1. Let $\left(a_n\right)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers such that $\sum_{n=1}^{\infty} a_n$ converges.

(a) Prove that

$$\sum_{n=1}^{\infty} a_n x^n$$

converges uniformly on the closed interval $[-1, 1]$.

(b) Given an example to show that this series need not converge uniformly on $[-2, 2]$.

Solution. (a) Given an $\epsilon > 0$, pick n_0 such that $\sum_{n\geq n_0} a_n < \epsilon$. Then for any $x \in [-1, 1]$, we have $\sum_{n\geq n_0} |a_n x^n| < \epsilon$.

(b) Let $a_n = (2/3)^n$ for all $n \in \mathbb{N}$, then the series $\sum_{n=1}^{\infty} a_n x^n$ diverges for $|x| \geq 3/2$.

2. For each of the following, either give an example or explain why no such example exists.

(a) An abelian (i.e. commutative) group with 30 elements which is not cyclic.

(b) A non-commutative group with $217 = 31 \times 7$ elements.

Solution. (a) According to the structure of finite abelian groups, every commutative group with 30 elements is isomorphic to $(\mathbb{Z}/2\mathbb{Z})\times(\mathbb{Z}/3\mathbb{Z})\times(\mathbb{Z}/5\mathbb{Z}) \cong \mathbb{Z}/30\mathbb{Z}$. The last isomorphism is a special case of the Chinese remainder theorem.

(b) No. By Sylow's theorem, every group G with 127 elements has a normal subgroup N with 31 elements and a subgroup H with 7 elements. The number of 7-Sylow subgroups divides 31 and is congruenet to 1 modulo 7, so H is also a normal subgroup. Therefore G is isomorphic to a $(\mathbb{Z}/31\mathbb{Z}) \times (\mathbb{Z}/7\mathbb{Z})$.

3. Let $f(x)$ be an infinitely differentiable real-valued function on the real line such that $-x^2 \leq f(x) \leq x^2$ for all non-zero real numbers x.

(a) Show that $f(0) = 0$.

(b) Show directly from the definition of derivative that $f'(0) = 0$.

Solution. (a) We have $f(0) = \lim_{x\to 0} f(x) = 0$ because f is smooth, and for every $\epsilon > 0$, $|f(x)| \leq \epsilon/2 < \epsilon$ for all x with $|x| \leq \min(1, \epsilon/2)$.

(b) For every $\epsilon > 0$ and every non-zero real number x with $|x| < \epsilon$, we have $\left|\frac{f(x)-f(0)}{x}\right| \leq \frac{x^2}{|x|} < \epsilon$.

4. Let V, W be finite dimensional vector spaces over $\mathbb{R}$ and consider their dual spaces $V^* := \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ and $W^* := \mathrm{Hom}_{\mathbb{R}}(W, \mathbb{R})$. For any linear transformation $T : V \to W$, and for any $f \in W^*$, let $T^*(f) := f \circ T$.

(a) Prove that for T and f as above, $T^*(f)$ is an element of V^*.

(b) Prove that T^* defines a linear transformation from W^* to V^*.

(c) Prove that if T is injective then T^* is surjective.

Solution. (a) For any $a, b \in \mathbb{R}$ and any $v, v' \in V$, we have

$$T^*(f)(av + bv') = f(T(av + bv')) = f(aT(v) + bT(v')) = aT^*(f)(v) + bT^*f(v').$$

(b) For any $a, b \in \mathbb{R}$ and any $\lambda, \mu \in W^*$, we have $T^*(a\lambda + b\mu) = aT^*(\lambda) + bT^*(\mu)$ because when evaluated at any $v \in V$ we get the same output $a\lambda(T(v)) + b\mu(T(v))$.

(c) Let U be a vector subspace of W such that $W = T(V) \oplus W$. Given $\lambda \in V^*$, define $\mu \in W^*$ by $\mu(T(v) + u) := \mu(v)$ for all $v \in V$ and all $u \in U$. Then $T^*(\mu) = \lambda$.

5. Let $a_0, a_1, a_2, \ldots$ be a sequence of positive real numbers such that $a_i > a_{i+1}$ for all i. For all $n \geq 0$, let $s_n = \sum_{i=0}^{n} (-1)^i a_i$.

(a) Prove that the sequence $s_0, s_2, s_4, \ldots$ converges.

(b) Prove that the sequence $s_1, s_3, s_5, \ldots$ converges.

(c) Determine whether the sequence $s_0, s_1, s_2, s_3, \ldots$ must converge. Give either a proof or a counter-example.

Solution. Note that $s_0, s_2, s_4, \ldots$ is a strictly decreasing sequence, while $s_1, s_3, s_5, \ldots$ is a strictly increasing sequence, and $s_{2i+1} < s_{2j}$ for all $i, j \in \mathbb{N}$. Assertions (a), (b) follow. For any strictly decreasing sequence $(a_i)_{i \in \mathbb{N}}$ of positive real numbers such that $\lim_{i \to \infty} a_i > 0$, we get a counter-example for (c), e.g. $a_i = 1 + \frac{1}{i+1}$.

6. Give $\mathbb{Q}$ the topology defined by the standard metric on $\mathbb{R}$.

(a) Does there exist a non-empty subset $Z \subsetneq \mathbb{Q}$ which is both open and closed in $\mathbb{Q}$? Either give such an example, or show that no such subset exists.

(b) Let S be a connected subset of $\mathbb{Q}$ which contains 0. Prove that $S = \{0\}$, i.e. S is a singleton.

Solution. (a) $Z = \mathbb{Q} \cap (\sqrt{2}, \infty)$ is such a subset: it is open, and its complement is $\mathbb{Q} \cap (-\infty, \sqrt{2}] = \mathbb{Q} \cap (-\infty, \sqrt{2})$ is also open.

(b) Suppose that S contains a non-zero rational a. Let c be an irrational number between 0 and a. Then S is the disjoint union of two non-empty open subsets $S \cap (c, \infty)$ and $S \cap (-\infty, c)$, a contradiction.

7. Let C be the oriented closed curve in $\mathbb{R}^2$ given by the parametrization

$$t \mapsto (3\cos t, 4\sin t), \quad t \in [0, 2\pi].$$

Compute the line integral

$$\int_C \frac{y\,dx - x\,dy}{x^2 + y^2}.$$

(Hint: you can use without proof the fact that $\operatorname{curl}\left(\frac{y}{x^2+y^2}\vec{\imath} - \frac{x}{x^2+y^2}\vec{\jmath}\right) = 0$.)

Solution. Let C' be the circle $\{(a, b) \in \mathbb{R}^2 \mid a^2 + b^2 = 1\}$ on the (x, y)-plane, oriented counter-clockwise. By Stokes/Green theorem,

$$\int_C \frac{y\,dx - x\,dy}{x^2 + y^2} = \int_{C'} \frac{y\,dx - x\,dy}{x^2 + y^2} = \int_{C'} y\,dx - x\,dy = -\int_0^{2\pi} d\,\theta = -2\pi.$$

8. Let J be the matrix $\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ in $\mathrm{M}_4(\mathbb{R})$.

(a) Does there exist a matrix $A \in \mathrm{M}_4(\mathbb{R})$ such that $A^2 = J$? Either give an example, or prove that such a matrix A does not exist.

(b) Does there exist a *symmetric* matrix $B \in \mathrm{M}_4(\mathbb{R})$ such that $B^2 = J$? Either give an example, or prove that such a matrix B does not exist.

Solution. (a) $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ satisfies $A^2 = J$.

(b) No such B exists: if $B \in \mathrm{M}_4(\mathbb{R})$ is symmetric and $B \cdot B^t = B^2 = J$, then $-4 = \operatorname{tr}(J) = \operatorname{tr}(B \cdot B^t) \geq 0$. Alternatively, by the spectral theorem B is diagonalizable with real eigenvalues.

9. Let f be a continuous real valued function on $\mathbb{R}^2$. Let D be the set of all points on $\mathbb{R}^2$ having distance at most 1 from the origin, and let $f(D) \subseteq \mathbb{R}$ be the set consisting of all values of f taken on at points of D. Prove that there exist real numbers a, b with $a \leq b$ such that $f(D)$ is equal to the closed interval $[a,b] = \{x \in \mathbb{R} \,|\, a \leq x \leq b\}$.

Solution. Since f is continuous and D is compact, there exists points $x_1, x_2 \in D$ such that $f(x_1) = \min\{f(x)|x \in D\} =: a$ and $f(x_1) = \min\{f(x)|x \in D\} =: b$. On the other hand D is connected, because it is the union of line segments in D passing through the origin, so $f(D)$ is also connected. Therefore $f(D) = [a,b]$.

10. Let $\vec{v}$ be the column vector $(1,2,2)^t$ in $\mathbb{R}^3$. Find an *orthogonal* matrix $A \in \mathrm{M}_3(\mathbb{R})$ such that $A \cdot \vec{v} = \vec{v}$, $A^4 = \mathrm{I}_3$ and $A^2 \neq \mathrm{I}_3$, where I_3 is the identity matrix in $\mathrm{M}_3(\mathbb{R})$.

(Recall that a 3×3 matrix B is orthogonal if $B \cdot B^t = B^t \cdot B = \mathrm{I}_3$. If your answer is a product of matrices, you do not have to carry out the multiplication explicitly.)

Solution. Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be an orthonormal basis of $\mathbb{R}^3$ with $\vec{v}_1 = \frac{1}{3}\vec{v}$. The linear operator U on $\mathbb{R}^3$ whose matrix representation with respect to the above orthonormal basis is

$$D := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

has the required property. Note that there is another orthogonal opertor S on $\mathbb{R}^3$ with $S(\vec{v}) = \vec{v}$, $S^4 = \mathrm{Id}_{\mathbb{R}^3}$ and $S^2 \neq \mathrm{Id}_{\mathbb{R}^3}$, namely $S = U^{-1} = U^3$. Clearly U and S are the two rotations by angles $\pm\pi/2$ about the line $\mathbb{R} \cdot \vec{v}$, and they are the only two orthogonal operators on $\mathbb{R}^3$ satisfying the required properties. Note also that there are infinitely many orthogonal operators T on $\mathbb{R}^3$ such that $T(\vec{v}) = \vec{v}$ and $T^2 = \mathrm{Id}_{\mathbb{R}^3}$, namely all reflections on $\mathbb{R}^3$ about a plane which contains $\mathbb{R} \cdot \vec{v}$.

To be more explicit, let $C = \begin{pmatrix} 1 & -2 & -2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{pmatrix}$, a 3×3 matrix such that $C \cdot C^t = C^t \cdot C = 9\,\mathrm{I}_3$, whose first column is $\vec{v}$. Let $A = C \cdot D \cdot C^{-1} = \frac{1}{9} C \cdot D \cdot C^t$, then $A^4 = \mathrm{I}_3$ and $A^2 \neq \mathrm{I}_3$.

11. Let f be a $\mathbb{R}$-valued infinitely differentiable function on $\mathbb{R}$ such that $f''(x) \leq 0$ for all $x \in [0,1]$, and $f(0) = f(1) = 0$. Show that $f(x) \geq 0$ for all $x \in [0,1]$. (Hint: Suppose that $f(a) < 0$ for some $a \in [0,1]$, and apply the mean value theorem to get a contradiction.)

Solution. Suppose that $f(a) < 0$ for some $a \in [0,1]$. By the mean value theorem there exist $b \in [0,a]$ with $f'(b) < 0$ and $c \in [a,1]$ with $f(c) > 0$. Applying the mean value theorem to f', one sees that there exists $d \in [b,c]$ such that $f''(d) > 0$, a contradiction.

12. Consider the polynomial $f(x) = x^6 + x^3 + 1$ in $\mathbb{Q}[x]$.

(a) Is $f(x)$ irreducible in $\mathbb{R}[x]$?

(b) Is $f(x)$ irreducible in $\mathbb{Q}[x]$? (Hint: Consider $f(x+1)$.)

Solution. (a) $f(x)$ is reducible, for every irreducible polynomial in $\mathbb{R}[x]$ has degree 1 or 2.

(b) $f(x+1) \equiv (x^3+1)^2 + (x^3+1) + 1 \equiv x^6 \pmod 3$, and the constant term of $f(x+1)$ is 3. So $f(x+1)$ is irreducible by Eisenstein's criterion.

Note that $f(x)$ is the ninth cyclotomic polynomial:

$$x^9 - 1 = (x^3 - 1)(x^6 + x^3 + 1) = (x-1)(x^2 + x + 1)(x^6 + x^3 + 1).$$

The roots of $f(x)$ are the 6 primitive ninth roots of unity. Note also that $(x+1)^9 - 1 \equiv x^9 \pmod 3$, and $(x+1)^3 - 1 \equiv x^3 \pmod 3$, so we get again that $f(x+1) \equiv x^6 \pmod 3$.