{-1/2}, |\xi^1|^{-1/2}, |\xi^2|^{-1/2}\} \lesssim 1$ by the assumption (1.2) that $|\lambda| \ge 1$. Combining (3.10), (3.11), and (3.12), we get

$$||(\tilde{r}X_i^*)^k a_{y,\xi}||_{L^\infty} \lesssim r_1^{-d/2} r_2^{-d/2} \tag{3.13}$$

Next, consider the term

$$\sum_{\ell=2}^{N-k} ||(\tilde{r}X_i^*)^\ell(\lambda\Phi + 2\pi\xi\cdot x)||_{L^\infty}^{(N-k)/\ell}$$

We have $|(\tilde{r}X_i)^\ell(\lambda\Phi)| = \tilde{r}^\ell|\lambda|\cdot|X_i^\ell\Phi| \lesssim \tilde{r}^\ell|\lambda|$ because of the uniform bounds on $|\nabla\rho|^{-1}$, the derivatives of ρ, and the derivatives of Φ. Similarly, we have $|(\tilde{r}X_i)^\ell(2\pi\xi\cdot x)| \lesssim \tilde{r}^\ell|\xi|$. Since $r \approx \min\{|\lambda|^{-1/2}, |\xi^1|^{-1/2}, |\xi^2|^{-1/2}\}$, we have $r^{-2} \approx \max\{|\lambda|, |\xi^1|, |\xi^2|\}$, and thus $r^{-2} \approx$

$|\lambda| + |\xi|$, as

$$\max\{|\lambda|, |\xi^1|, |\xi^2|\} \leq |\lambda| + |\xi^1| + |\xi^2| \leq 2(|\lambda| + |\xi|)$$
$$\max\{|\lambda|, |\xi^1|, |\xi^2|\} \geq \tfrac{1}{3}(|\lambda| + |\xi^1| + |\xi^2|) \gtrsim |\lambda| + |\xi|$$

Thus,

$$\begin{aligned} \sum_{\ell=2}^{N-k} ||(\tilde{r}X_i)^\ell(\lambda\Phi + 2\pi\xi \cdot x)||_{L^\infty}^{(N-k)/\ell} &\lesssim \sum_{\ell=2}^{N-k} \left(\tilde{r}^\ell(|\lambda| + |\xi|)\right)^{(N-k)/\ell} \\ &\lesssim \sum_{\ell=2}^{N-k} (\tilde{r}^\ell r^{-2})^{(N-k)/\ell} \\ &\lesssim 1 \end{aligned} \tag{3.14}$$

since $\ell \geq 2$ and again $\tilde{r} \leq r \lesssim 1$. Combining (3.8), (3.13), and (3.14) gives us so far that

$$\begin{aligned} |\mathcal{I}(y,\xi)| &\lesssim \frac{r_1^d r_2^d}{|K|^N r_{j_0}} \sum_{k=0}^{N} r_1^{-d/2} r_2^{-d/2} \left[||\tilde{r}X_i(\lambda\Phi + 2\pi\xi \cdot x) - K||_{L^\infty}^{N-k} + 1\right] \\ &= \frac{r_1^{d/2} r_2^{d/2}}{|K|^N r_{j_0}} \sum_{k=0}^{N} \left[||\tilde{r}X_i(\lambda\Phi + 2\pi\xi \cdot x) - K||_{L^\infty}^{N-k} + 1\right] \end{aligned} \tag{3.15}$$

and this holds for any $i = 1, \ldots, 2d$, any $j_0 = 1, 2$, and any $K \neq 0$ in $\mathbb{C}$.

To finish, for each choice of λ, y, and ξ, we will pick K accordingly and show that we get the desired inequality. For any λ, y, ξ, and i, define

$$K_i := \tilde{r}X_i(\lambda\Phi(x) + 2\pi\xi \cdot x)|_y$$
$$K := K_j \text{ for the minimum } j \text{ satisfying } |K_j| = \max_i |K_i|$$

If $|K| \leq 1$, then

$$1 + \frac{r^2}{\max_j r_j}\left(\sum_{i=1}^{2d} |K_i|^2\right) \lesssim 1 + \left(\sum_{i=1}^{2d} |K|^2\right)^{1/2} \lesssim 1$$

and so if $N \geq 0$, (3.7) immediately implies

$$|\mathcal{I}(y,\xi)| \lesssim r_1^{d/2} r_2^{d/2} r_{j_0}^{-1} \left(1 + \frac{r^2}{\max_j r_j} \left(\sum_{i=1}^{2d} |K_i|^2\right)^{1/2}\right)^{-N}$$

which is exactly the desired result. So suppose instead that $|K| \geq 1$. We turn now to estimating the expression in the sum in (3.15). Recall that for $||\tilde{r}X_i(\lambda\Phi + 2\pi\xi \cdot x) - K||_{L^\infty}^{N-k}$ we are taking the L^∞ norm over $M \cap I_{y,\xi} = M \cap \left[y + [-r_1/2, r_1/2]^d \times [-r_2/2, r_2/2]^d\right]$, so let $x \in M \cap I_{y,\xi}$ and pick $i = j$ for the minimum j satisfying $|K_j| = \max_i |K_i|$. Then

$$\begin{aligned} |\tilde{r}X_j(\lambda\Phi(x) + 2\pi\xi \cdot x) - K| &= |\tilde{r}X_j(\lambda\Phi(x) + 2\pi\xi \cdot x) - \tilde{r}X_j(\lambda\Phi(x) + 2\pi\xi \cdot x)|_y| \\ &\leq \tilde{r}\left[|\lambda||(X_j\Phi)(x) - (X_j\Phi)(y)| + 2\pi|X_j(\xi \cdot x) - X_j(\xi \cdot x)|_y|\right] \end{aligned} \tag{3.16}$$

By the Mean Value Theorem and the assumptions (1.4), (1.5), and (1.6) on the derivatives of Φ and ρ,

$$\begin{aligned} |(X_j\Phi)(x) - (X_j\Phi)(y)| &\leq |\nabla(X_j\Phi)(c)||x - y| \\ &\lesssim |x - y| \\ &\lesssim \max\{|x^1 - y^1|, |x^2 - y^2|\} \\ &\lesssim \max_j r_j \end{aligned} \tag{3.17}$$

In the first step, c is some point on the line segment between x and y, and the second to last step just follows from the fact that $(a^2+b^2)^{1/2} \approx \max\{|a|,|b|\}$. Similarly,

$$\begin{aligned}
|X_j(\xi \cdot x) - X_j(\xi \cdot x)|_y| &= \left| \left(\xi_j - \frac{\partial_j \rho}{|\nabla \rho|^2} \nabla \rho \cdot \xi\right)\Big|_x - \left(\xi_j - \frac{\partial_j \rho}{|\nabla \rho|^2} \nabla \rho \cdot \xi\right)\Big|_y \right| \\
&= \left| \left(\frac{\partial_j \rho(x)}{|\nabla \rho(x)|^2} \nabla \rho(x) - \frac{\partial_j \rho(y)}{|\nabla \rho(y)|^2} \nabla \rho(y)\right) \cdot \xi \right| \\
&\leq |\xi| \cdot \left| \frac{\partial_j \rho(x)}{|\nabla \rho(x)|^2} \nabla \rho(x) - \frac{\partial_j \rho(y)}{|\nabla \rho(y)|^2} \nabla \rho(y) \right| \\
&\lesssim |\xi||x-y| \\
&\lesssim |\xi| \max_j r_j
\end{aligned} \tag{3.18}$$

where here the the Mean Value Theorem argument is slightly more complicated as we have a vector-valued function, but the result is the same; to get from line 3 to line 4 above, we need to obtain an inequality of the form $|F(x) - F(y)| \lesssim |x-y|$ for the function $F : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ defined by $F(x) = (\partial_j \rho(x)/|\nabla \rho(x)|^2) \nabla \rho(x)$. If the function F has components $(F_1, \ldots, F_{2d})$ then the Mean Value Theorem says, among other things, that for each $i = 1, \ldots, 2d$, we can find a point c_i such that $|F_i(x) - F_i(y)| = |DF_i(c_i) \cdot (x-y)|$. Each F_i is a monomial in $|\nabla \rho|^{-2}$ and first partial derivatives of ρ, so we can get a bound on $|DF_i(c_i)|$ independent of i, given our assumptions on ρ. Thus, for any i, we have an inequality of the form $|F_i(x) - F_i(y)| \leq C|x-y|$ with C independent of i. Our desired inequality follows from these inequalities, as $|F(x) - F(y)| \lesssim \max_i\{|F_i(x) - F_i(y)|\}$.

Combining (3.16), (3.17), and (3.18), we get

$$\begin{aligned}
|\tilde{r} X_j(\lambda \Phi + 2\pi \xi \cdot x) - K| &\leq \tilde{r}\left[|\lambda||(X_j \Phi)(x) - (X_j \Phi)(y)| + 2\pi |X_j(\xi \cdot x) - X_j(\xi \cdot x)|_y|\right] \\
&\lesssim \tilde{r}\left[|\lambda| \max_j r_j + |\xi| \max_j r_j\right] \\
&= r^2(|\lambda| + |\xi|) \\
&\lesssim 1
\end{aligned} \tag{3.19}$$

because $r^{-2} \approx |\lambda| + |\xi|$, as noted earlier. Going back to our main inequality (3.15) and applying (3.19), we now have

$$
\begin{aligned}
|\mathcal{I}(y,\xi)| &\lesssim \frac{r_1^{d/2} r_2^{d/2}}{|K|^N r_{j_0}} \sum_{k=0}^{N} \Big[||\tilde{r} X_j(\lambda\Phi + 2\pi\xi\cdot x) - K||_{L^\infty}^{N-k} + 1 \Big] \\
&\lesssim r_1^{d/2} r_2^{d/2} r_{j_0}^{-1} |K|^{-N} \\
&\lesssim r_1^{d/2} r_2^{d/2} r_{j_0}^{-1} (1+|K|)^{-N} \\
&= r_1^{d/2} r_2^{d/2} r_{j_0}^{-1} (1+\max_i |K_i|)^{-N} \\
&\approx r_1^{d/2} r_2^{d/2} r_{j_0}^{-1} \left(1 + \left(\sum_{i=1}^{2d} |X_i(\lambda\Phi + 2\pi\xi\cdot x)|_y|^2 \right)^{1/2} \right)^{-N}
\end{aligned}
$$

which is the desired result. Note that the third line follows because in this case we are assuming $|K| \geq 1$ and therefore $1+|K| \lesssim |K|$, and the last line follows from the definition of K_i and the equivalence of ℓ^p norms on $\mathbb{R}^{2d}$.

It remains to prove the last statement that if $\mathcal{I}(y,\xi) \neq 0$ then $|\rho(y)| \lesssim \max_j r_j$. Recall that

$$
\mathcal{I}(y,\xi) := \int_M e^{i\lambda\Phi(x)} \varphi_{\xi^1}(x^1 - y^1) \varphi_{\xi^2}(x^2 - y^2) a(x) d\sigma(x)
$$

so (2.5) implies that if $\mathcal{I}(y,\xi) \neq 0$, then $|x^j - y^j| \lesssim r_j$ for $j = 1,2$. But then if $\mathcal{I}(y,\xi) \neq 0$, the Mean Value Theorem gives us that

$$
\begin{aligned}
|\rho(y)| &\leq |\rho(x)| + |\rho(y) - \rho(x)| \\
&= 0 + |\nabla\rho(c) \cdot (y-x)| \\
&\leq |\nabla\rho(c)| \cdot |y-x| \\
&\lesssim \max_j r_j
\end{aligned}
$$

by similar arguments as before. □

3.3. Rapid Decay Cases

In this section, we show that in certain regions of $(y, \xi) \in \mathbb{R}^{2d} \times \mathbb{R}^{2d}$, we can improve the estimate in Proposition 4 to the point where it implies rapid decay for the operator in the region. As a reminder, the only assumptions used up until this point and for the result in this section are the assumptions in the definition of the operator (1.1) and the initial size estimates (1.3), (1.4), (1.5), and (1.6) for the derivatives and support of a, Φ, and ρ.

Before introducing the specific regions in which rapid decay occurs, we first explain the significance of the choice of vector fields X_i in the estimate (3.6) for $\mathcal{I}(y, \xi)$. Note that

$$\begin{aligned} X_i(\lambda\Phi + 2\pi\xi \cdot x)|_y &= \left(\partial_i - \frac{\partial_i \rho}{|\nabla\rho|^2} \sum_{j=1}^{2d} (\partial_j \rho)\partial_j\right)(\lambda\Phi + 2\pi\xi \cdot x)\Big|_y \\ &= \left[(\lambda\partial_i\Phi + 2\pi\xi_i) - \frac{\partial_i\rho}{|\nabla\rho|^2}\sum_{j=1}^{2d}(\partial_j\rho)(\lambda\partial_j\Phi + 2\pi\xi_j)\right]\Bigg|_y \\ &= (\lambda\partial_i\Phi(y) + 2\pi\xi_i) - \frac{\partial_i\rho(y)}{|\nabla\rho(y)|^2}\nabla\rho(y)\cdot(\lambda\nabla\Phi(y) + 2\pi\xi) \end{aligned}$$

and so as a vector,

$$(X_i(\lambda\Phi + 2\pi\xi \cdot x)|_y)_{i=1}^{2d} = (\lambda\nabla\Phi(y) + 2\pi\xi) - \left(\frac{\nabla\rho(y)\cdot(\lambda\nabla\Phi(y) + 2\pi\xi)}{|\nabla\rho(y)|^2}\right)\nabla\rho(y) \tag{3.20}$$

which is exactly the projection of $(\lambda\nabla\Phi(y) + 2\pi\xi)$ onto the space orthogonal to $\nabla\rho(y)$. In other words, the expression

$$\left(\sum_{i=1}^{2d} |X_i(\lambda\Phi + 2\pi\xi \cdot x)|_y|^2\right)^{1/2}$$

in (3.6) is essentially the length of the portion of the gradient of the phase $(\lambda\nabla\Phi(y) + 2\pi\xi)$ which is orthogonal to $\nabla\rho(y)$.

It turns out that there are two regions in which rapid decay occurs: the first is when r_1 and r_2 are not comparable, i.e., one is always bigger than the other by at least some (large enough) fixed multiple. The kernel $\mathcal{I}(y,\xi)$ experiences rapid decay on this region essentially because the size difference between r_1 and r_2 ultimately implies that if, for example, $r_1 << r_2$, we have $|\xi^1| >> |\lambda|$ and $|\xi^1| >> |\xi^2|$. Since the gradient of the phase is $\lambda\nabla\Phi + 2\pi\xi$, these relations combined with the fact that the derivatives of Φ are controlled and all first partial derivatives of ρ are uniformly bounded below in magnitude means that the phase cannot be stationary with respect to the manifold $M = \{\rho(x) = 0\}$.

The other region in which rapid decay occurs is when r_1 and r_2 are comparable, but the portion of the gradient of the phase $(\lambda\nabla\Phi(y) + 2\pi\xi)$ that is *parallel* to $\nabla\rho(y)$ is not too large, i.e., the portion of $(\lambda\nabla\Phi(y)+2\pi\xi)$ that is *orthogonal* to $\nabla\rho(y)$ is not too small. Here, the condition that the gradient of the phase cannot be "too close to parallel" to $\nabla\rho$ again just means that the phase is non-stationary with respect to the manifold M.

We formalize this in the following proposition.

Proposition 5. *Let $\Xi_1 \subset \mathbb{R}^{2d}$ be defined by*

$$\Xi_1 := \{\xi \in \mathbb{R}^{2d} \mid \min_j r_j \leq c \max_j r_j\} \tag{3.21}$$

Then for all $c > 0$ small enough, depending only on admissible constants, and any fixed $N \leq 2d+2$, we have the following pointwise inequality for all $(y,\xi) \in B_1 \times \Xi_1$.

$$|\mathcal{I}(y,\xi)| \lesssim (|\lambda| + |\xi|)^{-(N-1)/2}|\lambda|^{-d/2} \tag{3.22}$$

Next, if $c' > 0$ is small enough and $c'' > 0$ is large enough, depending only on admissible constants, then for any $(y,\xi) \in B_1 \times (\mathbb{R}^{2d} \setminus \Xi_1)$ satisfying $|\xi| \geq c''|\lambda|$ and

$$\left|\frac{(\lambda\nabla\Phi(y) + 2\pi\xi)\cdot\nabla\rho(y)}{|\nabla\rho(y)|^2}\right| \leq c' r^{-2} \tag{3.23}$$

we have the inequality

$$|\mathcal{I}(y,\xi)| \lesssim (|\lambda| + |\xi|)^{-N/2}\lambda^{-(d-1)/2} \tag{3.24}$$

Finally, if $E \subset \mathbb{R}^{2d} \times \mathbb{R}^{2d}$ *is the set of all* (y, ξ) *where either* (3.22) *or* (3.24) *holds, then*

$$\left| \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \mathcal{I}(y,\xi)\chi_E(y,\xi) f_1(y^1,\xi^1) f_2(y^2,\xi^2) dy d\xi \right| \lesssim |\lambda|^{-\frac{d-1}{2}} ||f_1||_{L^2} ||f_2||_{L^2} \tag{3.25}$$

for all $f_j \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$.

Proof. To prove (3.22), suppose without loss of generality that $r_1 \leq r_2$, so that $r = r_1$ and $\max_j r_j = r_2$. If $\min_j r_j \leq c \max_j r_j$, i.e., $r_1 \leq c r_2$, then picking $c < 1$ means

$$r_1 < r_2 = \min\{|\lambda|^{-1/2}, (3|\xi^2|)^{-1/2}\} \leq |\lambda|^{-1/2}$$

so $|r_1| \neq |\lambda|^{-1/2}$ and therefore $r_1 = (3|\xi^1|)^{-1/2}$. Furthermore, $r_1 \leq c r_2$ implies

$$(3|\xi^1|)^{-1/2} \leq c \min\{|\lambda|^{-1/2}, (3|\xi^2|)^{-1/2}\} \leq c|\lambda|^{-1/2} \implies |\lambda| \leq 3c^2 |\xi^1|$$

Next, pick $k_1 \in \{1, \ldots, d\}$ and $k_2 \in \{d+1, \ldots, 2d\}$ to be the minimal indices such that $|\xi_{k_1}| = ||\xi^1||_\infty = ||(\xi_1, \ldots, \xi_d)||_\infty$ and $|\xi_{k_2}| = ||\xi^2||_\infty = ||(\xi_{d+1}, \ldots, \xi_{2d})||_\infty$. Then if $c < \min\{1, \left(\frac{\pi}{3C_\Phi \sqrt{d}}\right)^{1/2}\}$, we have by the triangle inequality that

$$\begin{aligned} |\partial_{k_1}(\lambda\Phi(x) + 2\pi\xi \cdot x)| &= |\lambda \partial_{k_1}\Phi(x) + 2\pi\xi_{k_1}| \\ &\geq 2\pi|\xi_{k_1}| - |\lambda||\partial_{k_1}\Phi(x)| \\ &\geq \tfrac{2\pi}{\sqrt{d}}|\xi^1| - C_\Phi|\lambda| \\ &\geq \tfrac{2\pi}{\sqrt{d}}|\xi^1| - 3c^2 C_\Phi |\xi^1| \\ &\geq \tfrac{\pi}{\sqrt{d}}|\xi^1| \\ &= \tfrac{\pi}{3\sqrt{d}} r_1^{-2} \end{aligned}$$

for any $x \in B_1$, where C_Φ is the constant in the initial size assumption (1.4) on Φ, and the third line comes from the fact that

$$|\xi_{k_1}| = ||\xi^1||_\infty = \tfrac{1}{\sqrt{d}}(d||\xi^1||_\infty^2)^{1/2} \geq \tfrac{1}{\sqrt{d}}(|\xi_1|^2 + \cdots + |\xi_d|^2)^{1/2} = \tfrac{1}{\sqrt{d}}||\xi^1||_2 = \tfrac{1}{\sqrt{d}}|\xi^1|$$

Also by the triangle inequality,

$$\begin{aligned}
|\partial_{k_2}(\lambda\Phi(x) + 2\pi\xi \cdot x)| &= |\lambda\partial_{k_2}\Phi(x) + 2\pi\xi_{k_2}| \\
&\leq |\lambda||\partial_{k_2}\Phi(x)| + 2\pi|\xi_{k_2}| \\
&\leq |\lambda|C_\Phi + 2\pi|\xi^2| \\
&\leq (C_\Phi + \tfrac{2\pi}{3})r_2^{-2} \\
&\leq c^2(C_\Phi + \tfrac{2\pi}{3})r_1^{-2}
\end{aligned}$$

Ultimately to estimate $\mathcal{I}(y, \xi)$ we will again apply (3.3), but this time with a different vector field, which is why we need the above inequalities. Define the vector field

$$X_{k_1k_2} = \frac{\partial_{k_2}\rho(x)}{\sqrt{\partial_{k_1}\rho(x)^2 + \partial_{k_2}\rho(x)^2}}\partial_{k_1} - \frac{\partial_{k_1}\rho(x)}{\sqrt{\partial_{k_1}\rho(x)^2 + \partial_{k_2}\rho(x)^2}}\partial_{k_2} \tag{3.26}$$

which we can do without any issues as by assumption $\partial_i\rho$ is bounded below for every i. By the above inequalities and another application of the triangle inequality,

$$\begin{aligned}
&|X_{k_1k_2}(\lambda\Phi(x) + 2\pi\xi \cdot x)| \\
&\geq \left|\frac{\partial_{k_2}\rho(x)}{\sqrt{\partial_{k_1}\rho(x)^2 + \partial_{k_2}\rho(x)^2}}\right| \cdot \frac{\pi}{3\sqrt{d}}r_1^{-2} - \left|\frac{\partial_{k_1}\rho(x)}{\sqrt{\partial_{k_1}\rho(x)^2 + \partial_{k_2}\rho(x)^2}}\right| \cdot c^2(C_\Phi + \tfrac{2\pi}{3})r_1^{-2} \\
&= \left|\frac{1}{\sqrt{1 + (\partial_{k_1}\rho(x)/\partial_{k_2}\rho(x))^2}}\right| \cdot \frac{\pi}{3\sqrt{d}}r_1^{-2} - \left|\frac{1}{\sqrt{1 + (\partial_{k_2}\rho(x)/\partial_{k_1}\rho(x))^2}}\right| \cdot c^2(C_\Phi + \tfrac{2\pi}{3})r_1^{-2} \\
&\geq \left(\frac{\pi}{3\sqrt{d}C_\rho C'_\rho} - c^2\left(C_\Phi + \frac{2\pi}{3}\right)\right)r_1^{-2} \\
&\gtrsim r_1^{-2}
\end{aligned} \tag{3.27}$$

if $c > 0$ is chosen small enough (and at least as small as specified earlier) depending only on admissible constants.

We also look at the other terms that will appear when we apply the stationary phase and size result; for the same reasons given in the proof of Proposition 4, we have

$$\sum_{\ell=2}^{N-k} |(rX_{k_1k_2})^{\ell}(\lambda\Phi + 2\pi\xi\cdot x)|^{(N-\ell)/k} \lesssim 1$$

Also, if η is a smooth function on B_1 that has similar support and size inequalities as the function $a_{y,\xi}$ from Proposition 4, namely, that η is supported in a box $\tilde{I}$ which is a product of $2d$ intervals, each of length at most r, and for $|\alpha| \leq N$, satisfies

$$|\partial^{\alpha}\eta(x)| \leq C_{\alpha} r^{-|\alpha|}$$

then by the same arguments as in Proposition 4, we have

$$|(rX^*_{k_1k_2})^k(a_{y,\xi}\eta)| \lesssim r_1^{-d/2} r_2^{-d/2}$$

With these estimates in hand, applying (3.3) gives us

$$\begin{aligned}
&\left|\int_M e^{i(\lambda\Phi(x)+2\pi\xi\cdot x)} a_{y,\xi}(x)\eta(x)d\sigma(x)\right| \\
&\quad\lesssim \frac{r^{2d-1}}{|K|^N}\sum_{k=0}^{N} ||(rX^*_{k_1k_2})^k a_{y,\xi}\eta||_{L^\infty}\Bigg[||rX_{k_1k_2}(\lambda\Phi + 2\pi\xi\cdot x) - K||_{L^\infty}^{N-k} \\
&\qquad\qquad + \sum_{\ell=2}^{N-k} ||(rX_{k_1k_2})^{\ell}(\lambda\Phi + 2\pi\xi\cdot x)||_{L^\infty}^{(N-k)/\ell}\Bigg] \\
&\quad\lesssim \frac{r^{2d-1}}{|K|^N} r_1^{-d/2} r_2^{-d/2}\sum_{k=0}^{N}\left[||rX_{k_1k_2}(\lambda\Phi + 2\pi\xi\cdot x) - K||_{L^\infty(M\cap I_{y,\xi}\cap\tilde{I})}^{N-k} + 1\right]
\end{aligned}$$

Note the difference between the quantity on the first line and $|\mathcal{I}(y,\xi)|$; we have an additional factor of $\eta(x)$ in the integral. This explains why we get a factor of $r^{2d-1}/|K|^N$ in front of the sum on the second line, as opposed to the factor we got when applying (3.3) in the proof of Proposition 4, which was $r_1^d r_2^d/(|K|^N r_{j_0})$; our amplitude here is $a_{y,\xi}\eta$, which is supported in $I_{y,\xi} \cap \tilde{I}$, as opposed to $a_{y,\xi}$, which is supported in $I_{y,\xi}$.

Next, let $K = rX_{k_1k_2}(\lambda\Phi+2\pi\xi\cdot x)|_z$ for any $z \in I_{y,\xi}\cap\tilde{I}$. Then $|K| = |rX_{k_1k_2}(\lambda\Phi+2\pi\xi\cdot x)|_z \gtrsim r \cdot r_1^{-2} = r^{-1}$ by (3.27), and by the Mean Value Theorem and the same arguments as in Proposition 4, we have

$$||rX_{k_1k_2}(\lambda\Phi(x) + 2\pi\xi \cdot x) - K||_{L^\infty(M\cap\tilde{I})} \lesssim r^2(|\lambda| + |\xi|) \lesssim 1$$

Note that it is important to multiply by η in the integral to restrict the domain of integration to a product of $2d$ intervals of length r (restricted from the previous domain of $I_{y,\xi}$, which is a product of d intervals of length $r_1 = r$ and d intervals of length $r_2 = \max_j r_j$), as otherwise this mean value argument would give us an upper bound of $r \cdot \max_j r_j(|\lambda| + |\xi|)$ here, as it does in Proposition 4.

Combining everything that we have so far, we get the estimate

$$\left|\int_M e^{i(\lambda\Phi(x)+2\pi\xi\cdot x)} a_{y,\xi}(x)\eta(x)d\sigma(x)\right| \lesssim r^{2d-1} r^N r_1^{-d/2} r_1^{-d/2}$$

To finish this piece, we just construct a partition of unity on $I_{y,\xi}$ adapted to boxes of side length r, where the cutoff functions η are smooth and satisfy $|\partial^\alpha\eta(x)| \leq C_\alpha r^{-|\alpha|}$ uniformly. This partition of unity can be constructed with the number of elements being at most a constant multiple of $r^{-2d}r_1^d r_2^d$ (the volume of $I_{y,\xi}$ divided by the volume of each $\tilde{I}$), so summing over the partition and using the inequality above gives $|\mathcal{I}(y,\xi)| \lesssim r^{N-1} r_1^{d/2} r_2^{d/2}$. This then implies (3.22) because $r_j \leq |\lambda|^{-1/2}$ for each j and $r^{-2} \approx |\lambda| + |\xi|$, as noted earlier.

Next, we prove (3.24). For this piece, the assumption $\xi \in \mathbb{R}^{2d} \setminus \Xi_1$ implies $r_1 \approx r_2$, and thus $\tilde{r} \approx r$, and this along with (3.20) gives

$$\tilde{r}\left(\sum_{i=1}^{2d} |X_i(\lambda\Phi + 2\pi\xi \cdot x)|_y)^2\right)^{1/2} \approx r\left|\lambda\nabla\Phi + 2\pi\xi - \nabla\rho\frac{(\lambda\nabla\Phi + 2\pi\xi)\cdot\nabla\rho}{|\nabla\rho|^2}\right|$$

From here, we apply our other defining assumptions for this region: $|\xi| \geq c''|\lambda|$ and (3.23) and use the triangle inequality to get

$$r\left|\lambda\nabla\Phi + 2\pi\xi - \nabla\rho\frac{(\lambda\nabla\Phi + 2\pi\xi)\cdot\nabla\rho}{|\nabla\rho|^2}\right| \geq r\left[2\pi|\xi| - \frac{|\nabla\Phi|}{c''}|\xi| - |\nabla\rho|c'r^{-2}\right]$$

But then because $r^{-2} \approx |\xi| + |\lambda|$ and $|\xi| \geq c''|\lambda|$, we have $r^{-2} \approx |\xi|$, so as long as c'' is sufficiently large and c' is sufficiently small (both depending only on admissible constants), we have that

$$\begin{aligned} 1 + \tilde{r}\left(\sum_{i=1}^{2d} |X_i(\lambda\Phi + 2\pi\xi\cdot x)|_y|^2\right)^{1/2} &\gtrsim r\left[2\pi|\xi| - \frac{|\nabla\Phi|}{c''}|\xi| - |\nabla\rho|c'r^{-2}\right] \\ &\gtrsim r|\xi| \\ &\approx |\xi|^{1/2} \\ &\gtrsim (|\lambda| + |\xi|)^{1/2} \end{aligned}$$

And so (3.24) follows from the above and (3.6), because $r_{j_0}^{-1} r_1^{d/2} r_2^{d/2} \approx r^{d-1} \leq |\lambda|^{-\frac{d-1}{2}}$.

Finally, we prove (3.25). Pick $N = 2d + 2$. (Note that this choice of N determines the number of derivatives that we need control on for the initial assumptions (1.3), (1.4), (1.5), and (1.6).) Then for $(y, \xi) \in E$, either (3.22) or (3.24) holds, and either of the two implies

$$|\mathcal{I}(y,\xi)| \lesssim (|\lambda| + |\xi|)^{-(2d+1)/2}|\lambda|^{-\frac{d-1}{2}}$$

because $|\lambda|$ is bounded below by assumption. Then, by Cauchy-Schwarz:

$$\begin{aligned}
&\left| \int_{B_1\times\mathbb{R}^{2d}} \mathcal{I}(y,\xi)\chi_E(y,\xi) f_1(y^1,\xi^1) f_2(y_2,\xi^2)\, dy d\xi \right| \\
&\qquad \lesssim |\lambda|^{-\frac{d-1}{2}} \left(\int_{B_1\times\mathbb{R}^{2d}} (|\lambda| + |\xi|)^{-(2d+1)} dy d\xi \right)^{1/2} ||f_1||_{L^2} ||f_2||_{L^2} \\
&\qquad \lesssim |\lambda|^{-\frac{d-1}{2}} \left(\int_{\mathbb{R}^{2d}} (|\lambda| + |\xi|)^{-(2d+1)} d\xi \right)^{1/2} ||f_1||_{L^2} ||f_2||_{L^2} \\
&\qquad \lesssim |\lambda|^{-\frac{d-1}{2}} \left(\int_{\mathbb{R}^{2d}} (|\lambda| + |\lambda||z|)^{-(2d+1)} |\lambda|^{2d} dz \right)^{1/2} ||f_1||_{L^2} ||f_2||_{L^2} \\
&\qquad \lesssim |\lambda|^{-\frac{d-1}{2}} \left(|\lambda|^{-1} \int_{\mathbb{R}^{2d}} (1 + |z|)^{-(2d+1)} dz \right)^{1/2} ||f_1||_{L^2} ||f_2||_{L^2} \\
&\qquad \lesssim |\lambda|^{-\frac{d-1}{2}} ||f_1||_{L^2} ||f_2||_{L^2}
\end{aligned}$$

where in the fourth line we are making the change of variables $\xi = |\lambda| z$. This completes the proof of Proposition 5. $\square$

CHAPTER 4

Main Contribution

In this chapter, we prove the general result, Theorem 1, and show how the same argument can be leveraged to prove two specific results, Theorem 2 and Theorem 3, for which the determinant condition does not hold, but does not hold only on a small set.

4.1. A General Bilinear Result

By the work in the previous chapter, we know that the operator in Theorem 1 exhibits maximum decay on certain regions, and we are left to analyze the remaining region. If we define

$$\tau_0(\lambda, y, \xi) := \frac{-\nabla\rho(y) \cdot (\lambda\nabla\Phi(y) + 2\pi\xi)}{|\nabla\rho(y)|^2} \tag{4.1}$$

and also define $\Xi_2 := \mathbb{R}^{2d} \setminus \Xi_1$, then the region we are left to consider is

$$E := \{(y, \xi) \subset B_1 \times \Xi_2 \,|\, |\xi| \leq c''|\lambda| \text{ or } |\tau_0| \geq c' r^{-2}\} \tag{4.2}$$

where we note that another condition on the region implied by the fact that $\xi \in \Xi_2$ is that $r_1 \approx r_2$. Before analyzing this region, we first transform the problem slightly, introducing another variable τ, so that later an appropriate change of variables will have its Jacobian determinant given by a multiple of the determinant (1.7) in the hypothesis of Theorem 1. The following proposition comes from [18], and as the proof is short we prove it again here, for clarity.

Proposition 6. *For any positive integer $N \leq 2d+2$, any sufficiently small $c''' > 0$ depending only on admissible constants, and any $(y, \xi) \in B_1 \times \Xi_2$, we have*

$$|\mathcal{I}(y, \xi)| \lesssim \chi_{|\rho(y)| \lesssim r} r^d \int_{|\tau - \tau_0| \leq c''' r^{-1}} \frac{d\tau}{(1 + r|\lambda\nabla\Phi(y) + 2\pi\xi + \tau\nabla\rho(y)|)^N} \tag{4.3}$$

Proof. This estimate is a transformation of the bound in Proposition 4. First, note that the factor of $\chi_{|\rho(y)|\lesssim r}$ on the right hand side follows from the observation in Proposition 4 that if $|\mathcal{I}(y,\xi)| \neq 0$ we necessarily have $|\rho(y)| \lesssim \max_j r_j$, because $\xi \in \Xi_2$ implies $\max_j r_j \lesssim r$.

Next, note that by definition of τ_0 in (4.1), the effect of the vector fields X_i shown in (3.20), and the fact that $\xi \in \Xi_2$ implies $\max_j r_j \approx r$, we have

$$1 + \frac{r^2}{\max_j r_j} \left(\sum_{i=1}^{2d} |X_i(\lambda\Phi + 2\pi\xi \cdot x)|_y|^2 \right)^{1/2} \approx 1 + r|\lambda\nabla\Phi(y) + 2\pi\xi + \tau_0\nabla\rho(y)|$$

And if, as in the bounds of the integral in (4.3), we have $|\tau - \tau_0| \lesssim r$, then by the triangle inequality,

$$\begin{aligned} 1 + r|\lambda\nabla\Phi(y) + 2\pi\xi + \tau\nabla\rho(y)| &= 1 + r|\lambda\nabla\Phi(y) + 2\pi\xi + \tau_0\nabla\rho(y) + (\tau - \tau_0)\nabla\rho(y)| \\ &\leq 1 + r|(\tau - \tau_0)\nabla\rho(y)| + r|\lambda\nabla\Phi(y) + 2\pi\xi + \tau_0\nabla\rho(y)| \\ &\lesssim 1 + r \cdot r^{-1} \cdot |\nabla\rho(y)| + r|\lambda\nabla\Phi(y) + 2\pi\xi + \tau_0\nabla\rho(y)| \\ &\lesssim 1 + r|\lambda\nabla\Phi(y) + 2\pi\xi + \tau_0\nabla\rho(y)| \end{aligned}$$

A completely symmetric proof shows the reverse inequality, so together we have

$$1 + r|\lambda\nabla\Phi(y) + 2\pi\xi + \tau_0\nabla\rho(y)| \approx 1 + r|\lambda\nabla\Phi(y) + 2\pi\xi + \tau\nabla\rho(y)|$$

Finally, if we start with Proposition 4 and apply the above estimates, along with the fact that $r_1 \approx r_2 \approx r$, we get

$$\begin{aligned} |\mathcal{I}(y,\xi)| &\lesssim \chi_{|\rho(y)|\lesssim \max_j r_j} \left(1 + \frac{r^2}{\max_j r_j} \left(\sum_{i=1}^{2d} |X_i(\lambda\Phi + 2\pi\xi \cdot x)|_y|^2 \right)^{1/2} \right)^{-N} r_{j_0}^{-1} r_1^{d/2} r_2^{d/2} \\ &\approx \chi_{|\rho(y)|\lesssim r}\, r^{d-1} (1 + r|\lambda\nabla\Phi(y) + 2\pi\xi + \tau_0\nabla\rho(y)|)^{-N} \\ &\approx \chi_{|\rho(y)|\lesssim r}\, r^d \int_{|\tau-\tau_0|\leq c''' r^{-1}} (1 + r|\lambda\nabla\Phi(y) + 2\pi\xi + \tau\nabla\rho(y)|)^{-N} d\tau \end{aligned}$$

which is exactly the desired estimate. $\square$

At the beginning of this section, we saw that our work was reduced to estimating

$$\int_E \mathcal{I}(y,\xi) f_1(y^1,\xi^1) f_2(y^2,\xi^2)\, dy d\xi$$

Now, if we apply the triangle inequality to the above integral and apply Proposition 6 with $N = 2d+1$, we can further reduce to estimating

$$\int_F \frac{r^d \chi_{|\rho(y)| \lesssim r} |f_1(y^1,\xi^1)||f_2(y^2,\xi^2)|}{(1 + r|\lambda \nabla\Phi(y) + 2\pi\xi + \tau\nabla\rho(y)|)^{2d+1}} dy d\xi d\tau \tag{4.4}$$

where

$$F := \{(y,\xi,\tau) \in B_1 \times \Xi_2 \times \mathbb{R} \,|\, |\xi| \le c''|\lambda| \text{ or } |\tau_0| \ge c' r^{-2}, \text{ and } |\tau - \tau_0| \le c''' r^{-1}\}$$

(We can pick any $N \le 2d+2$ in applying Proposition 6, but as we will see later, $N = 2d+1$ is the best choice in this case.) Before proceeding, we first expand the set F slightly so that its defining conditions are slightly easier to work with. First, note that for any ξ such that $(y,\xi,\tau) \in F$, we have $\xi \in \Xi_2$ and therefore

$$r_1^{-2} \approx r_2^{-2} \approx r^{-2} = \max\{r_1^{-2}, r_2^{-2}\} = \max\{|\lambda|, 3|\xi^1|, 3|\xi^2|\} \approx \max\{|\lambda|, |\xi|\} \approx |\lambda| + |\xi|$$

where we are using the equivalence of ℓ^p norms on $\mathbb{R}^d$ to get that $|\xi^1| \approx \max\{|\xi_1|, \dots, |\xi_d|\}$, and similarly for $|\xi^2|$, and then the equivalence of ℓ^p norms on $\mathbb{R}^{2d}$ to get that $\max\{|\xi_1|, \dots, |\xi_{2d}|\} \approx |\xi|$. Next, we claim that for any $(y,\xi,\tau) \in F$, we have $|\lambda| + |\xi| \approx |\lambda| + |\tau|$. To see this, note that if $|\tau_0| \ge c' r^{-2}$, then

$$\begin{aligned}
|\lambda| + |\tau| &\le |\lambda| + |\tau_0| + |\tau - \tau_0| \\
&\lesssim |\lambda| + |\xi| + r^{-1} \\
&\lesssim |\lambda| + |\xi| + r^{-2} \\
&\approx |\lambda| + |\xi|
\end{aligned}$$

because by definition of τ_0 in (4.1), the triangle inequality, the Cauchy-Schwarz inequality, and the bounds (1.4) and (1.6) on Φ and ρ, respectively, we have $|\tau_0| \lesssim |\lambda| + |\xi|$. Also,

$$\begin{aligned}
|\lambda| + |\tau| &\geq |\lambda| + |\tau_0| - |\tau - \tau_0| \\
&\geq |\lambda| + c' r^{-2} - c''' r^{-1} \\
&\geq |\lambda| + c' r^{-2} - c''' r^{-2} \\
&\gtrsim |\lambda| + r^{-2} \\
&\approx |\lambda| + |\xi|
\end{aligned}$$

provided c''' is taken to be small enough that $c' > 2c'''$ (which is fine to do, per Proposition 6). Note also that we are using the fact that $r^{-1} \geq |\lambda|^{1/2} \gtrsim 1$ and therefore $r^{-1} \lesssim r^{-2}$.

On the other hand, if $|\tau_0| \leq c' r^{-2}$ then by definition of F we must have $|\xi| \leq c''|\lambda|$, and therefore

$$|\lambda| + |\tau| \geq |\lambda| \gtrsim |\lambda| + |\xi|$$

and also

$$\begin{aligned}
|\lambda| + |\tau| &\leq |\lambda| + |\tau_0| + |\tau - \tau_0| \\
&\leq |\lambda| + c' r^{-2} + c''' r^{-1} \\
&\lesssim |\lambda| + r^{-2} \\
&\approx |\lambda| + |\xi|
\end{aligned}$$

So in either case we have $|\lambda| + |\tau| \approx |\lambda| + |\xi|$. By abuse of notation, we redefine the set F to be slightly larger, based on the above observations.

$$F := \left\{ (y, \xi, \tau) \in B_1 \times \Xi_2 \times \mathbb{R} \,|\, r_1^{-2} \approx r_2^{-2} \approx |\lambda| + |\xi| \approx |\lambda| + |\tau| \right\} \tag{4.5}$$

It remains now to consider (4.4) with this new set F, and show that it is bounded above by $|\lambda|^{-\frac{d-1}{2}}||f_1||_{L^2(\mathbb{R}^d\times\mathbb{R}^d)}||f_2||_{L^2(\mathbb{R}^d\times\mathbb{R}^d)}$. By interpolation, it suffices to bound (4.4) above by both $|\lambda|^{-\frac{d-1}{2}}||f_1||_{L^\infty}||f_2||_{L^1}$ and $|\lambda|^{-\frac{d-1}{2}}||f_1||_{L^1}||f_2||_{L^\infty}$. We will just show the first bound, as the other case is completely symmetric. To that end, we first use Hölder's inequality to get

$$\begin{aligned}
&\int_F \frac{r^d\chi_{|\rho(y)|\lesssim r}|f_1(y^1,\xi^1)||f_2(y^2,\xi^2)|}{(1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^{2d+1}}dyd\xi d\tau\\
&\quad\lesssim ||f_1||_{L^\infty}||f_2||_{L^1}\operatorname*{ess\,sup}_{y^2,\xi^2}\int\frac{r^d\chi_F(y,\xi,\tau)\chi_{|\rho(y)|\lesssim r}d\xi^1dy^1d\tau}{(1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^{2d+1}}\\
&\quad\approx ||f_1||_{L^\infty}||f_2||_{L^1}\operatorname*{ess\,sup}_{y^2,\xi^2}\int\frac{r_2^d\chi_F(y,\xi,\tau)\chi_{|\rho(y)|\lesssim r_2}d\xi^1dy^1d\tau}{(1+r_2|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^{2d+1}}
\end{aligned}$$

where in the last line, we are just using the fact that $r_2\approx r$ to make calculations simpler, as r_2 is a constant with respect to the variables of integration ξ^1, y^1, τ. Next, we integrate out the ξ^1 variables. Write

$$\nabla_1=\left(\tfrac{\partial}{\partial y_1},\dots,\tfrac{\partial}{\partial y_d}\right)\text{ and }\nabla_2=\left(\tfrac{\partial}{\partial y_{d+1}},\dots,\tfrac{\partial}{\partial y_{2d}}\right)$$

then

$$r_2|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|\approx r_2|\lambda\nabla_1\Phi(y)+2\pi\xi^1+\tau\nabla_1\rho(y)|+r_2|\lambda\nabla_2\Phi(y)+2\pi\xi^2+\tau\nabla_2\rho(y)|$$

as $|a|+|b|\approx\sqrt{a^2+b^2}$. If we use this and then make the change of variables $z=2\pi r_2\xi^1+r_2(\lambda\nabla_1\Phi(y)+\tau\nabla_1\rho(y))$ (so the first quantity on the right hand side above is $|z|$), we get

$$\begin{aligned}
\int\frac{d\xi^1}{(1+r_2|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^N}&\approx\int\frac{r_2^{-d}dz}{(1+r_2|\lambda\nabla_2\Phi(y)+2\pi\xi^2+\tau\nabla_2\rho(y)|+|z|)^{2d+1}}\\
&\lesssim\frac{r_2^{-d}}{(1+r_2|\lambda\nabla_2\Phi(y)+2\pi\xi^2+\tau\nabla_2\rho(y)|)^{d+1}}
\end{aligned}$$

To get the second line, we are just using polar coordinates and doing the calculation

$$\int_{\mathbb{R}^d}\frac{dz}{(A+|z|)^{2d+1}}=\int_0^\infty\int_{B_t(0)}\frac{t^{d-1}dSdt}{(A+t)^{2d+1}}\approx\int_0^\infty\frac{t^{d-1}dt}{(A+t)^{2d+1}}\le\int_0^\infty\frac{dt}{(A+t)^{d+2}}\approx\frac{1}{A^{d+1}}$$

with $A = 1 + r_2|\lambda\nabla_1\Phi(y) + 2\pi\xi^1 + \tau\nabla_1\rho(y)| > 0$. Combining this with the work above, we have now shown

$$\begin{aligned}
&\int_F \frac{r^d \chi_{|\rho(y)|\lesssim r}|f_1(y^1,\xi^1)||f_2(y^2,\xi^2)|}{(1 + r|\lambda\nabla\Phi(y) + 2\pi\xi + \tau\nabla\rho(y)|)^{2d+1}} dy d\xi d\tau \\
&\quad \lesssim ||f_1||_{L^\infty}||f_2||_{L^1} \operatorname*{ess\,sup}_{y^2,\xi^2} \int \frac{\chi_{\tilde{F}}(y^1,\tau)\chi_{|\rho(y)|\lesssim r_2} dy^1 d\tau}{(1 + r_2|\lambda\nabla_2\Phi(y) + 2\pi\xi^2 + \tau\nabla_2\rho(y)|)^{d+1}}
\end{aligned}$$

where now after integrating out ξ^1, we are left with the set $\tilde{F} = \{(y,\tau)\,|\,r_2^{-2} \approx |\lambda| + |\tau|\}$. Next, we make the change of variables

$$(u,s) = \varphi(y^1,\tau) = (\lambda\nabla_2\Phi(y) + 2\pi\xi^2 + \tau\nabla_2\rho(y), \rho(y))$$

which has Jacobian determinant given by

$$\begin{aligned}
&|\det(D\varphi)(y^1,\tau)| \\
&\quad = \left|\det\begin{bmatrix} \partial^2_{y_1,y_{d+1}}(\lambda\Phi(y) + \tau\rho(y)) & \cdots & \partial^2_{y_d,y_{d+1}}(\lambda\Phi(y) + \tau\rho(y)) & \partial_{y_{d+1}}\rho(y) \\ \vdots & \vdots & \ddots & \vdots \\ \partial^2_{y_1,y_{2d}}(\lambda\Phi(y) + \tau\rho(y)) & \cdots & \partial^2_{y_d,y_{2d}}(\lambda\Phi(y) + \tau\rho(y)) & \partial_{y_{2d}}\rho(y) \\ \partial_{y_1}\rho(y) & \cdots & \partial_{y_d}\rho(y) & 0 \end{bmatrix}\right| \\
&\quad = (|\lambda|^2 + |\tau|^2)^{\frac{d-1}{2}} \\
&\qquad \cdot \left|\det\begin{bmatrix} \partial^2_{y_1,y_{d+1}}(\omega_1\Phi(y) + \omega_2\rho(y)) & \cdots & \partial^2_{y_d,y_{d+1}}(\omega_1\Phi(y) + \omega_2\rho(y)) & \partial_{y_{d+1}}\rho(y) \\ \vdots & \vdots & \ddots & \vdots \\ \partial^2_{y_1,y_{2d}}(\omega_1\Phi(y) + \omega_2\rho(y)) & \cdots & \partial^2_{y_d,y_{2d}}(\omega_1\Phi(y) + \omega_2\rho(y)) & \partial_{y_{2d}}\rho(y) \\ \partial_{y_1}\rho(y) & \cdots & \partial_{y_d}\rho(y) & 0 \end{bmatrix}\right| \\
&\quad \geq c(|\lambda|^2 + |\tau|^2)^{\frac{d-1}{2}} \\
&\quad \approx c(|\lambda| + |\tau|)^{d-1} \qquad (4.6)
\end{aligned}$$

In the second equality, $\omega_1 = \lambda/(|\lambda|^2+|\tau|^2)$ and $\omega_2 = \tau/(|\tau|^2+|\lambda|^2)$, so $\omega_1^2+\omega_2^2 = 1$ and we are in the situation of (1.7) (up to reordering the columns, which doesn't change the magnitude of the determinant), which gives us the inequality on the line below. We can factor out $(|\lambda|^2+|\tau|^2)^{(d-1)/2}$ from the determinant in the second equality because this determinant is a sum of terms which are all a product of one (non-zero) entry in the last row, one (non-zero) entry in the last column, and $(d-1)$ entries of the form $\partial^2_{y_iy_j}(\lambda\Phi(y)+\tau\rho(y))$. Factoring out $(|\lambda|^2+|\tau|^2)^{(d-1)/2}$ from the determinant means that these terms being summed will instead be a product of one (non-zero) entry in the last row, one (non-zero) entry in the last column, and $(d-1)$ entries of the form $\partial^2_{y_iy_j}(\omega_1\Phi(y)+\omega_2\rho(y))$ with $\omega_1^2+\omega_2^2=1$.

Thus by hypothesis (1.7), this determinant is nonzero on the support of the integrand. It remains to ensure that φ is injective on the support of the integrand, and then we can apply our desired change of variables. There is a helpful lemma in [18] which guarantees that we can do this by subdividing the support of the integrand; we state the relevant portion below:

Lemma 4 ([18, Lemma 3]). *Suppose $\rho(u,v)$ and $\Phi(u,v)$ are C^3 functions on $U\times V\subset\mathbb{R}^d\times\mathbb{R}^d$, where U and V are both d-fold products of open intervals, and let $\omega:=(\omega_1,\omega_2)\in S^1$. Suppose*

$$\left|\det\begin{bmatrix} 0 & \partial_{u_1}\rho & \cdots & \partial_{u_d}\rho \\ \partial_{v_1}\rho & \partial^2_{v_1u_1}(\omega_1\Phi+\omega_2\rho) & \cdots & \partial^2_{v_1u_d}(\omega_1\Phi+\omega_2\rho) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{v_d}\rho & \partial^2_{v_du_1}(\omega_1\Phi+\omega_2\rho) & \cdots & \partial^2_{v_du_d}(\omega_1\Phi+\omega_2\rho)\end{bmatrix}\right|\geq c$$

at some point $p:=(u_p,v_p,\omega_p)\in U\times V\times S^1$. Then there exist metric balls U_0, V_0, and W_0, centered at u_p, v_p, and ω_p, respectively, such that for all $v\in V_0$ and all $\lambda\in\mathbb{R}$, the map

$$\Psi_{v,\lambda}(u,\tau):=(\lambda\nabla_v\Phi(u,v)+\tau\nabla_v\rho(u,v),\rho(u,v))\in\mathbb{R}^d\times\mathbb{R}$$

is injective on the open set

$$\mathcal{U}_\lambda:=\left\{(u,\tau)\in U_0\times\mathbb{R}\,:\,\frac{(\lambda,\tau)}{\sqrt{\lambda^2+\tau^2}}\in W_0\right\}$$

The radii of the balls U_0,V_0, and W_0 can be taken to depend only on c, d and the C^3-norms *of ρ and Φ on $U \times V$.*

Per this lemma, we can cover the (compact) set $[-b_1 \times b_1]^d \times S^1$ with a finite collection of sets of the form $U_0 \times W_0$, and as the radii of these balls can be taken to depend only on admissible constants, the size of this collection can be taken to depend only on admissible constants as well[1]. This means that the (same-sized) collection of corresponding sets of the form U_λ must cover $(-b_1, b_1)^d \times \mathbb{R}$, as for any $\tau \in \mathbb{R}$, $(\lambda, \tau)/\sqrt{\lambda^2 + \tau^2}$ must fall somewhere on S^1 and therefore into a set of the form W_0. Consider the collection formed by intersecting each of these subsets of the form U_λ with the set $\tilde{F}$ that we are integrating over; call this $\{\tilde{F}_k\}_{1 \le k \le N}$. The above reasoning says that φ is injective on each $\tilde{F}_k$ and N depends only on admissible constants. As each $\tilde{F}_k$ is a subset of $\tilde{F}$, the lower bound (4.6) on the determinant trivially still holds on $\tilde{F}_k$. This is exactly what we need to be able to apply the change of variables, and when we do this on each $\tilde{F}_k$, we get

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{y^2,\xi^2} \int \frac{\chi_{\tilde{F}_k}(y^1,\tau)\chi_{|\rho(y)|\lesssim r_2}\, dy^1 d\tau}{(1 + r_2|\lambda\nabla_2\Phi(y) + 2\pi\xi^2 + \tau\nabla_2\rho(y)|)^{d+1}} \\
&\quad = \operatorname*{ess\,sup}_{y^2,\xi^2} \int \frac{\chi_{\tilde{F}_k}(y^1,\tau)\chi_{|\rho(y)|\lesssim r_2}}{(1 + r_2|\lambda\nabla_2\Phi(y) + 2\pi\xi^2 + \tau\nabla_2\rho(y)|)^{d+1}} \cdot \frac{|\det D\varphi(y^1,\tau)|}{|\det D\varphi(y^1,\tau)|} dy^1 d\tau \\
&\quad \lesssim \operatorname*{ess\,sup}_{y^2,\xi^2} \int \frac{\chi_{\tilde{F}_k}(y^1,\tau)\chi_{|\rho(y)|\lesssim r_2}}{(1 + r_2|\lambda\nabla_2\Phi(y) + 2\pi\xi^2 + \tau\nabla_2\rho(y)|)^{d+1}} \cdot \frac{|\det D\varphi(y^1,\tau)|}{c(|\lambda| + |\tau|)^{d-1}} dy^1 d\tau \\
&\quad \lesssim \operatorname*{ess\,sup}_{y^2,\xi^2} \int \frac{\chi_{\tilde{F}_k}(y^1,\tau)\chi_{|\rho(y)|\lesssim r_2}}{(1 + r_2|\lambda\nabla_2\Phi(y) + 2\pi\xi^2 + \tau\nabla_2\rho(y)|)^{d+1}} \cdot \frac{|\det D\varphi(y^1,\tau)|}{c r_2^{-2(d-1)}} dy^1 d\tau \\
&\quad \lesssim \operatorname*{ess\,sup}_{y^2,\xi^2} r_2^{2d-2} \int \frac{\chi_{|s|\lesssim r_2}\, du ds}{(1 + r_2|u|)^{d+1}} \\
&\quad \lesssim \operatorname*{ess\,sup}_{y^2,\xi^2} r_2^{2d-1} \int \frac{du}{(1 + r_2|u|)^{d+1}} \\
&\quad \lesssim \operatorname*{ess\,sup}_{y^2,\xi^2} r_2^{d-1} \\
&\quad \lesssim |\lambda|^{-\frac{d-1}{2}}
\end{aligned}
$$

[1] We stated this lemma as it appears in [18], but in our case, we have something stronger than what is assumed in the hypothesis: the determinant in our case is bounded below by c everywhere, not just at a particular point. The implication for our situation is that the radii of the balls U_0, V_0, W_0 do not depend on c. In fact, when one looks at the details of the proof of the lemma, there is actually no need to subdivide our domain of integration to guarantee injectivity of φ. However, we still use the lemma here to streamline the proof and to not reproduce work from [18].

where the third line follows from applying (4.6), and the fourth line follows from the fact that $\tilde{F} \supset \tilde{F}_k$. This finishes the proof of Theorem 1, as if we continue from earlier,

$$\begin{aligned}
&\int_F \frac{r^d \chi_{|\rho(y)|\lesssim r} |f_1(y^1,\xi^1)||f_2(y^2,\xi^2)|}{(1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^{2d+1}} dy d\xi d\tau \\
&\quad \lesssim ||f_1||_{L^\infty}||f_2||_{L^1} \operatorname*{ess\,sup}_{y^2,\xi^2} \sum_{k=1}^N \int \frac{\chi_{\tilde{F}_k}(y^1,\tau)\chi_{|\rho(y)|\lesssim r_2} dy^1 d\tau}{(1+r_2|\lambda\nabla_2\Phi(y)+2\pi\xi^2+\tau\nabla_2\rho(y)|)^{d+1}} \\
&\quad \lesssim |\lambda|^{-\frac{d-1}{2}} ||f_1||_{L^\infty}||f_2||_{L^1}
\end{aligned}$$

As we indicated earlier, the bound $|\lambda|^{-\frac{d-1}{2}}||f_1||_{L^1}||f_2||_{L^\infty}$ for (4.4) follows by a completely symmetric proof, as the partial derivatives commute on Φ and ρ, so we get exactly the same Jacobian determinant. Note also that the choice of $N = 2d+1$ does not directly influence the power on λ in the final bound; it is just the minimal exponent needed to ensure that $\int_{\mathbb{R}^d} \frac{du}{(1+|u|)^{N-d}}$ is bounded, which is the last integral in the calculations estimating the integral on each set $\tilde{F}_k$.

As was the case in [18], the $|\lambda|$ decay in Theorem 1 is actually optimal for any operator of the form (1.1) that satisfies just a few smoothness and boundedness conditions on a, Φ, and ρ – fewer assumptions than are required by Theorem 1. We show this below, even though the argument in [18] suffices for the bilinear operator here as well. The key idea in proving this is that if we apply the operator to certain $L^2(\mathbb{R}^d)$ functions that are supported on a box with side length proportional to $|\lambda|^{-1/2}$ and that oscillate at a rate based on λ and the derivatives of the phase Φ, the complex exponential in the operator will not be able to oscillate enough to cause cancellation.

To make calculations simpler, suppose without loss of generality that $0 \in M$ and that $a(x)$ is real and positive on a small neighborhood of 0. Also suppose that $|\lambda| \geq 1$. We note a few key inequalities: first, if $\partial_{2d}\rho(x) \neq 0$ for all $x \in B_1$, the Implicit Function Theorem guarantees a function Ψ on a neighborhood U of $\overline{0}$ in $\mathbb{R}^{2d-1}$ such that M is the graph of $x_{2d} = \Psi(\overline{x})$ on U (where we take $\overline{x} = (x_1, \ldots, x_{2d-1})$). Moreover, as long as ρ is differentiable with bounded derivative on B_1 and $\partial_{2d}\rho(x) \geq c_0 > 0$ for some c_0 on B_1, then the Mean Value Theorem implies

$$|\Psi(\overline{x}) - \Psi(\overline{0})| = |\nabla\Psi(\overline{c}) \cdot (\overline{x} - \overline{0})| \leq c_1 \max_{j \neq 2d} |x_j|$$

where c_1 is some constant depending only on c_0 and the upper bound of $|\nabla\rho|$ on B_1. Next, note that if Φ is at least C^2 with a uniform bound on all second derivatives on B_1, then Taylor's Theorem implies that $|\Phi(x) - \Phi(0) - \sum_{j=1}^{2d} x_j \partial_j \Phi(0)|$ is $O(|x|^2)$ as $x \to 0$. This means that we can find some $c_2 > 0$ such that if $|x_j| < c_2|\lambda|^{-1/2}$ for $j \neq 2d$ and $|x_{2d}| < c_1c_2|\lambda|^{-1/2}$, we have $|\lambda\Phi(x) - \lambda\Phi(0) - \sum_{j=1}^{2d} \lambda x_j \partial_j \Phi(0)| < \pi/4$ and consequently $\mathrm{Re}[e^{i(\lambda\Phi(x) - \lambda\Phi(0) - \sum_{j=1}^{2d} \lambda x_j \partial_j \Phi(0))}] > 1/\sqrt{2}$. Finally, we also pick c_2 small enough so that $|x_j| < c_2$ for $j \neq 2d$ and $|x_{2d}| < c_1c_2$ implies $a(x) \geq \frac{1}{2}a(0)$. Note that both c_1 and c_2 are independent of λ and depend only on constants relating to a, Φ, ρ, and d.

With all of this in mind, pick

$$f_1(x^1) = e^{-i\sum_{j=1}^{d} \lambda x_j \partial_j \Phi(0)} \chi_{E_1}(x^1) \qquad f_2(x^2) = e^{-i\sum_{j=d+1}^{2d} \lambda x_j \partial_j \Phi(0)} \chi_{E_2}(x^2)$$

for

$$\begin{aligned}
E_1 &:= (-c_2|\lambda|^{-1/2}, c_2|\lambda|^{-1/2})^d \\
E_2 &:= (-c_2|\lambda|^{-1/2}, c_2|\lambda|^{-1/2})^{d-1} \times (-c_1c_2|\lambda|^{-1/2}, c_1c_2|\lambda|^{-1/2})
\end{aligned}$$

Then we have

$$
\begin{aligned}
|I_\lambda(f_1,f_2)| &= |e^{-i\lambda\Phi(0)}I_\lambda(f_1,f_2)| \\
&\geq |\operatorname{Re}[e^{-i\lambda\Phi(0)}I_\lambda(f_1,f_2)]| \\
&\gtrsim \int_M \chi_{E_1}(x^1)\chi_{E_2}(x^2)a(x)d\sigma(x) \\
&\gtrsim \int_M \chi_{E_1}(x^1)\chi_{E_2}(x^2)d\sigma(x) \\
&= \int_M \chi_{E_1}(x^1)\chi_{E_2}(x_{d+1},\ldots,x_{2d-1},\Psi(\overline{x}))(1+|\nabla\Psi(\overline{x})|^2)^{1/2}d\overline{x} \\
&\geq \int_M \chi_{E_1}(x^1)\chi_{E_2}(x_{d+1},\ldots,x_{2d-1},\Psi(\overline{x}))d\overline{x} \\
&\approx (|\lambda|^{-1/2})^{2d-1} \\
&\approx |\lambda|^{-\frac{d-1}{2}}||f_1||_{L^2}||f_2||_{L^2}
\end{aligned}
$$

because $||f_j||_{L^2} \approx |\lambda|^{-d/4}$. (The fourth line follows because $|x_j| < c_2|\lambda|^{-1/2} \leq c_2$ for $j \neq 2d$ and $|x_{2d}| < c_1c_2|\lambda|^{-1/2} \leq c_1c_2$, so $a(x) \gtrsim 1$ on the support of the integral by choice of c_2.) We reiterate that the constants implied in the inequality $|\tilde{I}_\lambda(f_1,f_2)| \gtrsim |\lambda|^{-\frac{d-1}{2}}||f_1||_{L^2}||f_2||_{L^2}$ depend only on admissible constants, per the initial definition of the notation $\gtrsim$, and in particular do not depend on λ.

Thus the decay in Theorem 1 is best possible for any operator of the form (1.1) that, by the above reasoning, satisfies:

- a is continuous on B_1
- Φ is C^2 with uniformly bounded second derivatives on B_1
- ρ is C^1 and there is some index i for which $\partial_i\rho(x) \geq c_0 > 0$ for all $x \in B_1$, and $|\nabla\rho|$ is bounded on B_1.

4.2. Two Specific Bilinear Results

4.2.1. Operator 1: Introduction and Maximum Decay

A natural question that arises when looking at Theorem 1 is: what happens when the determinant condition fails to hold? In this section, we prove Theorem 2, which is concerned with a specific operator for which the determinant condition fails to hold only on a "small" set: the plane $x_2 = 0$. As a reminder, here we are looking at the operator

$$\tilde{I}_\lambda(f_1, f_2) = \int_{\rho(x)=0} e^{i\lambda(\frac{1}{2}x_2^2x_4+x_3x_5)} f_1(x^1)f_2(x^2)a(x)d\sigma(x)$$

for

$$\rho(x) = -\frac{1}{2}x_2^2x_5 + x_3x_4 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

and f_1, f_2 measurable functions on $\mathbb{R}^3$.

If we calculate the determinant in (1.7), we get

$$\begin{aligned}
&\det\begin{bmatrix} \partial_{x_1}\rho & \partial^2_{x_1,x_4}(\omega_1\Phi+\omega_2\rho) & \partial^2_{x_1,x_5}(\omega_1\Phi+\omega_2\rho) & \partial^2_{x_1,x_6}(\omega_1\Phi+\omega_2\rho) \\ \partial_{x_2}\rho & \partial^2_{x_2,x_4}(\omega_1\Phi+\omega_2\rho) & \partial^2_{x_2,x_5}(\omega_1\Phi+\omega_2\rho) & \partial^2_{x_2,x_6}(\omega_1\Phi+\omega_2\rho) \\ \partial_{x_3}\rho & \partial^2_{x_3,x_4}(\omega_1\Phi+\omega_2\rho) & \partial^2_{x_3,x_5}(\omega_1\Phi+\omega_2\rho) & \partial^2_{x_3,x_6}(\omega_1\Phi+\omega_2\rho) \\ 0 & \partial_{x_4}\rho & \partial_{x_5}\rho & \partial_{x_6}\rho \end{bmatrix} \\
&= \det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-x_2x_5 & x_2\omega_1 & -x_2\omega_2 & 0 \\ 1+x_4 & \omega_2 & \omega_1 & 0 \\ 0 & 1+x_3 & 1-\frac{1}{2}x_2^2 & 1 \end{bmatrix} \\
&= x_2(\omega_1^2+\omega_2^2)
\end{aligned}$$

which is zero only when $x_2 = 0$, as by assumption $\omega_1^2 + \omega_2^2 = 1$. Notably, the magnitude of this determinant is bounded below by ϵ whenever $|x_2| \geq \epsilon$. As we said earlier, the question is: how does this affect the decay of the operator? As it turns out, it is no longer possible to have the optimal decay of $|\lambda|^{-1} = |\lambda|^{-\frac{3-1}{2}}$ that occurs for an operator in dimension $d = 3$ for which this determinant is nonzero. In this thesis, we are only able to prove decay of $|\lambda|^{-1/2} \log |\lambda|$, but we can show that the maximum possible decay for this operator is $|\lambda|^{-3/4}$.

Before we show this, we note that the operator may not actually achieve this maximum possible decay of $|\lambda|^{-3/4}$. As we will see below, this exponent just comes from testing the operator on a specific group of indicator functions indexed by λ; when we evaluate the operator at these functions we can get a lower bound that is a multiple of $|\lambda|^{-3/4}$. This means that it is impossible to prove any decay faster than $|\lambda|^{-3/4}$ for this operator, but doesn't necessarily mean that the operator will achieve decay of $|\lambda|^{-3/4}$.

Consider the following L^2 functions

$$f_1(x_1, x_2, x_3) = \chi_{|\cdot|<c_1}(x_1)\chi_{|\cdot|<c_2|\lambda|^{-1/2}}(x_2)\chi_{|\cdot|<c_3}(x_3)$$
$$f_2(x_4, x_5, x_6) = \chi_{|\cdot|<c_4}(x_4)\chi_{|\cdot|<c_5|\lambda|^{-1}}(x_5)\chi_{|\cdot|<c_6}(x_6)$$

where the constants c_i will be chosen later but will only depend on Φ, a, ρ and a lower bound for $|\lambda|$. Then $||f_1||_{L^2} \approx |\lambda|^{-1/4}$ and $||f_2||_{L^2} \approx |\lambda|^{-1/2}$.

By choice of f_1 and f_2 we have $|\Phi(x)| \leq (c_2^2 c_4 + c_3 c_5)|\lambda|^{-1}$ on the support of the integrand and so by choosing c_2, c_4, c_3, c_5 sufficiently small, we may assume $|\lambda\Phi(x)| \leq \pi/4$ and thus $\mathrm{Re}[e^{i\lambda\Phi(x)}] \geq 1/\sqrt{2}$ on the support of the integrand. Also, given a lower bound for $|\lambda|$, and if we assume without loss of generality that $a(x)$ is real and positive on a small neighborhood of 0, by choosing all the c_i sufficiently small we can guarantee $a(x) \approx 1$ on the support of the integrand. Finally, as we can write the hypersurface $\{\rho(x) = 0\}$ as the graph of $x_6 = \Psi(x_1, x_2, x_3, x_4, x_5) =: \Psi(\overline{x})$, by choosing c_1, c_2, c_3, c_4, c_5 sufficiently small depending on c_6 and other admissible constants, we may assume that $|\Psi(\overline{x})| < c_6$ on the support of the integrand.

Combining these estimates allows us to write

$$\begin{aligned}
&|\tilde{I}_\lambda(f_1,f_2)| \geq |\operatorname{Re}[\tilde{I}_\lambda(f_1,f_2)]| \\
&\quad \gtrsim \int_M f_1(x_1,x_2,x_3)f_2(x_4,x_5,x_6)a(x)d\sigma(x) \\
&\quad \gtrsim \int_M \chi_{|\cdot|<c_1}(x_1)\chi_{|\cdot|<c_2|\lambda|^{-1/2}}(x_2)\chi_{|\cdot|<c_3}(x_3)\chi_{|\cdot|<c_4}(x_4)\chi_{|\cdot|<c_5|\lambda|^{-1}}(x_5)\chi_{|\cdot|<c_6}(x_6)d\sigma(x) \\
&\quad \geq \int_{\mathbb{R}^5} \chi_{|\cdot|<c_1}(x_1)\chi_{|\cdot|<c_2|\lambda|^{-1/2}}(x_2)\chi_{|\cdot|<c_3}(x_3)\chi_{|\cdot|<c_4}(x_4)\chi_{|\cdot|<c_5|\lambda|^{-1}}(x_5)\chi_{|\cdot|<c_6}(\Psi(\overline{x}))d\overline{x} \\
&\quad = \int_{\mathbb{R}^5} \chi_{|\cdot|<c_1}(x_1)\chi_{|\cdot|<c_2|\lambda|^{-1/2}}(x_2)\chi_{|\cdot|<c_3}(x_3)\chi_{|\cdot|<c_4}(x_4)\chi_{|\cdot|<c_5|\lambda|^{-1}}(x_5)d\overline{x} \\
&\quad \approx |\lambda|^{-3/2} \\
&\quad \approx |\lambda|^{-3/4}||f_1||_{L^2}||f_2||_{L^2}
\end{aligned}$$

and thus $|\tilde{I}_\lambda(f_1,f_2)| \gtrsim |\lambda|^{-3/4}||f_1||_{L^2}||f_2||_{L^2}$. Note also that if we instead use the triangle inequality in the first step, we would get

$$\tilde{I}_\lambda(f_1,f_2) \leq \int_M |f_1(x_1,x_2,x_3)f_2(x_4,x_5,x_6)a(x)|d\sigma(x)$$

If we then make similar arguments (using this time that $|a(x)| \lesssim 1$ and $|\Psi(\overline{x})| \lesssim 1$), we get $|\tilde{I}_\lambda(f_1,f_2)| \lesssim |\lambda|^{-3/4}||f_1||_{L^2}||f_2||_{L^2}$, and thus in fact

$$|\tilde{I}_\lambda(f_1,f_2)| \approx |\lambda|^{-3/4}||f_1||_{L^2}||f_2||_{L^2}$$

which shows that decay better (or worse) than $|\lambda|^{-3/4}$ is not possible for this specific choice of functions f_1, f_2.

4.2.2. Operator 1: Dyadic Decomposition

We will show the desired bound in Theorem 2 by using a dyadic decomposition applied to the main contribution piece (4.4) for this specific operator. Because our specific operator satisfies all other hypotheses of Theorem 1 besides the determinant condition (1.7), we can apply all of the results in Section 3.3, and so we immediately get

$$\begin{aligned}
|\tilde{I}_\lambda(f_1,f_2)| &= \left|\int_{\mathbb{R}^6\times\mathbb{R}^6} \tilde{\mathcal{I}}(y,\xi) f_1(y^1,\xi^1) f_2(y^2,\xi^2)\,dyd\xi\right| \\
&\lesssim \left|\int_{(\mathbb{R}^6\times\mathbb{R}^6)\setminus E} \tilde{\mathcal{I}}(y,\xi) f_1(y^1,\xi^1) f_2(y^2,\xi^2)\,dyd\xi\right| + \left|\int_E \tilde{\mathcal{I}}(y,\xi) f_1(y^1,\xi^1) f_2(y^2,\xi^2)\,dyd\xi\right| \\
&\lesssim |\lambda|^{-1}||f_1||_{L^2}||f_2||_{L^2} + \int_F \frac{r^3\chi_{|\rho(y)|\lesssim r}|f_1(y^1,\xi^1)||f_2(y^2,\xi^2)|}{(1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^7}\,dyd\xi d\tau
\end{aligned}$$

where E,F are the sets defined in (4.2) and (4.5), respectively[2], and more specifically, the above follows from applying the decomposition and then using Proposition 5 and Proposition 6. As hinted at earlier, the issue in estimating the main contribution piece of our specific operator arises when $y_2=0$, as this is where the Jacobian determinant vanishes for the change of variables we wish to apply. To get around this, we use a dyadic decomposition in $|y_2|$. Below, we state and then prove the estimates we need for the different pieces in the dyadic decomposition.

Lemma 5. *Let $\epsilon>0$, let ρ,Φ be given as in* (1.9)*, and let F be defined by* (4.5)*. Then we have the following estimates*

$$\int_F \frac{r^3\chi_{|\rho(y)|\lesssim r}\chi_{|y_2|\geq\epsilon}|f_1(y^1,\xi^1)||f_2(y^2,\xi^2)|}{(1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^7}\,dyd\xi d\tau \lesssim \epsilon^{-1}|\lambda|^{-1}||f_1||_{L^\infty}||f_2||_{L^1} \tag{4.7}$$

$$\int_F \frac{r^3\chi_{|\rho(y)|\lesssim r}\chi_{|y_2|\geq\epsilon}|f_1(y^1,\xi^1)||f_2(y^2,\xi^2)|}{(1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^7}\,dyd\xi d\tau \lesssim \epsilon^{-1}|\lambda|^{-1}||f_1||_{L^1}||f_2||_{L^\infty} \tag{4.8}$$

$$\int_F \frac{r^3\chi_{|\rho(y)|\lesssim r}\chi_{|y_2|<\epsilon}|f_1(y^1,\xi^1)||f_2(y^2,\xi^2)|}{(1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^7}\,dyd\xi d\tau \lesssim \epsilon||f_1||_{L^\infty}||f_2||_{L^1} \tag{4.9}$$

$$\int_F \frac{r^3\chi_{|\rho(y)|\lesssim r}\chi_{|y_2|<\epsilon}|f_1(y^1,\xi^1)||f_2(y^2,\xi^2)|}{(1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^7}\,dyd\xi d\tau \lesssim ||f_1||_{L^1}||f_2||_{L^\infty} \tag{4.10}$$

Proof. The proofs of (4.7) and (4.8) are very similar to each other, and are also very similar to the proof for the general operator in Section 4.1, so we will just prove (4.8) here. First,

[2] For technical reasons in the proofs for these specific operators, we will assume that the set F originally defined in (4.5) has the additional condition $|\tau-\tau_0|\leq c'''r^{-1}$. There is no issue in doing so, as the set F is merely an enlargement of a set that we were integrating our (non-negative) function over, and this original set contained the condition $|\tau-\tau_0|\leq c'''r^{-1}$. Thus, the only thing we are doing differently in the proofs for these specific operators is using a slightly tighter upper bound in the step where we introduce the set F.

we proceed as in the proof in Section 4.1:

$$\begin{aligned}
&\int_F \frac{r^3 \chi_{|\rho(y)|\lesssim r}\chi_{|y_2|\geq\epsilon}|f_1(y^1,\xi^1)||f_2(y^2,\xi^2)|}{(1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^7}dyd\xi d\tau \\
&\quad\leq ||f_1||_{L^1}||f_2||_{L^\infty} \operatorname*{ess\,sup}_{y^1,\xi^1} \int \frac{r^3\chi_F(y,\xi,\tau)\chi_{|y_2|\geq\epsilon}\chi_{|\rho(y)|\lesssim r}}{(1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^7}d\xi^2 dy^2 d\tau \\
&\quad\approx ||f_1||_{L^1}||f_2||_{L^\infty} \operatorname*{ess\,sup}_{y^1,\xi^1} \chi_{|y_2|\geq\epsilon} \int \frac{r_1^3\chi_F(y,\xi,\tau)\chi_{|\rho(y)|\lesssim r_1}}{(1+r_1|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^7}d\xi^2 dy^2 d\tau \\
&\quad\lesssim ||f_1||_{L^1}||f_2||_{L^\infty} \operatorname*{ess\,sup}_{y^1,\xi^1} \chi_{|y_2|\geq\epsilon} \int \frac{\chi_{\tilde F}(y^2,\tau)\chi_{|\rho(y)|\lesssim r_1}dy^2 d\tau}{(1+r_1|\lambda\nabla_1\Phi(y)+2\pi\xi^1+\tau\nabla_1\rho(y)|)^4}
\end{aligned}$$

We want to make a similar change of variables as before, and we can write out the function explicitly because in this example, Φ and ρ are simple:

$$\begin{aligned}
(u,s) &= \varphi(y^2,\tau) \\
&= (\lambda\nabla_1\Phi(y)+2\pi\xi^1+\tau\nabla_1\rho(y), \rho(y)) \\
&= (2\pi\xi_1+\tau, \\
&\qquad \lambda y_2 y_4 + 2\pi\xi_2 + \tau - \tau y_2 y_5, \\
&\qquad \lambda y_5 + 2\pi\xi_3 + \tau + \tau y_4, \\
&\qquad -\tfrac{1}{2}y_2^2 y_5 + y_3 y_4 + y_1 + y_2 + y_3 + y_4 + y_5 + y_6)
\end{aligned}$$

And we can still proceed with the change of variables, because the Jacobian determinant is

$$\begin{aligned}
&|\det(D\varphi(y^2,\tau)| \\
&\quad = \left|\det\begin{bmatrix} 0 & 0 & 0 & 1 \\ \lambda y_2 & -\tau y_2 & 0 & 1-y_2y_5 \\ \tau & \lambda & 0 & 1+y_4 \\ 1+y_3 & 1-\frac{1}{2}y_2^2 & 1 & 0 \end{bmatrix}\right| \\
&\quad = |y_2(\lambda^2+\tau^2)| \\
&\quad \geq \epsilon(|\lambda|^2+|\tau|^2)
\end{aligned}$$

where in the last line we are using the fact that we are only taking the supremum over $\{(y^1,\xi^1)\,|\,|y_2|\geq\epsilon\}$. Furthermore

$$|\det(D\varphi(y^2,\tau))|\geq\epsilon(|\lambda|^2+|\tau|^2)\gtrsim\epsilon(|\lambda|+|\tau|)^2\approx\epsilon r_1^{-4}$$

because $|\lambda|^2+|\tau|^2+2|\lambda||\tau|\leq|\lambda|^2+|\tau|^2+2\max\{|\lambda|,|\tau|\}^2\lesssim|\lambda|^2+|\tau|^2$. By the same argument as in the proof in Section 4.1, we can sub-divide $\tilde{F}$ into subsets $\{\tilde{F}_k\}_{1\leq k\leq N}$ where N depends only on admissible constants, such that on each $\tilde{F}_k$, $\varphi(y^2,\tau)$ is injective. Then

$$\begin{aligned}
&\operatorname*{ess\,sup}_{y^1,\xi^1}\chi_{|y_2|\geq\epsilon}\int\frac{\chi_{\tilde{F}_k}(y^2,\tau)\chi_{|\rho(y)|\lesssim r_1}dy^2d\tau}{(1+r_1|\lambda\nabla_1\Phi(y)+2\pi\xi^1+\tau\nabla_1\rho(y)|)^4}\\
&\quad=\operatorname*{ess\,sup}_{y^1,\xi^1}\chi_{|y_2|\geq\epsilon}\int\frac{\chi_{\tilde{F}_k}(y^2,\tau)\chi_{|\rho(y)|\lesssim r_1}}{(1+r_1|\lambda\nabla_1\Phi(y)+2\pi\xi^1+\tau\nabla_1\rho(y)|)^4}\frac{|D\varphi(y^2,\tau)|}{|D\varphi(y^2,\tau)|}dy^2d\tau\\
&\quad\lesssim\operatorname*{ess\,sup}_{y^1,\xi^1}\chi_{|y_2|\geq\epsilon}\int\frac{\chi_{\tilde{F}_k}(y^2,\tau)\chi_{|\rho(y)|\lesssim r_1}}{(1+r_1|\lambda\nabla_1\Phi(y)+2\pi\xi^1+\tau\nabla_1\rho(y)|)^4}\frac{|D\varphi(y^2,\tau)|}{\epsilon r_1^{-4}}dy^2d\tau\\
&\quad\lesssim\operatorname*{ess\,sup}_{y^1,\xi^1}\epsilon^{-1}r_1^4\,\chi_{|y_2|\geq\epsilon}\int\frac{\chi_{|s|\lesssim r_1}duds}{(1+r_1|u|)^4}\\
&\quad\lesssim\operatorname*{ess\,sup}_{y^1,\xi^1}\epsilon^{-1}r_1^5\int\frac{du}{(1+r_1|u|)^4}\\
&\quad\lesssim\operatorname*{ess\,sup}_{y^1,\xi^1}\epsilon^{-1}r_1^2\\
&\quad\lesssim\epsilon^{-1}|\lambda|^{-1}
\end{aligned}$$

and thus

$$\int_F\frac{r^3\chi_{|\rho(y)|\lesssim r}\chi_{|y_2|\geq\epsilon}|f_1(y^1,\xi^1)||f_2(y^2,\xi^2)|}{(1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^7}dyd\xi d\tau\lesssim\epsilon^{-1}|\lambda|^{-1}||f_1||_{L^1}||f_2||_{L^\infty}$$

because we have the same estimate for each $\tilde{F}_k$. This proves (4.8). Next, we prove (4.9).

We proceed as before, but instead of the change of variables, we do a simple size estimate.

$$\begin{aligned}
&\int_F \frac{r^3\chi_{|\rho(y)|\lesssim r}\chi_{|y_2|<\epsilon}|f_1(y^1,\xi^1)||f_2(y^2,\xi^2)|}{(1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^7}dyd\xi d\tau \\
&\quad\lesssim ||f_1||_{L^\infty}||f_2||_{L^1}\operatorname*{ess\,sup}_{y^2,\xi^2}\int \frac{\chi_{\tilde{F}}(y^1,\tau)\chi_{|y_2|\le\epsilon}\chi_{|\rho(y)|\lesssim r_2}dy^1d\tau}{(1+r_2|\lambda\nabla_2\Phi(y)+2\pi\xi^2+\tau\nabla_2\rho(y)|)^4} \\
&\quad\le ||f_1||_{L^\infty}||f_2||_{L^1}\operatorname*{ess\,sup}_{y^2,\xi^2}\int \chi_{\tilde{F}}(y^1,\tau)\chi_{|y_2|\le\epsilon}\chi_{|\rho(y)|\lesssim r_2}dy^1d\tau \\
&\quad\lesssim ||f_1||_{L^\infty}||f_2||_{L^1}\operatorname*{ess\,sup}_{y^2,\xi^2} r_2^{-1}\int \chi_{\tilde{\tilde{F}}}(y_1,y_2,y_3)\chi_{|y_2|\le\epsilon}\chi_{|\rho(y)|\lesssim r_2}dy_1dy_2dy_3 \\
&\quad\lesssim ||f_1||_{L^\infty}||f_2||_{L^1}\operatorname*{ess\,sup}_{y^2,\xi^2} r_2^{-1}\cdot\epsilon\cdot r_2 \\
&\quad\lesssim \epsilon||f_1||_{L^\infty}||f_2||_{L^1}
\end{aligned}$$

Here, the jump from the first line to the second line combines all of the same initial reasoning in the analogous part of the proof of Theorem 1. The fourth line follows from integrating out τ, because $\tau\in\tilde{F}\implies|\tau-\tau_0|\lesssim r^{-1}\approx r_2^{-1}$. The fifth line follows from making the change of variables $(z_1,z_2,z_3)=(\rho(y),y_2,y_3)$, which has Jacobian determinant 1, and then integrating with respect to the z variables, keeping in mind that $(y_1,y_2,y_3)\in\tilde{\tilde{F}}\implies|y_1|,|y_2|,|y_3|<b_1$.

The proof of the last equation (4.10) is completely analogous to the proof of (4.9), except that in this case, we are integrating over the $y^2=(y_4,y_5,y_6)$ variables, so doing the size estimates in the final steps no longer picks up a factor of ϵ. $\square$

Now we proceed with the proof of Theorem 2. As noted earlier, we already have

$$|\tilde{I}_\lambda(f_1,f_2)|\lesssim|\lambda|^{-1}||f_1||_{L^2}||f_2||_{L^2}+\int_F\frac{r^3\chi_{|\rho(y)|\lesssim r}|f_1(y^1,\xi^1)||f_2(y^2,\xi^2)|}{(1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^7}dyd\xi d\tau \tag{4.11}$$

For the dyadic decomposition, we write

$$\int_F \frac{r^3\chi_{|\rho(y)|\lesssim r}|f_1(y^1\xi^1)||f_2(y^2,\xi^2)|}{1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^7}dyd\xi d\tau$$
$$= \sum_{j=-\infty}^{\infty}\int_{F_j} \frac{r^3\chi_{|\rho(y)|\lesssim r}|f_1(y^1\xi^1)||f_2(y^2,\xi^2)|}{1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^7}dyd\xi d\tau$$

where

$$\begin{aligned} F_j &= \{(y,\xi,\tau)\in F \,|\, 2^j \le |y_2| \le 2^{j+1}\} \\ &= \{(y,\xi,\tau)\in B_1\times\Xi_2\times\mathbb{R}\,|\, r_1^{-2}\approx r_2^{-2}\approx|\lambda|+|\xi|\approx|\lambda|+|\tau|, \text{ and } 2^j\le|y_2|\le 2^{j+1}\} \end{aligned} \tag{4.12}$$

Now, we apply the four results from Lemma 5, and we get

$$\int_{F_j} \frac{r^3\chi_{|\rho(y)|\lesssim r}|f_1(y^1,\xi^1)||f_2(y^2,\xi^2)|}{(1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^7}dyd\xi d\tau \lesssim 2^{-j}|\lambda|^{-1}||f_1||_{L^\infty}||f_2||_{L^1} \tag{4.13}$$
$$\int_{F_j} \frac{r^3\chi_{|\rho(y)|\lesssim r}|f_1(y^1,\xi^1)||f_2(y^2,\xi^2)|}{(1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^7}dyd\xi d\tau \lesssim 2^{-j}|\lambda|^{-1}||f_1||_{L^1}||f_2||_{L^\infty} \tag{4.14}$$
$$\int_{F_j} \frac{r^3\chi_{|\rho(y)|\lesssim r}|f_1(y^1,\xi^1)||f_2(y^2,\xi^2)|}{(1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^7}dyd\xi d\tau \lesssim 2^{j}||f_1||_{L^\infty}||f_2||_{L^1} \tag{4.15}$$
$$\int_{F_j} \frac{r^3\chi_{|\rho(y)|\lesssim r}|f_1(y^1,\xi^1)||f_2(y^2,\xi^2)|}{(1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^7}dyd\xi d\tau \lesssim ||f_1||_{L^1}||f_2||_{L^\infty} \tag{4.16}$$

(4.13) follows from (4.7) and the fact that $y_2\in F_j \implies |y_2|\ge 2^j$. (4.14) follows from (4.8) and the fact that $y_2\in F_j \implies |y_2|\ge 2^j$. (4.15) follows from (4.9) and the fact that $y_2\in F_j \implies |y_2|\le 2^{j+1}$, and gives us the upper bound $2^{j+1}||f_1||_{L^\infty}||f_2||_{L^1}\lesssim 2^j||f_1||_{L^\infty}||f_2||_{L^1}$. (4.16) follows from (4.10) and the fact that $y_2\in F_j \implies |y_2|\le 2^{j+1}$. Combining these, we have

$$\int_{F_j} \frac{r^3\chi_{|\rho(y)|\lesssim r}|f_1(y^1,\xi^1)||f_2(y^2,\xi^2)|}{(1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^7}dyd\xi d\tau \lesssim \min\{2^{-j}|\lambda|^{-1},2^j\}||f_1||_{L^\infty}||f_2||_{L^1}$$
$$\int_{F_j} \frac{r^3\chi_{|\rho(y)|\lesssim r}|f_1(y^1,\xi^1)||f_2(y^2,\xi^2)|}{(1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^7}dyd\xi d\tau \lesssim \min\{2^{-j}|\lambda|^{-1},1\}||f_1||_{L^1}||f_2||_{L^\infty}$$

And then interpolating gives us

$$\int_{F_j} \frac{r^3 \chi_{|\rho(y)| \lesssim r} |f_1(y^1, \xi^1)||f_2(y^2, \xi^2)|}{(1 + r|\lambda \nabla \Phi(y) + 2\pi\xi + \tau \nabla \rho(y)|)^7} dy d\xi d\tau$$
$$\lesssim \min\{2^{-j/2}|\lambda|^{-1/2}, 2^{j/2}\} \min\{2^{-j/2}|\lambda|^{-1/2}, 1\} ||f_1||_{L^2} ||f_2||_{L^2}$$

There are three possible ways of doing the multiplication above. We can have

$$\min\{2^{-j/2}|\lambda|^{-1/2}, 2^{j/2}\} \min\{2^{-j/2}|\lambda|^{-1/2}, 1\} \leq (2^{-j/2}|\lambda|^{-1/2}) \cdot (2^{-j/2}|\lambda|^{-1/2}) = 2^{-j}|\lambda|^{-1}$$

when j is large, i.e., for the upper tail of the sum $j > J_1$ (we will pick J_1 later). We can have

$$\min\{2^{-j/2}|\lambda|^{-1/2}, 2^{j/2}\} \min\{2^{-j/2}|\lambda|^{-1/2}, 1\} \leq (2^{j/2}) \cdot (1) = 2^{j/2}$$

when j is small, i.e., for the lower tail of the sum $j < J_2$ (we will pick J_2 later). Lastly, we can have

$$\min\{2^{-j/2}|\lambda|^{-1/2}, 2^{j/2}\} \min\{2^{-j/2}|\lambda|^{-1/2}, 1\} \leq (2^{j/2}) \cdot (2^{-j/2}|\lambda|^{-1/2}) = |\lambda|^{-1/2}$$

for summing finitely many terms, e.g., $j = J_2$ to $j = J_1$ if $J_1 > J_2$. In theory, we could also use the inequality

$$\min\{2^{-j/2}|\lambda|^{-1/2}, 2^{j/2}\} \min\{2^{-j/2}|\lambda|^{-1/2}, 1\} \leq (2^{-j/2}|\lambda|^{-1/2}) \cdot (1) = 2^{-j/2}|\lambda|^{-1/2}$$

but this is never an optimal choice: if $j \geq 0$ then $2^{-j/2}|\lambda|^{-1/2} \leq 1$, (because also $|\lambda| \geq 1$ by assumption), and so $2^{-j/2}|\lambda|^{-1/2} \geq (2^{-j/2}|\lambda|^{-1/2} \leq 1)^2 = 2^{-j}|\lambda|^{-1}$. On the other hand, if $j < 0$, then $2^{-j/2} > 1$ and thus $2^{-j/2}|\lambda|^{-1/2} > |\lambda|^{-1/2}$.

To complete the proof, we use these three estimates and sum over j:

$$
\begin{aligned}
&\int_F \frac{r^3\chi_{|\rho(y)|\lesssim r}|f_1(y^1\xi^1)||f_2(y^2,\xi^2)|}{1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^7}dyd\xi d\tau \\
&\quad = \sum_{j=-\infty}^{\infty}\int_{F_j}\frac{r^3\chi_{|\rho(y)|\lesssim r}|f_1(y^1,\xi^1)||f_2(y^2,\xi^2)|}{(1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^7}dyd\xi d\tau \\
&\quad \lesssim \sum_{j<J_2}2^{j/2}||f_1||_{L^2}||f_2||_{L^2}+\sum_{j=J_2}^{J_1}|\lambda|^{-1/2}||f_1||_{L^2}||f_2||_{L^2}+\sum_{j>J_1}2^{-j}|\lambda|^{-1}||f_1||_{L^2}||f_2||_{L^2} \\
&\quad = ||f_1||_{L^2}||f_2||_{L^2}\left(\sum_{j<J_2}2^{j/2}+\sum_{j=J_2}^{J_1}|\lambda|^{-1/2}+\sum_{j>J_1}2^{-j}|\lambda|^{-1}\right) \\
&\quad \lesssim ||f_1||_{L^2}||f_2||_{L^2}\left(2^{J_2/2}+(J_1-J_2)|\lambda|^{-1/2}+2^{-J_1}|\lambda|^{-1}\right)
\end{aligned}
$$

The middle term in the last line can never be smaller than $|\lambda|^{-1/2}$ (unless we pick $J_1=J_2$, but one can check that doing so would lead to an overall bound of $|\lambda|^{-1/3}$), so we pick J_1, J_2 so that

$$2^{J_2/2}=|\lambda|^{-1/2} \implies 2^{J_2}=|\lambda|^{-1} \implies J_2=-\log_2|\lambda|$$

and

$$2^{-J_1}|\lambda|^{-1}=|\lambda|^{-1/2} \implies 2^{-J_1}=|\lambda|^{1/2} \implies J_1=-\tfrac{1}{2}\log_2|\lambda|$$

which gives us

$$
\begin{aligned}
&\int_F \frac{r^3\chi_{|\rho(y)|\lesssim r}|f_1(y^1,\xi^1)||f_2(y^2,\xi^2)|}{(1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^7}dyd\xi d\tau \\
&\quad \lesssim ||f_1||_{L^2}||f_2||_{L^2}\left(2^{J_2/2}+(J_1-J_2)|\lambda|^{-1/2}+2^{-J_1}|\lambda|^{-1}\right) \\
&\quad \lesssim ||f_1||_{L^2}||f_2||_{L^2}\left(|\lambda|^{-1/2}+(-\tfrac{1}{2}\log_2|\lambda|+\log_2|\lambda|)|\lambda|^{-1/2}+|\lambda|^{-1/2}\right) \\
&\quad \lesssim |\lambda|^{-1/2}\log|\lambda|\,||f_1||_{L^2}||f_2||_{L^2}
\end{aligned}
$$

and this combined with (4.11) completes the proof of Theorem 2.

4.2.3. Operator 2: A Simpler Argument

In this final section, we prove Theorem 3. As a reminder, here we are dealing with the operator

$$\tilde{\tilde{I}}_\lambda(f_1, f_2) = \int_M e^{i\lambda(x_2x_4 + x_2x_3x_5 + \frac{1}{2}x_3x_5^2)} f_1(x^1) f_2(x^2) a(x) d\sigma(x) \tag{4.17}$$

where $M = \{x \in B_1 \mid \rho(x) = 0\}$ for

$$\rho(x) = \tfrac{1}{2}x_2^2x_5 + \tfrac{1}{2}x_2x_5^2 - x_3x_4 - \tfrac{1}{2}x_3^2x_5 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

and in this case, the determinant in (1.7) is

$$\begin{aligned}
&\det\begin{bmatrix} \partial_{x_1}\rho & \partial^2_{x_1,x_4}(\omega_1\Phi + \omega_2\rho) & \partial^2_{x_1,x_5}(\omega_1\Phi + \omega_2\rho) & \partial^2_{x_1,x_6}(\omega_1\Phi + \omega_2\rho) \\ \partial_{x_2}\rho & \partial^2_{x_2,x_4}(\omega_1\Phi + \omega_2\rho) & \partial^2_{x_2,x_5}(\omega_1\Phi + \omega_2\rho) & \partial^2_{x_2,x_6}(\omega_1\Phi + \omega_2\rho) \\ \partial_{x_3}\rho & \partial^2_{x_3,x_4}(\omega_1\Phi + \omega_2\rho) & \partial^2_{x_3,x_5}(\omega_1\Phi + \omega_2\rho) & \partial^2_{x_3,x_6}(\omega_1\Phi + \omega_2\rho) \\ 0 & \partial_{x_4}\rho & \partial_{x_5}\rho & \partial_{x_6}\rho \end{bmatrix} \\
&= \det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 + x_2x_5 + \frac{1}{2}x_5^2 & \omega_1 & \omega_1x_3 + \omega_2(x_2 + x_5) & 0 \\ 1 - x_4 - x_3x_5 & -\omega_2 & \omega_1(x_2 + x_5) - \omega_2x_3 & 0 \\ 0 & 1 - x_3 & 1 + \frac{1}{2}x_2^2 + x_2x_5 - \frac{1}{2}x_3^2 & 1 \end{bmatrix} \\
&= \omega_1(\omega_1(x_2 + x_5) - \omega_2x_3) - (-\omega_2)(\omega_1x_3 + \omega_2(x_2 + x_5)) \\
&= (x_2 + x_5)(\omega_1^2 + \omega_2^2)
\end{aligned}$$

Note that we get the third line by first expanding the determinant along the first row, then expanding the resulting determinant along the last column, which gives us that our original determinant is equal to the determinant of the center 2×2 block.

The proof of Theorem 3 is essentially a simpler version of the proof of Theorem 2, as in this case we benefit from the symmetry in the determinant in (1.7): for the operator in Theorem 2, the determinant in (1.7) is $x_2(\omega_1^2 + \omega_2^2)$, but for the operator in Theorem 3, the determinant is $(x_2 + x_5)(\omega_1^2 + \omega_2^2)$. As it turns out, we no longer need to do a dyadic decomposition, and just need to do a standard optimization argument to choose where to split the integral around the determinant's zero.

Before we proceed with the proof of Theorem 3, we note that the maximum possible decay for this operator is also $|\lambda|^{-3/4}$, the same as for the previous operator. Again, this operator may not necessarily achieve λ-decay of $|\lambda|^{-3/4}$; all that this argument shows is that it is impossible to prove that this operator has decay faster than $|\lambda|^{-3/4}$, so the actual decay of the operator must be $|\lambda|^{-3/4}$ or worse.

Proving that the maximum possible decay is $|\lambda|^{-3/4}$ follows from testing the operator on the functions

$$\begin{aligned} f_1(x_1, x_2, x_3) &= \chi_{|\cdot|<c_1}(x_1)\chi_{|\cdot|<c_2|\lambda|^{-1/2}}(x_2)\chi_{|\cdot|<c_3}(x_3) \\ f_2(x_4, x_5, x_6) &= \chi_{|\cdot|<c_4|\lambda|^{-1/2}}(x_4)\chi_{|\cdot|<c_5|\lambda|^{-1/2}}(x_5)\chi_{|\cdot|<c_6}(x_6) \end{aligned}$$

for all of the constants c_i chosen to be appropriately small. The argument showing that $\tilde{\tilde{I}}_\lambda(f_1, f_2) \gtrsim |\lambda|^{-3/4}||f_1||_{L^2}||f_2||_{L^2}$ is completely analogous to the argument in Section 4.2.1, so we omit it here.

To start the proof of Theorem 3, we note, as for the previous operator, that because our specific operator satisfies all other hypotheses of Theorem 1 besides the determinant condition (1.7), we can apply all of the results in Section 3.3, and so we immediately get

$$\begin{aligned} |\tilde{\tilde{I}}_\lambda(f_1, f_2)| &= \left| \int_{\mathbb{R}^6\times\mathbb{R}^6} \tilde{\tilde{\mathcal{I}}}(y,\xi) f_1(y^1,\xi^1) f_2(y^2,\xi^2)\, dy d\xi \right| \\ &\lesssim \left| \int_{(\mathbb{R}^6\times\mathbb{R}^6)\setminus E} \tilde{\tilde{\mathcal{I}}}(y,\xi) f_1(y^1,\xi^1) f_2(y^2,\xi^2)\, dy d\xi \right| + \left| \int_E \tilde{\tilde{\mathcal{I}}}(y,\xi) f_1(y^1,\xi^1) f_2(y^2,\xi^2)\, dy d\xi \right| \\ &\lesssim |\lambda|^{-1}||f_1||_{L^2}||f_2||_{L^2} + \int_F \frac{r^3 \chi_{|\rho(y)|\lesssim r} |f_1(y^1,\xi^1)||f_2(y^2,\xi^2)|}{(1 + r|\lambda\nabla\Phi(y) + 2\pi\xi + \tau\nabla\rho(y)|)^7} dy d\xi d\tau \end{aligned}$$

where E, F are the sets defined in (4.2) and (4.5), respectively[3]. Next, we prove a lemma that gives estimates for the integral above on the regions near $y_2 + y_5 = 0$ and away from $y_2 + y_5 = 0$.

Lemma 6. *Let* $\epsilon > 0$*, let* ρ, Φ *be given as in* (1.11)*, and let* F *be defined by* (4.5)*. Then we have the following estimates*

$$\int_F \frac{r^3 \chi_{|\rho(y)| \lesssim r} \chi_{|y_2+y_5| \geq \epsilon} |f_1(y^1, \xi^1)||f_2(y^2, \xi^2)|}{(1 + r|\lambda \nabla \Phi(y) + 2\pi\xi + \tau \nabla \rho(y)|)^7} dy d\xi d\tau \lesssim \epsilon^{-1} |\lambda|^{-1} ||f_1||_{L^2} ||f_2||_{L^2} \tag{4.18}$$

$$\int_F \frac{r^3 \chi_{|\rho(y)| \lesssim r} \chi_{|y_2+y_5| \leq \epsilon} |f_1(y^1, \xi^1)||f_2(y^2, \xi^2)|}{(1 + r|\lambda \nabla \Phi(y) + 2\pi\xi + \tau \nabla \rho(y)|)^7} dy d\xi d\tau \lesssim \epsilon ||f_1||_{L^2} ||f_2||_{L^2} \tag{4.19}$$

Proof. By completely analogous arguments to the ones in the proof of (4.7) and the proof of (4.8), we have

$$\int_F \frac{r^3 \chi_{|\rho(y)| \lesssim r} \chi_{|y_2+y_5| \geq \epsilon} |f_1(y^1, \xi^1)||f_2(y^2, \xi^2)|}{(1 + r|\lambda \nabla \Phi(y) + 2\pi\xi + \tau \nabla \rho(y)|)^7} dy d\xi d\tau \lesssim \epsilon^{-1} |\lambda|^{-1} ||f_1||_{L^\infty} ||f_2||_{L^1}$$

$$\int_F \frac{r^3 \chi_{|\rho(y)| \lesssim r} \chi_{|y_2+y_5| \geq \epsilon} |f_1(y^1, \xi^1)||f_2(y^2, \xi^2)|}{(1 + r|\lambda \nabla \Phi(y) + 2\pi\xi + \tau \nabla \rho(y)|)^7} dy d\xi d\tau \lesssim \epsilon^{-1} |\lambda|^{-1} ||f_1||_{L^1} ||f_2||_{L^\infty}$$

and interpolation gives us (4.18). To prove (4.19), we start out the same way as in the proof of (4.9): in the first step, we apply Hölder's inequality, using that $r \approx r_1 \approx r_2$, and integrate

[3] As explained earlier, here we are assuming that the set F originally defined in (4.5) has the additional condition $|\tau - \tau_0| \leq c''' r^{-1}$.

out the ξ^1 variables. We then use a simple size estimate.

$$\begin{aligned}
&\int_F \frac{r^3 \chi_{|\rho(y)|\lesssim r}\chi_{|y_2+y_5|\leq\epsilon}|f_1(y^1,\xi^1)||f_2(y^2,\xi^2)|}{(1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^7}dyd\xi d\tau \\
&\quad\lesssim ||f_1||_{L^\infty}||f_2||_{L^1} \operatorname*{ess\,sup}_{y^2,\xi^2} \int \frac{\chi_{\tilde{F}}(y^1,\tau)\chi_{|y_2+y_5|\leq\epsilon}\chi_{|\rho(y)|\lesssim r_2}dy^1 d\tau}{(1+r_2|\lambda\nabla_2\Phi(y)+2\pi\xi^2+\tau\nabla_2\rho(y)|)^4} \\
&\quad\leq ||f_1||_{L^\infty}||f_2||_{L^1} \operatorname*{ess\,sup}_{y^2,\xi^2} \int \chi_{\tilde{F}}(y^1,\tau)\chi_{|y_2+y_5|\leq\epsilon}\chi_{|\rho(y)|\lesssim r_2}dy^1 d\tau \\
&\quad\lesssim ||f_1||_{L^\infty}||f_2||_{L^1} \operatorname*{ess\,sup}_{y^2,\xi^2} r_2^{-1} \int \chi_{\tilde{\tilde{F}}}(y_1,y_2,y_3)\chi_{|y_2+y_5|\leq\epsilon}\chi_{|\rho(y)|\lesssim r_2}dy_1dy_2dy_3 \\
&\quad\lesssim ||f_1||_{L^\infty}||f_2||_{L^1} \operatorname*{ess\,sup}_{y^2,\xi^2} r_2^{-1}\cdot\epsilon\cdot r_2 \\
&\quad\lesssim \epsilon||f_1||_{L^\infty}||f_2||_{L^1}
\end{aligned}$$

To get from the fourth line to the fifth line, we're using the change of variables $(z_1,z_2,z_3) = (\rho(y), y_2+y_5, y_3)$, which has Jacobian determinant 1, and we're also using the fact that $\tilde{\tilde{F}} \subset B_1$.

But now, unlike in the previous example, the $L^1 - L^\infty$ estimate also has a factor of ϵ appearing:

$$\begin{aligned}
&\int_F \frac{r^3 \chi_{|\rho(y)|\lesssim r}\chi_{|y_2+y_5|\leq\epsilon}|f_1(y^1,\xi^1)||f_2(y^2,\xi^2)|}{(1+r|\lambda\nabla\Phi(y)+2\pi\xi+\tau\nabla\rho(y)|)^7}dyd\xi d\tau \\
&\quad\lesssim ||f_1||_{L^1}||f_2||_{L^\infty} \operatorname*{ess\,sup}_{y^1,\xi^1} \int \frac{\chi_{\tilde{F}}(y^2,\tau)\chi_{|y_2+y_5|\leq\epsilon}\chi_{|\rho(y)|\lesssim r_1}dy^2 d\tau}{(1+r_1|\lambda\nabla_1\Phi(y)+2\pi\xi^1+\tau\nabla_1\rho(y)|)^4} \\
&\quad\leq ||f_1||_{L^1}||f_2||_{L^\infty} \operatorname*{ess\,sup}_{y^1,\xi^1} \int \chi_{\tilde{F}}(y^2,\tau)\chi_{|y_2+y_5|\leq\epsilon}\chi_{|\rho(y)|\lesssim r_1}dy^2 d\tau \\
&\quad\lesssim ||f_1||_{L^1}||f_2||_{L^\infty} \operatorname*{ess\,sup}_{y^1,\xi^1} r_1^{-1} \int \chi_{\tilde{\tilde{F}}}(y_4,y_5,y_6)\chi_{|y_2+y_5|\leq\epsilon}\chi_{|\rho(y)|\lesssim r_1}dy_4dy_5dy_6 \\
&\quad\lesssim ||f_1||_{L^1}||f_2||_{L^\infty} \operatorname*{ess\,sup}_{y^1,\xi^1} r_1^{-1}\cdot\epsilon\cdot r_1 \\
&\quad\lesssim \epsilon||f_1||_{L^1}||f_2||_{L^\infty}
\end{aligned}$$

And so again, interpolation gives us our desired result (4.19) $\square$

With Lemma 6 proved, there is not much left to do to prove Theorem 3. To find the optimal ϵ to use to split up the main contribution, we just set the two bounds in Lemma 6 equal: $\epsilon^{-1}|\lambda|^{-1} = \epsilon$, which implies that we should choose $\epsilon = |\lambda|^{-1/2}$. If we apply Lemma 6 with this choice of ϵ, this finishes the proof of Theorem 3.

We note two things when comparing the result for this operator to the result for the operator in Theorem 2. First, if we tried to use this argument on the previous operator $\tilde{I}_\lambda$ and we immediately interpolated the results of Lemma 5, we would get the bounds $\epsilon^{-1}|\lambda|^{-1}||f_1||_{L^2}||f_2||_{L^2}$ and $\epsilon^{1/2}||f_1||_{L^2}||f_2||_{L^2}$. Moving forward with these estimates would mean that the optimal choice of ϵ would be such that $\epsilon^{-1}|\lambda|^{-1} = \epsilon^{1/2}$, i.e., $\epsilon = |\lambda|^{-2/3}$, and we would get an overall bound for the operator of $|\lambda|^{-1/3}||f_1||_{L^2}||f_2||_{L^2}$.

Second, we note that if we tried to use the previous dyadic argument on this operator, it wouldn't improve our result. To see why this is, note the $L^\infty - L^1$ and $L^1 - L^\infty$ estimates we get in the proof of Lemma 6. If we take these estimates and proceed as in the proof for the previous operator in Section 4.2.2, dividing the set F into sets $F_j := \{(y, \xi, \tau) \in F \,|\, 2^j \leq |y_2 + y_5| \leq 2^{j+1}\}$, we would get

$$\int_{F_j} \frac{r^3 \chi_{|\rho(y)| \lesssim r} |f_1(y^1, \xi^1)||f_2(y^2, \xi^2)|}{(1 + r|\lambda \nabla \Phi(y) + 2\pi\xi + \tau \nabla \rho(y)|)^7} dy d\xi d\tau$$
$$\lesssim \min\{2^{-j/2}|\lambda|^{-1/2}, 2^{j/2}\} \min\{2^{-j/2}|\lambda|^{-1/2}, 2^{j/2}\} ||f_1||_{L^2}||f_2||_{L^2}$$

There are three possible estimates we can pull out from this. First, we can do

$$\min\{2^{-j/2}|\lambda|^{-1/2}, 2^{j/2}\} \min\{2^{-j/2}|\lambda|^{-1/2}, 2^{j/2}\} \leq 2^{-j/2}|\lambda|^{-1/2} \cdot 2^{-j/2}|\lambda|^{-1/2} = 2^{-j}|\lambda|^{-1}$$

which is the best estimate to use when j is large, i.e. $j > J_1$ for some J_1. Next, we can do

$$\min\{2^{-j/2}|\lambda|^{-1/2}, 2^{j/2}\} \min\{2^{-j/2}|\lambda|^{-1/2}, 2^{j/2}\} \leq 2^{j/2} \cdot 2^{j/2} = 2^j$$

which is the best estimate to use when j is small, i.e., $j < J_2$ for some J_2. Finally, the only other option is

$$\min\{2^{-j/2}|\lambda|^{-1/2}, 2^{j/2}\}\min\{2^{-j/2}|\lambda|^{-1/2}, 2^{j/2}\} \leq 2^{-j/2}|\lambda|^{-1/2} \cdot 2^{j/2} = |\lambda|^{-1/2}$$

which can be used only for a finite number of terms in the sum. However, if this last option is used, it means that the best possible decay we can get through the dyadic approach is $|\lambda|^{-1/2}$. If this last estimate is not used, then we would only have two sums, and we would be using the first estimate above for the sum $\sum_{j>J}$ and the second estimate above for the sum $\sum_{j\leq J}$; one can check that proceeding with that setup will also give an exponent of $|\lambda|^{-1/2}$. In either case, the dyadic approach cannot improve on the result that we have already proved using the simpler method.

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