Name (PRINT): ________________________________ PennID: __________

Math 240
Final Exam
Spring 2019

Instructions:

Instructor (circle one): LICHTENFELZ MCGRATH

Name of your TA: ________________________________

Turn off and put away your cell phone. Please sign and date the pledge below to comply with the Code of Academic Integrity. No calculators or any other electronic devices are allowed during this exam. You may NOT consult any material during this exam, as it is self-contained.

Read each question carefully, and answer each question completely.

SHOW all of your work. Write all your solutions clearly and legibly.

If any question is not clear, ask for clarification.

GOOD LUCK!

#	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
Total	120/120	

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination.

________________________________ __________

Signature Date

Problem 1 (10pts): Consider the matrix

$$A(t) = \begin{bmatrix} t & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 2 & t \end{bmatrix}.$$

1. Find all the values of t for which the rank of $A(t)$ is less than 3.

2. For the above values of t, determine when is it possible to solve the following system:

$$A(t)\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

   You do not have to find the solutions.

Problem 2 (10 pts): Define a map $T : \mathbb{R}^3 \to \mathbb{R}$ by

$$T(\mathbf{v}) = \det \begin{pmatrix} 1 & 2 & | \\ 3 & -1 & \mathbf{v} \\ 4 & 0 & | \end{pmatrix}$$

Let A be the matrix $[T]_{\mathcal{B}}^{\mathcal{C}}$, where $\mathcal{B} = \{e_1, e_2, e_3\}$ is the standard basis for $\mathbb{R}^3$, and $\mathcal{C} = \{1\}$.

1. Find a basis for the kernel (i.e. Nullspace) of A.
2. Find $\text{rank}(A)$ and $\text{nullity}(A)$.

Problem 3 (10 pts): For each set S_i below, decide whether or not it is a subspace of the appropriate vector space.

$S_1 = \{$All polynomials $p(t)$ of degree 5 or less such that $p(-1) = 0\}$,
$S_2 = \{(x, y, z) \in \mathbb{R}^3 : 3x - y + z = 0\}$,
$S_3 = \{(x, y, z) \in \mathbb{R}^3 : x + yz = 0\}$,
$S_4 = \{$All 3×3 matrices A such that $(A - 3I)^2 = 0\}$, where I is the identity matrix.

S_1:

S_2:

S_3:

S_4:

Problem 4 (10pts): The displacement of a certain forced oscillator can be modeled by the differential equation

$$y''(t) + 5y'(t) + 6y(t) = \cos t. \tag{1}$$

1. Find all solutions to this differential equation.
2. Describe briefly the long-term behavior of solutions of equation (1) above. If $y(t)$ is a solution of (1), is it true that $\lim_{t\to+\infty} y(t) = 0$?

Problem 5 (10 pts): Suppose that $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2\}$ are two bases of $\mathbb{R}^2$ such that $\mathbf{v}_1 = 3\mathbf{w}_1 - 5\mathbf{w}_2$ and $\mathbf{v}_2 = 2\mathbf{w}_1 - 4\mathbf{w}_2$.

(a) Compute the change of basis matrix from $\mathcal{C}$ to $\mathcal{B}$.

(b) If

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

is the matrix of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ in the basis $\mathcal{B}$, find $[T]_{\mathcal{C}}^{\mathcal{C}}$.

Problem 6 (10 pts): Consider the set $\mathcal{U}_{2\times 2}$ of upper triangular two by two matrices and the basis $\mathcal{B}$ for $\mathcal{U}_{2\times 2}$ given by

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

Let $T : \mathcal{U}_{2\times 2} \to \mathcal{U}_{2\times 2}$ be given by $T(A) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} A - A \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$.

1. Show that T is a linear transformation.
2. Compute the matrix $[T]_{\mathcal{B}}^{\mathcal{B}}$.
3. Compute the kernel of $[T]_{\mathcal{B}}^{\mathcal{B}}$, and use it to find all of the matrices in $\mathcal{U}_{2\times 2}$ that commute with $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$.

Problem 7 (10 pts): This is a multiple choice problem. Only **ONE** of the options below is correct. You **MUST** justify your answer completely or eliminate **ALL** other answers for full points.

Concerning the matrices

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 1 & 1 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix},$$

which of the following is true?

(a) A is defective and B is diagonalizable.
(b) The null spaces of $(A - 3I)$ and $(B - 3I)$ are the same.
(c) The geometric multiplicity of the eigenvalue $\lambda = 3$ is the same in A and B.
(d) The Jordan form of A has fewer Jordan blocks than the Jordan form of B.
(e) $(A - 3I)^2 = 0$ and $(B - 3I)^2 \neq 0$.

Problem 8 (10 pts): Consider the matrix

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Find the general solution of the homogeneous system of ordinary differential equations $\mathbf{x}'(t) = A\mathbf{x}(t)$.

(b) Compute the matrix exponential e^{tA}.

Problem 9 (10 pts): Suppose λ is an eigenvalue of an $n \times n$ matrix A, and $\mathbf{v} \in \mathbb{R}^n$ is a vector such that

$$(A - \lambda I)^3 \mathbf{v} = 0 \quad \text{but} \quad (A - \lambda I)^2 \mathbf{v} \neq 0.$$

Show directly from the definition that the set $\{\mathbf{v}, (A - \lambda I)\mathbf{v}, (A - \lambda I)^2\mathbf{v}\}$ is linearly independent.

Problem 10 (10 pts): Find and classify all equilibrium points of the nonlinear system

$$x'(t) = x^2 - xy,$$
$$y'(t) = 2x - y + 1.$$

Problem 11 (10 pts): Find the general solution to

$$ty''(t) - 2y'(t) + (2 - t)y(t) = 0, \qquad t > 0,$$

given that one solution is $y_1(t) = e^t$.

Problem 12 (10 pts): For each statement below, decide whether it is true or false and **CIRCLE YOUR OPTION CLEARLY**, as indicated here:

(True)/False or True/(False)

You do not need to show your work in this problem.

1. (True/False) If the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then so is $\{2\mathbf{v}_1, 3\mathbf{v}_2, 4\mathbf{v}_3\}$.

2. (True/False) The solution set to the vector differential equation $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{b}$, where A is an $n \times n$ matrix and $\mathbf{b}$ is any vector in $\mathbb{R}^n$, is always an n-dimensional vector space.

3. (True/False) If a matrix A is invertible, then $-A^T$ must also invertible.

4. (True/False) If $T : \mathbb{R}^4 \to \mathbb{R}^5$ is a linear transformation given by $T\mathbf{v} = A\mathbf{v}$, where A is a 5×4 matrix with $\text{Rank}(A) = 2$, then $\text{Nullity}(A) = 3$.

5. (True/False) If A, B are $n \times n$ matrices with $AB = 0$, then $A = 0$ or $B = 0$.

6. (True/False) A spring mass system is undergoing simple harmonic motion. If the mass is increased while no other aspects of the system are changed, then the motion remains simple harmonic and the period decreases.

7. (True/False) If $\mathbf{v}$ is an eigenvector of A, then $\mathbf{v}$ is an eigenvector of A^T as well.

8. (True/False) If λ is an eigenvalue of A, then λ is an eigenvalue of A^T as well.

9. (True/False) If A is 2×2 matrix such that $\text{trace}(A) = 1$ and $\det(A) = -6$, then A must be diagonalizable.

10. (True/False) If A is a 3×3 matrix with eigenvalues $2 - i, 2 + i$, and 5, then A is invertible.

Scratch paper

Scratch paper