Signature | Printed Name

Preliminary Examination, Part I

Friday, May 1st, 2020 | 9:30-12:30

This examination is based on Penn's code of academic integrity

This part of the examination consists of six problems. You should work all of the problems. Show all of your work. Try to keep computations well-organized and proofs clear and complete — and justify your assertions.

If a problem has multiple parts, earlier parts may be useful for later parts. Moreover, if you skip some part, you may still use the result in a later part.

All problems have equal weight of 10 points.

Write your answers on letter-sized paper, either one or two pages for each problem; you can also write your answers on a printout. Please scan your answers, convert the scans to a single pdf file. Alternatively, you can use a tablet and write your answers in a digital format, then generate a single pdf file.

Please upload you answer, in pdf format, to the Canvas site for this exam, by 12:50 pm at the latest. Please also send a copy of the scan, by secure share, to Ms. Reshma Tanna. (So we will still receive your answers even if something goes wrong in the uploading process.)

Score	
1	
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5	
6	
Grader	

1. Let $\left(a_n\right)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers such that $\sum_{n=1}^{\infty} a_n$ converges.

(a) Prove that

$$\sum_{n=1}^{\infty} a_n x^n$$

converges uniformly on the closed interval $[-1, 1]$.

(b) Given an example to show that this series need not converge uniformly on $[-2, 2]$.

2. For each of the following, either give an example or explain why no such example exists.

(a) An abelian (i.e. commutative) group with 30 elements which is not cyclic.

(b) A non-commutative group with $217 = 31 \times 7$ elements.

3. Let $f(x)$ be an infinitely differentiable real-valued function on the real line such that $-x^2 \leq f(x) \leq x^2$ for all non-zero real numbers x.

(a) Show that $f(0) = 0$.

(b) Show directly from the definition of derivative that $f'(0) = 0$.

4. Let V, W be finite dimensional vector spaces over $\mathbb{R}$ and consider their dual spaces $V^* := \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ and $W^* := \mathrm{Hom}_{\mathbb{R}}(W, \mathbb{R})$. For any linear transformation $T : V \to W$, and for any $f \in W^*$, let $T^*(f) := f \circ T$.

(a) Prove that for T and f as above, $T^*(f)$ is an element of V^*.

(b) Prove that T^* defines a linear transformation from W^* to V^*.

(c) Prove that if T is injective then T^* is surjective.

5. Let $a_0, a_1, a_2, \ldots$ be a sequence of positive real numbers such that $a_i > a_{i+1}$ for all i. For all $n \geq 0$, let $s_n = \sum_{i=0}^{n} (-1)^i a_i$.

(a) Prove that the sequence $s_0, s_2, s_4, \ldots$ converges.

(b) Prove that the sequence $s_1, s_3, s_5, \ldots$ converges.

(c) Determine whether the sequence $s_0, s_1, s_2, s_3, \ldots$ must converge. Give either a proof or a counter-example.

6. Give $\mathbb{Q}$ the topology defined by the standard metric on $\mathbb{R}$.

(a) Does there exist a non-empty subset $Z \subsetneqq \mathbb{Q}$ which is both open and closed in $\mathbb{Q}$? Either give such an example, or show that no such subset exists.

(b) Let S be a connected subset of $\mathbb{Q}$ which contains 0. Prove that $S = \{0\}$, i.e. S is a singleton.

PRELIMINARY EXAMINATION, PART II

Friday, May 1st, 2020 2:00-5:00

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Please upload you answer, in pdf format, to the Canvas site for this exam, by 5:20 pm at the latest. Please also send a copy of the scan, by secure share, to Ms. Reshma Tanna. (So we will still receive your answers even if something goes wrong in the uploading process.)

Score	
7	
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GRADER	

7. Let C be the oriented closed curve in $\mathbb{R}^2$ given by the parametrization

$$t \mapsto (3\cos t, 4\sin t), \quad t \in [0, 2\pi].$$

Compute the line integral

$$\int_C \frac{y\,dx - x\,dy}{x^2 + y^2}.$$

(Hint: you can use withou proof the fact that $\operatorname{curl}\left(\frac{y}{x^2+y^2}\vec{\imath} - \frac{x}{x^2+y^2}\vec{\jmath}\right) = 0$.)

8. Let J be the matrix $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ in $\mathrm{M}_4(\mathbb{R})$.

(a) Does there exist a matrix $A \in \mathrm{M}_4(\mathbb{R})$ such that $A^2 = J$? Either give an example, or prove that such a matrix A does not exist.

(b) Does there exist a *symmetric* matrix $B \in \mathrm{M}_4(\mathbb{R})$ such that $B^2 = J$? Either give an example, or prove that such a matrix B does not exist.

9. Let f be a continuous real valued function on $\mathbb{R}^2$. Let D be the set of all points on $\mathbb{R}^2$ having distance at most 1 from the origin, and let $f(D) \subseteq \mathbb{R}$ be the set consisting of all values of f taken on at points of D. Prove that there exist real numbers a, b with $a \leq b$ such that $f(D)$ is equal to the closed interval $[a, b] = \{x \in \mathbb{R} \,|\, a \leq x \leq b\}$.

10. Let $\vec{v}$ be the column vector $(1, 2, 2)^t$ in $\mathbb{R}^3$. Find an *orthogonal* matrix $A \in \mathrm{M}_3(\mathbb{R})$ such that $A \cdot \vec{v} = \vec{v}$, $A^4 = \mathrm{I}_3$ and $A^2 \neq \mathrm{I}_3$, where I_3 is the identity matrix in $\mathrm{M}_3(\mathbb{R})$.

(Recall that a 3×3 matrix B is orthogonal if $B \cdot B^t = B^t \cdot B = \mathrm{I}_3$. If your answer is a product of matrices, you do not have to carry out the multiplication explicitly.)

11. Let f be a $\mathbb{R}$-valued infinitely differentiable function on $\mathbb{R}$ such that $f''(x) \leq 0$ for all $x \in [0, 1]$, and $f(0) = f(1) = 0$. Show that $f(x) \geq 0$ for all $x \in [0, 1]$. (Hint: Suppose that $f(a) < 0$ for some $a \in [0, 1]$, and apply the mean value theorem to get a contradiction.)

12. Consider the polynomial $f(x) = x^6 + x^3 + 1$ in $\mathbb{Q}[x]$.

(a) Is $f(x)$ irreducible in $\mathbb{R}[x]$?

(b) Is $f(x)$ irreducible in $\mathbb{Q}[x]$? (Hint: Consider $f(x+1)$.)