Name: ___________________________ PennID: ______________________

Math 240 Final Exam
December 14, 2017

Encircle your Professor, TA and recitation time below:
Professor       Simmons  Wibmer  Zhu
Nibras Islam    We 8-9    We 9-10  Fr 8-9  Fr 9-10
Patrick Emedom-Nnamdi  We 8-9    We 9-10  Fr 8-9  Fr 9-10
Brian Luo        Mo 8-9    Mo 9-10  We 8-9  We 9-10
Benjamin Foster  Tu 8:30-9:30  Tu 9:30-10:30  Th 8:30-9:30  Th 9:30-10:30

Instructions
Turn off and put away your cell phone. Please write your Name and PennID on the top of this page and encircle your professor, TA and recitation time. Please sign and date the pledge below. No calculators or any other electronic devices are allowed during this exam. You may use a cheat sheet but no other material. Read each question carefully, and answer each question completely. Show all of your work. Write all your solutions clearly and legibly; no credit will be given for illegible solutions. Please try to be as organized as possible and clearly indicate your final solution (in the space provided). If any question is not clear, ask for clarification.

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My signature below certifies that I have complied with the University of Pennsylvania’s Code of Academic Integrity in completing this examination.

_________________________________________________________  ____________________________
Signature                                                      Date
Problem 1 (10 pts): Decide whether the following statements are true or false. You do NOT have to justify your answer. (Please indicate your answer by encircling “true” or “false”.)

1. The determinant of a matrix whose entries are all positive must be positive.

   True  False

2. If \( \{v_1, \ldots, v_n\} \) is a basis for \( \mathbb{R}^n \) and \( A \) is an invertible \( n \times n \)-matrix, then \( \{Av_1, \ldots, Av_n\} \) must also be a basis for \( \mathbb{R}^n \).

   True  False

3. There is a unique linear transformation \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) that takes both \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) to \( \begin{bmatrix} 1 \\ -2 \end{bmatrix} \) and \( \begin{bmatrix} 2 \\ 3 \end{bmatrix} \).

   True  False

4. If \( L \) is a linear differential operator and \( y_1, y_2 \) are solutions to \( Ly = F(x) \), then \( y_1 - y_2 \) is a solution to \( Ly = 0 \).

   True  False

5. If \( A^2 \) is diagonalizable, then also \( A \) must be diagonalizable.

   True  False
Problem 2 (10 pts): True or False? You need to justify your answer to receive credit!

1. If $A$ and $B$ share a common eigenvector $v$, then $AB - BA$ is not invertible.
   Circle one: True       False

   Justification:

2. A polynomial differential operator of order 5, annihilates every polynomial of degree less than or equal to 4.
   Circle one: True       False

   Justification:
Problem 3 (10 pts): Compute the reduced row-echelon form, the rank and a basis for the nullspace for the following matrix $A$:

$$
A = \begin{bmatrix}
0 & 1 & 2 & 3 & 4 \\
0 & 1 & 2 & 4 & 6 \\
0 & 0 & 0 & 1 & 2
\end{bmatrix}
$$

RREF($A$) = ______________________ rank($A$) = ___ basis: __________________________
Problem 4 (10 pts): If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and $\det(A) = 5$, what is

$$\begin{vmatrix} 2a & d + 3a & g \\ 2b & e + 3b & h \\ 2c & f + 3c & i \end{vmatrix}$$

$\det(A) =$________
Problem 5 (10 pts): Let $S$ be the subset of $M_{2 \times 2}(\mathbb{R})$ consisting of all $2 \times 2$-matrices whose entries add up to zero. Show that $S$ is a subspace of $M_{2 \times 2}(\mathbb{R})$. 
Problem 6 (10 pts):

1. Let $T : \mathbb{R}^3 \rightarrow P_3(\mathbb{R})$ be the linear transformation defined by

$$T \left( \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = 2a - (a + b - c)x + (2c - a)x^3.$$

Compute $[T]^C_B$ relative to the standard bases $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $C = \{1, x, x^2, x^3\}$.

$$[T]^C_B = \ldots$$

2. Compute $[T(v)]_C$ for $v = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}.$

$$[T(v)]_C = \ldots$$
Problem 7 (10 pts): Decide whether or not the matrix $A$ is diagonalizable. If so, find an invertible matrix $S$ and a diagonal matrix $D$ such that $S^{-1}AS = D$.

$$A = \begin{bmatrix} -4 & -3 & 3 \\ 6 & 5 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

(Hint: $\lambda = 2$ is an eigenvalue of $A$.)

Diagonalizable? Yes  No

If yes, $S =$ ____________________ $D =$ ____________________
Problem 8 (10 pts): Suppose $A$ is a $4 \times 4$ defective matrix with characteristic polynomial

$$p(\lambda) = (\lambda + 2)^2(\lambda - 1)(\lambda + 1).$$

1. Find the algebraic multiplicity and geometric multiplicity of $-2$. Briefly explain your reasoning.

   algebraic multiplicity:________   geometric multiplicity:________

2. What is the rank of $A + 2I$? Briefly explain your reasoning.

   rank($A + 2I$) =__________

3. Find the rank of $A$. Briefly explain your reasoning.

   rank($A$) =__________
Problem 9 (10 pts): Solve the following initial value problem:

\[ y'' + y' - 6y = 10e^{2x}, \quad y(0) = 0, \quad y'(0) = 7. \]
Problem 10 (10 pts): Determine the general solution of $x' = Ax$ for

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$ 

(Hint: $\det(A - \lambda I) = -(\lambda - 4)^3.$)

$x(t) =$
Problem 11 (10 pts): Find the general solution to the following linear differential equation, given that $x^2$ is a solution.

$$x^2 y'' - 4xy' + 6y = 0, \ x > 0.$$

$$y(t) = \boxed{\text{solution}}$$
**Problem 12 (10 pts):** Characterize the equilibrium point of

\[ x'(t) = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix} x(t) \]

as:

- stable, unstable, neither stable or unstable and
- node, saddle point, proper node, degenerate node, center, spiral point.

You need to justify your answer.

circle one: stable unstable neither

circle one: node saddle point proper node degenerate node center spiral point
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Scratch paper