ANSWERS

Penn UNIVERSITY of PENNSYLVANIA

Math 240 Final Exam

12 - 2 pm, May 2, 2017

No books, paper or any electronic device may be used, other than a hand-written note sheet at most 8.5″ × 11″ in size. Please turn off your cell phones.

This examination consists of twelve (12) long-answer questions, each question is worth 10 points. Please show all your work. Merely displaying some formulas is not sufficient ground for receiving partial credits. Please **box your answers**.

NAME (PRINTED):

PENN ID:

INSTRUCTOR:

TA:

RECITATION TIME:

My signature below certifies that I have complied with the University of Pennsylvania's *code of academic integrity* in completing this examination.

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Your signature

1	2	3	4	5	6	7	8	9	10	11	12	Total

1. A certain 3×3 matrix H has the following property: The matrix equation

$$H\vec{\mathbf{x}} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

has exactly one solution. Based on this information, determine whether the matrix equation

$$H\vec{\mathbf{x}} = \begin{pmatrix} 1 \\ \sqrt{2} \\ \sqrt{3} \end{pmatrix}$$

I. has no solution   II. has a unique solution   III. has infinitely many solutions
IV. can't be determined

Justify your answer completely.

Answer: II

2. Can

$$M = \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$$

be written as a linear combination of the following matrices?

$$A = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix}$$

If so, find the linear combination. If not, then explain why it can't be written as the linear combination of A, B and C.

Answer:

$$M = A + 2B - 3C$$

3. In $\mathbb{R}^4$, let W be the subset of all vectors

$$\vec{\mathbf{v}} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

such that $w - z = y - x$.

(a) (5 points) Show that W is a subspace of $\mathbb{R}^4$.

(b) (5 points) Find a basis for W.

Answer:

(a) Show closed under addition and scalar multiplication

(b)

$$\vec{\mathbf{v}} = \begin{bmatrix} y+z-w \\ y \\ z \\ w \end{bmatrix}$$

so a basis would be

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

4. Consider a 3×3 matrix B satisfying the following properties:

$$\text{Nullspace}(B - I_3) = \text{Span}\left\{\begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix}\right\},$$

$$\text{Nullspace}(B + 2I_3) = \text{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\},$$

$$\text{Nullspace}(B - 5I_3) = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}.$$

Answer the following questions:

(a) (4 points) Find the characteristic polynomial of B.

(b) (3 points) Find a diagonal matrix D and an invertible matrix S so that

$$S^{-1}BS = D.$$

(c) (3 points) Determine the row reduced echelon form of B.

Answer:

(a) $p(\lambda) = (1 - \lambda)(-2 - \lambda)(5 - \lambda)$.

(b)

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \qquad S = \begin{pmatrix} 1 & 0 & 1 \\ 7 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) $RREF(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

5. Suppose A is a 3×3 matrix with eigenvalues $\lambda_1 = -1$, $\lambda_2 = 0$, $\lambda_3 = 1$ and corresponding eigenvectors

$$\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \vec{\mathbf{v}}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \vec{\mathbf{v}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

(a) (4 points) What is the second column of A?

(b) (2 points) What is the diagonal matrix similar to A?

(c) (4 points) Choose all the matrices that are equal to A? (Possibly more than one)

I. A^{-1} II. A^{2017} III. A^{240} IV. A^2 V. None of the above.

You need to show your work completely.

Answer:

(a)

$$\frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

(b)

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) II

6. Solve the initial value problem below for $y(x)$.

$$y''' - 2y'' + y' = 2 - 24e^x + 40e^{5x}$$
$$y(0) = \frac{1}{2}, y'(0) = \frac{5}{2}, y''(0) = -\frac{9}{2}$$

Answer : $y = 11 - 11e^x + 9xe^x + 2x - 12x^2 + \frac{1}{2}e^{5x}$.

7. Consider the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - a\frac{\mathrm{d}y}{\mathrm{d}t} - ay = 0.$$

(a) (5 points) For which values of the real parameter a is the system describing oscillations of a vertical spring-mass system (i.e. harmonic oscillator)? Briefly explain why.

I. $a > 0$ II. $a < 0$ III. $-4 < a < 0$ IV. $a \leqslant -4$ V. $a > 4$

(b) (5 points) Rewrite the differential equation as an equivalent first-order differential system.

Answer:

(a) II

(b)

$$u = y, v = y'$$

$$u' = v$$
$$v' = au + av$$

8. Solve the following initial value problem

$$\frac{\mathrm{d}\,\vec{\mathbf{x}}(t)}{\mathrm{d}\,t} = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \vec{\mathbf{x}}(t), \qquad \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Answer: $e^t \begin{pmatrix} \cos(3t) + \sin(3t) \\ -\sin(3t) + \cos(3t) \end{pmatrix}.$

9. (a) (4 points) Find the eigenvalues of the matrices $\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix}$.

(b) (3 points) Classify the type of equilibrium point for the differential equation

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \vec{x}.$$

I. Stable node II. Unstable node III. Saddle IV. Stable spiral V. Unstable spiral

(c) (3 points) Classify the type of equilibrium point for the differential equation

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix} \vec{x}.$$

I. Stable node II. Unstable node III. Saddle IV. Stable spiral V. Unstable

spiral

Answer:

(a) $1 \pm 3i$ and $-1 \pm 3i$ respectively.

(b) V

(c) IV

10. Given that the characteristic polynomial of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

is $p(\lambda) = (1-\lambda)^3$. Find a fundamental solution set (only one) of $\vec{x}' = A\vec{x}$. **You need to give complete solutions. Otherwise no credit even if you choose the right answer.**

I. $\left\{ e^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, e^t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

II. $\left\{ e^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, e^t \left(t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right), e^t \left(\frac{t^2}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \right\}$

III. $\left\{ e^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, e^t \left(t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right), e^t \left(\frac{t^2}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right) \right\}$

IV. $\left\{ e^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, e^t \left(t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right), e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

V. $\left\{ e^t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, e^t \left(t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right), e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

Answer: II

11. Given $y_1(x) = x^{-1}$ is a solution for the homogenous equation

$$xy'' + (2 + 2x)y' + 2y = 0.$$

Find the general solution to

$$xy'' + (2 + 2x)y' + 2y = 8e^{2x}.$$

(Hint: Use reduction of order. Substitute $y = uy_1$ into the non-homogeneous equation. To get the general solution, don't forget the constants of integration while you are computing integrals.)

Answer:

$$y = x^{-1}(e^{2x} + C_1 + C_2e^{-2x})$$

12. (a) (4 points) Determine all equilibria of the system

$$x' = x - 2y + 5xy$$
$$y' = 2x + y$$

(b) (6 points) For each equilibrium, choose the type from below. **You need to justify your choice.**

I. Stable node II. Unstable node III. Saddle IV. Stable spiral V. Unstable spiral

Answer :

(a) $(0,0)$ and $(\frac{1}{2}, -1)$.

(b) $(0,0)$ V. $(\frac{1}{2}, -1)$ III.

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