Name: ______________________________ PennID: ______________

Math 240
Final Exam
December 19th, 2018

Instructions:
Turn off and put away your cell phone. Please write your Name (legibly!) and PennID on the top of this page. Please sign and date the pledge below to comply with the Code of Academic Integrity. No calculators or any other electronic devices are allowed during this exam. You may NOT consult any material during this exam, as it is self-contained. Read each question carefully, and answer each question completely.
SHOW all of YOUR WORK Write all your solutions clearly and legibly.
Please try to be as organized as possible. If any question is not clear, ask for clarification.
GOOD LUCK!

#	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
Total	120/120	

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

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Signature Date

Problem 1 (10 pts): Find a basis for Nullspace(A) and a basis for Colspace(A) where

$$A = \begin{bmatrix} 2 & 4 & 1 & 11 & 14 \\ 1 & 2 & 1 & 8 & 10 \\ 4 & 8 & 2 & 22 & 28 \end{bmatrix}.$$

Problem 2 (10 pts): Suppose that $T : \mathbb{R}^4 \to \mathbb{R}^{2018}$ is a linear transformation. Let A denote the matrix associated to T with respect to the standard bases of $\mathbb{R}^4$ and $\mathbb{R}^{2018}$ respectively. Suppose also that the following hold:

- $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for $\mathbb{R}^4$.
- $T(\vec{v}_1) = T(\vec{v}_2)$ and $T(\vec{v}_3) = T(\vec{v}_4)$.
- $T(\vec{v}_1)$ is not a scalar multiple of $T(\vec{v}_3)$.
- $T(\vec{v}_3)$ is not a scalar multiple of $T(\vec{v}_1)$.

What is the rank of A and what is the nullity of A? Explain your reasoning.

Problem 3 (10 pts): Let A equal the following 5×5 matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}.$$

1. Compute the Rank and Nullity of the matrix A.
2. Compute all the eigenvalues of A.
3. Find a diagonal matrix similar to A.

Problem 4 (10pts): Find the general solution of the second order differential equation

$$y'' + 2y' + y = 5e^{-t}.$$

Problem 5 (10pts): Suppose $T : \mathbb{R}^3 \to P_2(\mathbb{R})$ is a linear transformation satisfying the following conditions:

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = x, \qquad T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = 2x^2, \qquad T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = x + x^2.$$

Compute $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$.

Problem 6 (10 pts): Consider the following linear differential system

$$X'(t) = \begin{bmatrix} -3 & 1 & 1 \\ 0 & -3 & 0 \\ -1 & 0 & -1 \end{bmatrix} X(t).$$

(a) Find its general solution.

(b) Must every solution satisfy

$$\lim_{t \to \infty} X(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} ?$$

Justify your answer.

Problem 7 (10 pts): Consider the following two bases for $P_2(\mathbb{R})$:
(Recall $P_2(\mathbb{R})$ is the set of polynomials over the real numbers with degree at most 2.)

$$B = \left\{1 + x + x^2,\ 1 + x,\ 1 + x^2\right\},$$
$$C = \left\{2 + 2x + 3x^2,\ -2 - x - x^2,\ -1 + x + 2x^2\right\}$$

Find the change of basis matrix $P_{B \leftarrow C}$.

Problem 8 (10 pts): Find the general solution of:

$$t^2y'' + 2ty' - 2y = t^2, \quad t > 0,$$

by first finding a solution of the form $y_1(x) = t^r$ to the homogeneous problem:

$$t^2y'' + 2ty' - 2y = 0,$$

and then using reduction of order.

Problem 9 (10 pts): Given that

$$\begin{bmatrix} 16 & -21 \\ 10 & -13 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 16 & -21 \\ 10 & -13 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix},$$

compute all the entries of $\begin{bmatrix} 16 & -21 \\ 10 & -13 \end{bmatrix}^{2018}$.

Problem 10 (10 pts): Solve the following system of differential equations:

$$x' - 4x + y'' = t^2$$
$$x' + x + y' = 0.$$

Hint: Eliminate one of the variables by expressing y in terms of x or x in terms of y.

Problem 11 (10 pts): The set $\mathcal{M}_{2\times 2}(\mathbb{R})$ of 2×2 matrices with real entries, equipped with matrix addition and scalar multiplication, is a vector space. The *trace* $T(A)$ of a 2×2 matrix A is defined to be the sum of its diagonal entries. In other words, the trace of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $a + d$.

1. Show that T is a linear transformation from $\mathcal{M}_{2\times 2}(\mathbb{R})$ to $\mathbb{R}^1$.
2. Compute the matrix $[T]_B^C$ associated to the linear transformation T. where

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}, \qquad C = \{1\}.$$

3. Compute the rank and nullity of the matrix $[T]_B^C$.

Problem 12 (10 pts): This problem consists of six true/false and four multiple choice questions. Do them all. You do not need to show your work.

True/False

1. (True/False) For every $n \times n$ invertible matrix A, the set of eigenvalues of A and A^{-1} are the same.

2. (True/False) For every $n \times n$ invertible matrix A, the set of eigenvectors of A and A^{-1} are the same.

3. (True/False) If we consider the two differential operators $P_1(D) = D^7 + 5D + 3$ and $P_2(D) = D^6 - 2D^4 + 1$, then $P_1P_2 = P_2P_1$.

4. (True/False) If A is an $n \times n$ defective matrix with real entries, then the linear differential system $A\vec{x}(t) = \dfrac{d\vec{x}(t)}{dt}$ cannot have n linearly independent solutions.

5. (True/False) Suppose a spring-mass system with spring constant k, mass m, damping constant c, and no external forces experiences critical damping. Suppose now the mass is increased while k and c remain fixed. Then the altered system experiences overdamping.

6. (True/False) If A and B are real $n \times n$ matrices with the same characteristic polynomial, then the solution sets to the vector differential equations $A\vec{x} = \dfrac{d\vec{x}(t)}{dt}$ and $B\vec{x} = \dfrac{d\vec{x}(t)}{dt}$ are the same.

Multiple choice questions

7. Consider the two following statements:

   Statement I: There exists a 2×2 matrix A such that Colspace(A) equals Nullspace(A).

   Statement II: There exists a 3×3 matrix A such that Colspace(A) equals Nullspace(A).

   Only one of the following options is correct. Choose that option.

   (a) Only Statement I is true.
   (b) Only Statement II is true.
   (c) Both statements are true.
   (d) Both statements are false.

8. Consider the two following statements:

   Statement I: If B is a 2×2 defective matrix with real entries, each of its eigenvalues must be real.

   Statement II: If B is a 3×3 defective matrix with real entries, each of its eigenvalues must be real.

   Only one of the following options is correct. Choose that option.

(a) Only Statement I is true.

(b) Only Statement II is true.

(c) Both statements are true.

(d) Both statements are false.

9. Consider a spring mass system with spring constant $k > 0$, damping constant $c > 0$ and no external force. Suppose that the mass attached to the spring is given an initial velocity 0.2 m/s at $t = 0$ s. Suppose that the mass attached to the spring passes through equilibrium at $t = 1$ s and $t = 2$ s. Only one of the following options is correct. Choose that option.

   (a) The spring-mass system is critically damped.

   (b) The spring-mass system is under-damped.

   (c) The spring-mass system is over-damped.

10. Let $P(D)$ denote $D^2 - 67D - 37$. Let A denote $\begin{bmatrix} 0 & 1 \\ 37 & 67 \end{bmatrix}$.

    Consider the following statements.

    Statement I: If $y(t)$ satisfies $P(D)y = 0$, then $\vec{X}(t) := \begin{bmatrix} y(t) \\ dy/dt \end{bmatrix}$ satisfies $\frac{d\vec{X}}{dt} = A\vec{X}$.

    Statement II: If $\vec{X}(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ satisfies $\frac{d\vec{X}}{dt} = A\vec{X}$, then $P(D)x_1 = 0$.

    Only one of the following options is correct. Choose that option.

    (a) Only Statement I is true.

    (b) Only Statement II is true.

    (c) Both statements are true.

    (d) Both statements are false.

Scratch paper

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