Math 116 Final Exam Wed., Dec. 17, 2008

Instructions: This exam consists of *ten problems.* Do all ten, showing your work and explaining your assertions. Each problem is worth 20 points, for a total of 200 points. Partial credit will be given as appropriate.

1. a) In terms of ε, δ, write down explicitly the meaning of the assertion that the function $f(x) = 4x + 1$ is continuous at $x = 1$.

b) Explicitly prove this assertion by finding an appropriate δ for each ε.

2. Let $f(x) = x^2|x|$ for $x \in \mathbb{R}$.

a) Find the set of $x \in \mathbb{R}$ at which f is continuous. Also find the set of x at which f is differentiable, and for each such x find $f'(x)$.

b) Determine whether $\int_{-1}^{1} f(x)\,dx$ exists. If it does, evaluate it.

3. In $\mathbb{R}^3$, with the usual dot product and cross product:

a) Find all vectors $v \in \mathbb{R}^3$ such that $v \cdot v = 0$.

b) Find all vectors $v \in \mathbb{R}^3$ such that $v \times v = 0$.

Explain.

4. In $\mathbb{R}^2$, let v, w, z be non-zero vectors.

a) If $v \perp w$ and $w \perp z$, show that v is *not* orthogonal to z.

b) What happens if instead we consider the analogous problem for three non-zero vectors v, w, z in $\mathbb{R}^3$?

5. a) Show that there is a unique hyperbola in $\mathbb{R}^2$ that has foci at $(2, 0)$ and at $(-2, 0)$ and that passes through the point $(1, 0)$.

b) Find all the lines through the origin that do *not* intersect this hyperbola.

6. Let $F : \mathbb{R} \to \mathbb{R}^3$ be a differentiable function, parametrizing a curve C, with positive speed.

a) Suppose that the velocity and acceleration vectors at some point $P \in C$ are linearly dependent. What is the curvature of C at P?

b) Suppose that the magnitude of velocity vector at each point is equal to 2. Find the arclength of the segment of the curve parametrized by the interval $[0, 3]$. Justify your assertion.

7. a) Describe the graph of the function $f(x, y) = x^2 + 3y^2$, and find all relative maxima and relative minima.

b) Find the volume above the square $0 \le x \le 1$, $0 \le y \le 1$ in the x, y-plane and below the graph of $z = x^2 + 3y^2$.

8. a) Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) + 2xf(x) = e^{x-x^2}$.

b) Determine whether the set of such functions forms a vector space under addition and scalar multiplication of functions.

(over)

9. a) Find all solutions to the differential equation

$$y'' + y' - 6y = 1.$$

b) Determine whether any solutions to this differential equation define bounded functions $\mathbb{R} \to \mathbb{R}$. If there are any, find them.

10. Consider the vector space $C(0,1)$ of continuous real-valued functions on $[0,1]$, with the inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$.

a) For each non-negative integer n, let $f_n(x) = x^n$. For each m, n evaluate $\langle f_m, f_n \rangle$ and $\|f_n\|$.

b) Find an orthonormal basis for the subspace of $C(0,1)$ spanned by f_0, f_1.