Math 116 Final Exam Tues., Dec. 22, 2009

Instructions: This exam consists of *ten problems*. Do all ten, showing your work and explaining your assertions. Each problem is worth 20 points, for a total of 200 points. Partial credit will be given as appropriate.

1. a) Determine if $\lim_{x \to 1} x^3 + x^2 + 1$ exists; and if so, determine its value.

b) In terms of ε, δ (or neighborhoods), write the explicit meaning of what you asserted in part (a). (You do not have to verify your assertion in this form.)

2. Let $f(x) = x$ for $0 \le x \le 1$, and $f(x) = 4x - 3$ for $1 < x \le 2$.

a) Show that f is integrable on the closed interval $[0, 2]$, and evaluate $\int_0^2 f(x)\,dx$.

b) For all x in the open interval $(0, 2)$, determine whether f is continuous at x and whether it is differentiable at x.

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $f(0) = -1$, such that $f(x) > 0$ whenever $|x| > 1$.

a) Show that $f(x) = 0$ for at least two distinct values of $x \in \mathbb{R}$.

b) Show that f achieves a minimum at some real number x_0, and also that $f(x_0) < 0$. [Hint: Does f achieve a minimum on $[-1, 1]$? Is a minimum for f on $[-1, 1]$ also a minimum for f on all of $\mathbb{R}$?]

4. a) Find all non-zero vectors $z \in \mathbb{R}^3$ that are orthogonal to the two vectors $v = (1, 2, 3)$ and $w = (2, -1, 0)$, with respect to the dot product on $\mathbb{R}^3$.

b) Show that if v, w, z are as in part (a), then these three vectors form an orthogonal basis of $\mathbb{R}^3$.

5. a) Using the definition of an ellipse as a locus of points, show that if E is an ellipse with foci F_1, F_2 then there is a circle centered at F_1 such that E is contained in the interior of the circle.

b) Using the definition of a parabola as a locus of points, show that if P is a parabola with directrix D and focus $F \notin D$, then P does not intersect D.

6. A particle moves counterclockwise around the unit circle $x^2 + y^2 = 1$ at positive (but variable) speed.

a) Show that the position and velocity vectors are always linearly independent.

b) Show that the position and principal normal vectors are never linearly independent.

7. a) Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) + (\cos x)f(x) = \cos x$ for all x. Does the set of all such functions form a vector space?

b) Find all such functions f satisfying $f(0) = 0$.

8. Let V be the set of solutions to the differential equation $y'' + y = 0$.

a) Show that V is a vector space, and find a basis for V.

b) For $f \in V$, define $D(f) = f'$. Show that D is a linear transformation $V \to V$, and find the matrix of D with respect to the basis you gave in part (a).

(continued)

9. Which of the following are linear transformations? For each one that is, determine whether it is one-to-one and whether it is onto.

a) $T : \mathbb{R}^2 \to \mathbb{R}^3$, $T(a, b) = (a, b, a + b)$.

b) $S : \mathbb{R}^3 \to \mathbb{R}^2$, $S(a, b, c) = (a, b + c)$.

10. Let W be the span of the vector $(1, 1, 1)$ in $\mathbb{R}^3$. For each $v \in \mathbb{R}^3$, let $T(v) \in \mathbb{R}^3$ be the orthogonal projection of v onto W.

a) For $v = (a, b, c) \in \mathbb{R}^3$, express $T(v)$ in terms of a, b, c.

b) Show that T is a linear transformation, and find its rank and nullity.