Penn
UNIVERSITY of PENNSYLVANIA

FINAL EXAM, MATH 240: CALCULUS III

MAY 8TH, 2018

No book, paper or electronic device may be used, other than a hand-written note sheet at most 8.5″ × 11″ in size. Cell phones should be **in your bags** and **turned off**.

This examination consists of eleven (11) questions; part c of question 10 is extra-credit. Please **show all your work**. Merely displaying some formulas is not sufficient ground for receiving partial credits. Please **box your answers**.

Name (PRINTED): ____________________________________

LECTURE SECTION (CIRCLE ONE):

001 PIMSNER   002 CHAI   003 POWERS

TA: ____________________________________

My signature below certifies that I have complied with the University of Pennsylvania's *code of academic integrity* in completing this examination.

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Your signature

1	2	3	4	5	6
7	8	9	10a,b	10c	11

TOTAL:

1. (10 pts) Let R be the 3×3 matrix

$$\begin{bmatrix} 1/3 & -2/3 & -2/3 \\ 2/3 & 2/3 & -1/3 \\ 2/3 & -1/3 & 2/3 \end{bmatrix}$$

Let

$$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

be a vector in $\mathbb{R}^3$ such that

$$R \cdot \vec{v} = \vec{v} \quad \text{and} \quad a - b + c = 1.$$

Determine the z-coordinate c of $\vec{v}$.

Answer. $c =$ ___________

2. (10 pts) Express the matrix $\begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$ as a linear combination $a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, with $a, b, c \in \mathbb{C}$.

Answer: $a =$__________, $b =$__________, $c =$__________.

3. (10 pts) Let $y(t)$ be a smooth function which satisfies the differential equation

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y(t) = 0$$

and the initial conditions

$$y(0) = 0, \quad \left.\frac{dy}{dt}\right|_{t=0} = 1.$$

Determine the value of $y(1)$.

Answer. $y(1) =$ __________________.

4. (10 pts) Find a particular solution $y(x)$ of the differential equation

$$\left(\frac{d}{dx}-1\right)^2\left(\frac{d}{dx}+1\right)y = \frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 24(x+1)e^x.$$

Answer. $y(x) =$ ____________________________________

5. (10 pts) Which ones of the following four systems of differential equations have the property that all of its solutions remain bounded as $t \to \infty$? Circle **all** systems which have this property.

Answer. A. B. C. D.

A. $\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 2 & 7 \end{bmatrix} \cdot \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

B. $\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

C. $\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

D. $\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

6. (10 pts) Suppose that B is a 4×4 matrix with eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -1$, $\lambda_4 = 2$ and corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

What is B? (Compute B explicitly by giving its 16 entries.)

Answer. $B =$

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7. (10 pts) Consider the differential equation of a function $y(t)$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 4\sin(y(t)) = 0.$$

(a) Write down an autonomous system of first order (non-linear) ODE which is equivalent to the above differential equation. (Hint: if $y(t)$ satisfies the above second order ODE, then the vector-valued function $\vec{x}(t) = \begin{bmatrix} y(t) \\ \frac{dy}{dt} \end{bmatrix}$ satisfies a first order ODE.)

(b) Find all equilibrium points of the autonomous system you gave in part (a) above.

(c) For each of the equilibrium points you found in (b), determine the nature of the equilibrium point using a suitable system of first order linear ODE with constant coefficients. (The possibilities include: stable node, unstable node, saddle point, unstable spiral, stable spiral when the determinant of the approximate linear system is non-zero.)

8. (10 pts) Let $x(t), y(t), z(t)$ be functions in t such that

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

Find $x(t) + y(t) + z(t)$.

Answer. $x(t) + y(t) + z(t) =$ ________________________

9. (10 pts) Suppose that A is an "unknown" 3×3 matrix with entries in $\mathbb{C}$ and x, y, z are complex numbers which satisfies

$$A \cdot \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & x \\ 1 & 2 & y \\ 1 & 1 & z \end{bmatrix}$$

(a) Express the third column of the explicit matrix above as a linear combination of the first two columns, i.e. find two complex numbers a, b such that $a \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Answer. $a =$ ____________________, $b =$ ____________________

(b) Find the triple (x, y, z) using (a) above.

Answer. $(x, y, z) =$ ____

10. (10 pts for parts a, b) The function $y_1(x) = e^x$ satisfies the differential equation

$$x\frac{d^2y_1}{dx^2} - 2(x+1)\frac{dy_1}{dx} + (x+2)y_1(x) = 0.$$

(a) Write down a differential equation for a function $u(x)$ so that if $u(x)$ satisfies this differential equation, then the function $y_p(x)$

$$y_p(x) := u(x)y_1(x) = u(x)e^x$$

satisfies the differential equation

$$x\frac{d^2y_p}{dx^2} - 2(x+1)\frac{dy_p}{dx} + (x+2)y_p(x) = x^3e^{2x}$$

(Note: This differential equation should be a second order ODE for $u(x)$, but a first order ODE for $\frac{du}{dx}$.)

(b) Find a solution $y_2(x) = v(x) \cdot y_1(x) = v(x)e^x$ for a non-constant function $v(x)$ such that

$$x\frac{d^2y_2}{dx^2} - 2(x+1)\frac{dy_2}{dx} + (x+2)y_2(x) = 0$$

(c) (problem 10 continued, 5 extra points) Find a particular solution y_p of the differential equation

$$x\frac{d^2y}{dx^2} - 2(x+1)\frac{dy}{dx} + (x+2)y_(x) = x^3e^{2x}$$

using the equation you found in (a).

11. (10 pts) Let A be the 5×5 matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 6 & 3 \\ 0 & 0 & 0 & -2 & 1 \end{bmatrix}$$

Note that A is composed of a 3×3 matrix and a 2×2 matrix put along the diagonal. Compute A^{40} explicitly. (The expression will involve high powers of some numbers.)

Answer. $A^{40} =$

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